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INVARIANT MEASURES AND THE EQUICONTINUOUS STRUCTURE RELATION I

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In this introductory paper we introduce and illistrate some notions and problems from Topological Dynamics. This discipline originated from the qualitative theory of differential equations (work of Poincaré, Lyapunov, Birkhoff and others). This paper concerns "abstract" Topological Dynamics: there is no direct relationship with differential equations. After the necessary definitions (Sections 1,2,3) we consider the problem when the regionally proximal relation of a minimal flow is an equivalence relation and when a minimal flow which has no non-trivial equicontinuous factors is weakly mixing. In the Sections 4 and 5 we state and prove a result of Mac Mahon's, namely that the answer to both problems is affirmative if the flow has an invariant measure. Although the the results are not new, the proofs are simpler than the existing ones.

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1. FLOWS, HOMOMORPHISMS AND FACTORS

Let T be a topological group, arbitrary but fixed. A *flow* is a pair $\underline{X} := <X,\pi>$ where X is a compact Hausdorff space and π is an *action* of T on X, i.e. a continuous mapping π: $(t,x) \mapsto tx$: $T\times X \to X$ (so we write alternatively $\pi(t,x)$ or tx or even t.x), satisfying the following conditions:

$$ex = x; \quad t(sx) = (ts)x \quad \text{for} \quad t,s \in T \quad \text{and} \quad x \in X.$$

EXAMPLES.

1. $T = \mathbb{Z}$, ϕ: $X \to X$ a homeomorphism and $n{\circ}x := \phi^n x$ for $n \in \mathbb{Z}$, $x \in X$ (*discrete flow*). So every homeomorphism generates a discrete flow. Important examples:
   (a) $X = \mathbb{S}^1 = \{z \in \mathbb{C} \,|\, |z| = 1\}$ and $\phi(z) := z\exp(2\pi i\theta)$ for $z \in \mathbb{S}^1$ (rotation). It is well-known, that the *orbit* $\{\phi^n z : n \in \mathbb{Z}\}$ of every point z of $\mathbb{S}^1$ is dense in $\mathbb{S}^1$ iff θ is irrational.
   (b) $X = \{0,1\}^{\mathbb{Z}}$, the space of all 2-sided infinite sequences of 0's and 1's, with the ordinary product topology. Let σ: $X \to X$ be given by $(\sigma x)_i := x_{i+1}$ (the sequence $\ldots x_{-1}x_0x_1x_2\ldots$ is shifted one position to the left).
2. $T = \mathbb{R}$, $X = S^1\times S^1$ (the 2-torus) and π: $T\times X \to X$ defined by

$$t.(z_1,z_2) := (z_1 \exp it,\; z_2 \exp i\theta t).$$

It follows readily from Example 1a that if θ is irrational, then every orbit $\{t.(z_1,z_2) : t \in \mathbb{R}\}$ is dense in the torus (Kronecker).

If $\underline{X} = <X,\pi>$ and $\underline{Y} = <Y,\sigma>$ are flows, then a *homomorphism* from $\underline{X}$ to $\underline{Y}$ is a continuous mapping ϕ: $X \to Y$ such that $\phi\circ\pi(t,-) = \sigma(t,-)\circ\phi$ for all $t \in T$ (thus, $\phi(tx) = t\phi(x)$ for all $t \in T$ and $x \in X$). Notation: ϕ: $\underline{X} \to \underline{Y}$. If ϕ: $\underline{X} \to \underline{Y}$ is a homomorphism and ϕ: $X \to Y$ is a homeomorphism of X onto Y, then ϕ is called an *isomorphism*. A homomorphism ϕ: $\underline{X} \to \underline{Y}$ such that ϕ: $X \to Y$ is a surjection is called an *extension* of $\underline{Y}$ (also $\underline{X}$ will be called an extension of $\underline{Y}$); in that case, $\underline{Y}$ is called a *factor* of $\underline{X}$, and ϕ is also called a factor mapping.

EXAMPLES.

3. If $\underline{X} = <X,\pi>$ is a flow and Y is a closed subset of X which is *invariant*, i.e. $ty \in Y$ for all $t \in T$ and $y \in Y$, then $\sigma(t,y) := \pi(t,y)$ for $t \in T$, $y \in Y$ defines an action σ of T on Y, and the inclusion mapping $Y \to X$ is a homomorphism of flows from $\underline{Y}$ to $\underline{X}$. ($\underline{Y}$ is called a *subflow* of $\underline{X}$ in this case.)

