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WEAK LIMITS OF RESIDUAL LIFE TIMES

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Introduction

Consider a light bulb. It has a certain life time X, which is a random variable with probability distribution function F. After having burned t hours, there remains a residual life time, with a distribution function F_t defined by

$$(1) \qquad 1 - F_t(x) = P\{X-t > x \mid X > t\}.$$

The residual life time is a random variable X_t which is defined on the conditional probability space $\{X > t\}$.

If X has an exponential distribution $F(x) = 1-e^{-\lambda x}$, then so has X_t. Indeed

$$1 - F_t(x) = P\{X > x+t \mid X > t\} = \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x}.$$

It is wellknown that this characterizes the exponential distribution: $F_t = F_0$ for all $t > 0$ implies that $F_0(x) = 1-e^{-\lambda x}$ for some $\lambda > 0$.

In this paper we shall be concerned with the limit behaviour of the residual life time for $t \to \infty$: For which distribution functions F does there exist a norming function a(t) such that $X_t/a(t)$ converges in distribution to a random variable with non-degenerate distribution function G?; What are the possible limit distributions G? (section 2); what are their domains of attraction? (section 3). In section 4 we prove the remarkable fact that in this situation weak convergence is equivalent to the convergence of some positive moment.

One of the reasons for publishing these investigations is that they give a probabilistic interpretation of the fundamental properties of regularly varying functions.

Although the label "residual life time" gives clear intuitive meaning to the random variables X_t associated with the distribution functions F_t defined above, the field of applications of this theory in probability theory is much wider. In many cases one is not so much interested in all values of a random variable X as in extremely large values (for instance in the study of heat-waves or storms). One restricts one's attention to the subset $\{X > t\}$ i.e. to the set of large values of X and in fact one is studying the limit behaviour of the probability distributions F_t defined above.

Results and techniques from extreme value theory are used. The possible limit types for X_t are

$$\Xi_\alpha(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1-(1+x)^{-\alpha} & \text{for } x \geq 0, \end{cases}$$

$$\Pi\ (x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{x} & \text{for } x \geq 0, \end{cases}$$

where α is a positive constant. The domains of attraction of these distribution functions are exactly the domains of attraction of the well-known (see [2]) limit distributions Φ_α and Λ of the partial maxima of i.i.d. random variables.

If we allow as norming functions for X_t both a scale transformation $a(t)$ and a shift $b(t)$, then it is possible to obtain limit distributions of $(X_t-b(t))/a(t)$ which are discrete. This remarkable result is closely linked to the problem of convergence of $(X_t-b(t))/a(t)$ where t tends to infinity through some discrete subset of R^+. The necessary theory to tackle this more general situation is being developed by A.A. Balkema (who also suggested the problem of limit distributions for residual life times). We hope to publish the theory in the general situation in a subsequent paper.

1. The possible limit distributions

We say that the distribution function F is in the domain of r.l.t. attraction of a non-degenerate distributions function G (notation $F \in D_r(G)$) if for some positive function a and all continuity points x of G

$$\lim_{t\to\infty} F_t(a(t)x) = G(x)$$

where F_t is defined by (1). To find the possible limit laws G we derive a functional equation for G.

Lemma 1

A non-degenerate distribution function G has a non-empty domain of r.l.t. attraction if and only if $G(x) < 1$ for all x and there exists a positive function A such that

$$(2) \qquad 1-G(y+x\,A(y)) = (1-G(x))(1-G(y))$$

for all positive continuity points x and y of G.

Proof

If (2) holds and $G(x) < 1$ for all positive x, then $G \in D_r(G)$ and hence G has a non-empty domain of r.l.t. attraction.

Conversely suppose for some distribution dunction F and for some positive function a we have

$$(3) \qquad \lim_{t\to\infty} F_t(a(t)x) = \lim_{t\to\infty} \frac{1-F(t+x\,a(t))}{1-F(t)} = 1-G(x)$$

for all positive continuity points x of G. Then $t+xa(t) \to \infty$ for all such x. Replacing t by $t+ya(t)$ in (3) (where y is some positive continuity point of G) we get

$$\lim_{t\to\infty} \frac{1-F(t+ya(t) + xa(t+y\,a(t)))}{1-F(t+y\,a(t))} = 1-G(x).$$

