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Preliminary Report on Dr. Rainbow's Treatment of Alternating Currents

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1. As Prof. v.d. Waerden pointed out in a previous memorandum (May '49), Dr. Rainbow's method of prospecting by means of alternating currents leads to the equations

$$(1) \qquad f(i\omega) = \int_0^\infty e^{-i\omega t}\, dF(t), \qquad f(2i\lambda^2) = \int_0^\infty e^{-(1+i)\lambda t}\, dH(t),$$

where $\omega/2\pi = \nu$ denotes the frequency in Hz. Assuming that the (complex) values of f($i\omega$) are given for a set of frequencies ν_i (i = 1...j), we shall put

$$(2) \qquad N_0 = \min_{(i)} \nu_i \quad , \quad N_1 = \max_{(i)} \nu_i \; , \qquad (N_0 > 0).$$

It is required to determine the function H, which will be supposed to be of bounded variation in every interval $0 \leq t \leq T < \infty$. Hence the first of the equations (1) does not manifest itself in the problem; yet it has a definite meaning, viz. that $f = f(i\omega)$ is a Laplace transform; hence f cannot be represented by every function.

Since this preliminary report gives only a very rough survey of some aspects of the problem, the following assertions may be stated without proof:

(a) f is analytic for any real $\omega \neq 0$;
(b) f has a first-order pole for $\omega = 0$ with positive residue;
(c) f admits an asymptotic expansion for $\omega \to \infty$.

When formulating (b) and (c), certain constants appear that can be approximately determined as soon as f is known within a range of frequencies, which is not <u>too</u> small. These constants enable us to calculate f, with some accuracy for values of ν outside the interval $N_0 \leq \nu \leq N_1$. This shows the importance of N_0 and N_1 : the larger the interval $(N_0\ N_1)$, the better our extrapolation will be.

Taking x and α as arbitrary parameters, which satisfy

$$(3) \qquad x > 0, \quad \alpha > 1,$$

integration of the second equation (1) yields

$$(3') \qquad \int_0^\infty \lambda^\alpha e^{-(1+i)\lambda x} f(2i\lambda^2)\, d\lambda = \int_0^\infty\int_0^\infty \lambda^\alpha e^{-(1+i)(x+t)\lambda}\, d\lambda\, dH(t).$$

Putting $(1+i)(x+t)\lambda = z$, we obtain, since $x+t > 0$:

$$\int_0^\infty \lambda^\alpha e^{-(1+i)(x+t)\lambda}\, d\lambda = \{(1+i)(x+t)\}^{-\alpha-1} \int_0^\infty z^\alpha e^{-z}\, dz = \frac{\Gamma(\alpha+1)}{\{(1+i)(x+t)\}^{\alpha+1}},$$

hence, by interchanging the order of integration in (3'),

$$\int_0^\infty \lambda^\alpha e^{-(1+i)\lambda x} f(2i\lambda^2)\, d\lambda = \frac{\Gamma(\alpha+1)}{(1+i)^{\alpha+1}} \int_0^\infty \frac{dH(t)}{(x+t)^{\alpha+1}}.$$

Introducing a new function $\gamma(\alpha)$ we can write

$$(3'') \qquad \gamma(\alpha) = \int_0^\infty \frac{dH(t)}{(x+t)^{\alpha+1}} = \frac{(1+i)^{\alpha+1}}{\Gamma(\alpha+1)} \int_0^\infty \lambda^\alpha e^{-(1+i)\lambda x} f(2i\lambda^2)\, d\lambda \qquad (x > 0, \alpha > 1).$$

The condition $\alpha > 1$ is made in order to make the integral in the right member convergent. For $f(i\omega)$ has a first-order pole when $\omega = 0$, with a certain residue c; hence $f(i\omega) \sim c(i\omega)^{-1}$ if $\omega \to 0$, and $f(2i\lambda^2) \sim c(2i\lambda^2)^{-1}$ if $\lambda \to 0$, which explains the condition $\alpha > 1$.

Since the left hand member in (3") is real, the right hand member must be real too: this gives us a check of the empirical values of f. For the rest $\gamma(\alpha)$ can be considered as given for any value of α.

