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ADJOINTS OF SEMIGROUPS ACTING ON VECTOR-VALUED FUNCTION SPACES

BY

G. GREINER*

Universität Tübingen
Auf der Morgenstelle 10, D-7400 Tübingen, Germany

AND

J. M. A. M. VAN NEERVEN

Centre for Mathematics and Computer Science
P.O. Box 4079, 1009 AB Amsterdam, The Netherlands

ABSTRACT

Let $T(t)$ be the translation group on $Y = C_0(\mathbb{R} \times K) = C_0(\mathbb{R}) \otimes C(K)$, K compact Hausdorff, defined by $T(t)f(x, y) = f(x + t, y)$. In this paper we give several representations of the sun-dial $Y^\odot$ corresponding to this group. Motivated by the solution of this problem, viz. $Y^\odot = L^1(\mathbb{R}) \otimes M(K)$, we develop a duality theorem for semigroups of the form $T_0(t)\otimes\mathrm{id}$ on tensor products $Z\otimes X$ of Banach spaces, where $T_0(t)$ is a semigroup on Z. Under appropriate compactness assumptions, depending on the kind of tensor product taken, we show that the sun-dial of $Z \otimes X$ is given by $Z^\odot \otimes X^*$. These results are applied to determine the sun-dials for semigroups induced on spaces of vector-valued functions, e.g. $C_0(\Omega; X)$ and $L^p(\mu; X)$.

Introduction

Suppose μ is a complex Borel measure of bounded variation on $\mathbb{R}$. For $t \in \mathbb{R}$ define the measure μ_t by $\mu_t(A) = \mu(A + t)$. Then a classical theorem due to Plessner [Pl] states that $\lim_{t\to 0} \|\mu - \mu_t\| = 0$ if and only if $\mu \ll m$, where m

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denotes the Lebesgue measure on $\mathbb{R}$. In Section 2 of this paper we derive the following analogue of this result for vector-valued measures: let X be a Banach space and let μ be an X-valued Borel measure of bounded variation on $\mathbb{R}$, then $\lim_{t\to 0}\|\mu-\mu_t\| = 0$ if and only if $\mu \in L^1(\mu; X)$. By the Radon–Nikodym theorem, the case $X = \mathbb{C}$ reduces to Plessner's theorem.

In case $X = Y^*$ is a dual space, this result can be restated in terms of the translation group in the following way: if $T(t)$ denotes the translation group on $C_0(\mathbb{R}; Y)$ then $L^1(\mathbb{R}; Y^*)$ is the maximal space of strong continuity of the adjoint $T^*(t)$ of $T(t)$. Now both $C_0(\mathbb{R}; Y)$ and $L^1(\mathbb{R}; Y^*)$ can be written as certain tensor products, namely $C_0(\mathbb{R}; Y) = C_0(\mathbb{R})\tilde{\otimes}_\epsilon Y$ and $L^1(\mathbb{R}; Y^*) = L^1(\mathbb{R})\tilde{\otimes}_\pi Y^*$ (the injective resp. projective tensor product), whereas the translation group on $C_0(\mathbb{R}; Y)$ can be regarded as the tensor product $T_0(t) \otimes \mathrm{id}$, with $T_0(t)$ denoting translation on $C_0(\mathbb{R})$. This suggests the following question:

Given two Banach spaces Z, X, a strongly continuous semigroup $T_0(t)$ on Z, with $Z^\odot$ the maximal space of strong continuity of $T_0^*(t)$, when is it true that we have a formula like $(Z \otimes X)^\odot = Z^\odot \otimes X^*$?

Here $(Z \otimes X)^\odot$ is the maximal space of strong continuity of the adjoint of the induced semigroup $T_0(t) \otimes \mathrm{id}$ on $Z \otimes X$. This question will be addressed in Section 3 for the injective and projective tensor product. These results can be applied to the vector-valued function spaces $L^1(\mu; X)$ and $C_0(\Omega; X)$. In order to treat also $L^p(\mu; X)$ for $1 < p < \infty$ we study in Section 4 the l-tensor product.

1. Adjoint Semigroups

In this section we will recall some of the standard results on adjoint semigroups. Proofs can be found in [BB, P]. Let $\{T_0(t)\}_{t\geq 0}$ (briefly, $T_0(t)$) be a C_0-semigroup on a Banach space X. The **adjoint** $T_0^*(t)$ of $T_0(t)$ is the semigroup on X^* defined by $T_0^*(t) := T_0(t)^*$. From

$$|\langle T_0^*(t)x^* - T_0^*(s)x^*, x\rangle| \leq \|x^*\|\, \|T_0(t)x - T_0(s)x\|$$

one sees that the map $t \mapsto T_0^*(t)x^*$ is weak*-continuous for every $x^* \in X^*$. Hence if X is reflexive, then $T_0^*(t)$ is weakly continuous and therefore strongly continuous. However in general $T_0^*(t)$ is not strongly continuous and it makes sense to define the sun-dual $X^\odot$ as the maximal subspace of X^* on which $T_0^*(t)$ acts in a strongly continuous manner:

$$X^\odot = \{x^* \in X^* : \lim_{t\downarrow 0}\|T_0^*(t)x^* - x^*\| = 0\}.$$

$X^\odot$ is a norm-closed, weak*-dense subspace of X^*. In fact, one has

$$X^\odot = \overline{D(A_0^*)},$$

where A_0^* is the adjoint of the generator A_0 of $T_0(t)$; the closure is taken with respect to the norm-topology of X^*. Letting $R(\lambda, A_0) = (\lambda - A_0)^{-1}$ be the resolvent of $T_0(t)$, then $R(\lambda, A_0^*) = R(\lambda, A_0)^*$ and $D(A_0^*) = R(\lambda, A_0^*)X^*$. Clearly $X^\odot$ is invariant under $T_0^*(t)$. By restricting $T_0^*(t)$ to $X^\odot$ one obtains a strongly continuous semigroup on $X^\odot$, which we will denote $T_0^\odot(t)$. Let $A_0^\odot$ be its generator, then one can show that $A_0^\odot$ is precisely the part of A_0^* in $X^\odot$.

PROPOSITION 1.1: *Let $k \geq 1$ and $\lambda \in \varrho(A_0)$. Then $X^\odot = \overline{R(\lambda, A_0^*)^k X^*}$.*

In fact, $R(\lambda, A_0^*)^k X^* = D((A_0^*)^k) \supset D((A_0^\odot)^k)$ and the latter is norm-dense in $X^\odot$ since $A_0^\odot$ is a generator on $X^\odot$.

Starting from $T_0^\odot(t)$ one can repeat the duality construction and define $T_0^{\odot *}(t)$ and $X^{\odot\odot} = (X^\odot)^\odot$. The canonical map $j : X \to X^{\odot *}$,

$$\langle jx, x^\odot \rangle := \langle x^\odot, x \rangle$$

is an embedding mapping X into $X^{\odot\odot}$. In case $jX = X^{\odot\odot}$ we say that X is **sun-reflexive with respect to $T_0(t)$**. It is well-known that this is the case if and only if $R(\lambda, A_0)$ is weakly compact [Pa2].

The spectra of A_0, A_0^* and $A_0^\odot$ coincide, see e.g. [Na, A-III]. This will be used throughout this paper, as well as more or less obvious identities like $R(\lambda, A_0)^* x^\odot = R(\lambda, A_0^\odot)x^\odot$ ($x^\odot \in X^\odot$), etc.

2. Translation in $C_0(\mathbb{R}; X)$

Let X be a Banach space. On $C_0(\mathbb{R}; X)$ the translation group $T(t)$ is defined by

$$T(t)f(s) = f(t+s), \qquad t \in \mathbb{R}.$$

In this section we prove in two different ways that the sun-dual on $C_0(\mathbb{R}; X)$ with respect to $T(t)$ is given by $L^1(\mathbb{R}; X^*)$.

Let $M(\mathbb{R}; X)$ denote the Banach space of all countably additive X-valued vector measures of bounded variation [DU]. If X is the scalar field we simply write $M(\mathbb{R})$. For $\mu \in M(\mathbb{R}; X)$ its **variation** $|\mu| \in M(\mathbb{R})$ is defined by

$$|\mu|(E) := \sup_\pi \{ \sum_{A \in \pi} \|\mu(E \cap A)\| \},$$

where the supremum is taken over all partitions π of $\mathbb{R}$ into finitely many disjoint subsets. If $\mu \in M(\mathbb{R};X)$ then $|\mu|$ is a finite positive measure in $M(\mathbb{R})$.

It is well-known (see [DU, pp. 181–182]) that the dual of $C_0(\mathbb{R};X)$ may be identified with $M(\mathbb{R};X^*)$ and we have

$$\| \int_{\mathbb{R}} f \, d\mu \| \leq \int_{\mathbb{R}} \|f\| \, d|\mu|, \qquad f \in C_0(\mathbb{R};X), \qquad \mu \in M(\mathbb{R};X^*).$$

The space $L^1(\mathbb{R};X)$ can be identified with a closed subspace of $M(\mathbb{R};X)$ in the following way: for $h \in L^1(\mathbb{R};X)$ define $\mu_h \in M(\mathbb{R};X)$ by

$$\mu_h(E) := \int_E h \, d\mu.$$

LEMMA 2.1: *Suppose $\mu \in M(\mathbb{R};X)$ and $f \in C(\mathbb{R})$ with $\lim_{t\to-\infty} f(t) = 0$. Define*

$$F(r) := \int_{-\infty}^{r} f(s) \, d\mu(s).$$

Then F is strongly measurable.

Proof: In order to apply Pettis' measurability theorem [DS], we must show that (i) F is weakly measurable, and (ii) F is essentially separably-valued.

To prove (i) first let m be a measure in $M(\mathbb{R})$. Then $\tilde{F}$ defined by

$$\tilde{F}(r) := \int_{-\infty}^{r} f(s) \, dm(s)$$

is measurable. (To see this, we may assume that μ and f are real-valued, split $f = f_+ - f_-$ and $m = m_+ - m_-$ and note that if f and m are **positive** then $\tilde{F}$ is monotone, hence measurable). Using this we see that for any $x^* \in X^*$ the function

$$r \mapsto \langle x^*, F(r) \rangle = \int_{-\infty}^{r} f(s) \, d\langle x^*, \mu \rangle(s)$$

is measurable. This proves (i).