Using (3) this reduces to

$$(4) \qquad \lim_{t\to\infty} \frac{1-F(t+y\,a(t) + x\,a(t+y\,a(t)))}{1-F(t)} = (1-G(x))(1-G(y)).$$

Now we use a device similar to that of the well-known Khinchine-Gnedenko lemma (see e.g. [1] p. 246). We shall prove that $a(t+y\,a(t))/a(t)$ has a limit $A(y)$ for all positive continuity points y of G as $t \to \infty$. Suppose for some y there is no limit. Then there exist tweo sequences $t_{1,n} \to \infty$ and $t_{2,n} \to \infty$ such that

$$\lim_{n\to\infty} \frac{a(t_{i,n} + y\,a(t_{i,n}))}{a(t_{i,n})} = A_i \text{ for } i = 1,2$$

with $0 \leq A_1 < A_2 \leq \infty$. Because the function a in (3) is defined up to an asymptotic equivalence, from (3) it follows

$$\lim_{n\to\infty} \frac{1-F(t_{i,n} + (y + x \frac{a(t_{i,n} + y\, a(t_{i,n}))}{a(t_{i,n})} a(t_{i,n}))}{1-F(t_{i,n})} = 1-G(y+xA_i)$$

for i = 1, 2 and hence by (4)

$$1-G(y + xA_1) = 1-G(y + xA_2) = (1-G(x))(1-G(y)).$$

As this is true for all positive continuity points x of G, we must have $0 < A_1 = A_2 < \infty$. So (2) is true and

$$\lim_{t\to\infty} \frac{a(t + y\, a(t))}{a(t)} = A(y) \tag{5}$$

Finally we prove $G(x) < 1$ for all x.
Suppose for some positive x_0 we have $G(x_0) = 1$ and $G(x) < 1$ for $x < x_0$. Then (2) gives

$$y + x\, A(y) < x_0 \qquad \text{for all } 0 < x,y < x_0$$

$$y + x_0\, A(y) \geq x_0 \qquad \text{for all } 0 < y < x_0$$

and hence $A(y) = 1 - x_0^{-1}y$. By (5) we have

$$\lim_{t\to\infty} \frac{a(t + y\, a(t))}{a(t)} = 1 - \frac{y}{x_0} .$$

Take for y a continuity point of G from $(0,x_0)$ and t_1 such that for $t \geq t_1$

$$\frac{1-F(t + y\, a(t))}{1-F(t)} < 1-G(y/2). \tag{6}$$

We define the sequence $\{t_n\}$ by

$$t_{n+1} = t_n + y\, a(t_n) \qquad \text{for } n = 1,2,\ldots .$$

Clearly this sequence is strictly increasing. Suppose t_n tends to some finite limit K for $n \to \infty$, then

$$\lim_{n\to\infty} \frac{1-F(t_n + y\, a(t_n))}{1-F(t_n)} = \frac{\lim_{n\to\infty} 1-F(t_{n+1})}{\lim_{n\to\infty} 1-F(t_n)} = 1$$

in contradiction with (6). Hence $t_n \to \infty$ for $n \to \infty$. Further we have

$$\frac{t_{n+1} - t_n}{t_n - t_{n-1}} = \frac{a(t_n)}{a(t_{n-1})} = \frac{a(t_{n-1} + y\, a(t_{n-1}))}{a(t_{n-1})} \to 1 - \frac{y}{x_0} \text{ for } n \to \infty.$$

Hence for $n \geq n_0$

$$t_{n+1} - t_n < (1 - \frac{y}{2x_0})(t_n - t_{n-1})$$

and by repeated application

$$t_{n+1} - t_n < (1 - \frac{y}{2x_0})^{n-n_0+1} (t_{n_0} - t_{n_0-1}).$$

Adding these inequations for $n = n_0, \ldots, N$ we obtain

$$t_{N+1} - t_{n_0} < (t_{n_0} - t_{n_0-1}) \sum_{n=n_0}^{N} (1 - \frac{y}{2x_0})^{n-n_0+1} \leq$$

$$\leq (t_{n_0} - t_{n_0-1}) \sum_{n=n_0}^{\infty} (1 - \frac{y}{2x_0})^{n-n_0+1} < \infty$$

which contradicts $t_N \to \infty$ for $N \to \infty$. By contradicton we now have proved $G(x) < 1$ for all x. □

Theorem 1

The distribution functions with non-empty domain of r.l.t. attraction are of the following types:

(7) $$\Xi_\alpha(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - (1+x)^{-\alpha} & \text{for } x \geq 0 \end{cases}$$

with $\alpha > 0$, or

(8) $$\Pi\,(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 - e^{-x} & \text{for } x \geq 0. \end{cases}$$

Proof

Suppose G has a non-empty domain of r.l.t.-attraction.
First we prove $G(0) = 0$ and $G(x) > 0$ for $x > 0$. By letting $x \downarrow 0$ in (2) we easily see $G(0) = 0$. Suppose for some $x > 0$ we have $G(x) = 0$, then from (2) it follows that $G(y) = G(y+xA(y))$ for all continuity points $y > 0$ and this is impossible.

Next we write (2) in the following form

$$G(y+xA(y)) - G(y) = G(x)(1-G(y)).$$

By lemma 1 we have $G(y) < 1$, hence the righthand side is positive for all positive continuity points x and y. Thus $G(y)$ is strictly increasing for all positive y.

Interchanging x and y in (2) we obtain

$$G(y+xA(y)) = G(x+yA(x))$$

for all positive continuity points x and y of G. Since G is strictly increasing, this gives

$$y + xA(y) = x + yA(x),$$

i.e.

$$\frac{A(x)-1}{x} = \text{constant}$$

and A has the form

$$A(x) = 1 + c \, . \, x$$

where c is a real number. Substitution in (2) gives

$$(9) \qquad 1 - G(x+y+cxy) = (1-G(x))(1-G(y)).$$

This relation holds for all positive continuity points x and y of G and hence for all positive x and y.

(i) Suppose $c < 0$. This is impossible because now for $y > -c^{-1}$ the lefthand side of (9) is a decreasing function of x.

(ij) Suppose $c = 0$. It is well-known (see e.g. [1] p. 8) that all solutions of (9) are of type Π.

(iij) Suppose $c > 0$. Define the distribution function $\tilde{G}$ by

$$\tilde{G}(x) = G(\frac{x-1}{c}) \qquad \text{for } x > 1.$$

Using the transformation $u = cx+1$ and $v = cy+1$ we get from (9)

$$1 - \tilde{G}(uv) = (1-\tilde{G}(u))(1-\tilde{G}(v))$$

for $u,v > 1$. Again from [1] p. 8 we get that G is of type Ξ_α for some positive α. $\square$

2. <u>The domain of attraction of Ξ_α</u>

To establish necessary and sufficient conditions for the domain of r.l.t. attraction of Ξ_α we need two lemma's.

<u>Lemma 2</u>

Suppose $\{t_n\}$ is an increasing sequence of real numbers. If for some positive y

$$(10) \qquad \lim_{n\to\infty} \frac{t_{n+1}-t_n}{t_n-t_{n-1}} = 1+y,$$

then

$$(11) \qquad \lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1+y.$$

<u>Proof</u>

Suppose $\varepsilon > 0$. There exists an n_0 such that for $n \geq n_0$ we have

$$(1+y-\varepsilon)(t_n-t_{n-1}) < t_{n-1} - t_n < (1+y+\varepsilon)(t_n-t_{n-1}).$$

Adding these inequalities for $n = n_0, \ldots, N$ we obtain

$$(1+y-\varepsilon)(t_N - t_{n_0-1}) < t_{N+1} - t_{n_0} < (1+y+\varepsilon)(t_N - t_{n_0-1}).$$

Now (10) implies that $t_N \to \infty$ for $N \to \infty$. Hence

$$1 + y - \varepsilon \leq \frac{t_{N+1}}{t_N} \leq 1 + y + \varepsilon$$

and since $\varepsilon > 0$ is arbitrary, (11) holds. □

Lemma 3.

Let U be a positive non-increasing function on $(0,\infty)$ and let ρ be real. Suppose that for each $\alpha > 1$ there exists a sequence $\{t_n\}$ diverging to ∞ such that

$$\limsup_{n\to\infty} \frac{U(t_n)}{U(t_{n+1})} \leq \alpha$$

and

$$\lim_{n\to\infty} \frac{U(t_n x)}{U(t_n)} = x^\rho$$

for all $x > 1$, then U varies regularly at infinity with exponent ρ, i.e.

$$\lim_{s\to\infty} \frac{U(sx)}{U(s)} = x^\rho \tag{12}$$

for all $x > 0$.