Putting

$$t = x\tau, \qquad dH(t) = (1+\tau)^3 \, dH^*(\tau), \tag{4}$$

we get

$$x^{\alpha+1}\gamma(\alpha) = \int_0^\infty \frac{dH^*(\tau)}{(1+\tau)^{\alpha-2}}. \tag{4'}$$

It will be assumed that H^* is of bounded variation in $\langle 0, \infty \rangle$, though, properly speaking, it would be necessary to give a proof of this plausible hypothesis.

Putting again

$$u = \frac{1}{1+\tau}, \quad \varphi(u) = H^*(\tau) - H^*(\infty), \quad c_k = -x^{k+3}\gamma(k+2) \qquad (k > -1), \tag{5}$$

we have

$$\varphi(0) = 0, \quad d\varphi(u) = -dH^*(\tau), \tag{5'}$$

if we agree that the Stieltjes differential will always correspond to <u>increasing</u> values of the argument. We thus obtain

$$\int_0^1 u^k \, d\varphi(u) = c_k \qquad (k > -1). \tag{5"}$$

If H^* is of bounded variation in $\langle 0, \infty \rangle$, φ will be so in $\langle 0, 1 \rangle$ by (5'). We cannot expect much of the Mellin inversion formula here, since the analytic continuation of the empirical function c_k of k is unknown; but, putting $k = 0, 1, 2, \ldots$, (5") becomes a moment problem of the Hausdorff type (cf. Hausdorff, Math. Zft. 16, 1923). As to the solution given by Hausdorff, different remarks could be made in connection with the practical purposes of the problem of Dr. Rainbow; this subject will be treated in a subsequent report.

If the quantities c_k can be determined with sufficient accuracy, it will be more efficient to take the k's as multiples of a smaller number than unity, e.g. $k = -\frac{1}{2}, 0, \frac{1}{2}, 1, \ldots$; the new problem will also be a Hausdorff moment problem.

2. In order to compute the moments c_k, we have to know $f(i\omega) = f(2i\lambda^2)$ for all positive values of λ. Now the formula

$$c_k = -\frac{(1+i)^{k+3}}{\Gamma(k+3)} \, x^{k+3} \int_0^\infty \lambda^{k+2} e^{-(1+i)\lambda x} f(2i\lambda^2) \, d\lambda \tag{6}$$

shows that only the values of f corresponding to a certain finite interval $0 < N_0 \leq \nu \leq N_1$ are significant. Putting

$$(6') \quad \int_0^\infty \lambda^{k+2} e^{-(1+i)\lambda x} f(2i\lambda^2)\, d\lambda = \int_0^{\Lambda_0} + \int_{\Lambda_0}^{\Lambda_1} + \int_{\Lambda_1}^{\infty} = J_0 + J + J_1 \,, \quad \Lambda_j = \sqrt{\pi N_j}\,,$$

the integrals J_0 and J_1 can be treated as mere corrections, and this is just what is made possible by our extrapolation of f. The ratio $|J_1| / |J|$ will tend to zero if $x \to \infty$; hence, there must be a smallest value ξ of x, for which, compared to the errors of observation, this ratio becomes negligible for all values of k, for which the moments c_k have been determined. If we wish to check our results, the calculation can be performed with different values of x, lying in the neighbourhood of ξ. As to the error caused by the integral J_0, it is difficult to get rid of it by an appropriate choice of x; but, if necessary, we can confine our analysis to moments c_k of sufficiently high order $k \geq k_1$, where k_1 is some number > 0. In this way the relative error caused by J_0 can be made arbitrarily small.

At any rate, the frequency range $(N_0\ N_1)$ turns out to be very important for our problem. One might be inclined to ask, whether this importance is not overestimated here. I think not. Of course other analytical methods, which only use the values of f in some finite interval $(\Lambda_0 \Lambda_1)$ might also be able to solve the problem of Dr. Rainbow. Yet, we must remember that this problem has many unknowns, and for topological reasons it seems impossible to solve it if only one frequency is used.

For, if frequencies are so low that displacement currents may be neglected, the electrical properties of the earth can be characterized by a single parameter $\mu\sigma$ in any point. Hence we have a threedimensional manifold of unknowns. On the other hand, if we use currents of a single frequency ν, we can measure but one amplitude in every point of the surface of the earth, which gives us ~~also~~ ∞^2 data, and the same can be done with phase-differences: this procures us also ∞^2 data, and the combination of these two twodimensional manifolds is insufficient to determine ∞^3 unknowns Only by varying the frequency we are able to get the ∞^3 data we require. In theoretical respect it would be sufficient if the range of frequencies were different from zero, however small it may be. But it will be clear that in practice we need so large a range that the measurements for different frequencies are noticeably influenced in different ways by the heterogeneity of the earth.

