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Hahn-Banach Type Theorems for Dual Semigroups

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In this paper it is shown that most of the Hahn-Banach theorems hold for the subspace $X^{\odot}$ of X^* of strong continuity of the dual of a C_0-semigroup. As an application we give a new proof of a recently discovered characterization of $\odot$-reflexivity.

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0. INTRODUCTION

Let X be a complex Banach space and A the generator of a C_0-semigroup $T(t)$. There exist real $M \geq 1$ and ω such that $\|T(t)\| \leq Me^{\omega t}$. It is well-known that $\{\lambda : Re\lambda > \omega\} \subset \varrho(A)$, the resolvent set of A. For such λ, we write $\mathrm{R}(\lambda,A)$ for $(\lambda I - A)^{-1}$.

It follows from the Hille-Yosida theorem that

$$\|\mathrm{R}(\lambda,A)\| \leq \frac{M}{Re\lambda - \omega}, \qquad (Re\lambda > \omega).$$

In this paper, we will use the symbol λ exclusively for *real* λ, $\lambda > \omega$.

The adjoint semigroup $T^*(t) = (T(t))^*$ is weak*-continuous; its weak*-generator is A^*, the adjoint of A. $T^*(t)$ need not be strongly continuous however, and therefore it makes sense to define the semigroup dual space $X^{\odot}$ as the subspace of X^* on which $T^*(t)$ is strongly continuous:

$$X^{\odot} = \{x^* \in X^* : \|T^*(t)x^* - x^*\| \to 0, \quad (t \downarrow 0)\}.$$

$X^{\odot}$ is the norm-closure of $D(A^*)$ and is a weak*-dense linear subspace of X^*, which is invariant under $T^*(t), \forall t \geq 0$. The restrictions $T^{\odot}(t)$ of $T^*(t)$ to $X^{\odot}$ form a C_0-semigroup on $X^{\odot}$, generated by $A^{\odot}$, the part of A^* in $X^{\odot}$. These facts are standard, see e.g. [1].

By applying the same construction to the semigroup $T^{\odot}(t)$, the second semigroup dual space $X^{\odot\odot}$ can be defined.

The map $j : X \to X^{\odot *}$,

$$< j(x), x^{\odot} > \quad = \quad < x^{\odot}, x >$$

is an embedding which maps X into $X^{\odot\odot}$ and hence we may regard X as a subspace of $X^{\odot\odot}$. X is called $\odot$-reflexive (with respect to $T(t)$) if $X = X^{\odot\odot}$. X is $\odot$-reflexive if and only if $X^{\odot}$ is; moreover, X is $\odot$-reflexive if and only if $\mathrm{R}(\lambda,A)$ is $\sigma(X, X^{\odot})$-compact [5]. Recently, B. de Pagter proved that X is $\odot$-reflexive if and only if $\mathrm{R}(\lambda,A)$ is weakly compact [4].

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Consider the trivial semigroup $T(t) = I$. It is easily seen that with respect to this semigroup the theorems above reduce to classical theorems about reflexivity. This observation suggests an analogy between the theories of X^* and $X^{\odot}$. One might ask if other theorems about duals have an analogon for $X^{\odot}$ too.

In this paper, it will be shown that for most of the Hahn-Banach theorems (see, for instance, [7]) this is indeed the case. *Invariance* under the semigroup turns out to be the relevant extra hypothesis to be imposed.

In the second part of this paper, the theory of the first part will be applied to study $\odot$-reflexivity. We will give a new proof of de Pagter's characterization of $\odot$-reflexivity.

1 EXTENSION AND SEPARATION THEOREMS

In this section some extension- and separation theorems for $X^{\odot}$ will be deduced.

Let F be a closed subspace of X. On F^* define a norm as usual:

$$\|f^*\| = \sup_{f\in F, \|f\|=1} |< f^*, f >| \quad (f^* \in F^*).$$

Denote by A_F the part of A in F; let $A_F^* : \; F^* \to F^*$ be its adjoint.

