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MONOTONIC APPROXIMATION OF INTEGRALS IN RELATION TO SOME INEQUALITIES FOR SUMS OF POWERS OF INTEGERS

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Monotonic approximation of integrals in relation to some inequalities for sums of powers of integers

by

J. van de Lune

ABSTRACT

This report mainly deals with the following question: For which (continuous) functions $f\colon [0,1] \to \mathbb{R}$ is the sequence of canonical trapezoidal approximations of $\int_0^1 f(x)dx$ monotonic? Most of the results are obtained by means of some new inequalities for sums and alternating sums of powers of integers.

KEY WORDS & PHRASES: *Approximation of integrals, Monotonicity, Inequalities, Sums of powers, Bernoullian polynomials, Gamma function.*

0. INTRODUCTION.

The subject of this note was inspired by the following problem (proposed by the author): For any positive integer n consider the regular n-gon $P_1,\dots,P_n$ where

$$(0.1) \qquad P_k = \exp(\tfrac{k}{n}\, 2\pi i), \qquad (k=1,\dots,n).$$

Let d_k be the distance from P_k to P_n, i.e.

$$(0.2) \qquad d_k = |P_k - 1|,$$

and let

$$(0.3) \qquad D_n = \frac{1}{n}\sum_{k=1}^{n} d_k.$$

Prove that the sequence $\{D_n\}_{n=1}^{\infty}$ tends *increasingly* to its limit $(=\frac{4}{\pi})$.
This problem may be solved as follows:
Since

$$D_n = \frac{1}{n}\sum_{k=1}^{n} |\exp(\tfrac{k}{n}\, 2\pi i) - 1| =$$

$$(0.4) \qquad = \frac{1}{n}\sum_{k=1}^{n} |(\cos\frac{2\pi k}{n} + i\sin\frac{2\pi k}{n}) - 1| =$$

$$= \frac{1}{n}\sum_{k=1}^{n} (2 - 2\cos\frac{2\pi k}{n})^{\frac{1}{2}} = \frac{2}{n}\sum_{k=1}^{n} \sin\frac{\pi k}{n},$$

it is already clear that

$$(0.5) \qquad \lim_{n\to\infty} D_n = 2\int_0^1 \sin \pi x\, dx = \frac{4}{\pi}$$

In order to show that $\{D_n\}_{n=1}^{\infty}$ is increasing we recall that

$$(0.6)\qquad \sum_{k=1}^{m} \sin kz = \frac{\sin \frac{mz}{2} \sin \frac{(m+1)z}{2}}{\sin \frac{z}{2}},$$

from which it is readily seen that

$$(0.7)\qquad D_n = \frac{2}{n} \frac{\cos \frac{\pi}{2n}}{\sin \frac{\pi}{2n}} = \frac{4}{\pi} \frac{\frac{\pi}{2n}}{\tan \frac{\pi}{2n}}.$$

Now consider the function ϕ: $(0,\frac{\pi}{2}] \to \mathbb{R}$ defined by

$$(0.8)\qquad \begin{cases} \phi(t) = \dfrac{t}{\tan t}, & (0<t<\frac{\pi}{2}) \\ \phi(\frac{\pi}{2}) = 0. \end{cases}$$

It is easily verified that ϕ is decreasing and since

$$(0.9)\qquad D_n = \frac{4}{\pi} \phi(\frac{\pi}{2n})$$

it follows that $\{D_n\}_{n=1}^{\infty}$ is increasing.

We note that D_n may be written as

$$(0.10)\qquad D_n = \frac{2}{n} \sum_{k=1}^{n} \frac{\sin \frac{k-1}{n}\pi + \sin \frac{k}{n}\pi}{2},$$

which is equivalent to saying that $\frac{1}{2}D_n$ is the n-th canonical trapezoidal approximation of $\int_0^1 \sin \pi x\, dx$. Thus we have proved that the sequence of canonical trapezoidal approximations of $\int_0^1 \sin \pi x\, dx$ is increasing. One might feel that this fact is not surprising since $\sin \pi x$ is concave on $[0,1]$.

However, an example such as

$$(0.11)\qquad f(x) = 1 - |x|, \qquad (-1\leq x\leq 1),$$

shows that the concavity of f is not a sufficient condition for this phenomenon. In this example f is concave but the corresponding sequence of trapezoidal approximations of $\int_{-1}^{1} f(x)\,dx$ is oscillating. We are thus led to the following question: *For which (continuous) functions* f: $[0,1] \to \mathbb{R}$, *say, is the sequence* $\{T_n(f)\}_{n=1}^{\infty}$, *defined by*

$$(0.12) \qquad T_n(f) = \frac{1}{n} \sum_{k=1}^{n} \frac{f(\frac{k-1}{n}) + f(\frac{k}{n})}{2},$$

monotonic? It seems that there is no simple general answer to this question.

However, using the Euler-Maclaurin summation formula, it is fairly easy to obtain a reasonably large class of functions f for which $T_n(f)$ is *eventually* monotonic.

During the investigation of the question just described, it turned out that in order to obtain any results of some general nature it is very helpful to settle the question first for the functions f_s and f_s^* defined by

$$(0.13) \qquad f_s(x) = x^s, \qquad (0 \leqq x \leqq 1 \ ; \ s>0)$$

and

$$(0.14) \qquad f_s^*(x) = |x|^s, \qquad (-1 \leqq x \leqq 1 \ ; \ s \in \mathbb{N}, \ s \geqq 2).$$

In section 1 of this note it will be shown that $T_n(f_s)$ is increasing (resp. decreasing) for any fixed $s \in (0,1)$ (resp. $s>0$), whereas in section 2 we will prove that $T_n(f_s^*)$ is decreasing for any fixed integer $s \geqq 2$.

There is little doubt that all these statements are easily conjectured. However, we have not been able to furnish any really simple proofs. All our proofs are based upon some inequalities concerning the sums $\sigma_n(s)$ and $\tau_n(s)$ defined by

$$(0.15) \qquad \sigma_n(s) = \sum_{k=1}^{n} k^s,$$

and

$$(0.16)\qquad \phi_n(s) = \sum_{k=0}^{n-1} (-1)^k (n-k)^s,$$

the most important ones being

$$(0.17)\qquad \sigma_n(s) > \frac{n^s(n+1)^{s+1} + n^{s+1}(n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>1)$$

and

$$(0.18)\qquad \phi_n(s) > \frac{n^s(n+1)^{s+1} + n^{s+1}(n+1)^s}{(n+1)^{s+1} + n^{s+1}}, \qquad (s\in\mathbb{N},\ s\geq 2)$$

In section 3 we will discuss some applications of the above results. As an example we mention here the intriguing fact that the function

$$(0.19)\qquad f(x) = |x|, \qquad (-1\leq x\leq 1)$$

can *not* be approximated pointwise (let alone uniformly) by a sequence of polynomials $\{P_n(x)\}_{n=1}^{\infty}$, every $P_n(x)$ being of the form

$$(0.20)\qquad C + \sum_{m=1}^{N} p_m\, x^m$$

where C is any real number whereas all coefficients p_m with *even* index m are non-negative.

1. APPROXIMATIONS OF $\int_0^1 x^s\, dx$, $(s>0)$.

For $n \in \mathbb{N}$ and $s \in \mathbb{C}$ we define

$$(1.1)\qquad \sigma_n(s) = \sum_{k=1}^{n} k^s.$$

In this section we will only be interested in the case $s>0$. Comparing $\frac{\sigma_n(s)}{n^{s+1}} = \frac{1}{n} \sum_{k=1}^{n} (\frac{k}{n})^s$ with $\int_0^1 x^s\, dx$, one readily finds the well known inequalities

$$(1.2) \qquad \frac{n^{s+1}}{s+1} < \sigma_n(s) < \frac{n^{s+1}}{s+1} + n^s, \qquad (s>0).$$

Also considering the canonical trapezoidal approximations of $\int_0^1 x^s\, dx$ it is easily seen that

$$(1.3) \qquad \sigma_n(s) < \frac{n^{s+1}}{s+1} + \frac{n^s}{2}, \qquad (0<s<1)$$

and

$$(1.4) \qquad \sigma_n(s) > \frac{n^{s+1}}{s+1} + \frac{n^s}{2}, \qquad (s>1),$$

the case $s=1$ being trivial: $\sigma_n(1) = \frac{n(n+1)}{2}$. Next we have the somewhat less trivial

PROPOSITION 1.1.

$$(1.5) \qquad \frac{n^{s+1}\,(n+1)^s}{(n+1)^{s+1} - n^{s+1}} < \sigma_n(s) < \frac{n^s\,(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>0).$$

PROOF. We first show that

$$(1.6) \qquad \sigma_n(s) > \frac{n^{s+1}\,(n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>0).$$

This inequality is easily seen to be true for $n = 1$ and all $s > 0$. Assume that (1.6) still holds for $n = 1,\dots,N$ and all $s > 0$. Then we have

$$(1.7) \qquad \sigma_{N+1}(s) = (N+1)^s + \sigma_N(s) > (N+1)^s + \frac{N^{s+1}\,(N+1)^s}{(N+1)^{s+1} - N^{s+1}},$$

so that it suffices to show that

$$(1.8)\qquad (N+1)^s + \frac{N^{s+1}\ (N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(N+1)^{s+1}\ (N+2)^s}{(N+2)^{s+1} - (N+1)^{s+1}},$$

or equivalently, that

$$(1.9)\qquad (N+1)^s(N+2)^{s+1} - (N+1)^{2s+1} \geqq (N+1)^{s+1}(N+2)^s - N^{s+1}(N+2)^s$$

for all $N \in \mathbb{N}$ and all $s > 0$. Putting $\frac{1}{N+1} = x$, we thus want to prove that

$$(1.10)\qquad (1+x)^{s+1} - 1 \geqq (1+x)^s - (1-x)^{s+1}\ (1+x)^s,$$

or equivalently, that

$$(1.11)\qquad \frac{(1+x)^{s+1} - 1}{x} \geqq \frac{1 - (1-x^2)^{s+1}}{x^2}\ .$$

Since for any fixed $s > 0$ the function x^{s+1} is convex for $x > 0$, it follows that (1.11) holds for all $x \in (0,1)$, completing the proof of (1.6). Next we show that

$$(1.12)\qquad \sigma_n(s) < \frac{n^s\ (n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>0).$$