Remark

For definition and properties of regularly varying functions see e.g. [3].

Proof

Suppose $\alpha > 1$. Let $\{t_n\}$ be a sequence of real numbers such that $t_n \to \infty$ for $n \to \infty$ and

$$\limsup_{n\to\infty} \frac{U(t_n)}{U(t_{n+1})} \leq \alpha.$$

For $s > 0$ we choose $n(s)$ such that

$$t_{n(s)} \leq s < t_{n(s)+1}.$$

Then for all $x > 1$ we have by the monotonicity of U that

$$\frac{U(t_{n(s)+1})}{U(t_{n(s)})} \cdot \frac{U(t_{n(s)+1} \cdot x)}{U(t_{n(s)+1})} \leq \frac{U(sx)}{U(s)} \leq \frac{U(t_{n(s)} \cdot x)}{U(t_{n(s)})} \cdot \frac{U(t_{n(s)})}{U(t_{n(s)+1})} .$$

Hence

$$\alpha^{-1} x^{\rho} \leq \liminf_{s \to \infty} \frac{U(sx)}{U(s)} \leq \limsup_{s \to \infty} \frac{U(sx)}{U(s)} \leq \alpha x^{\rho}.$$

Since $\alpha > 1$ is arbitrary this proves (12) for $x > 1$, which clearly implies (12) for all positive x. □

Theorem 2

A distribution function F with $F(x) < 1$ for all real x belongs to the domain of r.l.t.-attraction of Ξ_α if and only of $1 - F$ is regularly varying at infinity with exponent $-\alpha$.

Proof

Suppose $1 - F$ is $(-\alpha)$-varying at infinity i.e. for all positive x

$$\lim_{t \to \infty} \frac{1-F(tx)}{1-F(t)} = x^{-\alpha}.$$

Then clearly (3) holds with $a(t) = t$ and $G = \Xi_\alpha$.

Conversely suppose that for some positive function $a(t)$ we have

$$\lim_{t \to \infty} \frac{1-F(t+xa(t))}{1-F(t)} = 1 - \Xi_\alpha(x) = (1+x)^{-\alpha} \tag{13}$$

for all $x > 0$. From (5) we obtain

$$\lim_{t \to \infty} \frac{a(t+ya(t))}{a(t)} = A(y) = 1+y \tag{14}$$

for all $y > 0$. Fix $y > 0$. We choose t_1 such that

$$(15) \qquad \frac{1-F(t+ya(t))}{1-F(t)} < (1+y/2)^{-\alpha} < 1$$

for $t \geq t_1$. We now define the sequence $\{t_n\}_{n=1}^{\infty}$ by

$$(16) \qquad t_{n+1} = t_n + ya(t_n) \qquad \text{for } n = 1, 2, \ldots$$

First we show $t_n \to \infty$. Clearly the sequence is strictly increasing. Suppose $\lim_{n\to\infty} t_n = K < \infty$, then

$$\lim_{n\to\infty} \frac{1-F(t_n+ya(t_n))}{1-F(t_n)} = \frac{\lim_{n\to\infty} 1-F(t_{n+1})}{\lim_{n\to\infty} 1-F(t_n)} = 1$$

in contradiction with (15). Hence $t_n \to \infty$ for $n \to \infty$. From (14) and (16) we see that

$$\frac{t_{n+1}-t_n}{t_n-t_{n-1}} = \frac{a(t_n)}{a(t_{n-1})} \to 1+y \qquad \text{for } n \to \infty$$

and hence by lemma 2

$$\lim_{n\to\infty} \frac{t_{n+1}}{t_n} = 1+y.$$

With (16) this gives

$$\lim_{n\to\infty} \frac{a(t_n)}{t_n} = 1,$$

hence (as the function $a(t)$ in (13) is defined up to an asymptotic equivalence)

$$(17) \qquad \lim_{n\to\infty} \frac{1-F(t_n(1+x))}{1-F(t_n)} = (1+x)^{-\alpha}$$

for $x > 0$. Application of lemma 3 to (13) and (17) shows that $1 - F$ is $(-\alpha)$-varying at infinity. $\square$

3. The domain of attraction of Π

Again we first state a lemma.