To prove (ii) define

$$F_1(r) := \int_{-\infty}^{r} |f(s)| \, d|\mu|(s).$$

Since F_1 is monotone, F_1 is continuous except at a countable set E. For $r_0 \notin E$, $r \in \mathbb{R}$ we have

$$\|F(r) - F(r_0)\| = \| \int_{r_0}^{r} f(s) \, d\mu(s)\| \leq \int_{r_0}^{r} |f(s)| \, d|\mu|(s) = |F_1(r) - F_1(r_0)|.$$

From this it follows that F is continuous as well on $\mathbb{R}\backslash E$. Since moreover $\mathbb{R}\backslash E$ is separable it follows that $F(\mathbb{R}\backslash E)$ is separable. This proves (ii). ∎

THEOREM 2.2: *If $T(t)$ is the translation group on $C_0(\mathbb{R};X)$ then $C_0(\mathbb{R};X)^\odot = L^1(\mathbb{R};X^*)$.*

Proof: First we prove that $L^1(\mathbb{R};X^*) \subset C_0(\mathbb{R};X)^\odot$. Let $x^* \in X^*$ and $f \in L^1(\mathbb{R})$. Define $f \otimes x^* \in L^1(\mathbb{R};X^*)$ by

$$(f \otimes x^*)(s) = f(s)x^*.$$

Since translation is continuous on $L^1(\mathbb{R})$ it is clear that $f \otimes x^* \in C_0(\mathbb{R};X)^\odot$. Since the linear span of such functions is dense in $L^1(\mathbb{R};X^*)$, the inclusion $L^1(\mathbb{R};X^*) \subset C_0(\mathbb{R};X)^\odot$ follows. We now prove the reverse inclusion. Let A be the generator of $T(t)$. Since $C_0(\mathbb{R};X)^\odot = \overline{D(A^*)}$ it suffices to prove the inclusion $R(\lambda, A^*)M(\mathbb{R};X^*) \subset L^1(\mathbb{R};X^*)$. For $f \in C_0(\mathbb{R};X)$, $\mu \in M(\mathbb{R};X^*)$ we have

$$\begin{aligned}\langle R(\lambda, A^*)\mu, f\rangle = \langle \mu, R(\lambda, A)f\rangle &= \int_{\mathbb{R}} \int_0^\infty e^{-\lambda t} f(s+t)\ dt\ d\mu(s)\\ &= \int_{\mathbb{R}} \int_s^\infty e^{\lambda(s-t)} f(t)\ dt\ d\mu(s)\\ &= \int_{\mathbb{R}} \int_{-\infty}^t e^{\lambda(s-t)} f(t)\ d\mu(s)\ dt\\ &= \int_{\mathbb{R}} f(t)F(t)\ dt,\end{aligned}$$

where

$$F(t) := e^{-\lambda t} \int_{-\infty}^t e^{\lambda s}\ d\mu(s).$$

We will show that $F \in L^1(\mathbb{R};X^*)$. By Lemma 2.1, F is strongly measurable. But then we have

$$\begin{aligned}\|\int_{\mathbb{R}} F(t)\ dt\| &\le \int_{\mathbb{R}} \|F(t)\|\ dt\\ &= \int_{\mathbb{R}} e^{-\lambda t} \|\int_{-\infty}^t e^{\lambda s}\ d\mu(s)\|\ dt\\ &\le \int_{\mathbb{R}} \Big[\int_s^\infty e^{\lambda(s-t)}\ dt\Big]\ d|\mu|(s)\\ &= \frac{1}{\lambda}|\mu|(\mathbb{R}) < \infty.\end{aligned}$$

This proves that $F \in L^1(\mathbb{R};X^*)$. But since we had

$$\langle R(\lambda, A^*)\mu, f\rangle = \int_{\mathbb{R}} f(t)F(t)\ dt$$

for all f it is clear that $F = R(\lambda, A^*)\mu$ and the proof is finished. ∎

For $\mu \in M(\mathbb{R}; X)$ and $t \in \mathbb{R}$ we define $\mu_t \in M(\mathbb{R}; X)$ by $\mu_t(E) = \mu(E+t)$, where $E \subset \mathbb{R}$ is measurable. According to Theorem 2.2 we have, in case X is a dual space, that $\|\mu_t - \mu\| \to 0$ as $t \to 0$ if and only if $\mu \in L^1(\mathbb{R}; X)$. This easily extends to the case where X is an arbitrary Banach space.

COROLLARY 2.3: *Let $\mu \in M(\mathbb{R}; X)$. Then $\lim_{t\to 0} \|\mu_t - \mu\| = 0$ if and only if $\mu \in L^1(\mathbb{R}; X)$.*

Proof: Suppose $\|\mu_t - \mu\| \to 0$. Regarding μ as an X^{**}-valued vector measure, it follows from Theorem 2.2 that $\mu \in L^1(\mathbb{R}; X^{**})$. But since μ takes its values in X, the same must be true for the density function h_μ representing μ. In fact, by the Lebesgue differentiation theorem [DU, III. 12.8] we have, for almost all s,

$$h_\mu(s) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_s^{s+\varepsilon} h_\mu(\tau)\, d\tau = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \mu(s, s+\varepsilon).$$

Since $\mu(s, s+\varepsilon) \in X$ for all ε it follows that h_μ is X-valued. The converse assertion is clear. ∎

In the scalar case it is well-known that $C_0(\mathbb{R})^{\odot\odot} = BUC(\mathbb{R})$, the Banach space of bounded, uniformly continuous functions on $\mathbb{R}$. As might be expected, in the vector-valued case we get $C_0(\mathbb{R}; X)^{\odot\odot} = BUC(\mathbb{R}; X^{**})$. This follows from Theorem 3.11 below.

We will now investigate the special case of Theorem 2.2 where $X = C(K)$ with K compact Hausdorff (or $X = C_0(\Omega)$ with Ω locally compact Hausdorff). We have $C_0(\mathbb{R}; C(K)) \simeq C_0(\mathbb{R} \times K)$. The following lemma is more or less standard.

LEMMA 2.4: *Suppose $B \subset M(K)$ is separable. Then there is a positive $\mu \in M(K)$ such that $\nu \ll \mu$ for all $\nu \in B$.*

Proof: Let (ν_n) be a dense sequence in B and define

$$\mu := \sum_{n=1}^{\infty} \frac{|\nu_n|}{2^n \|\nu_n\|}.$$

Then $\nu_n \ll \mu$ for all n, so by closure also $\nu \ll \mu$ for all $\nu \in B$. ∎

Identifying $C_0(\mathbb{R}; C(K))$ with $C_0(\mathbb{R} \times K)$ the translation group from above is given by

$$T(t)f(x,y) = f(x+t, y).$$

The following result gives an alternative representation of the sun-dual of $C_0(\mathbb{R} \times K)$ with respect to this group. Lebesgue measure on $\mathbb{R}$ will be denoted by m; $\mu_1 \otimes \mu_2$ denotes the product measure of two measures μ_1, μ_2.

THEOREM 2.5: $C_0(\mathbb{R} \times K)^{\odot} = \bigcup_{0 \le \mu \in M(K)} L^1(\mathbb{R} \times K, m \otimes \mu)$.

Proof: By Theorem 2.2 we have $C_0(\mathbb{R} \times K)^{\odot} = L^1(\mathbb{R}; M(K))$. But any $f \in L^1(\mathbb{R}; M(K))$ is essentially separably valued. Therefore without loss of generality we may assume that $\{f(t) : t \in \mathbb{R}\}$ is a separable subset of $M(K)$. By Lemma 2.4 there is a positive $\mu \in M(K)$ such that $f(t) \ll \mu$ for all f. By the Radon–Nikodym theorem we may regard f as an element of $L^1(\mathbb{R}; L^1(K, \mu))$. By the Fubini theorem, the latter is isometric to $L^1(\mathbb{R} \times K, m \otimes \mu)$. This proves the inclusion $\subset$. For the reverse inclusion, let $\mu \ge 0$ and pick $f \in L^1(\mathbb{R} \times K, m \otimes \mu)$. Approximate f by a compactly supported $\tilde{f}$ in $C(\mathbb{R} \times K)$ and note that translation of $\tilde{f}$ is continuous in the L^1-norm. ∎

By Theorem 2.5, any $\nu \in C_0(\mathbb{R} \times K)^{\odot}$ belongs to some $L^1(\mathbb{R} \times K, m \otimes \mu)$ with $\mu \ge 0$. We will now give an explicit description of a possible choice for μ. For $\nu \in M(\mathbb{R} \times K)$ positive, define $\pi\nu \in M(K)$ by $\pi\nu(F) := \nu(\mathbb{R} \times F)$. Then for $f \in C(K)$ we have

$$\int_K f(y)\ d\pi\nu(y) = \int_K \int_{\mathbb{R}} f(y)\ d\nu(x, y).$$

We need the following lemma.

LEMMA 2.6: *Let λ, μ and ν be positive measures in $M(\mathbb{R})$, $M(K)$ and $M(\mathbb{R} \times K)$ respectively. If $\nu \ll \lambda \otimes \mu$ then $\nu \ll \lambda \otimes \pi\nu$.*

Proof: By assumption there is an $h \in L^1(\mathbb{R} \times K, \lambda \otimes \mu)$, $h \ge 0$ a.e., such that $d\nu = h\ d(\lambda \otimes \mu)$. Define

$$K_0 := \{y \in K : \int_{\mathbb{R}} h(x, y)\ d\lambda(x) = 0\};$$
$$K_1 := \{y \in K : \int_{\mathbb{R}} h(x, y)\ d\lambda(x) > 0\}.$$

By the Fubini theorem,

$$\nu(\mathbb{R} \times K_0) = \int_{K_0} \int_{\mathbb{R}} h(x, y)\ d\lambda d\mu = 0.$$

Now suppose $(\lambda \otimes \pi\nu)(A) = 0$. We have to show that $\nu(A) = 0$. But we have

$$\begin{aligned} 0 = (\lambda \otimes \pi\nu)(A) &= \int_K \int_{\mathbb{R}} \chi_A(x,y)\ d\lambda(x) d(\pi\nu)(y) \\ &= \int_K \int_{\mathbb{R}} \int_{\mathbb{R}} \chi_A(x,y) h(z,y)\ d\lambda(x) d\lambda(z) d\mu(y) \\ &= \int_K \int_{\mathbb{R}} \chi_A(x,y) \Big(\int_{\mathbb{R}} h(z,y)\ d\lambda(z) \Big)\ d\lambda(x) d\mu(y) \\ &= \int_{K_1} \int_{\mathbb{R}} \chi_A(x,y) \Big(\int_{\mathbb{R}} h(z,y)\ d\lambda(z) \Big)\ d\lambda(x) d\mu(y). \end{aligned}$$

Since $\int_{\mathbb{R}} h(z,y)\ d\lambda(z) > 0$ for $y \in K_1$, we see that $A \cap (\mathbb{R} \times K_1)$ is a $\lambda \otimes \mu$-null set, hence also a ν-null set (since by assumption $\nu \ll \lambda \otimes \mu$). Therefore $A \subset (A \cap (\mathbb{R} \times K_1)) \cup (\mathbb{R} \times K_0)$ is a ν-null set. ∎

Combination of Theorem 2.5 and Lemma 2.6 gives the following intrinsic characterization of those ν belonging to $C_0(\mathbb{R} \times K)^{\odot}$.