Theorem 1.1. *Let $T(t)$ be a C_0-semigroup, $\|T(t)\| \le Me^{\omega t}$. Suppose F is a closed subspace of X, invariant under $T(t), \forall t \ge 0$. Let $f^{\odot} \in F^{\odot}$. Then for each $\epsilon > 0$ there is an element $x^{\odot} \in X^{\odot}$ such that*

$$\|x^{\odot}\| < M\|f^{\odot}\| + \epsilon$$

and

$$x^{\odot}|_F = f^{\odot}.$$

Moreover, if $f^{\odot} \in D(A_F^*)$ then we may choose $x^{\odot} \in D(A^*)$.

Proof:

From the conditions on F it follows that $F^{\odot}$ is well-defined and is the closure of $D(A_F^*)$. Fix $f^{\odot} \in D(A_F^*)$ and $\epsilon > 0$. Since $\limsup_{\lambda>\omega,\lambda\to\infty} \|\lambda R(\lambda,A^*)\| \le M$ and $(I - A_F^*/\lambda)f^{\odot} \to f^{\odot} \quad (\lambda \to \infty)$ in the norm topology of F^*, we can choose $\lambda = \lambda(f^{\odot})$ such that

$$\|R(\lambda,A^*)\| \, \|(\lambda I - A_F^*)f^{\odot}\| = \|\lambda R(\lambda,A^*)\| \, \|(I - A_F^*/\lambda)f^{\odot}\| < M\|f^{\odot}\| + \epsilon.$$

Put $f^* = (\lambda I - A_F^*)f^{\odot}$. Then $f^* \in F^*$ and f^* can be extended to some $x^* \in X^*$ such that

$$|< x^*, x >| \quad \le \quad \|f^*\| \, \|x\| \quad \forall x \in X.$$

Put $x^{\odot} = R(\lambda,A^*)x^*$. Then $x^{\odot} \in D(A^*)$ extends $f^{\odot}$, and

$$\begin{aligned} < x^{\odot}, x > \;&=\; < x^*, R(\lambda,A)x > \;\le\; \|f^*\| \, \|R(\lambda,A^*)\| \, \|x\| \\ &<\; (M\|f^{\odot}\| + \epsilon) \, \|x\| \quad \forall x \in X. \end{aligned}$$

So

$$\|x^{\odot}\| < M\|f^{\odot}\| + \epsilon.$$

Now let $f^{\odot} \in F^{\odot}$. Without loss of generality assume that $\|f^{\odot}\| = 1$. Fix some $k > 2 + 4M/\epsilon$ and choose a sequence

$$(f_n^{\odot})_{n\ge1} \to f^{\odot}, \quad f_n^{\odot} \in D(A_F^*), \quad \|f_n^{\odot}\| = 1, \quad \forall n,$$

such that $\|f^{\odot}_{n+1} - f^{\odot}_n\| \leq 1/kn^2$, which is always possible since $F^{\odot}$ is the closure of $D(A^*_F)$. Choose $(y^{\odot}_n)_{n\geq 0} \subset D(A^*)$, such that $y^{\odot}_0$ extends $f^{\odot}_1$, $y^{\odot}_n$ extends $f^{\odot}_{n+1} - f^{\odot}_n$ $(n \geq 1)$,

$$\|y^{\odot}_0\| < M + \frac{\epsilon}{2}, \quad \|y^{\odot}_n\| < (M + \frac{\epsilon}{2})/kn^2 \quad (n \geq 1).$$

From this construction it follows that $\sum y^{\odot}_n$ converges to some $x^{\odot}$, which is in $X^{\odot}$, by the closedness of $X^{\odot}$. Since $\sum_{m=0}^{n-1} y^{\odot}_m$ is an extension of $f^{\odot}_n$, it follows that $x^{\odot}$ is an extension of $f^{\odot}$, which furthermore satisfies

$$\|x^{\odot}\| < (M + \frac{\epsilon}{2})(1 + \sum_{n=1}^{\infty} \frac{1}{kn^2}) < (M + \frac{\epsilon}{2})(1 + \frac{2}{k}) < M + \epsilon.$$

□

The following example shows that the inequality in Theorem 1.1 cannot be sharpened to $\|x^{\odot}\| \leq M\|f^{\odot}\|$.