4. If $\underline{X}$ and $\underline{Y}$ are flows, then a flow $\underline{X}\times\underline{Y}$ may be defined by $t.(x,y) := (tx,ty)$ for $t \in T$ and $(x,y) \in X\times Y$. The projections are homomorphisms from $\underline{X}\times\underline{Y}$ onto $\underline{X}$ and $\underline{Y}$, respectively, so $\underline{X}$ and $\underline{Y}$ are factors of $\underline{X}\times\underline{Y}$ in this sense defined above.

5. Let $\underline{X}$ be a flow and R an invariant closed equivalence relation in X, i.e. as a subset of $X\times X$ the set R is closed and invariant with respect to the coordinate-wise action of T on $X\times X$ (compare with Example 4 above). Then the quotient space X/R turns out to be a compact Hausdorff space (because R is closed), and an action of T on X/R can be defined by the rule

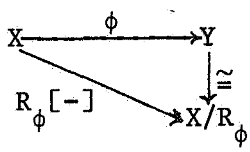
$$t.R[x] := R[tx] \text{ for } t \in T \text{ and } x \in X.$$

Because R is invariant, this definition is unambiguous; since the quotient mapping $R[-]: X \to X/R$ is perfect, this action is easily seen to be continuous. This flow on X/R will be denoted $\underline{X}/R$. Clearly, $R[-]: \underline{X} \to \underline{X}/R$ is a factor mapping in the sense above, and $\underline{X}/R$ is a factor of $\underline{X}$.

REMARK. Every factor arizes in the way, described in Example 5. Indeed, let $\phi: \underline{X} \to \underline{Y}$ be a factor mapping of flows. Then

$$R_\phi := \{(x_1,x_2) \in X\times X : \phi(x_1) = \phi(x_2)\}$$

is easily seen to be a closed invariant equivalence relation in X. The space X/R_ϕ is homeomorphic to Y and this homeomorphism establishes an isomorphism between $\underline{X}/R_\phi$ and $\underline{Y}$ in such a way, that $R_\phi[-]$ corresponds to ϕ:



2. MINIMAL FLOWS; EQUICONTINUITY

A flow $\underline{X}$ is called *minimal* whenever it has no proper closed invariant subsets. Equivalently, a flow $\underline{X}$ is minimal whenever each orbit $Tx(:=\{tx: t \in T\})$ for $x \in X$ is dense in X. By Zorn's lemma, every flow contains at least one minimal subflow. (As such a minimal subflow M is the closure of the orbit of any of its points, i.e. $M = \overline{Tx}$ for each $x \in M$, it is often called a *minimal orbit closure*.)

In the study of minimal sets it is convenient to restrict oneself to subclasses which have a richer structure. An example of such a subclass is the class of all equicontinuous minimal flows. An (arbitrary) flow $\underline{X}$ is called *equicontinuous* whenever the group of homeomorphisms $\{\pi^t : t \in T\}$ is (uniformly) equicontinuous with respect to the (unique!) uniformity $\hat{U}$ for X, that is,

$$\forall \alpha \in \hat{U}\ \exists \beta \in \hat{U} : (x_1,x_2) \in \beta \Rightarrow (tx_1,tx_2) \in \alpha \text{ for all } t \in T.$$

It is straightforward to show that in an equicontinuous flow X all orbit closures are minimal (indeed, if $x_1 \in \overline{Tx_2}$ then one shows $x_2 \in \overline{Tx_1}$), hence the orbit closures form a partition of the space X.

The equicontinuous minimal flows are fairly well understood. For example, if $\underline{X}$ is an equicontinuous minimal flow and X is metrizable, then X carries an invariant metric. Indeed, the closure $G := \overline{\{\pi^t : t \in T\}}$ of the given group of homeomorphisms in X^X is a compact topological group of homeomorphisms of X, acting continuously on X; it is quite easy to show that if a compact group acts on a metric space, then there is an invariant metric.

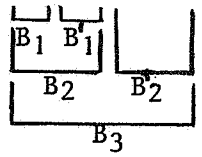
Another fact is, that every equicontinuous minimal flow has an invariant measure (see below for the precise definition). This follows easily from the existence of Haar measure on compact Hausdorff groups and the structure of such flows, which will be described in Theorem 1 below.