THEOREM 2.7: *$\nu \in C_0(\mathbb{R} \times K)^{\odot}$ if and only if $\nu \ll m \otimes \pi|\nu|$.*

One might wonder whether there is a more direct proof of Theorem 2.7. Indeed such a proof can be given. What may be more surprising is that it is possible to re-deduce Theorem 2.2 as a corollary from 2.7. Since we think that this approach is interesting in its own right, we will carry it out.

Direct proof of Theorem 2.7: If $\nu \in L^1(\mathbb{R} \times K, m \otimes \pi|\nu|)$ then as in the proof of Theorem 2.5 we have $\nu \in C_0(\mathbb{R} \times K)^{\odot}$. The proof of the converse proceeds in two steps. For Borel measures μ on $\mathbb{R}$ and ν on $\mathbb{R} \times K$ define the 'convolution' $\mu * \nu$ on $\mathbb{R} \times K$ by

$$\int_{\mathbb{R}\times K} f\ d(\mu * \nu) = \int_{\mathbb{R}\times K} \int_{\mathbb{R}} f(x+t,y)\ d\mu(t)\ d\nu(x,y).$$

Now let $\nu \in C_0(\mathbb{R} \times K)^{\odot}$.

STEP 1: For $T > 0$ let $m_{[0,T]}$ be the Borel measure on $\mathbb{R}$ defined by $m_{[0,T]}(E) = m(E \cap [0,T])$. For $f \in C_0(\mathbb{R} \times K)$ and $T > 0$ we have

$$\begin{aligned} \Big\langle \frac{1}{T} \int_0^T T^*(t)\nu\ dt, f \Big\rangle = \Big\langle \nu, \frac{1}{T} \int_0^T T(t) f\ dt \Big\rangle &= \frac{1}{T} \int_{\mathbb{R}\times K} \int_0^T f(x+t,y)\ dt\ d\nu(x,y) \\ &= \frac{1}{T} \langle m_{[0,T]} * \nu, f \rangle. \end{aligned}$$

This shows that the equality

$$\frac{1}{T}\int_0^T T^*(t)\nu\, dt = \frac{1}{T} m_{[0,T]} * \nu$$

holds. We claim that

$$m_{[0,T]} * \nu \ll m * |\nu|.$$

Indeed, let E be measurable such that $(m*|\nu|)(E) = 0$. This means by definition that

$$\int_{\mathbb{R}\times K}\int_{\mathbb{R}} \chi_E(x+t,y)\, dm(t)\, d|\nu|(x,y) = 0.$$

It follows that

$$\int_{\mathbb{R}\times K}\int_0^T \chi_E(x+t,y)\, dt\, d|\nu|(x,y) = 0.$$

Hence

$$\chi_E(x+t,y) = 0, \qquad m_{[0,T]}\otimes|\nu| - \text{a.e.}$$

From this it is clear that also

$$\chi_E(x+t,y) = 0, \qquad m_{[0,T]}\otimes\nu - \text{a.e.}$$

Rewriting this in terms of convolution, this is the same as $(m_{[0,T]} * \nu)(E) = 0$. Our claim is proved. By now we have shown that

$$\frac{1}{T}\int_0^T T^*(t)\nu\, dt \ll m * |\nu|.$$

Since by assumption

$$\lim_{T\downarrow 0}\frac{1}{T}\int_0^T T^*(t)\nu\, dt = \nu$$

strongly and since obviously $\{\mu : \mu \ll m*|\nu|\}$ is closed, it follows that $\nu \ll m*|\nu|$.

STEP 2: We claim that $m * |\nu| = m\otimes\pi|\nu|$. Let $\pi : \mathbb{R}\times K \to K$ be projection onto the second coordinate. We claim that the following equality holds:

$$\int_{\mathbb{R}\times K} f\circ\pi\, d|\nu| = \int_K f\, d\pi|\nu|.$$

Indeed, by the Riesz Representation Theorem the linear functional on $C(K)$ defined by

$$f \mapsto \int_{\mathbb{R}\times K} f\circ\pi\, d|\nu|$$

is represented by some $\mu \in C(K)^*$ and it is straightforward to check that $\mu = \pi|\nu|$. This proves the claim.

For $A \subset \mathbb{R} \times K$ measurable, put

$$A_{y_1} := A \cap \{(x,y) \in \mathbb{R} \times K : \ y = y_1\}.$$

Using our claim and the translation invariance of the Lebesgue measure m we see

$$\begin{aligned}(m * |\nu|)(A) &= \int_{\mathbb{R}\times K} \int_{\mathbb{R}} \chi_A(x+t,y) \ dm(t) \ d|\nu|(x,y) \\ &= \int_{\mathbb{R}\times K} m(A-x)_y \ d|\nu|(x,y) \\ &= \int_{\mathbb{R}\times K} m(A)_y \ d|\nu|(x,y) \\ &= \int_K m(A)_y \ d\pi|\nu|(y) \\ &= \int_K \int_{\mathbb{R}} \chi_A(t,y) \ dm(t) \ d\pi|\nu|(y) \\ &= \int_{\mathbb{R}\times K} \chi_A(t,y) \ d(m \otimes \pi|\nu|)(t,y) \\ &= (m \otimes \pi|\nu|)(A).\end{aligned}$$

This shows that $m * |\nu| = m \otimes \pi|\nu|$. Combining this with Step 1 we see that $\nu \ll m \otimes \pi|\nu|$ as was to be proved. ∎

Second proof of Theorem 2.2: Let X be an arbitrary Banach space. By the Banach–Alaoglu theorem the dual unit ball $K := B_{X^*}$ is weak*-compact. The map $i : X \to C(K)$ defined by $ix(x^*) = \langle x^*, x\rangle$ is an isometric embedding. Let $\tilde{i} : C_0(\mathbb{R}; X) \to C_0(\mathbb{R}; C(K)) = C_0(\mathbb{R} \times K)$ be the induced embedding. In this way we may regard $C_0(\mathbb{R}; X)$ as a closed, translation invariant subspace of $C_0(\mathbb{R} \times K)$. Let $y^\odot \in C_0(\mathbb{R}; X)^\odot$. We must show: $y^\odot \in L^1(\mathbb{R}; X^*)$. By the extension theorem for adjoint semigroups [Ne], $y^\odot$ can be extended to an element ν of $C_0(\mathbb{R} \times K)^\odot$. By Theorem 2.7 there is a density function $g \in L^1(\mathbb{R} \times K, m \otimes \pi|\nu|) = L^1(\mathbb{R}; L^1(K, \pi|\nu|))$ representing ν. We claim that $y^\odot = (\tilde{i})^*\nu$ can be regarded as an element of $L^1(\mathbb{R}; X^*)$. To see this, let $f \in C_0(\mathbb{R}; X)$ be arbitrary and note that

$$\int_{\mathbb{R}} f(\tau)\, dy^{\odot}(\tau) = \langle y^{\odot}, f\rangle = \langle \nu, \tilde{i}(f)\rangle$$
$$= \int_{\mathbb{R}} (\tilde{i}(f))(\tau)\, d\nu(\tau) = \int_{\mathbb{R}} g(\tau)\, (\tilde{i}(f))(\tau)\, d\tau$$
$$= \int_{\mathbb{R}} g(\tau)\, i(f(\tau))\, d\tau = \int_{\mathbb{R}} i^*(g(\tau))\, f(\tau)\, d\tau.$$

Hence $y^{\odot}$ can be represented by $\tilde{g}$, defined by $\tilde{g}(t) := i^*(g(t))$. Since $i^*(g(t)) \in X^*$ for all $t \in \mathbb{R}$ we see that $y^{\odot} \in L^1(\mathbb{R}; X^*)$ and the claim is proved. ∎

3. The Injective and Projective Tensor Product

Throughout this section X and Z will denote non-zero Banach spaces. We assume either both to be real or complex. $Z \otimes X$ denotes the algebraic tensor product (cf. [S1]).

The π-norm on $Z \otimes X$, often called the **projective** norm, is described most conveniently by its unit ball, which by definition is the convex closure of the set $B_Z \otimes B_X$, where B_Z and B_X are the unit balls of Z and X respectively. An analytic expression for the π-norm is given as follows:

$$\|u\|_\pi = \inf\{\sum_{i=1}^{n} \|z_i\|, \|x_i\| : u = \sum_{i=1}^{n} z_i \otimes x_i\}, \qquad u \in Z \otimes X.$$

The π-tensor product $Z\tilde{\otimes}_\pi X$ is the completion of $Z \otimes X$ with respect to this norm. Sometimes it is denoted by $Z\hat{\otimes}X$. The standard example for the π-tensor product is the following. Let Z be a space $L^1(\mu)$, where μ is some positive measure and X an arbitrary Banach space. Then $L^1(\mu)\tilde{\otimes}_\pi X$ can be identified in a canonical way with the space $L^1(\mu, X)$ of all X-valued Bochner integrable functions.

An element $u = \sum_{i=1}^{n} z_i \otimes x_i \in Z \otimes X$ can (algebraically) be identified with an operator $T_u \in \mathcal{L}(Z^*, X)$ by the formula

$$T_u z^* = \sum_{i=1}^{n} \langle z^*, z_i\rangle x_i.$$

The ε- or **injective** norm on $Z \otimes X$ is the norm induced by the operator norm on $\mathcal{L}(Z^*,X)$. Thus for $u=\sum_{i=1}^n z_i\otimes x_i$ the ε-norm is given by

$$\begin{aligned}\|u\|_\varepsilon &= \sup\left\{\left\|\sum_{i=1}^n \langle z^*,z_i\rangle x_i\right\| : \|z^*\|\le 1\right\}\\ &= \sup\left\{\left|\sum_{i=1}^n \langle z^*,z_i\rangle\langle x^*,x_i\rangle\right| : \|z^*\|\le 1\ ,\|x^*\|\le 1\right\}.\end{aligned}$$

The completion of $Z\otimes X$ with respect to this norm is denoted by $Z\tilde{\otimes}_\varepsilon X$. It is called the ε- or injective tensor product of Z and Y. Some authors denote it by $Z\check{\otimes}X$. The standard example is as follows: let $Z:=C_0(\Omega)$, Ω locally compact and X be an arbitrary Banach space. Then $C_0(\Omega)\tilde{\otimes}_\varepsilon X$ can be identified with $C_0(\Omega;X)$.