Example 1.2.

Let $X = C_0[0,\infty)$, the space of continuous complex-valued functions vanishing at infinity, provided with the supnorm. It is well-known [1] that

$$T(t)f(x) = f(x+t)$$

defines a C_0-contraction semigroup, whose semigroup dual space $X^{\odot}$ is $L^1[0,\infty)$, the action of $g \in X^{\odot}$ on $C_0[0,\infty)$ being given by

$$< g, f > \; = \int_0^{\infty} f(x)g(x)dm(x).$$

($m(x)$ denotes the Lebesgue measure on $[0,\infty)$). Put $F = F_1 \oplus F_2$; $F_1 = \{f \in X : f(x) = 0, \forall x \geq 1\}$, $F_2 =$ the one-dimensional subspace spanned by the function e^{-x}. F is closed and invariant under $T(t), \forall t \geq 0$. Put

$$< f^{\odot}, f > \; = \; f(1) \quad (f \in F)$$

then it is easily verified that $f^{\odot} \in F^{\odot}$ and $\|f^{\odot}\| = 1$. Let $g \in L^1[0,\infty)$ be any extension of $f^{\odot}$. Since g vanishes on F_1, it has support in $[1,\infty)$. Pick $\delta > 1$ such that

$$\int_1^{1+\delta} |g(x)|dm(x) < \|g\|.$$

Since g extends $f^{\odot}$, we have

$$e^{-1} = \; < f^{\odot}, e^{-x} > \; = \int_0^{\infty} g(x)e^{-x}dm(x) = \int_{1+\delta}^{\infty} g(x)e^{-x}dm(x) + \int_1^{1+\delta} g(x)e^{-x}dm(x)$$

$$\leq e^{-1} \int_1^{1+\delta} |g(x)|dm(x) \; + \; e^{-(1+\delta)} \int_{1+\delta}^{\infty} |g(x)|dm(x) < e^{-1}\|g\|.$$

Hence $\|g\| > 1 = \|f^{\odot}\|$. ◇

In typical situations, the requirement that F should be invariant is not fulfilled. Nevertheless, the following lemma shows that a certain class of funtionals still can be extended. First we need some definitions.

Let $F \subset X$ be a linear subspace such that $F \cap D(A)$ is dense in F. Define $D(A^*_F)$ as the collection of $x^* \in X^*$ for which there exists an element $f^* \in F^*$ such that

$$< f^*, f > \quad = \quad < x^*, Af >, \quad \forall f \in D(A) \cap F.$$

In this case, put $A^*_F : X^* \to F^*; \quad A^*_F x^* = f^*$. Since $D(A) \cap F$ is dense in F, $A^*_F x^*$ is well-defined as an element of F^*.

Lemma 1.3 . *Let $f^\odot \in D(A^*_{R(\lambda,A)F})$. Then there is a $x^\odot \in D(A^*)$ such that $x^\odot|_F = f^\odot|_F$.*

Proof:

Put $f^* = \lambda f^\odot - A^*_{R(\lambda,A)F} f^\odot$. Then $f^* \in (R(\lambda,A)F)^*$. By the Hahn-Banach theorem, f^* can be extended to an element $x^* \in X^*$. Put $x^\odot = R(\lambda,A^*)x^*$, then $x^\odot \in D(A^*)$ and it obviously extends $f^\odot$. □

Lemma 1.4 . *Let A be the generator of a C_0-semigroup $T(t)$ on a Banach space X. Let $G \subset X$ be a convex set. Then $\lambda R(\lambda,A)G \subset \overline{G}$ if and only if $T(t)G \subset \overline{G} \quad \forall t \geq 0$.*

Proof:

Suppose $T(t)G \subset \overline{G} \quad \forall t \geq 0$. It follows directly from

$$\lambda R(\lambda,A)x = \int_0^\infty \lambda e^{-\lambda t} T(t)x dt$$

that $\lambda R(\lambda,A)x \in \overline{G}$ if $x \in G$, since G is convex and $\lambda e^{-\lambda t} dt$ is a probability measure on $[0,\infty)$. The other half is proved analogously, using the inverse Laplace formula [6]

$$T(t)x = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\mu t} R(\mu, A) x d\mu \quad (\gamma > max(0,\omega)).$$

□

We will use Lemma's 1.3 and 1.4 to derive a semigroup version of a standard separation theorem.