One may verify directly that (1.12) holds for $n = 1$ and all $s > 0$. Assume (1.12) still holds for $n = 1,\dots,N$ and all $s > 0$. Then we have

$$(1.13)\qquad \sigma_{N+1}(s) = (N+1)^s + \sigma_N(s) < (N+1)^s + \frac{N^s\ (N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}}$$

so that it suffices to show that

$$(1.14)\qquad (N+1)^s + \frac{N^s\ (N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}} \leqq \frac{(N+1)^s\ (N+2)^{s+1}}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $N \in \mathbb{N}$ and all $s > 0$. Again, putting $\frac{1}{N+1} = x$, it is easily verified that (1.14) is equivalent to

$$(1.15)\qquad \frac{(1-x^2)^{s+1} - (1-x)^{s+1}}{x(1-x)} \leqq \frac{1-(1-x^2)^{s+1}}{x^2} .$$

Now observe that $(1-x^2) - (1-x) = x(1-x)$, that $0 < 1-x < 1-x^2 < 1$ and that for any fixed $s > 0$ the function x^{s+1} is convex for $x > 0$. It follows that (1.15) is true indeed, completing the proof. □

Defining

$$(1.16)\qquad U_n(s) = \frac{\sigma_n(s)}{n^{s+1}} , \qquad (n\in\mathbb{N}\ ;\ s\in\mathbb{C})$$

and

$$(1.17)\qquad L_n(s) = \frac{\sigma_n(s) - n^s}{n^{s+1}} , \qquad (n\in\mathbb{N}\ ;\ s\in\mathbb{C})$$

we have the following

PROPOSITION 1.2. *If* $s > 0$ *then the upper (resp. lower) Riemann-sum* $U_n(s)$ *(resp.* $L_n(s)$*) is decreasing (resp. increasing) in* n.

PROOF. In order to show that

$$(1.18)\qquad U_n(s) > U_{n+1}(s), \qquad (s>0)$$

we may just as well show that

$$(1.19)\qquad \frac{\sigma_n(s)}{n^{s+1}} > \frac{\sigma_{n+1}(s)}{(n+1)^{s+1}} , \qquad (s>0)$$

or, equivalently, that

$$(1.20)\qquad (n+1)^{s+1}\ \sigma_n(s) \geqq n^{s+1}\ \{(n+1)^s + \sigma_n(s)\}, \qquad (s>0)$$

which may also be written as

$$\sigma_n(s) > \frac{n^{s+1}(n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>0).$$

Similarly, the inequality

$$(1.21) \qquad L_n(s) < L_{n+1}(s), \qquad (s>0)$$

is seen to be equivalent to

$$\sigma_n(s) < \frac{n^s (n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>0).$$

Hence, proposition 1.2 is just a restatement of proposition 1.1. □

REMARK. During the preparation of this note J. H. VAN LINT notified the author that proposition 1.2 may be generalized as follows: If f: $[a,b] \to \mathbb{R}$ is monotonic and either convex or concave on $[a,b]$ then the corresponding sequence of canonical upper (resp. lower) Riemann-sums is decreasing (resp. increasing).

It should be noted that $T_n(f)$ need *not* be monotonic for a convex monotonic function f: $[a,b] \to \mathbb{R}$. As an example one may take an increasing convex function whose graph consists of two line segments joined at the point $x = \frac{a+b}{2}$.

Defining

$$(1.22) \qquad T_n(s) = \tfrac{1}{2}\{U_n(s) + L_n(s)\}, \qquad (n \in \mathbb{N}\,;\ s \in \mathbb{C}),$$

it is clear that if $s > 0$ then $T_n(s)$ is the n-th canonical trapezoidal approximation of $\int_0^1 x^s\,dx$.

Concerning $T_n(s)$ we have the following

THEOREM 1.1. *If* $0 < s < 1$ *(resp.* $s>1$*) then* $T_n(s)$ *is increasing (resp. decreasing) in* n.

PROOF.

Case 1. $s > 1$.

We want to prove that

$$T_n(s) > T_{n+1}(s) , \qquad (s>1). \tag{1.23}$$

Since

$$2T_n(s) = \frac{2\sigma_n(s) - n^s}{n^{s+1}} \tag{1.24}$$

we may just as well prove that

$$\frac{2\sigma_n(s) - n^s}{n^{s+1}} > \frac{2\sigma_{n+1}(s) - (n+1)^s}{(n+1)^{s+1}} , \qquad (s>1), \tag{1.25}$$

or equivalently, that

$$2\sigma_n(s) > \frac{n^{s+1}\ (n+1)^s + n^s\ (n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}} , \qquad (s>1). \tag{1.26}$$

In order to prove (1.26) we proceed by induction. If $n = 1$ we have to check whether

$$2 > \frac{2^s + 2^{s+1}}{2^{s+1} - 1} , \qquad (s>1) \tag{1.27}$$

or equivalently, whether

$$2^s > 2, \qquad (s>1). \tag{1.28}$$

It follows that (1.26) holds for $n = 1$ and all $s > 1$. Assume that (1.26) still holds for $n = 1,\dots,N$ and all $s > 1$. Then we have

$$2\sigma_{N+1}(s) = 2(N+1)^s + 2\sigma_N(s) > 2(N+1)^s + \frac{N^{s+1}\ (N+1)^s + N^s\ (N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}} \tag{1.29}$$

so that it suffices to show that

$$(1.30)\qquad 2(N+1)^s + \frac{N^{s+1}\ (N+1)^s + N^s\ (N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}} \geqq$$

$$\geqq \frac{(N+1)^{s+1}\ (N+2)^s + (N+1)^s\ (N+2)^{s+1}}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $N \in \mathbb{N}$ and all $s > 1$.

After some simplifications in (1.30) we see that we may just as well prove that

$$(1.31)\qquad \frac{N^s + 2(N+1)^{s+1}}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(N+1)(N+2)^s + (N+2)^{s+1}}{(N+2)^{s+1} - (N+1)^{s+1}} ,$$

or equivalently, that (as before, we write $\frac{1}{N+1} = x$)

$$(1.32)\qquad \frac{x(1-x)^s + 2}{1 - (1-x)^{s+1}} \geqq \frac{(1+x)^s + (1+x)^{s+1}}{(1+x)^{s+1} - 1}$$

which may be rewritten as

$$(1.33)\qquad 2(1-x^2)^s + x\{(1+x)^s - (1-x)^s\} - 2 \geqq 0.$$

In case s is an integer greater than 1 we may prove (1.33) as follows:

Using the binomial theorem the left hand side of (1.33) may be written as

$$(1.34)\qquad 2\sum_{r=0}^{s} \binom{s}{r}(-1)^r\ x^{2r} + x\sum_{r=0}^{s} \binom{s}{r}\ x^r\{1-(-1)^r\} - 2 =$$

$$= 2\{\sum_{r=1}^{\infty} \binom{s}{r}(-1)^r\ x^{2r} + \sum_{r=1}^{\infty} \binom{s}{2r-1}\ x^{2r}\}.$$

Now replace t^2 by z, $0 < z < 1$, so that it suffices to show that

$$(1.35)\qquad \sum_{r=1}^{\infty} \binom{s}{r}(-1)^r\ z^r + \sum_{r=1}^{\infty} \binom{s}{2r-1}\ z^r =$$

$$= \sum_{r=1}^{\infty} \binom{s}{2r} z^{2r} + \sum_{r=1}^{\infty} \binom{s}{2r-1}(z^r - z^{2r-1}) \geqq 0,$$

for all $z \in (0,1)$. Since $0 < z < 1$ we have

$$(1.36) \qquad z^r - z^{2r-1} \geqq 0$$

for all $r \in \mathbb{N}$.

Since in addition all binomial coefficients in (1.35) are positive, the proof of (1.23) is complete in case $s \in \mathbb{N}$, $s \geqq 2$. In order to prove theorem 1.1 for a general $s > 1$ we consider two cases

Case 1.a. $\quad 1 < s < 2.$

Observe that (1.33) is equivalent to

$$(1.37) \qquad x \frac{(1+x)^s - (1-x)^s}{2} \geqq 1 - (1-x^2)^s, \qquad (0<x<1),$$

which may also be written as

$$(1.38) \qquad \frac{x}{2} \sum_{r=0}^{\infty} \binom{s}{r} x^r\{1-(-1)^r\} \geqq 1 - \sum_{r=0}^{\infty} \binom{s}{r}(-1)^r x^{2r}.$$

This last inequality is equivalent to

$$(1.39) \qquad \sum_{r=1}^{\infty} \binom{s}{2r-1} x^{2r} + \sum_{r=1}^{\infty} \binom{s}{r}(-1)^r x^{2r} \geqq 0,$$

or, putting $x^2 = z$, $0 < z < 1$, to

$$(1.40) \qquad \sum_{r=2}^{\infty} \{\binom{s}{2r-1} + (-1)^r \binom{s}{r}\} z^r \geqq 0, \qquad (0<z<1).$$

Clearly (1.40) may be rewritten as

$$(1.41) \qquad \sum_{r=1}^{\infty} \{\binom{s}{4r-1} + \binom{s}{2r}\} z^{2r} + \sum_{r=1}^{\infty} \{\binom{s}{4r+1} - \binom{s}{2r+1}\} z^{2r+1} \geqq 0.$$

Now we observe that if $1 < s < 2$ then

(1.42) $\qquad \binom{s}{2r} > 0 \quad \text{for all } r \in \mathbb{N}$

and

(1.43) $\qquad \binom{s}{2r+1} < 0 \quad \text{for all } r \in \mathbb{N}.$

Moreover it is easily seen that $\left|\binom{s}{r}\right|$ is decreasing in r.

Hence, for all $r \in \mathbb{N}$ we have

(1.44) $\qquad \binom{s}{4r-1} + \binom{s}{2r} = \binom{s}{2r} - \left|\binom{s}{4r-1}\right| > 0$

and

(1.45) $\qquad \binom{s}{4r+1} - \binom{s}{2r+1} = \left|\binom{s}{2r+1}\right| - \left|\binom{s}{4r+1}\right| > 0$

and the proof of case 1.a is complete.