3. The question of the frequency-range that can be obtained in practice requires careful examination. It is hard to say for a theorist, how large the interval $(N_0\ N_1)$ must be, in order to obtain practical results. But, in view of (6'), a ratio $\Lambda_1/\Lambda_0 = 10$ seems to be the very least that can be admitted. Hence, by (6'), we get

$$(7) \qquad N_1 / N_0 \geq 100\,.$$

On the other hand, an interval that satisfies $N_1 / N_0 = 100$ seems to be the very best result that can be obtained empirically. This means that our measurements will require the utmost care: if the idea of Dr. Rainbow has any practical value (which possibility I do not exclude), his method will certainly not be an easy one in practice, since it will ask for much refinement. I will try to explain this opinion.

The range of useful frequencies is limited by 3 circumstances, viz.:

(a) by the skin effect and the damping of potential waves in the interior of the earth; these are related phenomena, which result in an upper bound of N_1;

(b) by technical difficulties in generating currents of very low frequencies; they give a lower bound of N_0;

(c) by disturbances due to earth currents; they also give a lower bound of N_0.

Among these, (a) seems to be the worst. The skin effect means bad penetration of the electric waves, and there is indeed no remedy against it. Different upper limits of N_1 have been proposed on rather accidental occasior but I did not get the impression that much attention has been spent on this question. Dr. Rainbow (Quart. Progress Repprts 1945, 58E, 186) took $N_1 < 5000$ so as to be able to neglect displacement currents. Dr. Baars propsed $N_1 = 100$ in view of the skin effect. Muzzey, who actually performed electrical measurements, took $N_1 = 50$ (Quart. Progress Reports 1944, 44E, 62); the presumption does not seem too bold, that this value has been taken bacause higher frequencies did not penetrate sufficiently. Now all these values seem much too high, at least if we wish to get information about structures at a depth of more than 1000 m. A frequency of 3 Hz seems to be the very highest that can be considered, and even this limit may be optimistic, as will be shown in the next section.

As to the technical side of the problem, this is not a question of energy. The required amount of energy can readily be supplied; but the difficulty is how to transform it into a current of very low frequency. A special apparatus will be needed; as a matter of fact, any frequency can be obtained by means of a vacuum tube generator; but, as the frequency goes down, the size of the instrument increases, and it must be transportable. This is a point the electrical engineer has to decide on.

If the artificial frequencies exceed those of the earth currents, the influence of the latter on the records can be eliminated by taking a condensor of sufficient capacity in the oscillograph circuit. Here too the size of the instrument is to be considered. But this device fails if it becomes necessary to extend the range of artificial frequencies so far downward that it overlaps the range of natural frequencies; and probably this will be the case. The difficulty might be overcome in different ways. The intensity of the artificia current can be varied, measurements at different spots and at different times can procure comparable observations, records that are no pure sinusoids can be treated by Fourier analysis. It would seem the disturbance caused by earth currents involves no serious difficulty, but its elimination might take a lot of work. The effect of the artificial current on the records must at least be of the same order as that of the earth current: this is the only condition that must be satisfied.

Since we must be prepared for the use of very low frequencies, it will be necessary to discuss the effect of polarization. No doubt polarization will be strong in the neighbourhood of the current electrode, causing a deformation of the sinusoidal current. This means that higher harmonics will arise; but, since the damping rapidly increases with frequency, it is not to be expected their influence on the records will be appreciable. On the other hand, polarization will cause a change of amplitude and phase in the field. Now, at some distance of the elctrode, say at 100 m, polarization might be negligible, sinc we are still using alternating current. Hence we may expect that ratios of amplitudes and phase-differences will not be involved in polarization effects at least if we only compare observations at points sufficiently remote from the current electrode; but perhaps a further analysis will be required. I must regret I had no experimental results at my disposal concerning polarization in the case of slowly alternating currents.

Summarizing, a frequency range from 1/30 Hz till 3 Hz might, for the present, be considered as suitable. But evidently experience must decide on the real limits N_0 and N_1 that can be obtained.

4. Though the case of a homogeneous earth has little interest in itself it can give us an idea of the magnitude of different effects, which is important in connection with the measurements that have to be performed.