Theorem 1.5. *Let F be a closed subspace of X, invariant under $T(t), \forall t \geq 0$. Let $y \notin F$. Then there is a $x^\odot \in D(A^*)$ such that*

$$< x^\odot, x > \quad = \quad 0 \quad \forall x \in F; \qquad < x^\odot, y > \quad = \quad 1.$$

Proof:

Let $f^\odot \in X^*$ be any functional for which

$$< f^\odot, f + ty > \quad = \quad t \quad (f \in F, t \in \mathbb{C}).$$

By the Hahn-Banach theorem such functionals exist. Let G be the subspace spanned by F and y. We claim that $f^{\odot} \in D(A^*_{\mathrm{R}(\lambda,A)G})$. Define $g^* \in (\mathrm{R}(\lambda,A)G)^*$ by

$$< g^*, \mathrm{R}(\lambda,A)(f+ty) > \quad = \quad t < f^{\odot}, A\mathrm{R}(\lambda,A)y > .$$

By the closedness and invariance of F it follows from Lemma 1.4 that $\mathrm{R}(\lambda,A)f \in F$, hence $A\mathrm{R}(\lambda,A)f \in F$ and so $< f^{\odot}, A\mathrm{R}(\lambda,A)f >= 0$ for all $f \in F$. Therefore $f^{\odot} \in D(A^*_{\mathrm{R}(\lambda,A)G})$ and

$$A^*_{\mathrm{R}(\lambda,A)G} f^{\odot} = g^*.$$

Now Lemma 1.3 applies. □

Theorem 1.5 can be obtained also as a simple consequence of the following more general separation theorem.

Theorem 1.6. *Let A be the generator of a C_0-semigroup $T(t)$ on X. Let $G \subset X$ be a closed convex set, invariant under $T(t)$, $\forall t \geq 0$. Let K be a convex compact set, $G \cap K = \emptyset$. Then there are $x^{\odot} \in D(A^*)$ and real constants $\gamma_1 < \gamma_2$ such that for all $x \in G, y \in K$:*

$$Re < x^{\odot}, x > \quad \leq \gamma_1 < \gamma_2 \leq \quad Re < x^{\odot}, y > .$$

Moreover, if G is balanced, then $x^{\odot}$ can be chosen such that

$$| < x^{\odot}, x > | \quad \leq \gamma_1 < \gamma_2 \leq \quad | < x^{\odot}, y > |.$$

Proof:

Define the set $G_\lambda = (I - A/\lambda)(G \cap D(A))$. Take $y \in K$. Since $\lambda\mathrm{R}(\lambda,A)y \to y \quad (\lambda \to \infty)$, there is a λ such that $\lambda\mathrm{R}(\lambda,A)y \notin G$. Since K is compact, we may even choose λ so that this holds for all $y \in K$, i.e., $G \cap \lambda\mathrm{R}(\lambda,A)K = \emptyset$. By the Hahn-Banach separation theorem, there are $x^* \in X^*$ and real constants $\gamma_1 < \gamma_2$ such that for all $x \in G$ and $y \in K$,

$$Re < x^*, x > \quad \leq \gamma_1 < \gamma_2 \leq \quad Re < x^*, \lambda\mathrm{R}(\lambda,A)y > .$$

In particular this is true for elements $x \in G \cap D(A)$. Hence, defining $z = (I - A/\lambda)x \in G_\lambda$, we have

$$Re < \lambda\mathrm{R}(\lambda,A^*)x^*, z > \quad = \quad Re < x^*, x > \quad \leq \gamma_1 < \gamma_2 \leq$$

$$Re < x^*, \lambda\mathrm{R}(\lambda,A)y > \quad = \quad Re < \lambda\mathrm{R}(\lambda,A^*)x^*, y > .$$