Case 1.b. $\quad s > 2.$

First observe that

(1.46) $\qquad (1-x^2)^s > 1 - sx^2, \qquad (0<x<1;\ s>1).$

Hence, in order to prove (1.33) it is sufficient to show that

(1.47) $\qquad 2(1-sx^2) + x\{(1+x)^s - (1-x)^s\} - 2 \geq 0,$

or, equivalently, that

(1.48) $\qquad (1+x)^s - (1-x)^s - 2sx \geq 0.$

The left-hand side of (1.48) takes the value 0 at x = 0. Hence, it suffices to show that its derivative is positive for 0 < x < 1, which is equivalent to proving that

$$(1.49) \qquad (1+x)^{s-1} + (1-x)^{s-1} - 2 > 0, \qquad (0<x<1).$$

Since (1.49) may be rewritten as

$$(1.50) \qquad \frac{(1+x)^{s-1} - 1}{x} > \frac{1 - (1-x)^{s-1}}{x}, \qquad (0<x<1),$$

and since x^{s-1} is convex for $x > 0$ if $s > 2$, we see that (1.50) holds, completing the proof of case 1.b.

Case 2. $0 < s < 1$.

In order to prove that $T_n(s) < T_{n+1}(s)$ if $0 < s < 1$ it suffices to prove that

$$(1.51) \qquad 2(1-x^2)^s + x\{(1+x)^s - (1-x)^s\} - 2 \leqq 0, \qquad (0<x<1).$$

This inequality may be established similarly as in case 1. Just observe that if $0 < s < 1$ then for all $r \in \mathbb{N}$ we have

$$(1.52) \qquad \binom{s}{2r} < 0 \quad \text{and} \quad \binom{s}{2r-1} > 0,$$

whereas $\left|\binom{s}{r}\right|$ is again decreasing in r. This completes the proof of theorem 1.1. □

From the above proof we obtain the following

THEOREM 1.2.

$$(1.53) \qquad 2\sigma_n(s) > \frac{n^{s+1}\,(n+1)^s + n^s\,(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>1)$$

$$(1.54) \qquad 2\sigma_n(s) < \frac{n^{s+1}\,(n+1)^s + n^s\,(n+1)^{s+1}}{(n+1)^{s+1} - n^{s+1}}, \qquad (0<s<1).$$

REMARK. One might suggest to study the behaviour of $T_n(s)$ by means of the Euler-Maclaurin summation formula.

In order to avoid notational ambiguities concerning Bernoullian numbers and polynomials we give the following definitions.

The Bernoullian polynomials $b_n(t)$ are defined by

$$(1.55)\qquad \frac{ze^{zt}}{e^z - 1} = \sum_{n=0}^{\infty} b_n(t)\, z^n, \qquad (|z|<2\pi).$$

The Bernoullian numbers b_n are defined by

$$(1.56)\qquad b_n = b_n(0), \qquad (n=0,1,2,3,\ldots)$$

or, equivalently, by

$$(1.57)\qquad \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} b_n\, z^n, \qquad (|z|<2\pi).$$

Putting

$$(1.58)\qquad B_n = n!\, b_n, \qquad (n=0,1,2,3,\ldots),$$

we have

$$(1.59)\qquad \frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n, \qquad (|z|<2\pi).$$

It is well known that for any polynomial $f(x)$

$$(1.60)\qquad \sum_{k=a}^{b-1} f(k) = \int_a^b f(x)\, dx - \tfrac{1}{2}\{f(b) - f(a)\} + \sum_{r=1}^{N} b_{2r}\{f^{(2r-1)}(b) - f^{(2r-1)}(a)\},$$

where

$$(1.61)\qquad N = [\tfrac{m}{2}], \qquad m \text{ denoting the degree of } f(x).$$

Choosing $a = 0$, $b = n$, $n \in \mathbb{N}$, $f(x) = x^m$, $m \in \mathbb{N}$, $m \geq 2$, we obtain from (1.60) that

(1.62) $$T_n(m) = \frac{1}{n}\sum_{k=1}^{n}\frac{f(\frac{k-1}{n}) + f(\frac{k}{n})}{2} = \frac{1}{n}\{\sum_{k=0}^{n-1} f(\frac{k}{n}) + \frac{f(1) - f(0)}{2}\} =$$

$$= \frac{1}{n}\{\int_0^n f(\frac{x}{n})\,dx + \sum_{r=1}^{[\frac{m}{2}]} b_{2r}\frac{f^{(2r-1)}(1) - f^{(2r-1)}(0)}{n^{2r-1}}\} =$$

$$= \int_0^1 f(x)\,dx + \sum_{r=1}^{[\frac{m}{2}]}\frac{b_{2r}}{n^{2r}}\{f^{(2r-1)}(1) - f^{(2r-1)}(0)\} =$$

$$= \frac{1}{m+1} + \sum_{r=1}^{[\frac{m}{2}]}\frac{b_{2r}}{n^{2r}}\frac{m!}{(m-2r+1)!} = \frac{1}{m+1} + \frac{1}{m+1}\sum_{r=1}^{[\frac{m}{2}]}\binom{m+1}{2r}\frac{B_{2r}}{n^{2r}}\,.$$

Since $B_2 = \frac{1}{6} > 0$ it already follows that for $m \in \mathbb{N}$, $m \geqq 2$, $T_n(m)$ is *eventually* decreasing in n.

It is also well known that

(1.63) $$b_n(t) = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k} B_k\, t^{n-k}\,,$$

(1.64) $$B_0 = 1\,,\quad B_1 = -\tfrac{1}{2}$$

and that

(1.65) $$B_{2k+1} = 0\,,\quad \text{for all } k \in \mathbb{N}\,.$$

Using these facts it is easily seen that

(1.66) $$T_n(m) = \frac{m!\; b_{m+1}(n)}{n^{m+1}} + \frac{1}{2n}\,,\qquad (m \geqq 2).$$

From (1.66) and the fact that $T_n(m)$ is decreasing in n one is tempted to conjecture that

(1.67) $$\frac{b_{m+1}(x)}{x^{m+1}} + \frac{1}{m!\;2x}\,,\qquad (x>0)$$

is decreasing in x for $x \geqq 1$.

One may verify that this conjecture is true indeed for some small values of m. However, we can show that it is not true in general. If the conjecture is true then we must have

$$(1.68)\qquad \frac{d}{dx}\left\{\frac{b_{m+1}(x)}{x^{m+1}} + \frac{1}{m!\,2x}\right\} \leqq 0, \qquad (x\geqq 1)$$

which is equivalent to

$$(1.69)\qquad x\, b'_{m+1}(x) - (m+1)\, b_{m+1}(x) - \frac{1}{2.\ m!}\, x^m \leqq 0, \qquad (x\geqq 1).$$

Since for $k \geqq 1$, $b'_k(x) = b_{k-1}(x)$, we have

$$(1.70)\qquad x\, b_m(x) - (m+1)\, b_{m+1}(x) - \frac{1}{2.\ m!}\, x^m \leqq 0, \qquad (x\geqq 1)$$

or, equivalently,

$$(1.71)\qquad m!\ x\, b_m(x) - (m+1)!\ b_{m+1}(x) - \tfrac{1}{2}x^m \leqq 0, \qquad (x\geqq 1).$$

In particular (1.71) must hold for $x = 1$, so that

$$(1.72)\qquad m!\ b_m(1) - (m+1)!\ b_{m+1}(1) - \tfrac{1}{2} \leqq 0.$$

It is well known that

$$(1.73)\qquad b_m(1) = b_m(0), \qquad (m\geqq 2)$$

so that we arrive at

$$(1.74)\qquad m!\ b_m - (m+1)!\ b_{m+1} - \tfrac{1}{2} \leqq 0,$$

which may also be written as

$$(1.75)\qquad B_m - B_{m+1} - \tfrac{1}{2} \leqq 0.$$

This inequality is true indeed for $2 \leqq m \leqq 13$. However, choosing m = 14, we have

$$(1.76)\qquad B_{14} - B_{15} - 1 = \frac{7}{6} - 0 - 1 = \frac{1}{6} > 0,$$

so that (1.75) is *not* true in general. Nevertheless, from (1.66) and theorem 1.1 we obtain that the sequence

$$(1.77)\qquad \left\{\frac{b_{m+1}(n)}{n^{m+1}} + \frac{1}{m!\ 2n}\right\}_{n=1}^{\infty}$$

is decreasing whenever $m \in \mathbb{N}$, $m \geqq 2$.

For later use we will now derive some estimates for

$$(1.78)\qquad w_n(s) \overset{\text{def}}{=} \sum_{k=1}^{n} (2k-1)^s, \qquad (n \in \mathbb{N}\ ;\ s>0).$$

Defining

$$(1.79)\qquad t_n(s) = \frac{2\ w_n(s)}{(2n)^{s+1}}, \qquad (n \in \mathbb{N}\ ;\ s>0),$$

and observing that $t_n(s)$ may be interpreted as the n-th canonical tangent approximation of $\int_0^1 x^s\,dx$, we immediately obtain the inequalities

$$(1.80)\qquad 2\ w_n(s) > \frac{(2n)^{s+1}}{s+1}, \qquad (0<s<1)$$

and

$$(1.81)\qquad 2\ w_n(s) < \frac{(2n)^{s+1}}{s+1}, \qquad (s>1).$$

In order to obtain better estimates for $w_n(s)$ we prove the following

PROPOSITION 1.3. *If* $0 < s < 1$ *(resp.* $s>1$*) then* $t_n(s)$ *is decreasing (resp. increasing) in* n.

PROOF.

Case 1. $\quad 0 < s < 1$

We want to prove that

$$(1.82) \qquad t_n(s) > t_{n+1}(s) \ , \qquad (0<s<1).$$

One may verify that (1.82) is equivalent to

$$(1.83) \qquad w_n(s) > \frac{n^{s+1}\ (2n+1)^s}{(n+1)^{s+1} - n^{s+1}} \ , \qquad (0<s<1).$$

Again we proceed by induction: For n = 1 inequality (1.83) reads

$$(1.84) \qquad 1 > \frac{3^s}{2^{s+1} - 1} \ , \qquad (0<s<1)$$

which may be written as

$$(1.85) \qquad \frac{1^s + 3^s}{2} < 2^s \ , \qquad (0<s<1).$$

Since x^s is concave on $1 \leqq x \leqq 3$ it follows that (1.85) **holds**, so that (1.83) is true for n = 1 and all $s \in (0,1)$.