For the sake of simplicity the radius of the earth can be taken infinite. Let the current elctrode be a sphere of radius a with centre O; take O as origin. Take (x_1, x_2, x_3) as co-ordinates of a point P, and let $x_3 = 0$ be the surface of the earth; hence the earth will be in contact with the lower half of the electrode. If we denote the distance between P and O by r, it will be supposed that the potential V at P depends on r and t only:

$$V = V(r,t) \;, \quad r^2 = x_1^2 + x_2^2 + x_3^2 . \tag{8}$$

It must be emphasized that only a very rough calculation of the damping of potential waves will be obtained in this way, since the skin effect has been neglected, which is certainly not allowed. The currents at the surface of the earth may be more powerful than those which will be calculated, whereas the currents in deeper layers will be weaker. Hence we must obtain a too favourable picture of the real penetration of the current, but nevertheless it will suffice for our purpose. At any rate, this explains why the upper bound $N_1 = 3$ Hz, which will be obtained, is perhaps still too optimistic.

A stationary potential wave of frequency $\omega/2\pi$ can be represented by

$$V = \varphi(r)\, e^{i\omega t} . \tag{8'}$$

The differential equation Dr. Rainbow starts from:

$$\Delta V = \mu\sigma \frac{\partial V}{\partial t} \qquad \left(\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right), \tag{9}$$

yields

$$\Delta\varphi = i\mu\sigma\omega\varphi . \tag{9'}$$

Putting

$$\varphi = r^{-1}u, \quad \mu\sigma\omega = 2k^2, \tag{10}$$

we have

$$u'' = 2ik^2u .$$

Hence we have two integrals

$$u = e^{\pm(1+i)kr} . \tag{11}$$

Since the amplitude cannot increase indefinitely if $r \to \infty$, we get a solution

$$u = e^{-(1+i)kr} ,$$

and the general expression of V(r,t) is given by

$$V = A\, r^{-1} e^{-kr + i(\omega t - kr)} , \tag{12}$$

where A is a constant. Taking real parts we can write, by an appropriate choice of the zero point of t:

(13) $$v(r,t) = Ar^{-1}e^{-kr}\sin(\omega t - kr) \qquad (\omega = 2\pi\nu,\ k = \sqrt{\pi\mu\sigma\nu}).$$

By (13), the phase difference expressed in radians between two points P_1 and P_2 is $k(r_1 - r_2)$. Thus the potential wave has a wave length $\lambda = 2\pi k^{-1}$ and a velocity $c = 2\pi\nu k^{-1}$. The amplitude is

(14) $$A(r) = Ar^{-1}e^{-kr},$$

so the damping increases rapidly with increasing frequency.

Take a couple of potential electrodes P_1 and P_2 collinear with the current electrode at O. Let the distances OP_1 and OP_2 be denoted by r and $r + \Delta r$, and let ΔV be the difference of potential between P_1 and P_2. By (13) we have

(15) $$\frac{dV}{dr} = -Ar^{-2}e^{-kr}\sin(\omega t - kr) - Akr^{-1}e^{-kr}\{\sin(\omega t - kr) + \cos(\omega t - kr)\}.$$

Neglecting the first term in the right hand member, which is small since r is very large, we obtain

(15') $$\frac{dV}{dr} = -\sqrt{2}\,Akr^{-1}e^{-kr}\sin(\omega t - kr + \tfrac{\pi}{4}).$$

Since $\Delta r \ll \lambda = 2\pi k^{-1}$, as it will be shown afterwards, we can put

(16) $$\Delta V = \frac{dV}{dr}\Delta r = -\sqrt{2}\,Akr^{-1}e^{-kr}\sin(\omega t - kr + \tfrac{\pi}{4}).\Delta r,$$

hence

(16') $$\Delta A = |\Delta V|_{max} = \sqrt{2}\,Akr^{-1}e^{-kr}\Delta r.$$

Summarizing, we have the following data:

- a = radius of the spherical current electrode;
- A(a) = amplitude of the potential at the current electrode;
- ΔA = amplitude of the difference of potentials between potential electrodes;
- r = mean distance from current electrode to potential electrodes;
- Δr = distance of potential electrodes;
- ν = frequency;
- σ = conductivity;
- μ = permeability.

Giorgi units will be used throughout.