Since $D(A) \cap G$ is closed in $D(A)$ (with respect to the norm-topology that $D(A)$ inherits from X) and $\mathrm{R}(\lambda,A)$ is a continuous map from X onto $D(A)$, $G_\lambda = (\lambda\mathrm{R}(\lambda,A))^{-1}(D(A) \cap G)$ is closed in X. It then follows from Lemma 1.4 (applied to G_λ) that $G \subset G_\lambda$. Therefore $\lambda\mathrm{R}(\lambda,A^*)x^*$ has the required properties. Finally, if G is convex and balanced, then note that the image of G under $x^{\odot}$ is also convex and balanced in $\mathbb{C}$ and does not contain $< x^{\odot}, y > \quad (y \in K)$. Hence it must be a multiple of the unit disc. From this it is clear that

$$| < x^{\odot}, x > | \quad \leq \gamma_1 < \gamma_2 \leq \quad | < x^{\odot}, y > | \qquad (x \in G, y \in K).$$

□

Let now G in Theorem 1.6 be a closed *subspace* of X, invariant under $T(t), \forall t \geq 0$ and $K = \{y\}, y \notin G$. It follows that there is some $x^{\odot} \in D(A^*)$ under which G has a bounded image. On the other hand, this image must be a subspace of $\mathbb{C}$, which forces $< x^{\odot}, G >= 0$. Hence $< x^{\odot}, y >\neq 0$. Multiplying $x^{\odot}$ with the right scalar gives $< x^{\odot}, y >= 1$.

Example 1.7 .

Let X and $T(t)$ be as in Example 1.2. Put $F = \{f \in X : f(0) = 0\}$. Then F is a closed subpace of X. If g is any L^1-function that vanishes on F, then it vanishes on X, i.e., $g = 0$ a.e., as is easily seen from Lebesgue's dominated convergence theorem. We conclude that *invariance* cannot be omitted from the hypotheses in Theorems 1.5 and 1.6. ◇

The topology that $X^{\odot}$ induces on X will be denoted by the $\odot$- *topology*. Since $X^{\odot}$ separates points on X (apply Theorem 1.5 with $F = \{0\}$!), this topology makes X into a locally convex topological vector space. In referring to this topology we will use notions like $\odot$-closed, $\odot$- compact, etc.

Corollary 1.8. *Let $G \subset X$ be convex and invariant under $T(t), \forall t \geq 0$. Then G is closed if and only if it is $\odot$-closed.*

Proof:

Immediate from Theorem 1.6. □

Bounded sequences of continuous functions in $C[0,1]$ that converge pointwise to some continuous function admit convex combinations that converge uniformly [7, Thm 3.13]. We will apply Corollary 1.8 to deduce the analogon for *almost everywhere* pointwise convergent sequences of functions.

Theorem 1.9. *Let (x_n) be a sequence that converges to some $x \in X$ in the $\odot$-topology. Then there are numbers $\alpha_{in} \geq 0$ and $t_{in} \geq 0$ such that*

$$y_i = \sum_{n=1}^{\infty} \alpha_{in} T(t_{in}) x_n \to x \quad \textit{strongly},$$

and for each i, $\sum_n \alpha_{in} = 1$ and only finitely many α_{in} are nonzero.