Assume (1.83) still holds for n = 1,...,N and all $s \in (0,1)$. Then we have

$$(1.86) \qquad w_{N+1}(s) = (2N+1)^s + w_N(s) > (2N+1)^s + \frac{N^{s+1}\ (2N+1)^s}{(N+1)^{s+1} - N^{s+1}} =$$
$$= \frac{(N+1)^{s+1}\ (2N+1)^s}{(N+1)^{s+1} - N^{s+1}} \ ,$$

so that it suffices to prove that

$$(1.87) \qquad \frac{(N+1)^{s+1}\ (2N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(N+1)^{s+1}\ (2N+3)^s}{(N+2)^{s+1} - (N+1)^{s+1}} \ ,$$

or, equivalently, that

$$(1.88) \qquad \frac{(2N+1)^s}{(N+1)^{s+1} - N^{s+1}} \geqq \frac{(2N+3)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $N \in \mathbb{N}$ and all $s \in (0,1)$.

Putting $\frac{1}{2N+3} = x$ it is clear that (1.88) is equivalent to

$$(1.89) \qquad \frac{(1-2x)^s}{(1-x)^{s+1} - (1-3x)^{s+1}} \geqq \frac{1}{(1+x)^{s+1} - (1-x)^{s+1}}$$

which may also be written as

$$(1.90) \qquad \frac{(1+x)^{s+1} - (1-x)^{s+1}}{2x} \geqq \frac{(1+\frac{x}{1-2x})^{s+1} - (1-\frac{x}{1-2x})^{s+1}}{\frac{2x}{1-2x}} .$$

Now observe that

$$(1.91) \qquad \frac{(1+x)^{s+1} - (1-x)^{s+1}}{2x} = \sum_{r=0}^{\infty} \binom{s+1}{2r+1} x^{2r} = s + 1 + \sum_{r=1}^{\infty} \binom{s+1}{2r+1} x^{2r}$$

and that

$$(1.92) \qquad \binom{s+1}{2r+1} < 0 \qquad \text{for all } r \in \mathbb{N} \text{ and all } s \in (0,1).$$

Since $\frac{x}{1-2x} > x$ for $0 < x < \frac{1}{2}$ it follows that (1.90) holds, completing the proof of case 1.

Case 2. $s > 1$.

Now we have to show that

$$(1.93) \qquad t_n(s) < t_{n+1}(s), \qquad (s>1),$$

or equivalently, that

$$(1.94) \qquad w_n(s) < \frac{n^{s+1}(2n+1)^s}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>1).$$

For $n = 1$ this reads

$$(1.95) \qquad 1 < \frac{3^s}{2^{s+1} - 1}, \qquad (s>1)$$

or, equivalently,

$$(1.96) \qquad 2^s < \frac{1^s + 3^s}{2}, \qquad (s>1)$$

which is true because of the convexity of x^s on the interval $1 \leqq x \leqq 3$.

Hence (1.94) holds for $n = 1$ and all $s > 1$.

Assume (1.94) still holds for $n = 1,\ldots N$ and all $s > 1$. Then we have

$$(1.97) \qquad w_{N+1}(s) = (2N+1)^s + w_N(s) < (2N+1)^s + \frac{N^{s+1}(2N+1)^s}{(N+1)^{s+1} - N^{s+1}} = $$
$$= \frac{(N+1)^{s+1}\ (2N+1)^s}{(N+1)^{s+1} - N^{s+1}}$$

and it clearly suffices to show that

$$(1.98) \qquad \frac{(N+1)^{s+1}\ (2N+1)^s}{(N+1)^{s+1} - N^{s+1}} \leqq \frac{(N+1)^{s+1}\ (2N+3)^s}{(N+2)^{s+1} - (N+1)^{s+1}}$$

for all $N \in \mathbb{N}$ and all $s > 1$.

It follows that we are done as soon as we show that the function

$$(1.99) \qquad \frac{(2x+1)^s}{(x+1)^{s+1} - x^{s+1}}, \qquad (x>0)$$

is increasing in x for any fixed $s > 1$.

Considering the derivative of (1.99) it suffices to prove that

$$(1.100) \qquad (s+1)(1+2x)\ \{(1+x)^s - x^s\} < 2s\ \{(1+x)^{s+1} - x^{s+1}\}$$

for $x > 0$ and $s > 1$.

In order to prove (1.100) we replace x by $\frac{t}{1-t}$, $0 < t < 1$. This yields the equivalent inequality

$$(1.101) \qquad (s+1)(1+t)(1-t^s) < 2s(1-t^{s+1}), \qquad (0<t<1;\ s>1),$$

which may be rewritten as

$$(1.102) \qquad (s-1) - (s+1)t + (s+1)t^s - (s-1)t^{s+1} > 0, \qquad (0<t<1;\ s>1).$$

The left hand side of (1.102) takes the value 0 for $t = 1$ and hence it suffices to show that its derivative is negative for $0 < t < 1$. Hence, we want to show that

$$(1.103) \qquad -(s+1) + s(s+1)\, t^{s-1} - (s-1)(s+1)\, t^{s} < 0$$

or, equivalently, that

$$(1.104) \qquad -1 + s\, t^{s-1} - (s-1)\, t^{s} < 0, \qquad (0<t<1;\ s>1).$$

The left hand side of (1.104) takes the value 0 at $t = 1$ and hence it suffices to show that its derivative is positive for $0 < t < 1$. Hence, we want to show that

$$(1.105) \qquad s(s-1)\, t^{s-2} - s(s-1)\, t^{s-1} > 0, \qquad (0<t<1;\ s>1),$$

or, equivalently, that

$$(1.106) \qquad t^{-1} > 1, \qquad (0<t<1;\ s>1).$$

Since this is trivially true the proof of proposition 1.3 is complete. □

From the above proof we obtain

PROPOSITION 1.4.

$$(1.107) \qquad w_n(s) > \frac{n^{s+1}\,(2n+1)^{s}}{(n+1)^{s+1} - n^{s+1}}, \qquad (0<s<1)$$

and

$$(1.108) \qquad w_n(s) < \frac{n^{s+1}\,(2n+1)^{s}}{(n+1)^{s+1} - n^{s+1}}, \qquad (s>1).$$

2. APPROXIMATIONS OF $\int_{-1}^{1} |x|^s \, dx$, $(s \in \mathbb{N},\ s \geq 2)$.

Throughout this section s will denote an arbitrary but fixed integer greater than 1 (unless explicitly stated otherwise).

DEFINITION. Let $T_n^*(s)$ (resp. $t_n^*(s)$) be the n-th canonical trapezoidal (resp. tangent) approximation of $\int_{-1}^{1} |x|^s \, dx$, i.e., more explicitly,

$$(2.1) \qquad T_n^*(s) = \frac{2}{n} \sum_{k=1}^{n} \tfrac{1}{2}\{|-1+(k-1)\tfrac{2}{n}|^s + |-1+k\tfrac{2}{n}|^s\} = \frac{2}{n} \sum_{k=1}^{n} |-1+\tfrac{2k}{n}|^s,$$

and

$$(2.2) \qquad t_n^*(s) = \frac{2}{n} \sum_{k=1}^{n} |-1+\tfrac{2k-1}{n}|^s.$$

PROPOSITION 2.1.

$$(2.3) \qquad T_n^*(s) + t_n^*(s) = 4\, T_n(s),$$

where $T_n(s)$ *has the same meaning as in* (1.22).

PROOF.

$$(2.4) \qquad T_n^*(s) + t_n^*(s) = \frac{2}{n^{s+1}} \sum_{k=1}^{n} \{|n-2k|^s + |n+1-2k|^s\} =$$

$$= \frac{2}{n^{s+1}} \{2\,\sigma_n(s) - n^s\} = 4\, T_n(s)$$

(compare (1.24)). □

Combining this result with theorem 1.1 we obtain the following

PROPOSITION 2.2. $T_n^*(s) + t_n^*(s)$ *is decreasing (in* n*)*.

DEFINITION. For every $n \in \mathbb{N}$ let

$$(2.5) \qquad \phi_n(s) = \sum_{k=0}^{n-1} (-1)^k (n-k)^s = n^s - (n-1)^s + \ldots + (-1)^{n-2}\, 2^s + (-1)^{n-1},$$

and

(2.6) $$\delta_n^*(s) = T_n^*(s) - t_n^*(s).$$

From (2.5) it is clear that

(2.7) $$\phi_n(s) + \phi_{n+1}(s) = (n+1)^s, \qquad (n \in \mathbb{N}).$$

Moreover, we have the following

PROPOSITION 2.3.

(2.8) $$\delta_n^*(s) = \frac{2}{n^{s+1}} \{2\phi_n(s) - n^s\}.$$

PROOF.

(2.9) $$\delta_n^*(s) = \frac{2}{n^{s+1}} \{ \sum_{k=1}^{n} |n-2k|^s - \sum_{k=1}^{n} |n+1-2k|^s \} = \frac{2}{n^{s+1}} \{2\phi_n(s) - n^s\}. \qquad \square$$

As a counterpart of proposition 2.2 we have

PROPOSITION 2.4. $\delta_n^*(s)$ *is decreasing (in* n*).*

PROOF. In order to prove that

(2.10) $$\delta_n^*(s) > \delta_{n+1}^*(s)$$

we may just as well show that

(2.11) $$\frac{1}{n^{s+1}} \{2\phi_n(s) - n^s\} > \frac{1}{(n+1)^{s+1}} \{2\phi_{n+1}(s) - (n+1)^s\},$$

or, equivalently, that

(2.12) $$(n+1)^{s+1} \{2\phi_n(s) - n^s\} > n^{s+1} \{(n+1)^s - 2\phi_n(s)\},$$

which may be rewritten as

(2.13) $$2\phi_n(s) > \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} + n^{s+1}}.$$

A direct proof of (2.13) by induction seems to be practically unfeasible. Therefore we make the following somewhat unethical detour.