It is well-known that dual spaces of tensor products can be identified with certain operator ideals. For $u^*\in(Z\tilde{\otimes}_\varepsilon X)^*$ or $u^*\in(Z\tilde{\otimes}_\pi X)^*$, define $T_{u^*}\in\mathcal{L}(Z,X^*)$ by

$$\langle u^*,u\rangle=\sum_{i=1}^n\langle T_{u^*}z_i,x_i\rangle,$$

where $u=\sum_{i=1}^n z_i\otimes x_i\in Z\otimes X$. In particular, the dual of $Z\tilde{\otimes}_\pi X$ can be identified with the space $\mathcal{L}(Z,X^*)$. On the other hand, the dual of $Z\tilde{\otimes}_\varepsilon X$ can be identified with the set of all **integral** operators $Z\to X^*$ [DU], which we denote by $\mathcal{L}^i(Z,X^*)$.

A bounded linear operator $T\in\mathcal{L}(Z)$ induces a linear operator $T\otimes\mathrm{id}:Z\otimes X\to Z\otimes X$ by the formula

$$(T\otimes\mathrm{id})(z\otimes x):=Tz\otimes x.$$

The operator $T\otimes\mathrm{id}$ is bounded for both the ε- and the π-norm. In fact, in both cases one has $\|T\otimes\mathrm{id}\|=\|T\|$. The unique continuous extensions to $Z\tilde{\otimes}_\varepsilon X$ and $Z\tilde{\otimes}_\pi X$ will be denoted by $T\tilde{\otimes}_\varepsilon\mathrm{id}$ and $T\tilde{\otimes}_\pi\mathrm{id}$ respectively.

LEMMA 3.1: $\sigma(T\tilde{\otimes}_\varepsilon\mathrm{id})=\sigma(T\tilde{\otimes}_\pi\mathrm{id})=\sigma(T)$.

Proof: We prove a slightly more general result: Suppose $\|\cdot\|$ is a reasonable crossnorm (in the sense of [DU; Def. VIII.1.1]) on $Z\otimes X$ with the additional property that every bounded linear operator $T:Z\to Z$ extends to a bounded linear operator $T\tilde{\otimes}\mathrm{id}$ on the completion $Z\tilde{\otimes}X$ of $Z\otimes X$ with respect to $\|\cdot\|$. Then $\sigma(T\tilde{\otimes}\mathrm{id})=\sigma(T)$.

$\sigma(T\tilde{\otimes}\mathrm{id})\subset\sigma(T)$: Suppose $\lambda-T$ is invertible. Then $(\lambda-T)^{-1}\tilde{\otimes}\mathrm{id}$ is a bounded operator on $Z\tilde{\otimes}X$ and it is obvious that on the dense subspace $Z\otimes X$,

$(\lambda - T)^{-1} \otimes \mathrm{id}$ is a two-sided inverse for $\lambda - (T \otimes \mathrm{id})$. By density it follows that $(\lambda - T)^{-1}\tilde{\otimes}\mathrm{id} = (\lambda - (T\tilde{\otimes}\mathrm{id}))^{-1}$, so $\lambda \in \varrho(T\tilde{\otimes}\mathrm{id})$.

$\sigma(T) \subset \sigma(T\tilde{\otimes}\mathrm{id})$: Suppose $\lambda \in \sigma(T)$. If $\lambda \in \sigma_{ap}(T)$, the approximate point spectrum of T (cf. [Na]), then by definition we can choose an approximate eigenvector $(z_n)_{n=1}^{\infty}$, i.e., $\|z_n\| = 1$ for all n and

$$\lim_{n\to\infty} \|Tz_n - \lambda z_n\| = 0.$$

We claim that $(z_n \otimes x)_{n=1}^{\infty}$ is an approximate eigenvector of $T\tilde{\otimes}\mathrm{id}$ for every norm-1 vector $x \in X$. Indeed, we have $\|z_n \otimes x\| = \|z_n\|\;\|x\| = 1$ and moreover

$$\begin{aligned}\|(T\tilde{\otimes}\mathrm{id})(z_n \otimes x) - \lambda(z_n \otimes x)\| &= \|(Tz_n - \lambda z_n) \otimes x\| \\ &= \|Tz_n - \lambda z_n\|\;\|x\| \to 0, \qquad n \to \infty.\end{aligned}$$

Thus $\lambda \in \sigma(T\tilde{\otimes}\mathrm{id})$. If $\lambda \in \sigma(T)\backslash\sigma_{ap}(T)$ then the range of $\lambda - T$ cannot be dense. According to the Hahn–Banach theorem, $\lambda \in \sigma_p(T^*)$. Choose a norm-1 vector z^* such that $T^*z^* = \lambda z^*$. We claim that $\lambda \in \sigma_p((T\tilde{\otimes}\mathrm{id})^*)$ with eigenvector $z^* \otimes x^*$, where $x^* \neq 0$ is arbitrary in X^*. Indeed, for any $z \otimes x$ we have

$$\begin{aligned}\langle (T\tilde{\otimes}\mathrm{id})^*(z^* \otimes x^*), z \otimes x\rangle &= \langle z^* \otimes x^*, Tz \otimes x\rangle \\ &= \langle z^*, Tz\rangle\langle x^*, x\rangle \\ &= \langle T^*z^*, z\rangle\langle x^*, x\rangle \\ &= \lambda\langle z^*, z\rangle\langle x^*, x\rangle \\ &= \lambda\langle z^* \otimes x^*, z \otimes x\rangle.\end{aligned}$$

The claim now follows from a density argument. Hence $\lambda \in \sigma((T\tilde{\otimes}\mathrm{id})^*) = \sigma(T\tilde{\otimes}\mathrm{id})$. The second inclusion is proved and the lemma follows. ∎

Given a strongly continuous semigroup $T_0(t)$ on Z with generator A_0 then $T(t) := T_0(t) \otimes \mathrm{id}$ extends to a one-parameter semigroup of bounded linear operators on $Z\tilde{\otimes}_\varepsilon X$ and $Z\tilde{\otimes}_\pi X$ respectively. In fact it is easy to see that it is strongly continuous as well. Moreover, spectrum and resolvent can be described. We state these facts in the following proposition, in which $\tilde{\otimes}$ denotes either the ε- or the π-tensor product.

PROPOSITION 3.2: *$T(t)$ is a strongly continuous semigroup. If we denote its generator by A then $\sigma(A) = \sigma(A_0)$. For λ in the resolvent set we have $R(\lambda, A) = R(\lambda, A_0)\tilde{\otimes} id$.*

Proof: By the spectral mapping formula (cf. [Na]) we have

$$\sigma(R(\lambda, A_0))\backslash\{0\} = (\lambda - \sigma(A_0))^{-1}$$

and similarly for A. Hence, to prove the first assertion, we see that it suffices to show that $\sigma(R(\lambda, A)) = \sigma(R(\lambda, A_0)\tilde{\otimes}\mathrm{id})$, but this follows from the previous lemma. The second assertion is obvious (e.g. apply a density argument). ∎

Our next aim is to give a description of the adjoints of $T(t)$ and $R(\lambda, A)$. In order to do this, we identify the dual spaces of $Z\tilde{\otimes}_\pi X$ and $Z\tilde{\otimes}_\varepsilon X$ with $\mathcal{L}(Z, X^*)$ and $\mathcal{L}^i(Z, X^*)$ respectively. Given a bounded operator on Z, we want to determine the adjoint of $S\tilde{\otimes}\mathrm{id}$, where $\tilde{\otimes}$ is either $\tilde{\otimes}_\varepsilon$ or $\tilde{\otimes}_\pi$. Given $z \otimes x \in Z \otimes X$ and $R \in \mathcal{L}(Z, X^*)$ or $R \in \mathcal{L}^i(Z, X^*)$, then

$$\langle R, (S\tilde{\otimes}\mathrm{id})(z \otimes x)\rangle = \langle R, (Sz) \otimes x\rangle = \langle RSz, x\rangle = \langle RS, z \otimes x\rangle.$$

This shows that we have $(S\tilde{\otimes}\mathrm{id})^*(R) = RS$. We summarize this observation in the following proposition.

PROPOSITION 3.3: *The adjoint operators $T^*(t)$ and $R(\lambda, A)^* : \mathcal{L}(Z, X^*) \to \mathcal{L}(Z, X^*)$ are given as follows :*

$$T^*(t)(S) = ST_0(t), \qquad S \in \mathcal{L}(Z, X^*);$$
$$R(\lambda, A)^*(S) = SR(\lambda, A_0), \qquad S \in \mathcal{L}(Z, X^*).$$

The same assertions are valid for the $\tilde{\otimes}_\varepsilon$ tensor product, with $\mathcal{L}(Z, X^*)$ replaced by $\mathcal{L}^i(Z, X^*)$.

Let us recall that the integral operators form a two-sided operator ideal, i.e. given $R \in \mathcal{L}^i(Z, X^*)$ and bounded linear operators $S_1 \in \mathcal{L}(Z)$ and $S_2 \in \mathcal{L}(X^*)$ then $S_2 \circ R \circ S_1$ is integral as well and $\|S_2 \circ R \circ S_1\|_i \le \|S_2\| \cdot \|R\|_i \cdot \|S_1\|$. Here $\|\cdot\|_i$ is the norm induced by $(Z\tilde{\otimes}_\varepsilon X)^*$.

Both dual spaces $\mathcal{L}(Z, X^*)$ and $\mathcal{L}^i(Z, X^*)$ contain $Z^* \otimes X^*$ as a subspace. In order to identify the closure of $Z^* \otimes X^*$ with appropriate subspaces of $\mathcal{L}(Z, X^*)$ and $\mathcal{L}^i(Z, X^*)$ respectively we make for the rest of Section 3 the following assumption:

ASSUMPTION 3.4: *Z^* has the approximation property (a.p.).*

The classical Banach spaces ℓ^p, $C_0(\Omega)$, $L^p(\mu)$ satisfy Assumption 3.4. Z^* having the a.p. implies that the closure of $Z^* \otimes X^*$ in $\mathcal{L}^i(Z, X^*)$ can be identified with $Z^* \tilde{\otimes}_\pi X^*$. Operators belonging to this closure are called **nuclear operators**. Moreover, since Z^* has the a.p., so does Z [DU]. The latter implies that the closure of $Z^* \otimes X^*$ in $\mathcal{L}(Z, X^*)$, which is $Z^* \tilde{\otimes}_\varepsilon X^*$, is precisely the set of all compact operators from Z into X^*.