Proof:

Let H_1 be the set $\{T(t)x_n : n \in \mathbb{N}, t \geq 0\}$. Let H be the convex hull of H_1. Then both H and its closure are convex and invariant under all $T(t)$, and by Corollary 1.8 its norm-closure and its $\odot$-closure are the same. Now x belongs to the $\odot$-closure by assumption, and it follows from metric space theory that there is some sequence $(y_i) \subset H$ norm-converging to x. □

Define on $C_0[0,1] = \{f \in C[0,1] : f(1) = 0\}$ the C_0-semigroup $T(t)$ of left-translations by

$$T(t)f(x) = \begin{cases} f(x+t), & x \leq 1-t; \\ 0, & \text{elsewhere}. \end{cases}$$

Corollary 1.10 . *Let $(f_n) \subset C_0[0,1]$ be a bounded sequence of functions, converging almost everywhere (with respect to the Lebesgue measure) to some $f \in C_0[0,1]$. Then there is a sequence of convex combinations of left-translates of f_n that converges uniformly to f.*

Proof:

The semigroup adjoint space $X^{\odot}$ is $L^1[0,1]$. By Lebesgue's dominated convergence theorem, a.e. pointwise convergence implies $\odot$-convergence, and the result follows from Theorem 1.9. □

2. $\odot$-REFLEXIVITY

The ideas of section 1 will now be applied to study $\odot$-reflexivity. We will give a new proof to the theorem that X is $\odot$-reflexive iff $R(\lambda,A)$ is *weakly* compact [4]. From now on let B ($B^{\odot *}$) denote the *closed* unit ball of X ($X^{\odot *}$).

It is well-known [5] that X is $\odot$-reflexive iff $R(\lambda,A)$ is $\sigma(X,X^{\odot})$-compact. From this the following lemma follows easily.

Lemma 2.1 . *Let $F \subset X$ be a closed subspace, invariant under $T(t), \forall t \geq 0$. If X is $\odot$-reflexive, then F is $\odot$-reflexive too.*

Proof:

By assumption the image $R(\lambda,A)B$ of the unit ball B of X is relatively $\odot$-compact and so is $(R(\lambda,A)B) \cap F$, since F is $\odot$-closed by Theorem 1.5. By Lemma 1.4, $R(\lambda,A)(B \cap F) \subset (R(\lambda,A)B) \cap F$ and so $R(\lambda,A)(B \cap F)$ is relatively $\odot$-compact. Since the topology induced by $F^{\odot}$ on F is weaker than the one induced by $X^{\odot}$ on F, $R(\lambda,A)(B \cap F)$ is relatively compact in the $F^{\odot}$-topology of F. □

Lemma 2.2 . *If $X^{\odot}$ is separable, then X is separable.*

Proof:

Let $B^{\odot}$ be the unit ball of $X^{\odot}$ and let $(x_n^{\odot}) \subset B^{\odot}$ be a countable dense set. Choose $(x_n) \subset X, \|x_n\| = 1$ such that $| < x_n^{\odot}, x_n > | \;\; > \;\; \frac{1}{2}$. Let F be the closed subspace spanned by the set $\{T(t)x_n : n \in \mathbb{N}, t \geq 0\}$. F is separable and invariant under $T(t), \forall t \geq 0$. Suppose there is some $y \notin F$. By Theorem 1.5, there is an element $x^{\odot} \in B^{\odot}$ that annihilates F and is nonzero at y. But then

$$\frac{1}{2} \leq | < x_n^{\odot}, x_n > | \leq | < x^{\odot} - x_n^{\odot}, x_n > | + | < x^{\odot}, x_n > | =$$

$$= | < x^{\odot} - x_n^{\odot}, x_n > | \leq \|x^{\odot} - x_n^{\odot}\|,$$

a contradiction to the density of $(x_n^{\odot})$ in $B^{\odot}$. This shows $F = X$ and hence X is separable. □

Theorem 2.3 . *If X is $\odot$-reflexive, then B is relatively weak*-sequentially compact in $X^{\odot *}$.*

Proof:

Let $(x_n) \subset B$ be a countable set. We have to show that there is an element $x^{\odot *} \in X^{\odot *}$ and a subsequence (x_{n_i}) such that for $i \to \infty$,

$$< x^{\odot}, x_{n_i} > \;\; \to \;\; < x^{\odot *}, x^{\odot} > \quad \forall x^{\odot} \in X^{\odot}.$$