Observe that

$$\frac{2n^s\ (n+1)^s}{(n+1)^s + n^s} > \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} + n^{s+1}} . \tag{2.14}$$

Indeed, putting $\frac{1}{n} = x$, it is easily seen that (2.14) is equivalent to

$$2\{(1+x)^{s+1} + 1\} > (2+x)\{(1+x)^s + 1\} \tag{2.15}$$

which may be simplified to

$$(1+x)^s > 1. \quad \square \tag{2.16}$$

It follows that proposition 2.4 is a consequence of the following

PROPOSITION 2.5.

$$\frac{n^s\ (n+1)^s}{n^s + (n+1)^s} < \phi_n(s) \leq \frac{n^{2s}}{(n-1)^s + n^s} . \tag{2.17}$$

PROOF. One may verify that

$$\phi_n(2) = \frac{n(n+1)}{2} \tag{2.18}$$

and

$$\frac{n^2\ (n+1)^2}{n^2 + (n+1)^2} < \frac{n(n+1)}{2} \leq \frac{n^4}{(n-1)^2 + n^2} \tag{2.19}$$

so that (2.17) holds for $s = 2$ and all $n \in \mathbb{N}$. Hence it suffices to prove (2.17) for $s \geq 3$ and all $n \in \mathbb{N}$. It is easily verified that (2.17) holds for $n = 1$ and all $s \geq 3$.

Assume (2.17) still holds for $n = 1,\dots,N$ and all $s \geq 3$. Then we have

$$\phi_{N+1}(s) = (N+1)^s - \phi_N(s) < (N+1)^s - \frac{N^s\ (N+1)^s}{N^s + (N+1)^s} = \frac{(N+1)^{2s}}{N^s + (N+1)^s} \tag{2.20}$$

so that the right-hand inequality in (2.17) also holds if n is replaced by $N + 1$.

We also have

$$(2.21)\qquad \phi_{N+1}(s) = (N+1)^s - \phi_N(s) \geqq (N+1)^s - \frac{N^{2s}}{(N-1)^s + N^s} =$$

$$= \frac{(N-1)^s(N+1)^s + N^s(N+1)^s - N^{2s}}{(N-1)^s + N^s},$$

so that it is sufficient to prove that

$$(2.22)\qquad \frac{(N-1)^s(N+1)^s + N^s(N+1)^s - N^{2s}}{(N-1)^s + N^s} > \frac{(N+1)^s(N+2)^s}{(N+1)^s + (N+2)^s}$$

for all $N \in \mathbb{N}$.

Putting $\frac{1}{N} = x$, it is easily seen that (2.22) is equivalent to

$$(2.23)\qquad \frac{(1-x)^s(1+x)^s + (1+x)^s - 1}{(1-x)^s + 1} > \frac{(1+x)^s(1+2x)^s}{(1+x)^s + (1+2x)^s}.$$

Crossmultiplication in (2.23) and some simplification leads to the equivalent inequality

$$(2.24)\qquad (1-x)^s(1+x)^{2s} + (1+x)^{2s} - (1+x)^s - (1+2x)^s > 0.$$

which in its turn is equivalent to

$$(2.25)\qquad (1-x^2)^s + (1+x)^s - 1 - (1+\frac{x}{1+x})^s > 0.$$

Using the binomial theorem in (2.25) we see that we still have to show that

$$(2.26)\qquad \sum_{r=1}^{s} \binom{s}{r} \{(-1)^r x^{2r} + x^r - \frac{x^r}{(1+x)^r}\} > 0.$$

First consider the first *two* terms of this sum.

One may verify that the following list of inequalities is such that everyone of them (except the last one) is a consequence of the next one:

$$(2.27)\qquad s\{-x^2 + x - \frac{x}{1+x}\} + \frac{s(s-1)}{2}\{x^4 + x^2 - \frac{x^2}{(1+x)^2}\} > 0,$$

$$(2.28)\qquad \{-x + 1 - \frac{1}{1+x}\} + \frac{s-1}{2}\{x^3 + x - \frac{x}{(1+x)^2}\} > 0,$$

$$(2.29)\qquad \frac{-2x}{1+x} + (s-1)\{x^2 + 1 - \frac{1}{(1+x)^2}\} > 0,$$

$$(2.30)\qquad \frac{-2x}{1+x} + (s-1)\{1 - \frac{1}{(1+x)^2}\} \geqq 0,$$

$$(2.31)\qquad -2x(1+x) + (s-1)\{(1+x)^2 - 1\} \geqq 0,$$

$$(2.32)\qquad -2 - 2x + (s-1)(2+x) \geqq 0,$$

$$(2.33)\qquad (s-3)x \geqq 4 - 2s\,.$$

Since $s \geqq 3$ we have that $4 - 2s < 0$ and $s - 3 \geqq 0$ so that (2.33) and hence (2.27) is true.

For every remaining term in (2.26) corresponding to an even r ($r=2a$, say) we clearly have

$$(2.34)\qquad x^{4a} + x^{2a} - \frac{x^{2a}}{(1+x)^{2a}} > 0,$$

so that it suffices to show that every term in (2.26) corresponding to an odd $r \geqq 3$ ($r=2a+1$, say) is non-negative.

Hence, we want to prove that

$$(2.35)\qquad -x^{2(2a+1)} + x^{2a+1} - \frac{x^{2a+1}}{(1+x)^{2a+1}} \geqq 0.$$

It is clear that (2.35) does *not* hold for $x = 1$ but we may check directly that (2.25) is true in this case. When $x = 1$, (2.25) reads

$$(2.36)\qquad 2^s - 1 - (1+\tfrac{1}{2})^s > 0$$

which is equivalent to

$$(2.37)\qquad (\tfrac{4}{3})^s > 1 + (\tfrac{2}{3})^s.$$

Since (2.37) is true for $s = 2$ it certainly holds for $s \geqq 3$.

Hence, it suffices to show (2.35) for $0 < x \leqq \frac{1}{2}$. Again, one may check that in the following list of inequalities everyone of them (except the last one) is a consequence of the next one ($r \in \mathbb{N}$, $r \geqq 3$):

(2.38) $$-x^{2r} + x^r - \frac{x^r}{(1+x)^r} \geqq 0,$$

(2.39) $$-x^r + 1 - \frac{1}{(1+x)^r} \geqq 0,$$

(2.40) $$(1-x^r)(1+x)^r \geqq 0,$$

(2.41) $$(1-x^r)(1+rx) \geqq 0,$$

(2.42) $$r \geqq x^{r-1} + rx^r,$$

(2.43) $$r \geqq x^2 + rx^3,$$

(2.44) $$r \geqq (r+1)\, x^2,$$

(2.45) $$x^2 \leqq \frac{r}{r+1} \quad (\geqq \tfrac{3}{4})$$

(2.46) $$0 < x \leqq \tfrac{1}{2}\sqrt{3}$$

(2.47) $$0 < x \leqq \tfrac{1}{2}.$$

Since (2.47) is true by assumption, the proof of proposition 2.5 and hence that of proposition 2.4 is complete. □

COROLLARY 1. $\frac{\phi_n(s)}{n^s}$ *is decreasing (in* n*).*

PROOF. We have to prove that

(2.48) $$\frac{\phi_n(s)}{n^s} > \frac{\phi_{n+1}(s)}{(n+1)^s}$$

or, equivalently, that

$$(2.49)\qquad (n+1)^s\ \phi_n(s) > n^s\ \{(n+1)^s - \phi_n(s)\}\ .$$

Since (2.49) is equivalent to

$$(2.50)\qquad \phi_n(s) > \frac{n^s(n+1)^s}{n^s + (n+1)^s}\ ,$$

the corollary follows from (2.17). □

REMARK. It is easily verified that

$$(2.51)\qquad \frac{\phi_n(1)}{n} = \frac{[\frac{n}{2}] + \frac{1}{2}\{1+(-1)^{n+1}\}}{n}$$

so that corollary 1 does *not* hold for s = 1.

This led us to the following open question: *For which* real s *is corollary* 1 *true?*

COROLLARY 2.

$$(2.52)\qquad \lim_{n\to\infty} \frac{\phi_n(s)}{n^s} = \tfrac{1}{2}.$$

PROOF. This is an immediate consequence of (2.17). □

In the formulation of corollary 2, s has to be interpreted as an integer greater than 1. However, it may be shown that (2.52) holds for any $s > 0$.

PROPOSITION 2.6. (Compare POLYA and SZEGÖ, Aufgaben und Lehrsätze aus der Analysis I, Springer, 1970, p.40, problem 27)

$$(2.53)\qquad \lim_{n\to\infty} \frac{\phi_n(s)}{n^s} = \tfrac{1}{2}, \qquad (s>0).$$

PROOF.

Case 1. $0 < s < 1$,

In this case x^s is convex on $x \geq 0$ so that

$$(2.54)\qquad \sum_{0\leqq k\leqq \frac{n-1}{2}} \{(n+1-2k)^s + (n-1-2k)^s\} < 2 \sum_{0\leqq k\leqq \frac{n-1}{2}} (n-2k)^s$$

from which it is readily seen that

$$(2.55)\qquad -1 + \phi_{n+1}(s) < \phi_n(s).$$

Consequently, we have

$$(2.56)\qquad 2\phi_n(s) > (n+1)^s - 1$$

so that

$$(2.57)\qquad \liminf_{n\to\infty} \frac{\phi_n(s)}{n^s} \geqq \tfrac{1}{2}.$$

From (2.56) we obtain

$$(2.58)\qquad \phi_{n+1}(s) = (n+1)^s - \phi_n(s) < (n+1)^s - \tfrac{1}{2}\{(n+1)^s - 1\} = \tfrac{1}{2}(n+1)^s + \tfrac{1}{2}$$

so that

$$(2.59)\qquad \limsup_{n\to\infty} \frac{\phi_n(s)}{n^s} \leqq \tfrac{1}{2}.$$

Case 2. $s \geqq 1$.