<u>EXAMPLES</u>.

<u>6</u>. The flows, described in Examples 1a (with θ irrational) and 2 (also with θ irrational) are minimal and equicontinuous.

<u>7</u>. In the example of 1b, the points x with minimal orbit closure are easy to characterize. This follows from a result which is known as BIRKHOFF's Recurrence Theorem); see for example [2], 2.5. Using this, one can easily show that a point x in the shift dynamical system has minimal orbit closure iff for every finite block in x (i.e. every finite segment $x_k \ldots x_{k+j}$ in x) there exists a natural number ℓ such that every block of length ℓ in x contains a copy of the given block. A famous point with minimal orbit closure is the following (the MORSE minimal sequence): $x = \overline{B}B$, where $\overline{B}$ denotes the mirror-image of B, and

B = 0 1 1 0 1 0 0 1 1 0 0 1 0 1 1 0 . . . .



where each B_{n+1} is obtained as the concatenation of B_n and the dual B'_n of B_n.

8. Let ψ: $T \to G$ be a continuous homomorphism of topological groups with G compact Hausdorff and $\psi[T]$ dense in G. Let H be a closed subgroup of G, and let X be the (compact Hausdorff!) space of left cosets gH of H in G. Define an action π of T on X by the rule

$$\pi(t,gH) := \psi(t)gH \text{ for } t \in T,\ g \in G.$$

Then $\underline{X} := \langle X,\pi\rangle$ turns out to be an equicontinuous minimal flow.

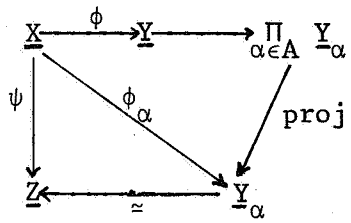
The Example 8 above gives a method to obtain *all* equicontinuous minimal flows:

THEOREM 1. *Let* $\underline{X}$ *be an equicontinuous minimal flow. Then there exists a closed subgroup* H *of the Bohr compactification* bT *of* T *such that* $\underline{X}$ *is isomorphic to the flow on* bT/H, *defined according to Example* 8 *above.*

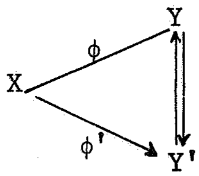
(The proof is a rather easy consequence of what was remarked earlier, viz. that the closure of $\{\pi^t: t \in T\}$ in X^X is a compact topological group. For details and references, cf. [10], Theorem 2.4.) In particular, if $T = \mathbb{Z}$ or $T = \mathbb{R}$, then an equicontinuous minimal flow $\underline{X}$ has the structure of a compact monothetic, resp. solenoidel, topological group.

3. EQUICONTINUOUS FACTORS

Let $\underline{X}$ be a minimal flow. Clearly, $\underline{X}$ has equicontinuous factors, viz. the trivial flow (*), consisting of a one-point space with the obvious action of T on it. Since there can be only *a set* of isomorphy classes of equicontinuous factors, there is a *maximal equicontinuous factor* (sometimes called *universal* equicontinuous factor). This is an equicontinuous factor of $\underline{X}$ over which every possible equicontinuous factor of $\underline{X}$ factorizes. In order to prove its existence, consider a set $\{\phi_\alpha: \underline{X} \to \underline{Y}_\alpha\}_{\alpha\in A}$ of representatives of such equivalence classes. Let $\phi: X \to \prod_{\alpha\in A} Y_\alpha$ be the induced mapping, and $Y := \phi[X]$. Then Y is a closed invariant subset of $\prod_{\alpha\in A} Y_\alpha$. Moreover, the flow $\underline{Y}$ is equicontinuous because the full product $\prod \underline{Y}_\alpha$ is so, and $\underline{Y}$ is minimal, because it is a factor of the minimal flow $\underline{X}$. So ϕ: $\underline{X} \to \underline{Y}$ is a factor such that $\underline{Y}$ is equicontinuous and minimal. Now let ψ: $\underline{X} \to \underline{Z}$ be any factor of $\underline{X}$ with $\underline{Z}$ equicontinuous and minimal. This factor is isomorphic to one of the ϕ_α: $\underline{X} \to \underline{Y}_\alpha$. From this, it follows that there is a homomorphism $\overline{\psi}$: $\underline{Y} \to \underline{Z}$ such that $\psi = \overline{\psi}\circ\phi$ (corresponding to the projection $\underline{Y} \to \underline{Y}_\alpha$). Since ϕ is surjective, $\overline{\psi}$ is unique.