Let Y be the closed linear span of $\{T(t)x_n : n \in \mathbb{N}, t \geq 0\}$. Y is separable and invariant under $T(t), \forall t \geq 0$. By Lemma 2.1, $Y^{\odot\odot} = Y$ is separable and hence $Y^{\odot}$ is separable, by Lemma 2.2. Let $H = (y_m^{\odot})$ be a countable dense set in Y. Since (x_n) is bounded, by a diagonalization argument we find a subsequence (x_{n_i}) such that $< y_m^{\odot}, x_{n_i} >$ converges for all m. By considering the x_{n_i} as elements of $X^{\odot *}$ it is seen from the Banach-Steinhaus theorem that there is a $y^{\odot *} \in Y^{\odot *}$ such that

$$< y_m^{\odot}, x_{n_i} > \;\; \to \;\; < y^{\odot *}, y_m^{\odot} > \quad \forall y_m^{\odot} \in H.$$

From the denseness of $(y^{\odot}_m)$ in H it follows that

$$< y^{\odot}, x_{n_i} > \quad \rightarrow \quad < y^{\odot *}, y^{\odot} > \quad \forall y^{\odot} \in Y^{\odot}.$$

Now define a functional $x^{\odot *}$ on $X^{\odot}$ by

$$< x^{\odot *}, x^{\odot} > \quad = \quad < y^{\odot *}, x^{\odot}|_Y >,$$

$x^{\odot}|_Y$ denoting the restriction of $x^{\odot}$ to Y. Then $x^{\odot *}$ is linear and continuous: If $x^{\odot}_n \rightarrow x^{\odot}$ in $X^{\odot}$, then also $x^{\odot}_n|_Y \rightarrow x^{\odot}|_Y$ in $Y^{\odot}$ and hence

$$< x^{\odot *}, x^{\odot}_n > \quad = \quad < y^{\odot *}, x^{\odot}_n|_Y > \quad \rightarrow \quad < y^{\odot *}, x^{\odot}|_Y > \quad = \quad < x^{\odot *}, x^{\odot} > .$$

So $x^{\odot *} \in X^{\odot *}$. Since each $x_{n_i} \in Y$, we also have

$$< x^{\odot}, x_{n_i} > \quad = \quad < x^{\odot}|_Y, x_{n_i} > \quad \rightarrow \quad < y^{\odot *}, x^{\odot}|_Y > \quad = \quad < x^{\odot *}, x^{\odot} > \quad \forall x^{\odot} \in X^{\odot}.$$

□

Before turning to the characterization of $\odot$-reflexivity, we note that from Theorem 2.3 two natural questions arise:

1. Is $B^{\odot *}$ itself weak*-sequentially compact?
2. Is $B^{\odot *}$ the weak*-sequential closure of B in $X^{\odot *}$?

The next theorem supplies a (partial) answer.

Theorem 2.4 . *Suppose X is separable and $\odot$-reflexive. Then $B^{\odot *}$ is weak*-sequentially compact. Moreover, $B^{\odot *}$ is the weak*-sequential closure of B in $X^{\odot *}$.*

Proof:

$X^{\odot\odot} = X$ is separable and so is $X^{\odot}$ by Lemma 2.2. Hence $B^{\odot *}$ is metrizable, by a well-known metrizability theorem [7]. Since $B^{\odot *}$ is also weak*-compact by the Banach-Alaoglu theorem, it follows that $B^{\odot *}$ is weak*-sequentially compact. Since $B \subset B^{\odot *}$ is weak*-dense (this is proved in much the same way as the weak*-denseness of the inclusion $B \subset B^{**}$), the second statement is just a simple consequence of metric space theory. □

If X is separable, the proof of Theorem 2.3 is much simpler. Indeed, we now just have to appeal to the first part of Theorem 2.4.