The case $s = 1$ follows from (2.51). Because of the concavity of x^s on $x \geqq 0$ we have similarly as before

$$(2.60)\qquad -1 + \phi_{n+1}(s) > \phi_n(s)$$

from which we obtain

$$(2.61)\qquad 2\phi_n(s) < (n+1)^s$$

and

(2.62) $$\phi_{n+1}(s) = (n+1)^s - \phi_n(s) > \tfrac{1}{2}(n+1)^s.$$

It follows that (2.53) is true. □

Returning to our study of $T_n^*(s)$ we have

THEOREM 2.1. $T_n^*(s)$ *is decreasing (in* n*).*

PROOF. Combine propositions 2.2 and 2.4. □

COROLLARY. *For* m, n ∈ ℕ, m ≥ 2 *we have*

(2.63) $$(2n)^{m+1}\{2\sum_{k=1}^{n-1}(2k-1)^m + (2n-1)^m\} > (2n-1)^{m+1}\{2\sum_{k=1}^{n-1}(2k)^m + (2n)^m\}$$

and

(2.64) $$(2n+1)^{m+1}\{2\sum_{k=1}^{n-1}(2k)^m + (2n)^m\} > (2n)^{m+1}\{2\sum_{k=1}^{n}(2k-1)^m + (2n+1)^m\}.$$

PROOF. (2.63) is just another way of writing $T_{2n-1}^*(m) > T_{2n}^*(m)$. Similarly (2.64) is equivalent to $T_{2n}^*(m) > T_{2n+1}^*(m)$. □

REMARK. A *direct* proof of (2.63) and (2.64) seems to be difficult.

We will now investigate the behaviour of the canonical upper Riemann-sums $U_n^*(s)$ corresponding to f_s^*.

It is easily verified that

(2.65) $$U_{2n-1}^*(s) = \frac{2}{(2n-1)^{s+1}}\{-1 + 2w_n(s)\}$$

and

(2.66) $$U_{2n}^*(s) = \frac{2\sigma_n(s)}{n^{s+1}}.$$

From the last two formulas we easily deduce that

(2.67) $$U_{2n}^*(s) = T_{2n}^*(s) + \frac{1}{n}$$

and

$$(2.68)\qquad U^*_{2n+1}(s) = T^*_{2n+1}(s) + \frac{2}{2n+1} - \frac{1}{(2n+1)^{s+1}} .$$

We are now in a suitable position to prove

PROPOSITION 2.7. *The sequence* $\{U^*_n(s)\}^\infty_{n=2}$ *is decreasing.*

PROOF. We first consider

$$(2.69)\qquad U^*_{2n}(s) > U^*_{2n+1}(s)$$

or, equivalently,

$$(2.70)\qquad T^*_{2n}(s) + \frac{1}{n} > T^*_{2n+1}(s) + \frac{2}{2n+1} - \frac{1}{(2n+1)^{s+1}} .$$

Since $T^*_{2n}(s) > T^*_{2n+1}(s)$, it suffices to show that

$$(2.71)\qquad \frac{1}{n} > \frac{2}{2n+1} - \frac{1}{(2n+1)^{s+1}} .$$

Since this is trivially true we are done with (2.69).

Next we consider

$$(2.72)\qquad U^*_{2n-1}(s) > U^*_{2n}(s), \qquad (n\geqq 2),$$

or, equivalently,

$$(2.73)\qquad T^*_{2n-1}(s) + \frac{2}{2n-1} - \frac{1}{(2n-1)^{s+1}} > T^*_{2n}(s) + \frac{1}{n} , \qquad (n\geqq 2).$$

It clearly suffices to show that

$$(2.74)\qquad \frac{2}{2n-1} - \frac{1}{(2n-1)^{s+1}} \geqq \frac{1}{n} , \qquad (n\geqq 2),$$

which is equivalent to

$$(2.75)\qquad (2n-1)^s \geqq n, \qquad (n\geqq 2).$$

Since this is obviously true our proof is complete. □

In order to be able to deal with the canonical lower Riemann-sums corresponding to f_s^* we first prove the following

PROPOSITION 2.8. $t_{2n+1}^*(s)$ *is increasing (in* n*).*

PROOF. First observe that

$$(2.76)\qquad t_{2n+1}^*(s) = \frac{2^{s+2}}{(2n+1)^{s+1}} \cdot \sigma_n(s)$$

Hence, we may just as well prove that

$$(2.77)\qquad \frac{\sigma_n(s)}{(2n+1)^{s+1}} < \frac{\sigma_{n+1}(s)}{(2n+3)^{s+1}},$$

which may be shown to be equivalent to

$$(2.78)\qquad \sigma_n(s) < \frac{(n+1)^s(2n+1)^{s+1}}{(2n+3)^{s+1} - (2n+1)^{s+1}}.$$

We will prove (2.78) by induction: For n = 1, (2.78) is equivalent to

$$(2.79)\qquad 5^{s+1} - 3^{s+1} < 3 \cdot 6^s.$$

It may be checked directly that (2.79) is true for s = 2.
Now observe that (2.79) is equivalent to

$$(2.80)\qquad 1 - (\tfrac{3}{5})^{s+1} < \tfrac{3}{5}(1+\tfrac{1}{5})^s$$

so that it suffices to show that

$$(2.81)\qquad \tfrac{3}{5}(1+\tfrac{1}{5})^s \geqq 1, \qquad (s \geqq 3).$$

Since

$$(2.82)\qquad (1+\tfrac{1}{5})^s > 1 + \frac{s}{5} + \frac{s(s-1)}{2\cdot 5^2} \geqq 1 + \frac{3}{5} + \frac{3}{25} = \frac{43}{25} > \frac{5}{3}$$

it follows that (2.78) holds for $n = 1$ and all $s \geqq 2$.

In (2.78) we replace n by $n - 1$, so that we still have to show that

$$(2.83)\qquad (\sigma_{n-1}(s) =)\ \ \sigma_n(s) - n^s < \frac{n^s(2n-1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}}\ ,$$

or, equivalently, that

$$(2.84)\qquad \sigma_n(s) < \frac{n^s(2n+1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}}\ , \qquad \text{for all } n \in \mathbb{N}\ .$$

(2.84) is obviously true for $n = 1$.

Assume (2.84) true for $n = 1,\ldots,N$ and all $s \geqq 2$. Then we have

$$(2.85)\qquad \sigma_{N+1}(s) = (N+1)^s + \sigma_N(s) < (N+1)^s + \frac{N^s(2N+1)^{s+1}}{(2N+1)^{s+1} - (2N-1)^{s+1}}\ ,$$

so that it suffices to show that

$$(2.86)\qquad (N+1)^s + \frac{N^s(2N+1)^{s+1}}{(2N+1)^{s+1} - (2N-1)^{s+1}} \leqq \frac{(N+1)^s(2N+3)^{s+1}}{(2N+3)^{s+1} - (2N+1)^{s+1}}$$

for all $N \in \mathbb{N}$.

Putting $\frac{1}{N} = x$ we arrive at the equivalent inequality

$$(2.87)\qquad (1+x)^s + \frac{(2+x)^{s+1}}{(2+x)^{s+1} - (2-x)^{s+1}} \leqq \frac{(1+x)^s(2+3x)^{s+1}}{(2+3x)^{s+1} - (2+x)^{s+1}}\ .$$

In (2.87) replace x by $2x$ (so that from now on $0 < x \leqq \frac{1}{2}$) in order to arrive at the equivalent inequality

$$(2.88)\qquad (1+2x)^s + \frac{(1+x)^{s+1}}{(1+x)^{s+1} - (1-x)^{s+1}} \leqq \frac{(1+2x)^s(1+3x)^{s+1}}{(1+3x)^{s+1} - (1+x)^{s+1}}\ .$$

After crossmultiplication and some simplification it turns out that we may just as well prove that

$$(2.89)\qquad (1+2x)^s\{(1+x)^{s+1} - (1-x)^{s+1}\} \geqq (1+3x)^{s+1} - (1+x)^{s+1}\ ,$$

which is equivalent to

$$(2.90)\qquad \frac{(1+x)^{s+1} - (1-x)^{s+1}}{2x} \geqq \frac{(1+\frac{x}{1+2x})^{s+1} - (1-\frac{x}{1+2x})^{s+1}}{\frac{2x}{1+2x}} .$$

Since

$$(2.91)\qquad \frac{(1+x)^{s+1} - (1-x)^{s+1}}{2x} = \frac{1}{2x}\sum_{r=0}^{s+1}\binom{s+1}{r} x^r \{1-(-1)^r\} = \sum_{r=0}^{\infty}\binom{s+1}{2r+1} x^{2r},$$

it follows that the left hand side of (2.90) is increasing in x on $x > 0$.

Observing that $x > \frac{x}{1+2x}$ for all $x > 0$, (2.90) follows and our proof is complete. □

From the above proof we obtain the following.

PROPOSITION 2.9.

$$(2.92)\qquad \sigma_n(s) < \frac{n^s(2n+1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}} .$$

In the remaining part of this section we will investigate the behaviour of the canonical lower Riemann-sums $L_n^*(s)$ corresponding to f_s^*. One may verify that

$$(2.93)\qquad L_{2n-1}^*(s) = \frac{4}{2n-1}\sum_{k=1}^{n-1}\left(\frac{2k-1}{2n-1}\right)^s = \frac{4\{w_n(s) - (2n-1)^s\}}{(2n-1)^{s+1}}$$

and

$$(2.94)\qquad L_{2n}^*(s) = 2L_n(s).$$

PROPOSITION 2.10. *The sequence* $\{L_n^*(s)\}_{n=2}^{\infty}$ *is increasing.*

PROOF. We first consider

$$(2.95)\qquad L_{2n+1}^*(s) < L_{2n+2}^*(s),$$

which may be shown to be equivalent to

$$(2.96)\qquad 2(n+1)^{s+1}\, w_n(s) < (2n+1)^{s+1}\, \sigma_n(s).$$

Using the inequalities (1.53) and (1.108) we see that it suffices to prove that

$$(2.97) \qquad 4(n+1)^{s+1} \frac{n^{s+1}(2n+1)^s}{(n+1)^{s+1} - n^{s+1}} < (2n+1)^{s+1} \frac{n^{s+1}(n+1)^s + n^s(n+1)^{s+1}}{(n+1)^{s+1} - n^s} ,$$

or, equivalently, that $4n(n+1) < (2n+1)^2$. It follows that (2.95) holds.