Lemma 4

If a distribution function F with $F(x) < 1$ for all real x belongs to the domain of r.l.t.-attraction of Π, then

$$\lim_{t\to\infty} \frac{1-F(t-0)}{1-F(t)} = 1. \tag{18}$$

Proof

Suppose $F \in D_r(\Pi)$ and (18) does not hold. Then there exist a sequence $\{s_n\}$ with $s_n \to \infty$ for $n \to \infty$ and a constant $1 < c \leq \infty$ such that

$$\lim_{n\to\infty} \frac{1-F(s_n-0)}{1-F(s_n)} = c. \tag{19}$$

Fix y such that $0 < y < \log c$. Since $F \in D_r(\Pi)$, by (3) there exists a t_1 such that

$$\frac{1-F(t+ya(t))}{1-F(t)} < e^{-y/2} < 1$$

for $t \geq t_1$. We define the sequence $\{t_m\}$ by

$$t_{m+1} = t_m + ya(t_m) \qquad \text{for } m = 1, 2, \ldots \tag{20}$$

In the same way as in the proof of theorem 2 we can show that $t_m \to \infty$ for $m \to \infty$. We define a subsequence $\{m(n)\}$ of the positive integers such that

$$t_{m(n)-1} < s_n \leq t_{m(n)}.$$

By the monotonicity of F we have

$$\frac{1-F(s_n-0)}{1-F(s_n)} \leq \frac{1-F(t_{m(n)-1})}{1-F(t_{m(n)})} .$$

By (3) and (20) the righthand member tends to $e^y < c$ and hence (19) cannot be true. □

Now we characterize $D_r(\Pi)$ by specifying the auxiliary function a in (3).

Theorem 3

If a distribution function F with $F(x) < 1$ for all real x belongs to the domain of r.l.t. attraction of Π, then $\int_0^\infty t dF(t) < \infty$ and

$$(21) \qquad \lim_{t\to\infty} \frac{1-F(t+xa(t))}{1-F(t)} = e^{-x}$$

for all real x with

$$(22) \qquad a(t) = \frac{\int_t^\infty (1-F(s)ds}{1-F(t)} .$$

Proof

Suppose $F \in D_r(\Pi)$. We define the function U by

$$U(s) = \inf\{z \mid 1-F(z) \leq s\} \qquad \text{for } 0 < s < 1.$$

We shall prove

$$(23) \qquad \lim_{s\downarrow 0} \frac{U(sx)-U(s)}{U(sy)-U(s)} = \frac{\log x}{\log y} \qquad \text{for all } x,y > 0 \ (y\neq 1),$$

then by the theorems 2.4.1 and 2.5.1 of [3] we have (21) with (22).

From the definition of U it follows

$$(24) \qquad 1 - F(U(s)) \leq s \leq 1 - F(U(s)-0)$$

and hence by lemma 4

$$\lim_{s\downarrow 0} \frac{1-F(U(s))}{s} = 1.$$

If we replace t by U(s) in (3), it follows

$$\lim_{s\downarrow 0} \frac{1-F(U(s)+xa(U(s)))}{1-F(U(s))} = \lim_{s\downarrow 0} \frac{1-F(U(s)+xa(U(s)))}{s} = e^{-x}$$

for $x > 0$. Hence for positive x and ε there exists an $s_0(x,\varepsilon)$ such that for $s \geq s_0(x,\varepsilon)$

(25) $$s^{-1}\{1-F(U(s)+(x+\varepsilon)a(U(s)))\} < e^{-x} < s^{-1}\{1-F(U(s)+(x-\varepsilon)a(u(s)))\}.$$

On the other hand by (24) we have

(26) $$s^{-1}\{1-F(U(se^{-x}))\} \leq e^{-x} \leq s^{-1}\{1-F(U(se^{-x})-0)\}.$$

Combining (25) and (26) we get

$$U(s) + (x-\varepsilon)\ a(U(s)) < U(se^{-x}) \leq U(s) + (x+\varepsilon)\ a(U(s)),$$

hence

$$\lim_{s\downarrow 0} \frac{U(se^{-x})-U(s)}{a(U(s))} = x$$

for all positive x. So for all x and y from (0,1) we have (23). As it is shown in the proof of theorem 2.4.1 of [3], this implies the validity of (23) for all positive x and y ($y\neq 1$). □

Remark

We have thus shown that $F \in D_r(\Pi)$ if and only if there exist sequences of real numbers $a_n > 0$ and b_n such that

$$\lim_{n\to\infty} F^n(a_n x+b_n) = \exp(-e^{-x})$$

for all real x. A similar remark can be made concerning the domain of r.t.l. attraction of Ξ_α.