Now we are going to show that in case of sun-reflexivity the sun-dual of the ε-tensor product can be described easily. We already noted in section 1 that a semigroup is sun-reflexive if and only if the resolvent of the generator is weakly compact.

THEOREM 3.5: *Let Z be sun-reflexive with respect to $T_0(t)$. Then the sun-dual of the semigroup $T(t)$ induced on $Z \tilde{\otimes}_\varepsilon X$ is the closure in $Z^* \tilde{\otimes}_\pi X^*$ of $Z^\odot \otimes X^*$.*

Proof: Given $z^* \in Z^*$ and $x^* \in X^*$ then $T^*(t)(z^* \otimes x^*) = (T_0^*(t) z^*) \otimes x^*$. It follows that

$$\|T^*(t)(z^* \otimes x^*) - z^* \otimes x^*\| = \|(T_0^*(t)z^* - z^*)\| \cdot \|x^*\|.$$

This shows that if $z^* \in Z^\odot$ then $z^* \otimes x^* \in (Z \tilde{\otimes}_\varepsilon X)^\odot$. Hence also the closed linear subspace of $Z^* \tilde{\otimes}_\pi X^*$ generated by $\{z^* \otimes x^* : z^* \in Z^\odot, x^* \in X^*\}$ is contained in $(X \tilde{\otimes}_\varepsilon Z)^\odot$.

To prove the reverse inclusion, we first claim that $(Z \tilde{\otimes}_\varepsilon X)^\odot \subset Z^* \tilde{\otimes}_\pi X^*$. For the rest of the proof we fix one $\lambda \in \varrho(A_0)$. For $S \in (Z \tilde{\otimes}_\varepsilon X)^* = \mathcal{L}^i(Z, X^*)$ we have by Proposition 3.3 $R(\lambda, A)^*(S) = S \circ R(\lambda, A_0)$. Since Z is sun-reflexive with respect to $T_0(t)$, it follows that $R(\lambda, A_0)$ is weakly compact. From a theorem of Grothendieck (see [DU, Thm VIII.4.12]) it follows that $S \circ R(\lambda, A_0)$ is nuclear. Thus $R(\lambda, A)^*(S) \in Z^* \tilde{\otimes}_\pi X^*$ and by Proposition 1.1 the claim is proved.

Thus if we fix $S \in \mathcal{L}^i(Z, X^*)$, then for arbitrary $\varepsilon > 0$ there exist $z_i \in Z^*$, $x_i \in X^*$ such that

$$\|S \circ R(\lambda, A_0) - \sum_{i=1}^n z_i^* \otimes x_i^*\|_i < \varepsilon.$$

It follows that

$$\left\| S \circ R(\lambda, A_0)^2 - \sum_{i=1}^n R(\lambda, A_0)^* z_i^* \otimes x_i^* \right\|_i$$
$$= \left\| \left(S \circ R(\lambda, A_0) - \sum_{i=1}^n z_i^* \otimes x_i^* \right) \circ R(\lambda, A_0) \right\|_i < \varepsilon \cdot \|R(\lambda, A_0)\|.$$

Since $R(\lambda, A_0)^* z_i^* \in Z^{\odot}$ it follows that $R(\lambda, A)^{*2}(S) = S \circ R(\lambda, A_0)^2$ is in the closed linear subspace of $Z^* \tilde{\otimes}_\pi X^*$ generated by $\{z^* \otimes x^* : z^* \in Z^{\odot}, x^* \in X^*\}$. The conclusion now follows from Proposition 1.1. ∎

We point out that the π-tensor product is not injective, i.e. given a subspace Y of Z^*, then in general $Y \tilde{\otimes}_\pi X^*$ cannot be identified with the closed linear subspace of $Z^* \tilde{\otimes}_\pi X^*$ generated by $\{y \otimes x^* : y \in Y,\ x^* \in X^*\}$. There are special cases where this is true, e.g. if Y is complemented in Z^* or if X is a $C_0(\Omega)$-space respectively. Thus we have the following corollary.

COROLLARY 3.6: *If in addition $Z^{\odot}$ is complemented in Z^* or $X = C_0(\Omega)$, Ω locally compact, then $(Z \tilde{\otimes}_\varepsilon X)^{\odot} = Z^{\odot} \tilde{\otimes}_\pi X^*$.*

If $T_0(t)$ is a positive semigroup on a Banach lattice Z whose dual has order continuous norm, then by a result of de Pagter (to be published), $Z^{\odot}$ is a projection band in Z^*. This applies in particular to the case $Z = C_0(\Omega)$ and we obtain:

COROLLARY 3.7: *Suppose $T_0(t)$ is a positive semigroup on $C_0(\Omega)$. Then there exists a measure space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{\mu})$ such that $C_0(\Omega; X)^{\odot} = L^1(\tilde{\mu}; X^*)$.*

Now we consider the case of the π-tensor product. We are looking for conditions, ensuring that the sun-dual of $X \tilde{\otimes}_\pi Z$ can be identified with $Z^{\odot} \tilde{\otimes}_\varepsilon X^*$. In contrast to Theorem 3.5 now sun-reflexivity (weak compactness of the resolvent) is not sufficient as Example 3.10 below shows. If we require compactness of the resolvent however, then the sun-dual can be described in a nice way.

THEOREM 3.8: *Assume that the generator of the semigroup $T_0(t)$ on Z has compact resolvent, then for the semigroup induced on $Z \tilde{\otimes}_\pi X$ we have $(Z \tilde{\otimes}_\pi X)^{\odot} = Z^{\odot} \tilde{\otimes}_\varepsilon X^*$.*

Proof: As in the proof of Theorem 3.5 it can be shown that $Z^{\odot} \tilde{\otimes}_\varepsilon X^*$ is contained in the sun-dual of $Z \tilde{\otimes}_\pi X$. To prove the converse inclusion we observe that $R(\lambda, A_0)$ being compact implies that for $\varepsilon > 0$ there exist $z_i \in Z$ and $z_i^* \in Z^*$ such that

$$\|R(\lambda, A_0) - \sum_{i=1}^{m} z_i^* \otimes z_i\| < \varepsilon.$$

Thus given $S \in \mathcal{L}(Z, X^*)$ then

$$\begin{aligned}&\left\| S \circ R(\lambda, A_0)^2 - \sum_{i=1}^{m} R(\lambda, A_0)^* z_i^* \otimes S z_i \right\| \\ &= \left\| S \circ \left(R(\lambda, A_0) - \sum_{i=1}^{m} z_i^* \otimes z_i \right) \circ R(\lambda, A_0) \right\| \\ &\leq \varepsilon \|S\| \|R(\lambda, A_0)\|.\end{aligned}$$

It follows that $R(\lambda, A)^{*2}(S)$ can be approximated with respect to the operator norm by elements of $Z^{\odot} \otimes X^*$. Since the operator norm induces the ε-norm it follows that $R(\lambda, A)^{*2}(S) \in Z^{\odot} \tilde{\otimes}_{\varepsilon} X^*$ for every $S \in \mathcal{L}(Z, X^*)$. Then from Proposition 1.1 we can conclude that $(Z \tilde{\otimes}_{\pi} X)^{\odot} \subset Z^{\odot} \tilde{\otimes}_{\varepsilon} X^*$. ∎

The case $Z = L^1(\mu)$ was already proved in [Pa1]. On spaces $C_0(\Omega)$, Ω locally compact, or spaces $L^1(\mu)$, a resolvent is weakly compact if and only it is compact (see [Pa2]). Therefore the following corollary is an immediate consequence of Theorem 3.8.

COROLLARY 3.9: *Assume that Z is either a space $L^1(\mu)$ or a space $C_0(\Omega), \Omega$ locally compact. If the semigroup $T_0(t)$ is sun-reflexive then*

$$(Z \tilde{\otimes}_{\pi} X)^{\odot} = Z^{\odot} \tilde{\otimes}_{\varepsilon} X^*.$$

In general weak compactness of the resolvent is not enough in Theorem 3.8, as the following example shows.

Example 3.10: Consider the semigroup of translations on $Z = L^p(\mathbb{R})$. For $1 < p < \infty$ we have $L^p(\mathbb{R})^{\odot} = L^p(\mathbb{R})^* = L^q(\mathbb{R})$ with $1/p + 1/q = 1$ and the resolvent is weakly compact, Z being reflexive. Assuming that

$$(L^p(\mathbb{R}) \tilde{\otimes}_{\pi} X)^{\odot} = L^q(\mathbb{R}) \tilde{\otimes}_{\varepsilon} X^* = \{T \in \mathcal{L}(L^p(\mathbb{R}), X^*) : T \text{ is compact } \}$$

then from Proposition 3.3 and Proposition 1.1 we conclude that $S \circ R(\lambda, A_0)$ is compact for every $S \in \mathcal{L}(L^p(\mathbb{R}), X^*)$. Choosing $X = L^q(\mathbb{R})$ and S the identity on $L^p(\mathbb{R})$ shows that $R(\lambda, A_0)$ has to be compact, which is not the case (for then A_0 must have countable spectrum, but it is well-known that $\sigma(A_0) = i\mathbb{R}$).