Next we consider

$$(2.98) \qquad L^*_{2n}(s) < L^*_{2n+1}(s)$$

or, equivalently,

$$(2.99) \qquad (2n+1)^{s+1} \sigma_{n-1}(s) < 2n^{s+1} w_n(s)$$

which may also be written as

$$(2.100) \qquad \frac{\sigma_n(s)}{n^{s+1}} - \frac{\sigma_{2n}(s)}{(2n)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}} \frac{1 - (1+\frac{1}{2n})^{s+1}}{2} + \frac{1}{2n} (1+\frac{1}{2n})^{s+1} .$$

A direct proof of either of the last three inequalities seems to be quite cumbersome. Therefore we proceed as follows.

We already know that $U^*_{2n-1}(s) > U^*_{2n}(s)$, $(n\geq 2)$. This last inequality is equivalent to

$$(2.101) \qquad \frac{1}{(2n-1)^{s+1}} \{-1 + 2w_n(s)\} > \frac{\sigma_n(s)}{n^{s+1}} ,$$

which may be rewritten as

$$(2.102) \qquad \frac{\sigma_n(s)}{n^{s+1}} - \frac{\sigma_{2n}(s)}{(2n)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}} \frac{1 - (1-\frac{1}{2n})^{s+1}}{2} - \frac{1}{2(2n)^{s+1}} .$$

Hence, in order to prove (2.100) it suffices to prove that

$$(2.103) \qquad \frac{\sigma_n(s)}{n^{s+1}} \frac{1 - (1-\frac{1}{2n})^{s+1}}{2} - \frac{1}{2(2n)^{s+1}} < \frac{\sigma_n(s)}{n^{s+1}} \frac{1 - (1+\frac{1}{2n})^{s+1}}{2} + \\ + \frac{1}{2n} (1+\frac{1}{2n})^{s+1} .$$

It is easily seen that (2.103) is equivalent to

$$(2.104)\qquad \sigma_n(s) < \frac{n^s + n^s(2n+1)^{s+1}}{(2n+1)^{s+1} - (2n-1)^{s+1}} .$$

Since (2.104) is true by proposition 2.9, our proof of proposition 2.10 is complete. □

3. SOME APPLICATIONS.

PROPOSITION 3.1. *Let* $f\colon [0,1] \to \mathbb{R}$ *be such that* $T_n(f)$ *is decreasing. Define* $\tilde{f}\colon [0,1] \to \mathbb{R}$ by

$$\tilde{f}(x) = f(1-x), \qquad (0 \leq x \leq 1).$$

Then $T_n(-f)$ *is increasing and* $T_n(\tilde{f})$ *is decreasing.*

PROOF. $T_n(-f) = -T_n(f)$ and $T_n(\tilde{f}) = T_n(f)$. □

PROPOSITION 3.2. *For every* $k \in \mathbb{N}$ *let* $f_k\colon [a,b] \to \mathbb{R}$ *be such that* $T_n(f_k)$ *is non-increasing in* n *and let* $\lim_{k\to\infty} f_k(x) = 0$ *for every* $x \in [a,b]$ *(strictly speaking we only need this for all* $x \in [a,b]$ *which are of the form* $x = a + p(b-a)$, p *being rational). Then* $\lim_{k\to\infty} T_n(f_k) = 0$ *for all* $n \in \mathbb{N}$.

PROOF. Obvious. □

PROPOSITION 3.3. *For every* $k \in \mathbb{N}$ *let* $f_k\colon [a,b] \to \mathbb{R}$ *be such that* $f_k(x)$ *tends to a finite limit (=*$f(x)$*, say) when* $k \to \infty$*. Also, for every* $k \in \mathbb{N}$, *let* $T_n(f_k)$ *be non-increasing in* n*. Then* $T_n(f)$ *is also non-increasing.*

PROOF. Since T_n acts as a linear functional we have

$$T_n(f) = T_n(f-f_k) + T_n(f_k) \geq T_n(f-f_k) + T_{n+1}(f_k) =$$

$$= T_n(f-f_k) + T_{n+1}(f_k-f) + T_{n+1}(f).$$

Now let $k \to \infty$ and apply proposition 3.2. □

PROPOSITION 3.4. *For every* $k \in \mathbb{N}$ *let* $f_k: [a,b] \to \mathbb{R}$ *be such that* $T_n(f_k)$ *is decreasing in* n. *Then* $T_n(\sum_{k=1}^{\infty} p_k f_k)$ *is also decreasing in* n *whenever* $\sum_{k=1}^{\infty} p_k f_k$ *converges pointwise on* [a,b], *the coefficients* p_k *being non-negative (at least one of them being positive).*

PROOF. Exercise. □

In all the examples which follow we will assume that $a < b$.

APPLICATION 1. Let $f(x) = \log x$, $(0<a\leq x\leq b)$. It is clear that

$$T_n(f) = T_n(\phi), \text{ where } \phi(t) = \log(a+(b-a)t), \quad (0\leq t\leq 1).$$

Writing $p = \frac{a}{b-a}$ (>0) we have

$$\phi(t) = \log(b-a) + \log(p+t) = \log(b-a) + \log(p+1) + \log(1-\frac{1-t}{p+1}).$$

Since

$$\log(1-\frac{1-t}{p+1}) = -\sum_{k=1}^{\infty} \frac{(1-t)^k}{k(p+1)^k}$$

it follows that $T_n(\phi)$ is increasing.

Exercise. Show that $\{\frac{(2n)!}{n!\, n^n}\}^{\frac{1}{n}}$ is increasing in n.

APPLICATION 2. Let $f(x) = e^{cx}$, $(a\leq x\leq b)$, where c is some real constant.

Case 1. $c < 0$.

Put $-c = p$ (>0) and $x = a + (b-a)t$, $(0\leq t\leq 1)$, and observe that

$$e^{-px} = e^{-p(a+(b-a)t)} = e^{-pb}\, e^{p(b-a)(1-t)} =$$

$$= e^{-pb} \sum_{k=0}^{\infty} \frac{p^k(b-a)^k}{k!} (1-t)^k .$$

It follows that $T_n(f)$ is decreasing in n.

Case 2. $c > 0$. Again, put $x = a + (b-a)t$, $(0 \leq t \leq 1)$.

Since

$$e^{cx} = e^{c(a+(b-a)t)} = e^{ca} \sum_{k=0}^{\infty} \frac{c^k (b-a)^k}{k!} t^k$$

it follows that $T_n(f)$ is decreasing in n.

Exercise. If $f(x) = \dfrac{1}{e^x - 1}$, $(0<a \leq x \leq b)$, then $T_n(f)$ is decreasing in n.

<u>REMARK</u>. So far we have only invoked theorem 1.1 and the easy propositions stated at the beginning of this section.

<u>APPLICATION 3</u>. For any $s \in \mathbb{R}$ let $f_s(x) = x^s$, $(0<a \leq x \leq b)$.

Putting $x = a + (b-a)t$, $(0 \leq t \leq 1)$ we have

$$f_s(x) = (a + (b-a)t)^s = (b-a)^s (p+t)^s$$

where $p = \frac{a}{b-a}$ (>0), so that we may just as well study the behaviour of $T_n(\phi_s)$ where $\phi_s(t) = (p+t)^s$, $(p>0;\ 0 \leq t \leq 1)$.

Case 1. $s < 0$.

Then we may write

$$(p+t)^s = (p+1+t-1)^s = (p+1)^s (1-\frac{1-t}{p+1})^s =$$

$$= (p+1)^s \sum_{r=0}^{\infty} \binom{s}{r} (-1)^r \frac{(1-t)^r}{(p+1)^r} .$$

Since $s < 0$ we have $\binom{s}{r}(-1)^r > 0$ for $r = 0,1,2,3,\ldots$ and it follows that $T_n(\phi_s)$ is decreasing in n.

Case 2. $0 \leq s \leq 1$.

It is clear that if $s = 0$ or if $s = 1$ then $T_n(\phi_s)$ is constant. If $0 < s < 1$ then, as before, we have

$$(p+t)^s = (p+1)^s \sum_{r=0}^{\infty} \binom{s}{r} (-1)^r \frac{(1-t)^r}{(p+1)^r} .$$

Observing that

$$\binom{s}{r}(-1)^r = \begin{cases} 1 \text{ if } r = 0 \\ < 0 \text{ if } r \in \mathbb{N} \end{cases}$$

it follows that $T_n(\phi_s)$ is increasing in n.

Case 3. $s > 1$.

Case 3a. $s \in \mathbb{N}$.

Then we simply have

$$(p+t)^s = \sum_{r=0}^{s} \binom{s}{r} p^{s-r} t^r$$

and it follows that $T_n(\phi_s)$ is decreasing.

Case 3b. $2m+1 < s < 2m+2$ for some $m \in \mathbb{N} \cup \{0\}$.

Then we have

$$(p+t)^s = (p+\tfrac{1}{2}+t-\tfrac{1}{2})^s = (p+\tfrac{1}{2}+\frac{z}{2})^s,$$

where $z = 2t-1$, so that $-1 \leq z \leq 1$.

It follows that we may just as well trace the behaviour of T_n for the function

$$(2p+1+z)^s = (2p+1)^s(1+\frac{z}{2p+1})^s.$$

Since

$$(1+\frac{z}{2p+1})^s = \sum_{r=0}^{\infty} \binom{s}{r} \frac{z^r}{(2p+1)^r}$$

and $\binom{s}{2r} > 0$ for all $r \in \mathbb{N}$, (note that the odd powers of z are irrelevant!) it follows from theorem 2.1 and proposition 3.4 that $T_n(f_s)$ is decreasing.

Case 3c. $2m < s < 2m+1$ for some $n \in \mathbb{N}$.

We *conjecture* that $T_n(f_s)$ is decreasing in n if $2m < s < 2m+1$ for some $m \in \mathbb{N}$.

However, so far we were unable to prove this.

APPLICATION 4. Let $s = 2m$ for some $m \in \mathbb{N}$ and define $f_s(x) = x^s$, $(a \leqq x \leqq b)$.