Let us study the relation between r and ν under the most favourable conditions. The technique of measuring imposes upper bounds for a and A(a). Next, Δr has to be small compared to r, since the earth has to be approximately homogeneous between the potential electrodes, for reasons that will be discussed in a subsequent report. This provides an upper bound for Δr too. On the hand, ΔA must be sufficiently large; the least difference of potential that can still be measured is about 1 μV. Yet, since we are interested in the variations of the difference ΔA caused by heterogeneities of the earth, and not in the mean differences ΔA as shown by a homogeneous earth, the lower bound of ΔA must be considerably higher than 1 μV. I shall put

(17) $$\begin{cases} a = 1\ m, \quad A(a) = 300\ V, \quad \Delta r = 50\ m, \quad \Delta A = 20\ \mu V = 2.10^{-5}\ V, \\ \mu = 4\pi.10^{-7}, \quad \sigma = 3{,}6 \cdot 10^{-2}\ \Omega^{-1}m^{-1}, \end{cases}$$

and I hope these values will be considered reasonable. The accepted value of μ is approximately right in the case of any non-ferromagnetic material. A mean value of σ has been taken, which can be obtained from the theory of terrestrial magnetism. Probably all values (17) can be replaced by better ones; but since only the order of magnitude of the effect matters, this needs not trouble us. Moreover, the values of a, A(a), ΔA and Δr have but little influence on r; it is the frequency that is preponderant, since it stands in (16') in an exponential expression.

Substituting, we have by (13), (17) :

(18) $$k = 12\pi \cdot 10^{-5}\sqrt{\nu},$$

hence k is a very small number; this justifies the simplification made in (16). By (14) we have

$$A(a) = A a^{-1} e^{-ka},$$

and since $ka \ll 1$, we can put

$$A = a\,A(a),$$

hence by (16')

(19) $$\Delta A = \sqrt{2}\, a\, A(a)\, k\, r^{-1} e^{-kr}\,\Delta r.$$

Putting

(20) $$kr = x$$

and substituting, we have approximately

(21) $$150\,\nu = x\,e^{x}.$$

The results have been tabulated hereby.

(22)

x =	1,5	2	2,5	3	3,5	4	4,5	5	5,5	6	6,5
ν (Hz) =	0,045	0,099	0,20	0,40	0,77	1,46	2,70	4,95	9,0	16,7	28,8
r (km) =	18,8	16,9	14,7	12,6	10,6	8,8	7,3	6,0	4,9	4,0	3,2

5. Supposing that all electrical energy is converted into Joule heat, it is y to calculate the power required. If E is the electric force, j the current density, we have, according to our simplifications,

$$|j| = \sigma\,|E| = \sigma\,\left|\frac{dV}{dr}\right|,$$

hence the current i through a half sphere of radius r satisfies

$$|i| = 2\pi r^2 |j| = 2\pi\sigma r^2 \left|\frac{dV}{dr}\right|.$$

The Joule heat produced in one half of a spherical shell of thickness dr during a time dt will be

$$|i\,dV|\,dt = 2\pi\sigma r^2 \left(\frac{dV}{dr}\right)^2 dr\,dt,$$

hence, by (15), the total heat W(t)dt produced in a time dt is determined b

$$W(t) = 2\pi\sigma \int_a^{\infty} r^2 \left(\frac{dV}{dr}\right)^2 dr.$$

We cannot apply (15') now, since most of the heat is produced for small values of r. Hence we have, by (15):

$$W(t) = 2\pi\sigma A^2 \int_a^{\infty} e^{-2kr}\left[r^{-1}\sin(\omega t - kr) + k\left\{\sin(\omega t - kr) + \cos(\omega t - kr)\right\}\right]^2 dr.$$

The power, required for generating the waves excited by the electrode at 0, will be the mean value

$$\bar{w} = \lim_{T\to\infty} \frac{1}{T}\int_0^T w(t)\,dt,$$

hence

$$(23) \qquad \bar{w} = 2\pi\sigma A^2\left(\tfrac{1}{2} J_{-2} + k J_{-1} + k^2 J_0\right),$$

if we put

$$(23') \qquad J_{-n} = \int_a^\infty r^{-n} e^{-2kr}\,dr.$$

Now, by (18), k is a small number, so we can make use of an asymptotic expansion of (23), valuable for $k\to 0$. Putting $2kr = v$, we have

$$J_{-n} = (2k)^{n-1}\int_{2ka}^\infty v^{-n}e^{-v}\,dv = (2k)^{n-1}\left\{\int_{2ka}^1 + \int_1^\infty\right\}$$

$$= (2k)^{n-1}\left[\int_{2ka}^1 v^{-n}\{1+O(v)\}\,dv + O(1)\right]$$

$$= (2k)^{n-1}\int_{2ka}^1 v^{-n}\,dv + (2k)^{n-1}O\left(\int_{2ka}^1 v^{-n+1}\,dv\right) + O(k^{n-1}).$$