In case $p = 1$ the resolvent of the translation group even fails to be weakly compact and the conclusion of Theorem 3.8 again does not hold, as we will now show. ∎

THEOREM 3.11: *If $T_0(t)$ is the translation group on $L^1(\mathbb{R})$ then $L^1(\mathbb{R};X)^{\odot} = BUC(\mathbb{R};X^*)$.*

Proof: First we claim that $R(\lambda, A_0)$ is representable [Pa1]. For almost all s we have

$$(R(\lambda, A_0)f)(s) = \int_0^\infty e^{-\lambda t} f(s+t)\, dt$$
$$= \int_{-\infty}^\infty e^{-\lambda(t-s)} \chi_{[s,\infty)}(t) f(t)\, dt.$$

Define $g : \mathbb{R} \to L^1(\mathbb{R})$ by $(g(t))(s) = e^{-\lambda(t-s)}\chi_{[s,\infty)}(t)$. We have

$$\|g(t)\|_{L^1(\mathbb{R})} = \int_{-\infty}^\infty e^{-\lambda(t-s)}\chi_{[s,\infty)}(t)\, ds = \int_{-\infty}^t e^{-\lambda(t-s)}\, ds = \frac{1}{\lambda}.$$

Since also g is continuous as a map $\mathbb{R} \to L^1(\mathbb{R})$, hence in particular strongly measurable, this shows that $g \in L^\infty(\mathbb{R}; L^1(\mathbb{R}))$ and our claim is proved. From Proposition 2.2 in [Pa1] we deduce that $L^1(\mathbb{R};X)^{\odot} \subset L^\infty(\mathbb{R};X^*)$. Let $h \in L^1(\mathbb{R};X)^{\odot}$. We claim that h is continuous. Let ϕ_n be any continuous function with compact support such that $\phi_n(t) = 1$ for all $t \in [-n,n]$. Clearly it suffices to prove that $h\phi_n$ is continuous for all n. Since each $h\phi_n$ is compactly supported and since obviously $h \in L^1(\mathbb{R};X)^{\odot}$ implies $h\phi_n \in L^1(\mathbb{R};X)^{\odot}$, we may consider $h\phi_n$ as an element of $L^1([-N_n,N_n];X)^{\odot}$ for some N_n large enough. Since $L^1([-N_n,N_n])$ is $\odot$-reflexive with respect to translation (see e.g. [HPh]) we have by Theorem 3.9 that

$$L^1([-N_n,N_n];X)^{\odot} = L^1([-N_n,N_n])^{\odot}\tilde{\otimes}_\varepsilon X^* \subset C([-N_n,N_n])\tilde{\otimes}_\varepsilon X^*$$
$$= C([-N_n,N_n];X^*).$$

Hence $h\phi_n \in C([-N_n,N_n];X^*)$. This proves that $L^1(\mathbb{R};X)^{\odot} \subset C(\mathbb{R};X^*)$. But then we must have that actually $h \in BUC(\mathbb{R};X^*)$: h is bounded as an element of $L^\infty(\mathbb{R};X^*)$, and uniformly continuous since otherwise the map $t \mapsto T^*(t)h$ is easily seen not to be norm-continuous. This shows $L^1(\mathbb{R};X)^{\odot} \subset BUC(\mathbb{R};X^*)$. The reverse inclusion holds trivially. ∎

This theorem is the L^1-analogue of Theorem 2.2. Now in general it is not true that

$$BUC(\mathbb{R};X) = BUC(\mathbb{R})\tilde{\otimes}_\varepsilon X$$

holds. In fact, any function in $BUC(\mathbb{R})\tilde{\otimes}_\varepsilon X$ must have relatively compact range whereas it is easy to construct functions in $BUC(\mathbb{R}; C_0(\mathbb{R}))$ not having relatively compact range. Just let $f \in C_0(\mathbb{R})$ be any non-zero function. Then the set of translates $\{T(t)f : t \in \mathbb{R}\}$ is not relatively compact, so by defining $F(t) = T(t)f$ we obtain an $F \in BUC(\mathbb{R}; C_0(\mathbb{R}))$ which does not have relatively compact range.

Remark 3.12: (a) The above examples show that for translation on $Z = L^p(\mathbb{R})$, $1 \le p < \infty$ the conclusion of Theorem 3.8 does not hold for every X.

In fact, let Z be *any* fixed Banach space and let $T_0(t)$ be a C_0-semigroup on Z with generator A_0. We claim that if for every X the formula $(Z\tilde{\otimes}_\pi X)^\odot = Z^\odot\tilde{\otimes}_\varepsilon X^*$ holds, then $R(\lambda, A_0)$ must be compact. Take $X = Z^*$. Let $X = Z^*$ and assume $(Z\tilde{\otimes}_\pi X)^\odot = Z^\odot\tilde{\otimes}_\varepsilon X^*$. Then $R(\lambda, A)^*(T) = T \circ R(\lambda, A_0)$ is a compact operator for every $T \in (Z\tilde{\otimes}_\pi X)^* = \mathcal{L}(Z, X^*) = \mathcal{L}(Z, Z^{**})$. In particular, letting $T : Z \to Z^{**}$ be the canonical embedding, it follows that $R(\lambda, A_0)$ itself is compact. See also [Pal], where $X = l^\infty$ is taken.

(b) Concerning 3.5 the situation is different and weak compactness of $R(\lambda, A_0)$ is not necessary in order that $(Z\tilde{\otimes}_\varepsilon X)^\odot = \overline{Z^\odot \otimes X^*}^{Z^*\tilde{\otimes}_\pi X^*}$ holds for every Banach space X. In fact, an inspection of the proof of Theorem 3.5 shows that a necessary and sufficient condition for this is that $T \circ R(\lambda, A_0)$ is nuclear for every operator $T \in \mathcal{L}^i(Z, X^*)$. An example of a semigroup without weakly compact resolvent but satisfying this condition (by Theorem 2.2 !) is translation in $C_0(\mathbb{R})$.

By combining 3.5 and 3.8 one can under suitable assumptions describe the bi-sun-dual of the ε- and the π-tensor product. In order to apply 3.5 and 3.8 we formally need the assumption that $Z^{\odot *}$ has the a.p. The proof below however shows that it suffices to have that Z^* has the a.p. ∎

For $L^1(\mu)\tilde{\otimes}_\pi X$ the following result was first proved by de Pagter (unpublished).

PROPOSITION 3.13: *Suppose $R(\lambda, A_0)$ is compact. Then:*

(i) *$(Z\tilde{\otimes}_\pi X)^{\odot\odot}$ is the closure in $Z^{\odot *}\tilde{\otimes}_\pi X^{**}$ of $Z \otimes X^{**}$. If either Z is complemented in $Z^{\odot *}$ or X is an $L^1(\mu)$-space then $(Z\tilde{\otimes}_\pi X)^{\odot\odot} = Z\tilde{\otimes}_\pi X^{**}$.*

(ii) *If either $Z^\odot$ is complemented in Z^* or $X = C_0(\Omega)$, Ω locally compact Hausdorff, then $(Z\tilde{\otimes}_\varepsilon X)^{\odot\odot} = Z\tilde{\otimes}_\varepsilon X^{**}$.*

Proof: First we prove (ii). By Corollary 3.6 we have $(Z\tilde{\otimes}_\varepsilon X)^\odot = Z^\odot\tilde{\otimes}_\pi X^*$. The conclusion now follows from Theorem 3.8 in case $Z^{\odot *}$ has the a.p. However, inspection of the proof of Theorem 3.8 shows that the a.p. was needed for showing that $R(\lambda, A_0)$ could be approximated by finite rank operators in the uniform operator topology. Hence what we must show in the present case is that $R(\lambda, A_0^{\odot})$ *can be approximated by finite rank operators.* That this is true when Z^* has the a.p., i.e. under Assumption 3.4 (regardless whether $Z^{\odot *}$ has the a.p.), is shown by the following argument. Fix $\lambda \in \varrho(A_0)$. Since Z^* has the a.p., $R(\lambda, A_0)$ is the uniform limit of finite rank operators $\Phi_n \in Z^* \otimes Z$. Then for $\mu \in \varrho(A_0)$, $R(\lambda, A_0)R(\mu, A_0)$ is the uniform limit of $\Phi_n R(\mu, A_0)$. Since $R(\mu, A_0)^* Z^* \subset Z^{\odot}$ it follows that $\Phi_n R(\mu, A_0) \in Z^{\odot} \otimes Z$. Moreover,

$$\|R(\lambda, A_0)^* R(\mu, A_0)^* - (\Phi_n R(\mu, A_0))^*\| = \|R(\mu, A_0)R(\lambda, A_0) - \Phi_n R(\mu, A_0)\|,$$

hence $\mu R(\lambda, A_0^{\odot})R(\mu, A_0^{\odot}) = \mu R(\lambda, A_0)^* R(\mu, A_0)^*|_{Z^{\odot}}$ is the uniform limit of $\mu \Phi_n R(\mu, A_0)^*|_{Z^{\odot}} \in Z \otimes Z^{\odot} \subset Z^{\odot *} \otimes Z^{\odot}$. Since

$$R(\lambda, A_0^{\odot}) = \lim_{\mu \to \infty} \mu R(\lambda, A_0^{\odot}) R(\mu, A_0^{\odot})$$

in the uniform operator topology (this follows from the resolvent equation for $A_0^{\odot}$), we can conclude that $R(\lambda, A_0^{\odot})$ can be approximated by finite rank operators. As we noted above, from these considerations we can conclude that

$$(Z^{\odot} \tilde{\otimes}_\pi X^*)^{\odot} = Z^{\odot\odot} \tilde{\otimes}_\varepsilon X^{**},$$

and since $R(\lambda, A_0)$ is compact we have $Z^{\odot\odot} = Z$, and (ii) is proved.

The first assertion of (i) is proved by a similar argument. Now suppose that Z is complemented in $Z^{\odot *}$. Then trivially every $T \in \mathcal{L}(Z, X^*)$ admits an extension to an operator in $\mathcal{L}(Z^{\odot *}, X^*)$. Also, if X is an $L^1(\mu)$-space, then X^* is injective [LT] and this again implies that every $T \in \mathcal{L}(Z, X^*)$ admits an extension to an operator in $\mathcal{L}(Z^{\odot *}, X^*)$. In other words, in either case the natural map (induced by restriction $\pi : Z^{\odot *} \to Z$)

$$\pi : \mathcal{L}(Z^{\odot *}, X^*) \to \mathcal{L}(Z, X^*)$$

is **surjective**. But since $\mathcal{L}(Y, X^*) = (Y \tilde{\otimes}_\pi X)^*$ this shows that the canonical inclusion map

$$j : Z \tilde{\otimes}_\pi X \to Z^{\odot *} \tilde{\otimes}_\pi X$$

is an embedding. Applying this to X^{**} instead of X (and noting that X^{***} is an $L^1(\mu)$-space if X^* is) we obtain that $Z \tilde{\otimes}_\pi X^{**}$ can be regarded as a closed subspace of $Z^{\odot *} \tilde{\otimes}_\pi X^{**}$ and this proves the second assertion. ∎

4. The ℓ-Tensor Product

It is not possible to identify the space $L^p(\mu; X)$, $1 < p < \infty$, with either a ε- or a π-tensor product. In this case the so-called l-tensor product solves the problem. It was introduced about 1970 by Chaney, Fremlin, Levin and Schaefer [Ch, Fr1, S3]. In order to define it, first of all one has to introduce the class of cone absolutely summing operators. The following result is taken from [S2, IV.3].