As $\underline{Y}$ is well-understood, in order to say something about $\underline{X}$, one would like to know something about the factor mapping $\phi\colon \underline{X} \to \underline{Y}$, or, what amounts to the same (see the final remark in Section 1), about the equivalence relation R_ϕ. It should be observed that a maximal equicontinuous minimal factor of a given minimal flow $\underline{X}$ is unique up to isomorphism, so that the corresponding equivalence relation in $\underline{X}$ is uniquely determined:



$R_\phi \subseteq R_{\phi'}$ and $R_{\phi'} \subseteq R_\phi$.

The closed, invariant equivalence relation in X, corresponding to the maximal equicontinuous factor of $\underline{X}$ is called the *equicontinuous structure relation*, and it is denoted by $E_{\underline{X}}$ (or, if no confusion arises, just E).

<u>THEOREM 2</u>. *Let* $\underline{X}$ *be a minimal flow, and let the subset* $Q_{\underline{X}}$ *of* X×X *be defined as*

$$Q_{\underline{X}} := \bigcap_{\alpha\in\hat{\mathcal{U}}} \overline{T\alpha}.$$

Then $Q_{\underline{X}}$ *is a closed invariant subset of* X×X, *and* $E_{\underline{X}}$ *is the smallest closed invariant equivalence relation on* X *in which* $Q_{\underline{X}}$ *is included.*

<u>PROOF</u>. It is easy to see, that $Q_{\underline{X}}$ is closed and invariant in X×X. Moreover, if $\psi\colon \underline{X} \to \underline{Y}$ is a factor, then uniform continuity of ψ implies, that $\psi\times\psi[Q_{\underline{X}}] \subseteq Q_{\underline{Y}}$. If $\underline{Y}$ is equicontinuous, then it is obvious that $Q_{\underline{Y}} = \Delta_Y$, hence $\psi\times\psi[Q_{\underline{X}}] \subseteq \Delta_Y$, that is, $Q_{\underline{X}} \subseteq R_\psi$. From this it follows that $Q_{\underline{X}} \subseteq E_{\underline{X}}$. So if S_0 denotes the smallest closed invariant equivalence relation in X in which $Q_{\underline{X}}$ is included, then $S_0 \subseteq E_{\underline{X}}$.
In order to prove the converse inclusion it is sufficient to show that $\underline{X}/S_0$ is equicontinuous: then the universal property of the maximal equicontinuous factor $\underline{X}/E_{\underline{X}}$ implies, that $\underline{X} \to \underline{X}/S_0$ factorizes over $\underline{X} \to \underline{X}/E_{\underline{X}}$, so that $E_{\underline{X}} \subseteq S_0$. The proof that $\underline{X}/S_0$ is equicontinuous is rather deep, and uses ELLIS' joint continuity theorem. See [2], 4.20. □

REMARK. In the above proof of the following was used: if $\underline{Y}$ is a flow, then $\underline{Y}$ is equicontinuous iff $Q_{\underline{Y}} = \Delta_Y$. The proof of this is straightforward.
In examples, the set $Q_{\underline{X}}$ is often fairly easy to determine. Helpfull is the following description: for a point $(x_1,x_2) \in X \times X$ one has $(x_1,x_2) \in Q_{\underline{X}}$ iff there are nets $(x_1(\lambda))_{\lambda\in\Lambda}$ and $(x_2(\lambda))_{\lambda\in\Lambda}$ in X and $(t_\lambda)_{\lambda\in\Lambda}$ in T such that

$$(x_1(\lambda),x_2(\lambda)) \rightsquigarrow (x_1,x_2) \quad \text{in } X \times X$$

$$(t_\lambda x_1(\lambda),t_\lambda x_2(\lambda)) \rightsquigarrow (z,z) \quad \text{for some } z \in X.$$

EXAMPLES.