Now, for any $x\to +0$, and for any integer n, we have

$$\int_x^1 v^n\,dv = O\left(x^{n+1}|\log x|\right) + O(1),$$

hence

$$J_{-n} = (2k)^{n-1}\frac{1-(2ka)^{-n+1}}{-n+1} + (2k)^{n-1}\left\{O(k^{-n+2}|\log k|) + O(1)\right\} + O(k^{n-1})$$

$$= \frac{a^{-n+1}}{n-1} + O(k^{n-1}) + O(k|\log k|),$$

if $n \neq 1$, and

$$J_{-1} = \int_{2ka}^1 v^{-1}\,dv + O(1) = O(|\log k|).$$

Substituting, we obtain from (23):

$$\bar{w} = 2\pi\sigma A^2\left[\tfrac{1}{2}\{a^{-1} + O(k) + O(k|\log k|)\} + k\,O(|\log k|) + k^2\{-a + O(k^{-1}) + O(k|\log k|)\}\right]$$

$$= \pi\sigma A^2 a^{-1} + O(k|\log k|).$$

Since we have always to do with a couple of current electrodes, placed at a very large distance from one another, and both causing a system of potential waves in the earth, it is reasonable to suppose that the power W^*

we need will be the double of $\overline{W}$. Hence we obtain

(24) $$w^* \sim 2\pi\sigma A^2 a^{-1}.$$

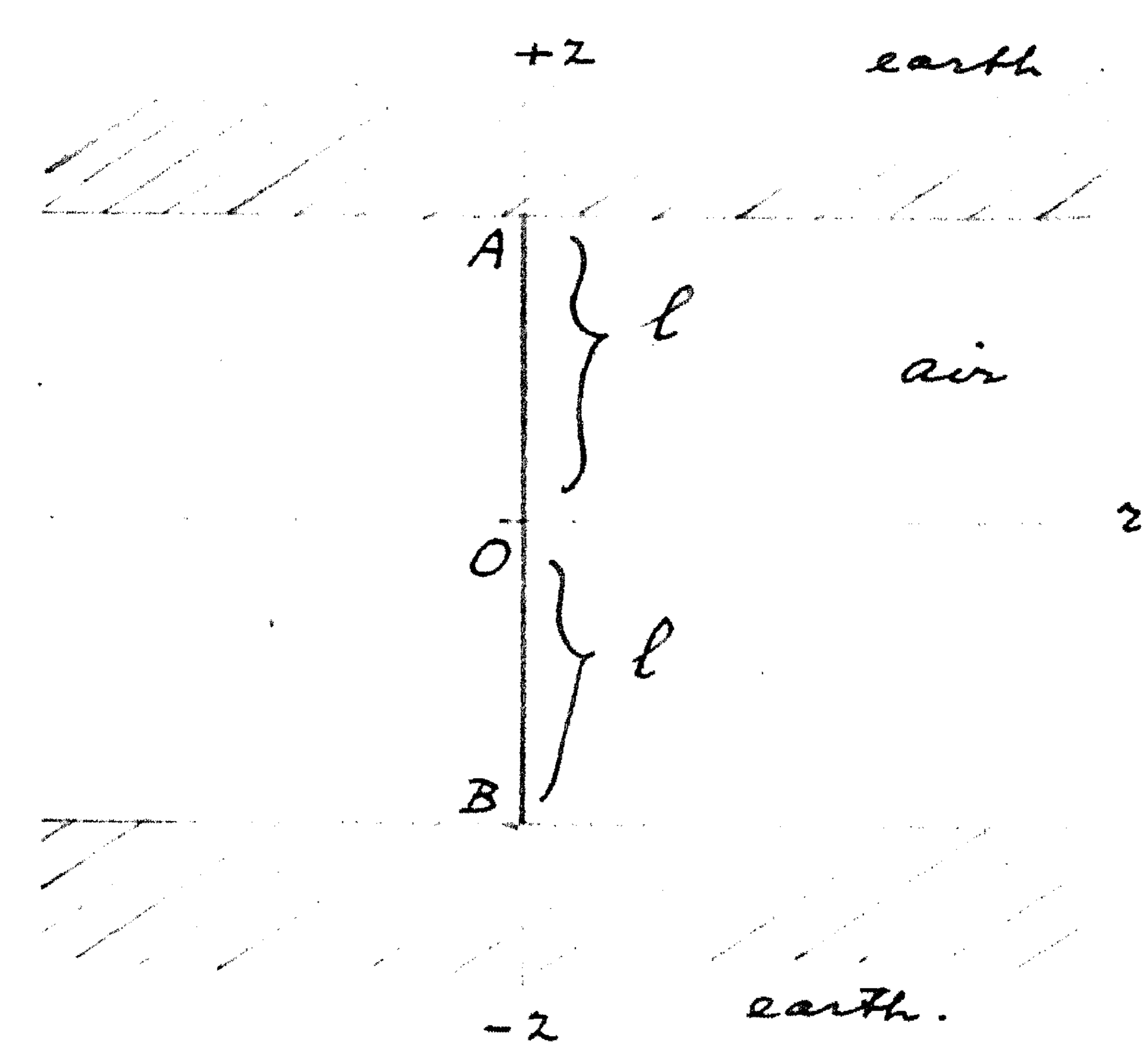
Taking the values (17), we get the approximate value