PROPOSITION 4.1: *Let Z be a Banach lattice, X a Banach space. For a bounded linear map $T : Z \to X$ the following are equivalent:*

(i) $\exists C > 0$ *such that for every* $0 \le f_1, ..., f_n \in Z$, $\sum_{i=1}^n \|Tf_i\| \le C\|\sum_{i=1}^n f_i\|$;
(ii) *For every positive sequence* (f_i) *in* Z *such that* $\sum_{i=1}^\infty f_i$ *converges, the sum* $\sum_{i=1}^\infty \|Tf_i\|$ *converges;*
(iii) *There is an* $L^1(\mu)$*-space such that* T *admits a factorization* $Z \xrightarrow{T_1} L^1(\mu) \xrightarrow{T_2} X$ *with* $T_1 \ge 0$;
(iv) $\exists\ 0 \le \phi \in Z^*$ *such that for all* $f \in Z$, $\|Tf\| \le \langle \phi, |f| \rangle$;
(v) *The set* $\{T^*x^* : \ \|x^*\| \le 1\}$ *is order bounded in* Z^*.

Definition 4.2: $T : Z \to X$ is called **cone absolutely summing** (c.a.s.) if one of the equivalent assertions of Proposition 4.1 is satisfied. The set of all c.a.s operators is denoted by $\mathcal{L}^l(Z, X)$. For $T \in \mathcal{L}^l(Z, X)$ define

$$\|T\|_l := \inf\{C : \text{ (i) in Proposition 4.1 holds with constant } C\}.$$

$\mathcal{L}^l(Z, X)$ is a Banach space and contains the finite-rank operators. If X is a Banach lattice then $\mathcal{L}^l(Z, X)$ is a Banach lattice as well.

The l-**nuclear** operators $\mathcal{N}^l(Z, X)$ are defined as the closure of the finite rank operators in $\mathcal{L}^l(Z, X)$.

As a subspace of $\mathcal{L}(Z, X)$, $\mathcal{L}^l(Z, X)$ has the following ideal property: given $T \in \mathcal{L}^l(Z, X)$, $R \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Z)$ such that its modulus $|S|$ exists, then $R \circ T \circ S \in \mathcal{L}^l(Z, X)$ and

$$\|R \circ T \circ S\|_l \le \|R\| \, \|T\|_l \, \| \, |S| \, \|.$$

Let $u = \sum_{i=1}^n z_i \otimes x_i$. By the formula $T_u z^* := \sum_{i=1}^n \langle z^*, z_i \rangle x_i$ we regard $Z \otimes X$ as a linear subspace of $\mathcal{L}^l(Z^*, X)$. On $Z \otimes X$ we define the l-**norm** $\| \cdot \|_l$ to be the norm induced by $\mathcal{L}^l(Z^*, X)$. The Banach space $Z \tilde{\otimes}_l X$ is defined to be the completion of $Z \otimes X$ with respect to the l-norm. In this way $Z \tilde{\otimes}_l X$ can be identified with the closure of $Z \otimes X$ in the space $\mathcal{L}^l(Z^*, X)$.

In this way $Z^*\tilde{\otimes}_l X$ can be identified with the closure of $Z^*\otimes X$ in $\mathcal{L}^l(Z^{**},X)$. Now elements $u=\sum_{i=1}^n z_i^*\otimes x_i \in Z^*\otimes X$ can also be identified with an operator $\tilde{T}_u : Z \to X$ (rather than $Z^{**}\to X$), by

$$\tilde{T}_u(z)=\sum_{i=1}^n \langle z_i^*, z\rangle x_i.$$

The following proposition states that indeed $Z^*\tilde{\otimes}_l X$ becomes in this way the closure of $Z^*\otimes X$ in $\mathcal{L}^l(Z,X)$. In fact, the $\mathcal{L}^l(Z,X)$-closure of $Z^*\otimes X$ is precisely $\mathcal{N}^l(Z,X)$.

PROPOSITION 4.3: *$Z^*\tilde{\otimes}_l X$ can be identified isometrically with $\mathcal{N}^l(Z,X)$.*

Proof: By definition, $\mathcal{N}^l(Z,X)$ is the closure of the finite rank operators in $\mathcal{L}^l(Z,X)$. Regarding a finite rank operator $Z\to X$ as an element of $Z^*\otimes X$ as above, we see that $\mathcal{N}^l(Z,X)$ is the closure of $Z^*\otimes X$ in $\mathcal{L}^l(Z,X)$. On the other hand, by definition $Z^*\tilde{\otimes}_l X$ is the $\mathcal{L}^l(Z^{**},X)$-closure of $Z^*\otimes X$. Therefore it suffices to show that the $\mathcal{L}^l(Z,X)$-norm and the $\mathcal{L}^l(Z^{**},X)$-norm agree on $Z^*\otimes X$. To this end, let $u\in Z^*\otimes X$ be given. On the one hand, we can consider u as a c.a.s. map $T_u : Z^{**}\to X$. This map is also c.a.s. as a map $Z^{**}\to X^{**}$ and

$$\|T_u\|_{\mathcal{L}^l(Z^{**},X)}=\|T_u\|_{\mathcal{L}^l(Z^{**},X^{**})}.$$

On the other hand we may regard u as a c.a.s. map $\tilde{T}_u : Z\to X$. In this case $\tilde{T}_u^{**} : Z^{**}\to X^{**}$ is c.a.s. [S2, IV Cor. 3.8] and

$$\|\tilde{T}_u\|_{\mathcal{L}^l(Z,X)}=\|\tilde{T}_u^{**}\|_{\mathcal{L}^l(Z^{**},X^{**})}.$$

But clearly as maps $Z^{**}\to X^{**}$ we have $T_u=\tilde{T}_u^{**}$, so combining the two above equalities gives the desired result. ∎

The map $j : L^p(\mu)\otimes X \to L^p(\mu;X)$, $1\le p<\infty$, defined by $j(f\otimes x)(t)=f(t)x$ extends to an isometric isomorphism from $L^p(\mu)\tilde{\otimes}_l X$ onto $L^p(\mu;X)$. In a similar way one has $C_0(\Omega)\tilde{\otimes}_l X = C_0(\Omega;X)$. This is summarized in the following proposition [S2, IV.7 Examples 1,4].

PROPOSITION 4.4: *One has $L^p(\mu;X)=L^p(\mu)\tilde{\otimes}_l X$, $1\le p<\infty$, and $C_0(\Omega;X)=C_0(\Omega)\tilde{\otimes}_l X$.*

One of the surprising properties of the l-tensor product is that the dual is given by the same class of operators which is used to define it (the l-norm is 'self-dual').

More precisely, one has [S2, IV.7.4]

$$(Z\tilde{\otimes}_l X)^* = \mathcal{L}^l(Z, X^*).$$

Now we want to describe the sun-dual of $Z\tilde{\otimes}_l X$ with respect to semigroups induced by a semigroup on one of the factors. Since (in contrast to the ε- and π-tensor product) the l-tensor product is not symmetric (even when X is a Banach lattice as well) we have to distinguish the two cases where $T_0(t)$ is given on Z or on X.

First we consider the case where we are given a C_0-semigroup $T_0(t)$ on X with generator A_0. As in Section 3, $\mathrm{id}\otimes T_0(t) := \mathrm{id}_Z \otimes T_0(t)$ extends to a C_0-semigroup on $Z\tilde{\otimes}_l X$.

THEOREM 4.5: *Each of the following conditions implies* $(Z\tilde{\otimes}_l X)^{\odot} = Z^*\tilde{\otimes}_l X^{\odot}$:

(i) $R(\lambda, A_0)$ *is compact;*

(ii) $R(\lambda, A_0)$ *is weakly compact and* Z *does not contain a sublattice isomorphic to* ℓ^1.

Proof: The inclusion $\supset$ can be proved as in 3.5.

For $T \in \mathcal{L}^l(Z, X^*)$ one has as in Proposition 3.3 that

$$R(\lambda, A)^*(T) = R(\lambda, A_0)^* \circ T.$$

Hence to prove the converse inclusion by Proposition 4.3 we have to show that $R(\lambda, A_0)^* \circ T$ is l-nuclear as a mapping $Z \to X^{\odot}$.

(i) Since $T : Z \to X^*$ is c.a.s, by Proposition 4.1(iii) T has a factorization

$$Z \xrightarrow{T_1} L^1(\mu) \xrightarrow{T_2} X$$

with $T_1 \geq 0$. Hence $R(\lambda, A_0)^* \circ T$ factorizes as

$$Z \xrightarrow{T_1} L^1(\mu) \xrightarrow{T_2'} X,$$

with $T_2' = R(\lambda, A_0)^* \circ T_2$ compact and taking values in $X^{\odot}$. Thus by [S2, Prop. IV.8.2] $R(\lambda, A_0)^* \circ T : Z \to X^{\odot}$ is l-nuclear.

(ii) By a result due to Schlotterbeck–Lotz (personal communication), if Y is reflexive and Z contains no sublattice isomorphic to ℓ^1, then $\mathcal{N}^l(Z,Y) = \mathcal{L}^l(Z,Y)$. Since by assumption $R(\lambda, A_0)^* : X^* \to X^{\odot}$ is weakly compact, by a well-known result of Davis–Figiel–Johnson–Pelczynski [DFJP] there exists a reflexive space Y such that $R(\lambda, A_0)^*$ admits the factorization

$$X^* \xrightarrow{R_1} Y \xrightarrow{R_2} X^{\odot}.$$

Since T is c.a.s., the operator $R_1 \circ T : Z \to Y$ is c.a.s. as well and we conclude that $R_1 \circ T$ is l-nuclear. Then $R(\lambda, A_0)^* \circ T = R_2 \circ R_1 \circ T$ is l-nuclear as well. ∎

Note that both $Z = C_0(\Omega)$ and $Z = L^p(\mu)$, $1 < p < \infty$ do not contain ℓ^1 as a sublattice.

Now we will discuss the case where we are given a C_0-semigroup $T_0(t)$ on Z. In general for a bounded linear operator T on Z, the operator $T \otimes \mathrm{id}$ does not admit an extension to a bounded operator on $Z\tilde{\otimes}_l X$. If however T possesses a modulus $|T|$, then the extension exists and

$$\|T\tilde{\otimes}_l \mathrm{id}\| \leq \| \, |T| \, \|.$$

Therefore in order to be sure that $T_0(t)\otimes$ id admits an extension to a C_0-semigroup $T(t) = T_0(t)\tilde{\otimes}_l\mathrm{id}$ of bounded operators on $Z\tilde{\otimes}_l X$, we will assume that $T_0(t)$ is a *positive* semigroup (see [Na]). Then for λ sufficiently large $R(\lambda, A_0)$ is positive, hence $R(\lambda, A_0)\otimes\mathrm{id}$ extends to a bounded linear operator on $Z\tilde{\otimes}_l X$. One easily shows that this extension equals $R(\lambda, A)$, the resolvent of the generator A of $T(t)$. Similarly as in Proposition 3.3 one has that $R(\lambda, A)^*$ considered as an operator on $\mathcal{L}^l(Z, X^*) = (Z\tilde{\otimes}_l X)^*$ is given by

$$R(\lambda, A)^*(T) = T \circ R(\lambda, A_0).$$

In order to be able to identify $(Z\tilde{\otimes}_l X)^{\odot}$ with $Z^{\odot}\tilde{\otimes}_l X^*$ we need a certain compactness property of $R(\lambda, A_0)$ which we will describe next.