Again, putting $x = a + (b-a)t$, $(0 \leqq t \leqq 1)$ we have

$$f_s(x) = (b-a)^s \left(\frac{a}{b-a} + t\right)^s ,$$

so that we may just as well study the behaviour of T_n for the function

$$\phi(t) = (\alpha+t)^s, \qquad (0 \leqq t \leqq 1;\ \alpha \in \mathbb{R}).$$

Case 1. $\alpha \geqq 0$.

Then

$$(\alpha+t)^s = \sum_{r=0}^{s} \binom{s}{r} \alpha^{s-r} t^r$$

and it is clear from theorem 1.1 and proposition 3.4 that $T_n(f_s)$ is decreasing in n.

Case 2. $\alpha < 0$.

Put $-\alpha = p$. Then we may just as well study the behaviour of T_n for the function

$$\psi(t) = (p-t)^s, \qquad (0 \leqq t \leqq 1;\ p>0).$$

Similarly as before it is easily seen that (putting $z=2t-1$) we may just as well study the behaviour of T_n for the function

$$\lambda(z) = \left(p-\tfrac{1}{2}+\frac{z}{2}\right)^s, \qquad (-1 \leqq z \leqq 1),$$

or, equivalently, for the function

$$\mu(z) = (\beta+z)^s, \qquad (-1 \leqq z \leqq 1;\ \beta \in \mathbb{R}).$$

Since

$$\mu(z) = \sum_{r=0}^{s} \binom{s}{r} \beta^{s-r} z^r$$

and since the odd powers of z are irrelevant for our purpose, it follows that the relevant part of the above expansion of $\mu(z)$ is

$$\sum_{r=0}^{m} \binom{2m}{2r} \beta^{2m-2r} z^{2r} ,$$

so that $T_n(\mu)$ and hence $T_n(f_s)$ is decreasing by theorem 2.1 and proposition 3.4.

REMARK. Again, let $s = 2m$ for some $m \in \mathbb{N}$ and let

$$\phi(x) = (\alpha+x)^s, \qquad (0\leq x\leq 1)$$

where α is some real constant.

Using the Euler-Maclaurin summation formula we find

$$T_n(\phi) = \int_0^1 \phi(x)dx + \sum_{r=1}^{m} b_{2r} \frac{\phi^{(2r-1)}(1) - \phi^{(2r-1)}(0)}{n^{2r}} =$$

$$= \int_0^1 f(x)dx + \frac{1}{2m+1} \sum_{r=1}^{m} \binom{2m+1}{2r} B_{2r} \frac{(\alpha+1)^{2m-2r+1} - \alpha^{2m-2r+1}}{n^{2r}} .$$

From this and application 4 we obtain that for any $m \in \mathbb{N}$ and any $\alpha \in \mathbb{R}$

$$\sum_{r=1}^{m} \binom{2m+1}{2r} B_{2r} \frac{(\alpha+1)^{2m-2r+1} - \alpha^{2m-2r+1}}{n^{2r}}$$

is decreasing in n.

APPLICATION 5. The example described in the introduction (p.4) is left as an exercise to the reader.

APPLICATION 6. We will prove that the sequence $\{(\frac{n^n}{n!})^{\frac{1}{n}}\}_{n=1}^{\infty}$ is increasing.

Taking logarithms we obtain

$$\log\left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = \frac{1}{n}\log\frac{n^n}{n!} = \frac{1}{n}\sum_{k=1}^{n}\log\frac{n}{k} =$$

$$= -\frac{1}{n}\sum_{k=1}^{n}\log\frac{k}{n} = -\frac{1}{n}\sum_{k=0}^{n-1}\log\frac{n-k}{n} =$$

$$= -\frac{1}{n}\sum_{k=0}^{n-1}\log\left(1-\frac{k}{n}\right) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{s=1}^{\infty}\frac{1}{s}\left(\frac{k}{n}\right)^s =$$

$$= \sum_{s=1}^{\infty}\frac{1}{s}L_n(s)$$

and our assertion follows easily (compare proposition 1.2).

Since $0 \leqq L_n(s) < \frac{1}{s+1}$, $(s>0)$ and $\lim_{n\to\infty} L_n(s) = \frac{1}{s+1}$, $(s>0)$ we also obtain an alternative proof of the following well-known result

$$\lim_{n\to\infty}\left(\frac{n^n}{n!}\right)^{\frac{1}{n}} = \exp\sum_{s=1}^{\infty}\frac{1}{s(s+1)} = e.$$

<u>REMARK</u>. Actually, we can prove more than asserted in the previous application.

If we define

$$f(s) = \left\{\frac{s^s}{\Gamma(s+1)}\right\}^{\frac{1}{s}}, \qquad (s>0),$$

then f is increasing on $\mathbb{R}^+$.

This may be seen as follows. For $s > 0$ we have

$$\Gamma(s+1) = s^s\, e^{-s}\sqrt{2\pi s}\; e^{\mu(s)}$$

where $\mu(s)$ is Binet's function which may be represented by

$$\mu(s) = \int_0^{\infty}\frac{e^{-st}}{t}\left\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\right\}dt, \qquad (s>0).$$

See, for example, G. SANSONE and J. GERRETSEN, *Lectures on the Theory of Functions of a Complex Variable*, Noordhoff-Groningen, 1960, p.216.

In order to prove that f(s) is increasing we may just as well show that log f(s) is increasing. Taking logarithms we obtain

$$\log f(s) = \frac{1}{s} \log \frac{s^s}{s^s\, e^{-s}\, \sqrt{2\pi s}\; e^{\mu(s)}} = 1 - \frac{\mu(s) + \frac{1}{2}\log 2\pi s}{s}.$$

It clearly suffices to show that

$$\frac{d}{ds}\,\frac{\mu(s) + \frac{1}{2}\log 2\pi s}{s} = \frac{s\{\mu'(s) + \frac{1}{2s}\} - \{\mu(s) + \frac{1}{2}\log 2\pi s\}}{s^2} < 0, \ (s>0),$$

or, equivalently, that

$$\phi(s) \overset{\text{def}}{=} s\{\mu'(s) + \frac{1}{2s}\} - \{\mu(s) + \tfrac{1}{2}\log 2\pi s\} < 0, \qquad (s>0).$$

Since

$$e^{\mu(s)}\sqrt{2\pi s} = \frac{e^s\, \Gamma(s+1)}{s^s}$$

it follows that

$$\lim_{s\downarrow 0} \{\mu(s) + \tfrac{1}{2}\log 2\pi s\} = \lim_{s\downarrow 0} \log \frac{e^s\, \Gamma(s+1)}{s^s} = \log 1 = 0.$$

From the integral representation of $\mu(s)$ we obtain

$$\mu'(s) = -\int_0^\infty e^{-st}\{\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\}dt = -\frac{1}{2s} + \int_0^\infty e^{-st}\{\frac{1}{t} - \frac{1}{e^t - 1}\}dt$$

so that

$$\mu'(s) + \frac{1}{2s} = \int_0^\infty e^{-st}\{\frac{1}{t} - \frac{1}{e^t - 1}\}dt.$$

Since

$$\lim_{t\to\infty} \{\frac{1}{t} - \frac{1}{e^t - 1}\} = 0$$

it follows from the general theory of Laplace transforms that

$$\lim_{s\downarrow 0} s\{\mu'(s) + \frac{1}{2s}\} = 0.$$

Consequently we have that $\lim_{s\downarrow 0} \phi(s) = 0$, so that it suffices to show that $\phi'(s) < 0$ for $s > 0$.

Since

$$\phi'(s) = s\frac{d}{ds}\{\mu'(s) + \frac{1}{2s}\} = s\frac{d}{ds}\int_0^\infty e^{-st}\{\frac{1}{t} - \frac{1}{e^t - 1}\}dt =$$

$$= -s\int_0^\infty e^{-st}\{1 - \frac{t}{e^t - 1}\}dt$$

and

$$1 - \frac{t}{e^t - 1} > 0, \qquad (t>0)$$

it follows that indeed $\phi'(s) < 0$ for $s > 0$, completing the proof of our assertion.

<u>APPLICATION 7</u>. If $\alpha < 0$ then the sequence $\{\frac{\sigma_n(\alpha)}{n^{\alpha+1}}\}_{n=1}^\infty$ is increasing.

We may prove this as follows: since

$$\frac{\sigma_n(\alpha)}{n^{\alpha+1}} = \frac{1}{n}\sum_{k=1}^{n}(\frac{k}{n})^\alpha = \frac{1}{n}\sum_{k=0}^{n-1}(\frac{n-k}{n})^\alpha = \frac{1}{n}\sum_{k=0}^{n-1}(1-\frac{k}{n})^\alpha =$$

$$= \frac{1}{n}\sum_{k=0}^{n-1}\sum_{s=0}^{\infty}\binom{\alpha}{s}(-1)^s(\frac{k}{n})^s =$$

(observe that $\binom{\alpha}{s}(-1)^s > 0$ for $s = 0,1,2,3,\ldots$)

$$= \sum_{s=0}^{\infty}\binom{\alpha}{s}(-1)^s\frac{1}{n}\sum_{k=0}^{n-1}(\frac{k}{n})^s = \sum_{s=0}^{\infty}\binom{\alpha}{s}(-1)^s L_n(s),$$

our assertion follows from proposition 1.2 and the observation that if $\alpha < 0$ then $\binom{\alpha}{s}(-1)^s > 0$ for all $s \in \mathbb{N} \cup \{0\}$.

<u>APPLICATION 8</u>. Let $0 < \mu < 1$ and define

$$f(x) = (1-x^2)^\mu, \qquad (-1\leq x\leq 1).$$

(Note that this includes the case of a half circle). Then $T_n(f)$ is increasing.

In order to see this, one may argue as follows: for $-1 \leqq x \leqq 1$, we have

$$(1-x^2)^\mu = \sum_{s=0}^{\infty} \binom{\mu}{s} (-1)^s x^{2s} = 1 + \sum_{s=1}^{\infty} \binom{\mu}{s} (-1)^s x^{2s} .$$

Now observe that if $0 < \mu < 1$ then

$$\binom{\mu}{s} (-1)^s < 0 \text{ for all } s \in \mathbb{N}$$

so that

$$T_n(f) = 2 + \sum_{s=1}^{\infty} \binom{\mu}{s} (-1)^s T_n^*(2s).$$

Invoking theorem 2.1 it follows that $T_n(f)$ is increasing.