(24') $$w^* = 20 \text{ kW},$$

and this amount of electrical power can be supplied by a motor of about 50 HP. The difficulty remains how to transform the energy into oscillations of the frequencies we need.

6. An attempt has been made to estimate the influence of the skin effect. The problem has been put as simple as possible, and it has been treated with the Sommerfeld method (see A. Sommerfeld, Vorl. über Theoretische Physik VI, 1947, p.239-259), which has also been used by Dr. Rainbow in the many layer problem (Quart. Progress Reports 1945, 58E, p.87).

The situation is supposed to be symmetric in relation to a vertical axis O. If x, y are the horizontal co-ordinates, we put



(25) $$x = r\cos\varphi, \quad y = r\sin\varphi.$$

Any electromagnetical vector $\vec{f}$ can be expressed as a sum of a vertical component f_z a radial component f_r and a circular one f_φ; according to our assumption, f_z, f_r and f_φ will be the functions of z, r and t only.

Two flat, infinite, horizontal and homogeneous "earths" are placed opposite to one another. Their surfaces are $z = l$ and $z = -l$, the space between which is filled with air. A couple of electrodes at A and B is connected by a vertical wire trough O. The electromagnetic field will be supposed to be singular along the line AB.

If $\vec{E}$ and $\vec{H}$ denote the electric and magnetic forces, the system admits two independent solutions, characterized by

(26) $$E_\varphi = H_z = H_r = 0$$

and

(26') $$E_z = E_r = H_\varphi = 0.$$

Since there is no source of energy exciting the second solution, we can confine our attention to the first one.

Let us consider oscillations with angular frequency ω. Any field quantity f can then be represented by

$$f = g e^{i\omega t},$$

g denoting some complex-valued function of z and r only. In particular, if we put

(27) $$H_\varphi = q e^{i\omega t},$$

the functions E_z and E_r can be expressed by q.

The complete solution of this problem, which is rather simple indeed, will not be given now. I only mention the final result. If physical constants without accents refer to the air, while constants with accents refer to the earth, the quantities Z, A, h, h', ν, ν', and Δ will have the following meanings:

(28)
- Z is the depth of a point beneath the surface of the (lower) earth, so $Z = -z - l > 0$;
- A is a factor independent of r and Z;
- $h^2 = \mu\omega(\varepsilon\omega - i\sigma)\,, \quad -\frac{\pi}{2} \le \arg h \le 0;$
- $\nu = \sqrt{\lambda^2 - h^2}\,, \quad \mathrm{Re}\,\nu \ge 0, \quad \arg \nu = \arg ih \text{ if } \mathrm{Re}\,\nu = 0;$
- (h', ν') are defined in the same way, the constants having accents;
- $\Delta = h^2\nu'(e^{\nu l} + e^{-\nu l}) + h'^2\nu(e^{\nu l} - e^{-\nu l}).$

We then have

(28') $$q(r, Z) = A \int_0^\infty \frac{\lambda}{\nu}\, \frac{e^{\nu l} - e^{-\nu l}}{\Delta}\, J_0(\lambda r)\, e^{-\nu' Z}\, d\lambda .$$

Since we can put $\varepsilon' = 0$, $\mu = \mu'$, $\sigma = 0$, we can write

(28") $$h^2 = \mu\varepsilon\omega^2, \quad h'^2 = -i\mu\sigma\omega,$$

if we replace σ' by σ.