Definition 4.6: An operator $T \in \mathcal{L}(Z)$ is called r-compact if its modulus $|T|$ exists and there is a sequence of finite rank operators $\Phi_n \in Z^* \otimes Z$ such that

$$\lim_{n\to\infty} \| \, |T - \Phi_n| \, \| = 0.$$

The adjoint of an r-compact operator is r-compact again. Since $\|T\| \leq \| \, |T| \, \|$, every r-compact operator is compact. In case $Z = L^1(\mu)$ or $Z = C_0(\Omega)$ the converse is true (see [S2]). For $Z = L^2(\mu)$ the situation is different. In [Fr2] an example is given of a positive compact operator on $L^2(\mu)$ which is not r-compact. However, in $L^2(\mu)$ every Hilbert–Schmidt operator is r-compact.

Note that a sufficient condition for r-compactness for a positive T is the existence of a positive sequence Φ_n of finite rank operators satisfying $0 \leq \Phi_n \leq T$ and $\|T - \Phi_n\| \to 0$. This is a convenient criterion to check, e.g., whether kernel operators are r-compact.

THEOREM 4.7: *Suppose $T_0(t)$ is a positive C_0-semigroup on a Banach lattice Z whose resolvent $R(\lambda, A_0)$ is r-compact for sufficiently large λ. Then $(Z\tilde{\otimes}_l X)^{\odot}$ is the closure in $Z^*\tilde{\otimes}_l X^*$ of $Z^{\odot} \otimes X^*$. If $Z^{\odot}$ is a sublattice of Z^* then $(Z\tilde{\otimes}_l X)^{\odot} = Z^{\odot}\tilde{\otimes}_l X^*$.*

Proof: As before, we will show that $R(\lambda, A)^{2*}(\mathcal{L}^l(Z, X^*)) \subset \overline{\operatorname{span}}(Z^{\odot} \otimes X^*)$, the closure taken in $Z^*\tilde{\otimes}_l X^*$. By assumption there are finite rank operators Φ_n satisfying $\| \, |R(\lambda, A_0) - \Phi_n| \, \| \to 0$. Given $T \in \mathcal{L}^l(Z, X^*)$ it follows that

$$\begin{aligned}\|R(\lambda, A)^{2*}(T) - T \circ \Phi_n \circ R(\lambda, A_0)\|_l &= \|T \circ (R(\lambda, A_0) - \Phi_n) \circ R(\lambda, A_0)\|_l \\ &\leq \|T\|_l \, \| \, |R(\lambda, A_0) - \Phi_n| \, \| \, \|R(\lambda, A_0)\| \\ &\to 0.\end{aligned}$$

Moreover if $\Phi_n = \sum_{i=1}^m z_i^* \otimes z_i$ then $T \circ \Phi_n \circ R(\lambda, A_0) = \sum_{i=1}^m R(\lambda, A_0)^* z_i^* \otimes T z_i \in Z^{\odot} \otimes X^*$ and the first part of the theorem is proved. The additional statement is a consequence of the left-injectivity of the l-tensor product in the sense that if Z_1 is a sublattice of Z_2, then $Z_1\tilde{\otimes}_l X$ can be identified with a closed subspace of $Z_2\tilde{\otimes}_l X$ (see [S2]). ∎

By the result of de Pagter mentioned after 3.6, the second statement of 4.7 applies to the case where Z^* has order continuous norm.

COROLLARY 4.8: *Suppose Z is a Banach lattice with Z^* having order continuous norm and let $T_0(t)$ be a positive semigroup on Z. If $R(\lambda, A_0)$ is r-compact for sufficiently large λ, then $(Z\tilde{\otimes}_l X)^{\odot\odot} = Z\tilde{\otimes}_l X^{**}$.*

Proof: Since $R(\lambda, A_0)$ is r-compact, hence compact, we have $Z^{\odot\odot} = Z$. Now since Z^* has order continuous norm, by the result of de Pagter $Z^{\odot}$ is a projection band in Z^*. Hence we can apply Theorem 4.7 to find that $(Z\tilde{\otimes}_l X)^{\odot} = Z^{\odot}\tilde{\otimes}_l X^*$. Moreover, the canonical embedding $Z \to Z^{\odot *}$ factorizes as $Z \to Z^{**} \to Z^{\odot *}$ where the second map is the adjoint of the inclusion map $i : Z^{\odot} \to Z^*$. But since $Z^{\odot}$ is a band, i^* is a lattice homomorphism. Combining this with the embedding $Z \to Z^{**}$ it follows that $Z^{\odot\odot} = Z$ is a sublattice of $Z^{\odot *}$. Hence we can apply 4.7 to the positive semigroup $T_0^{\odot}(t)$ on $Z^{\odot}$. Note that this semigroup has r-compact resolvent as well. Indeed, $R(\lambda, A_0)^* : Z^* \to Z^*$ is r-compact and $Z^{\odot}$ is complemented in Z^* by a positive projection. ∎

Weak compactness is not sufficient for the conclusion of Theorem 4.7 to hold: take any uniformly continuous semigroup on $L^p(\mu)$, $1 < p < \infty$ and note that in general $L^p(\mu; X)^* = (L^p(\mu)\tilde{\otimes}_l X)^* \neq L^q(\mu)\tilde{\otimes}_l X^* = L^q(\mu; X^*)$.

Remark 4.9: An inspection of the proof of Theorem 4.7 shows that the assumption of r-compactness of the resolvent can be weakened to the following assumption: *$T \circ R(\lambda, A_0)$ is l-nuclear for every $T \in \mathcal{L}^l(Z, X^*)$*. This condition is satisfied when e.g. $Z = L^p(\mu)$ $(1 < p < \infty)$ and the resolvent $R(\lambda, A_0)$ is represented by a positive measurable kernel k, i.e.,

$$(R(\lambda, A_0)f)(x) = \int k(x,y)f(y)\, d\mu(y) \qquad \text{for } \mu\text{-a.a. } x,$$

where k satisfies the condition

$$\sup_x \int k(x,y)^q\, d\mu(y) < \infty, \qquad \frac{1}{p} + \frac{1}{q} = 1.$$

This can be seen as follows. If $T \in \mathcal{L}^l(L^p(\mu), X^*)$ then by 4.1(iv) there exists a function $\phi \in L^q(\mu)$, $\phi \geq 0$ such that $\|Tf\| \leq \langle \phi, |f| \rangle$ for all $f \in L^p(\mu)$. Thus T has an extension to a bounded operator on $L^1(\phi d\mu)$, which we denote by T_1. Let $i : L^p(\mu) \to L^1(\phi d\mu)$ be the canonical embedding. Then $i \circ R(\lambda, A_0)$ is also represented by k. In order to show that $i \circ R(\lambda, A_0)$ is l-nuclear we have to verify that $k \in L^q(\mu)\tilde{\otimes}_l L^1(\phi d\mu) = L^q(\mu; L^1(\phi d\mu))$. By Jensen's inequality,

$$\begin{aligned}\int \Big| \int k(x,y)\phi(x)\, d\mu(x)\Big|^q d\mu(y) &\leq \int\int k(x,y)^q \phi(x)^q d\mu(x) d\mu(y) \\ &= \int \Big(\int k(x,y)^q d\mu(y)\Big) \phi(x)^q d\mu(x) \\ &\leq \Big(\sup_x \int k(x,y)^q d\mu(y)\Big) \cdot \|\phi\|_q^q.\end{aligned}$$

Thus $k \in L^q(\mu; L^1(\phi d\mu))$ and hence $i \circ R(\lambda, A_0)$ is l-nuclear. Then $T \circ R(\lambda, A_0) = T_1 \circ i \circ R(\lambda, A_0)$ is l-nuclear as well.

This criterion can be used for the translation group on $L^p(\mathbb{R})$ $(1 < p < \infty)$. In this case $R(\lambda, A_0)$ is given by

$$(R(\lambda, A_0)f)(x) = \int_x^\infty e^{\lambda(x-y)} f(y)\, dy,$$

so $k(x,y) = e^{\lambda(x-y)}\chi_{(x,\infty)}$. Hence for each x,

$$\int_{\mathbb{R}} k(x,y)^q\, dy = \int_x^\infty e^{\lambda q(x-y)}\, dy = \frac{1}{\lambda q}.$$

Therefore we obtain:

THEOREM 4.10: *Let $T_0(t)$ be the translation group on $L^p(\mathbb{R})$, $1 < p < \infty$. Then $L^p(\mathbb{R}; X)^\odot = L^q(\mathbb{R}; X^*)$.*

This example shows that the criterion from Remark 4.9 is weaker that the one of Theorem 4.7: for the translation group on $L^p(\mathbb{R})$ the resolvent is not compact and therefore certainly not r-compact.

We close with an application of Theorems 4.5 and 4.7 to vector valued $L^p(\mu)$-spaces.

THEOREM 4.11: *Consider a space $L^p(\mu)$, $1 < p < \infty$, and an arbitrary Banach space X.*

(i) *Given a C_0-semigroup $T_0(t)$ on X which is sun-reflexive, then the induced semigroup on $L^p(\mu; X)$ is sun-reflexive as well. Moreover,*

$$L^p(\mu, X)^\odot = L^q(\mu; X^\odot).$$

(ii) *Given a positive C_0-semigroup on $L^p(\mu)$ with r-compact resolvent, then for the semigroup induced on $L^p(\mu; X)$ we have $L^p(\mu; X)^\odot = L^q(\mu; X^*)$ and $L^p(\mu; X)^{\odot\odot} = L^p(\mu; X^{**})$.*

Proof: (i) ℓ^1 does not embed into the reflexive space $L^p(\mu)$. (ii) Since $L^p(\mu)$ is reflexive, $L^p(\mu)^\odot = L^q(\mu)$ is a sublattice of $L^q(\mu)$. ∎

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