4. Equivalence of weak convergence and moment convergence

In this section we state the conditions for the domains of attraction in an alternative form using certain moments of the distributions.

Lemma 5.

Suppose F is a distribution function with $F(x) < 1$ for all x and α is a positive constant.

a) If 1- F varies regurlarly at infinity with exponent $-\alpha$, then for all ξ from $(0,\alpha)$ the integral $\int_0^\infty x^\xi dF(x)$ converges and

$$\lim_{x\to\infty} \frac{\int_x^\infty y^\xi dF(y)}{x^\xi\{1-F(x)\}} = (1-\xi/\alpha)^{-1}.$$

b) If for some $\xi > 0$ and $c > 1$ the integral $\int_0^\infty x^\xi dF(x)$ converges and

$$\lim_{x\to\infty} \frac{\int_x^\infty y^\xi dF(y)}{x^\xi\{1-F(x)\}} = c,$$

then 1 - F varies regularly at infinity with exponent $-\xi c(c-1)^{-1}$.

Proof

Partial integration of the numerator gives

$$\frac{\int_x^\infty y^\xi dF(y)}{x^\xi\{1-F(x)\}} = \xi \frac{\int_x^\infty y^{\xi-1}\{1-F(y)\}dy}{x^\xi\{1-F(x)\}} + 1.$$

The statement of the lemma now follows from Karamata's theorem for regularly varying functions (see e.g. [3] theorem 1.2.1 and remark 1.2.1). □

Theorem 4

Suppose X is a real-valued random variable with distribution function F and $F(x) < 1$ for all real x.

a) (i) If $F \in D_r(\Xi_\alpha)$, i.e. if

$$\lim_{t\to\infty} P\{\frac{X}{t} > x \mid X > t\} = \Xi_\alpha(x-1)$$

for all $x > 0$, then for all $0 < \xi < \alpha$ the integral $\int_0^\infty y^\xi dF(y)$ converges and

$$\lim_{t\to\infty} E((\frac{X}{t})^\xi \mid X > t) = \int_0^\infty x^\xi d\Xi_\alpha(x-1) = (1-\xi/\alpha)^{-1}.$$

(ij) If for some $\xi > 0$ the integral $\int_0^\infty y^\xi dF(y)$ converges and for some $c > 1$

$$\lim_{t\to\infty} E((\frac{X}{t})^\xi \mid X > t) = c,$$

then $F \in D_r(\Xi_\alpha)$ with $\alpha = \xi c(c-1)^{-1}$.

b) We have $F \in D_r(\Pi)$, i.e.

$$\lim_{t\to\infty} P\{\frac{X-t}{a(t)} > x \mid X > t\} = \Pi(x)$$

for all positive x with (by theorem 3)

$$a(t) = \frac{\int_t^\infty \{1-F(s)\}ds}{1-F(t)} = E(X - t \mid X > t)$$

if and only if $\int_0^\infty x^2 dF(y)$ converges and

$$\lim_{t\to\infty} E((\frac{X-t}{a(t)})^2 \mid X > t) = \int_0^\infty x^2 d\Pi(x) = 2.$$

<u>Proof</u>

a) This part is a simple consequence of theorem 2 and lemma 5.

b) By theorem 3 and the theorem 2.5.1 and 2.5.2 of [3] we have $F \in D_r(\Pi)$ if and only if

$$\lim_{t\to\infty} \frac{\{1-F(t)\}\{\int_t^\infty \int_v^\infty (1-F(s))dsdv\}}{\{\int_t^\infty (1-F(s))ds\}^2} = 1.$$

By partial integration we obtain

$$\int_t^\infty \int_v^\infty (1-F(s))dsdv = \frac{1}{2} \int_t^\infty (s-t)^2 dF(s)$$

and

$$\lim_{t\to\infty} E\left(\left(\frac{X-t}{a(t)}\right)^2 \middle| X>t\right) = \lim_{t\to\infty} \frac{\int_t^\infty (s-t)^2 dF(s)}{1-F(t)} \cdot \frac{\{1-F(t)\}^2}{\{\int_t^\infty (1-F(s))ds\}^2} = 2. \square$$

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