Theorem 2.5. *X is $\odot$-reflexive if and only if $R(\lambda,A)$ is weakly compact.*

Proof:

If $R(\lambda,A)$ is weakly compact, then it certainly is $\sigma(X, X^{\odot})$-compact, and therefore X is $\odot$-reflexive. Conversely, if X is $\odot$-reflexive, then $R(\lambda,A)B$ is relatively weakly sequentially compact. To see this, let (x_n) be a countable subset in $R(\lambda,A)B$. Write $x_n = R(\lambda,A)y_n$, $y_n \in B$. By Theorem 2.3 there is a $y^{\odot *} \in X^{\odot *}$ and a subsequence (y_{n_i}) of (y_n) such that

$$< x^{\odot}, y_{n_i} > \quad \rightarrow \quad < y^{\odot *}, x^{\odot} > \quad \forall x^{\odot} \in X^{\odot}.$$

Applying this to elements $R(\lambda,A^*)x^* \in D(A^*) \subset X^{\odot}$ we see that

$$< x^*, R(\lambda, A)x_{n_i} > \quad \rightarrow \quad < R(\lambda, A^{\odot *})y^{\odot *}, x^* > \quad \forall x^* \in X^*.$$

But $R(\lambda, A^{\odot *})y^{\odot *} \in D(A^{\odot *}) \subset X^{\odot\odot} = X$. This proves our claim. By the Eberlein-Shmulyan theorem, $R(\lambda,A)B$ is relatively weakly compact, i.e., $R(\lambda,A)$ is weakly compact. □

Note that weak limits of subsequences in $R(\lambda,A)B$ are found to lie in $D(A^{\odot *})$.

It is tempting to conjecture that X is $\odot$-reflexive iff B is (relatively) (sequentially) $\odot$-compact. We will show that only the ' if'-part is true. In fact we have the following

Example 2.6 .

Let X and $T(t)$ be as in Corollary 1.10. It is well-known that $X^{\odot} = L^1[0,1]$ and X is $\odot$-reflexive with respect to $T(t)$ [2]. Let f_n be the function

$$f_n(x) = \begin{cases} 1, & x \leq \frac{1}{2}; \\ \frac{n}{2} + 1 - nx, & \frac{1}{2} \leq x \leq \frac{1}{2} + \frac{1}{n}; \\ 0, & \text{else.} \end{cases}$$

By Lebesgue's dominated convergence theorem, each subsequence $f_{n_i} \rightarrow \chi_{[0,\frac{1}{2}]}$ in the $\odot$-topology of X. But $\chi_{[0,\frac{1}{2}]}$ does not belong to X (however, it does belong to $L^{\infty}[0,1] = X^{\odot *}$!). Thus $\odot$-reflexivity does not imply relative sequential $\odot$-compactness of B. ◇

Theorem 2.7 . *If B is relatively sequentially $\odot$-compact, then X is $\odot$-reflexive.*

Proof:

Let $R(\lambda,A)(x_n) \subset R(\lambda,A)B$ be a sequence. By assumption there is a subsequence (x_{n_i}) of (x_n) and an element $x_0 \in X$ such that

$$< x^{\odot}, x_{n_i} > \quad \rightarrow \quad < x^{\odot}, x_0 > \quad \forall x^{\odot} \in X^{\odot}.$$

In particular this is true for elements $R(\lambda,A^*)x^* \in D(A^*)$. Thus

$$< x^*, R(\lambda,A)x_{n_i} > \quad \rightarrow \quad < x^*, R(\lambda,A)x_0 > \quad \forall x^* \in X^*.$$

This shows that $R(\lambda,A)B$ is relatively weakly sequentially compact, and therefore $R(\lambda,A)$ is weakly compact by the Eberlein-Shmulian theorem. □

The hypothesis of Theorem 2.7 can be weakened to relative $\odot$-compactness of B, as is seen from the following theorem:

Theorem 2.8 . *The implications i ⇒ ii ⇒ iii hold:*
i.B is relatively $\odot$-compact
ii. Every countable set in B has a $\odot$-limit point in X
iii.B is relatively sequentially $\odot$-compact.

Proof:

i ⇒ *ii*: Trivial. *ii* ⇒ *iii*: Using our semigroup versions of the Hahn-Banach theorems, the proof of the corresponding theorem for weak compactness, as e.g. given in Dunford and Schwartz [3], can be carried over almost word for word. □

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