9. Let T be the free group on two generators t_1 and t_2. Define an action of T on $\mathbb{S}^1$ as follows

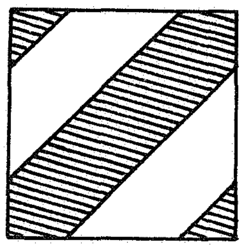
$$t_1.z := z\exp(2\pi i\theta) \text{ for } z \in \mathbb{S}^1$$

$$t_2.\exp(2\pi i\zeta) := \exp(2\pi i\zeta^2) \quad \text{for } \zeta \in [0;1).$$

If θ is irrational, then the flow is minimal. Using the tranformation t_2 and its iterates one sees readily, that $t_2^n(z_1,z_2) \rightsquigarrow (1,1)$ if $n \rightsquigarrow \infty$. Hence $Q = \mathbb{S}^1 \times \mathbb{S}^1$.

10. Same as above, but now

$$t_2.\exp(2\pi i\zeta) := \begin{cases} \exp(8\pi i\zeta^2) & \text{for } \zeta \in [0;\tfrac14] \\ \exp 2\pi i(\tfrac14+4(\zeta-\tfrac14)^2) & \text{for } \zeta \in [\tfrac14;\tfrac12] \\ \exp 2\pi i(\tfrac24+4(\zeta-\tfrac24)^2) & \text{for } \zeta \in [\tfrac12;\tfrac34] \\ \exp 2\pi i(\tfrac34+4(\zeta-\tfrac34)^2) & \text{for } \zeta \in [\tfrac34;1] \end{cases}.$$



Now it is a good exercise to show, that

$$Q = \{(\exp 2\pi i\zeta_1,\exp 2\pi i\zeta_2) : |\zeta_1-\zeta_2| \le \tfrac14\}.$$

REMARK. If $Q_{\underline{X}} = \Delta_X$, then $\underline{X}$ is itself equicontinuous (cf. the remark above). The other extreme is, that $Q_{\underline{X}} = X \times X$. In that case, also $E_{\underline{X}} = X \times X$, so $\underline{X}/E_{\underline{X}} = (*)$; consequently, $\underline{X}$ has no non-trivial equicontinuous factors. Of course, this is also the case if $Q_{\underline{X}} \neq X \times X$, and yet $E_{\underline{X}} = X \times X$ (see e.g. Example 10).
A minimal flow $\underline{X}$ is called *weakly mixing* whenever $\underline{X} \times \underline{X}$ is topologically ergodic, that is, invariant subsets of $X \times X$ are either dense or nowhere dense (so if $A \subseteq X \times X$ is closed, has non-empty interior and is invariant, then $A = X \times X$). If X is metric, then this property is equivalent to $X \times X$ having a point with a dense orbit.

THEOREM 3. *If the minimal flow $\underline{X}$ is weakly mixing, then $\underline{X}$ has no non-trivial equicontinuous factors; in fact,* $Q_{\underline{X}} = X\times X$.

PROOF. For every $\alpha \in \hat{U}$, $\overline{T\alpha}$ is closed, invariant has non-empty interior, so $\overline{T\alpha} = X\times X$. Hence $Q_{\underline{X}} := \bigcap_{\alpha\in\hat{U}} \overline{T\alpha} = X\times X$. □

QUESTION 1. Under which additional conditions for a minimal flow $\underline{X}$ is it true, that if $\underline{X}$ has no non-trivial equicontinuous factors then $\underline{X}$ is weakly mixing? (Note, that Example 10 shows that in general the converse of Theorem 3 is not true: $Q_{\underline{X}}$ is a closed invariant subset of X×X which has non-empty interior and is not equal to X×X.) Several people have studied this problem; cf. [7] and the references given there, and also [1].

QUESTION 2. Related is the question: under which conditions for a minimal flow $\underline{X}$ is $Q_{\underline{X}}$ an equivalence relation, i.e. $E_{\underline{X}} = Q_{\underline{X}}$? Also to this problem much research has been devoted; see for instance [9] and Chapter VIII in [12].

In the next section, a partial answer to these question will be described.

4. INVARIANT MEASURES

Let $\underline{X}$ be a flow, and let M(X) denote the set of probability measures on X, endowed with the weak topology. So either $M(X) := \{\mu \in C_u(X)' : \mu \geq \& \ \mu(1_X) = 1\}$, a closed convex set of the (compact!) unit ball in $C_u(X)'$ with its weak topology, or, alternatively, M(X) is the set of all regular Borel measures μ in X with total mass $\mu(X) = 1$.