For practical purposes the formula (28') is rather unmanageable. Hence, putting

(29) $$r = R \sin\vartheta, \quad Z = R\cos\vartheta,$$

it would be desirable to obtain an asymptotic expansion of q(r,Z) for $R \to \infty$, which holds uniformly ~~for~~ in some interval $\frac{\pi}{2} - \delta \le \vartheta \le \frac{\pi}{2}$.

The same problem arises in the case of an integral treated by Sommerfeld and Weyl (Ann.d.Physik 60), which integral is very similar to the right hand member of (28'); an asymptotic value, containing a factor R^{-1}exp(ikR) has been obtained. However, Dr. Bouwkamp of the Philips Laboratory, made some critical remarks on the subject (see also his note in the Math. Reviews, vol.11 (1950), p.143), so a further discussion of (28') seems to be necessary. Anyhow, the factor R^{-1}exp (ikR) also occurs in our formula(13).

√Hence, at least the order of magnitude, both of frequency-range and power, such as these have been calculated in nrs. 4 and 5, seems to be right.

In our fictitious example the wire AB between the electrodes was perpendicular. Probably the potential waves in the interior of the earth are but little influenced by this connecting wire. However, the Sommerfeld method is also able to solve the problem in the case of a rectangular wire; but of course this would mean some new complication of our formulae.

The discussion of more complicated problems, e.g. the determination of the approximate value of disturbances, caused by well-defined heterogeneities in the interior of the earth, would require still more work. Prof. v.d. Waerden suggested to solve such questions by means of model experiments, where the earth could be replaced by a tank of salt water of suitable conductivity.

7. In nr.4 we have obtained an upper bound N_1 of the useful frequencies ν, corresponding to a given distance r. It seems to be a common opinion among geophysicists that r must be 5z at least, when z denotes the depth that has to be explored. Now, the question was, whether the electrical method might still ~~xx~~ give information on the tectonics when z exceeds 1000 m, say when $1000 < z < 1500$ m. This would yield r = 7,5 km, which result would correspond, according to our table (22), to a value of about 3 Hz for N_1. Hence, by (7) N_0 must be 1/30 Hz at most. Of course auxiliary information on more shallow layers can be obtained by using higher frequencies, say up to 30 Hz, and probably such data will be necessary in order to interpret rightly the extremely low frequency results, but they will never suffice by themselves.

It is a well-known fact that earth currents can be followed up over long distances of hundreds of kilometers and more. Hence these currents can have only very low frequencies, all higher ones being sieved out by the earth. Electrical prospecting has no need of such enormous distances; those from 10 till 20 km seem to be enough. So there must be a spectral range, reaching to that of the earth current frequencies, which will satisfy our conditions. The only question left is, whether the former range will be large enough. According to our result this will not be the case, so the only escape is to use such frequencoes as occur in the earth currents themselves. One might even be inclined to use earth currents for prospecting; but this would require a separate theory, and the performance of a great number of simultaneous measurements.

Concluding, there seems to be but one way of extending the range of useful frequencies, viz. extending it in the direction of the very low values. If, in order to eliminate the disturbances caused by earth currents, we are prepared to perform a simplified Fourier analysis of every record, where only one sinusoidal component has to be considered, and if the technical equipment for generating very low frequencies is available, the idea of Dr. Rainbow seems to have a good chance of being realized. Close collaboration with the geophysicist and the electrical engineer will be necessary in order to overcome the difficulties mentioned here. It is the geophysicist who must give a clear idea of the intensity of earth currents in different spectral ranges; and the electrical engineer must give a definite opinion about the technical possibilities. I have tried to get some information on these two points; I am sorry to say: without success. As to the mathematical side of the problems, a further discussion of the skin effect and of the formula (28') seems to be necessary. But the present impression is that the physical difficulties of Dr. Rainbow's ideas are more serious than the mathematical ones.