The action of T on X induces an action of T on M(X). This action is given either by

$$t\mu(f) := \mu(f\circ\pi^t) = \int_X f(tx)d\mu(x),$$

for $f \in C_u(X)$, or by

$$t\mu(A) = \mu(t^{-1}A),$$

for a Borel set A in X. A measure μ on X is called *invariant* whenever $\mu \in M(X)$ and $t\mu = \mu$ for all $t \in T$. The following result is a special case of a theorem in [7]. It generalizes results from [3], [6] and [8].

THEOREM 4. *Let* $\underline{X}$ *be a minimal flow, and let* $\underline{X}$ *have an invariant measure. Then* $Q_{\underline{X}}$ *is an equivalence relation, that is,* $E_{\underline{X}} = Q_{\underline{X}}$.
Moreover, if $\underline{X}$ *has no non-trivial equicontinuous factors, that is, if* $E_{\underline{X}} = X\times X$, *then* $\underline{X}$ *is weakly mixing.*

PROOF. Below, some details will be given. □

REMARK. There are also several results of this nature which do not require an invariant measure. See for instance [5]. The "relativation" of the above result will be described in [11].

5. THE PROOF OF THEOREM 4

Let $\underline{X}$ be a minimal flow with invariant measure μ. The proof of the theorem will be given by providing a suitable class of continuous invariant pseudometrics on X. Indeed, if ρ is such a pseudometric and

$$D_\rho := \{(x_1,x_2) \in X\times X \,\vdots\, \rho(x_1,x_2) = 0\},$$

then D_ρ is a closed and invariant equivalence relation in X. Hence the flow $\underline{X}/D_\rho$ is well-defined. Since ρ is an invariant pseudometric on X, ρ induces an *invariant* metric on X/D_ρ. It is easy to show that this metric is compatible with the (compact!) quotient-topology of X/D_ρ. Since T acts on X/D_ρ by isometrics, the flow $\underline{X}/D_\rho$ is equicontinuous. So

$$E_{\underline{X}} \subseteq D_\rho$$

by the definition of $E_{\underline{X}}$. We shall now indicate a construction which will produce a set S of continuous invariant pseudometrics such that

$$D_{\underline{X}} := \bigcap_{\rho\in S} D_\rho \subseteq Q_{\underline{X}}.$$

This completes the proof of the first part of the theorem: $Q_{\underline{X}} \subseteq E_{\underline{X}} \subseteq D_{\underline{X}} \subseteq Q_{\underline{X}}$, so $Q_{\underline{X}} = E_{\underline{X}}$.
The construction of S is as follows. Let for every subset N of X×X and every point $x \in X$ the section of N at x be denoted by

$$N[x] := \{x' \in X \,\vdots\, (x,x') \in N\}.$$

Clearly, if N is a closed subset of X×X, then N[x] is closed in X, and we will see below, that if N is non-empty, closed and invariant, then $N[x] \neq \emptyset$.

LEMMA. *Let* N *be a non-empty closed invariant subset of* X×X, *and define the mapping* ρ_N: X×X → $\mathbb{R}^+$ *by*

$$\rho_N(x_1,x_2) := \mu(N[x_1] \,\Delta\, N[x_2]) \text{ for } (x_1,x_2) \in X\times X.$$

Then ρ_N *is an invariant continuous pseudometric on* X.

PROOF. It is straightforward to check that ρ_N is an invariant pseudometric on X×X. The proof that ρ_N is continuous on X×X is in several steps.
1. For every $x \in X$, $N[x] \neq \emptyset$. This follows from the fact that the image of N under the projection of X×X onto the first coordinate (this projection is a homomorphism of flows) is a non-empty closed invariant subset of X. So by minimality of X, this image is all of X, i.e. for all $x \in X$ there is a point of the form (x,x') in N, hence $x' \in N[x] \neq \emptyset$.

Now let 2^X denote the space of all closed, non-empty subsets of X, endowed with the Vietoris topology. We claim:
2. The mapping $x \mapsto N[x]: X \to 2^X$ is *upper semicontinuous*, that is, for every $x \in X$ and every open nbd U of the closed set N[x] in X there exists a nbd V of x such that $N[x'] \subseteq U$ for all $x' \in V$. The easy proof of this claim is left to the reader.
3. If $x_1,x_2 \in X$, then $\mu(N[x_1]) = \mu(N[x_2])$.
To prove this, let $\varepsilon > 0$, and let U be an open nbd of $N[x_2]$ in X such that $\mu(U) < \mu(N[x_2]) + \varepsilon$ (regularity of μ). By 2, there is a nbd V of x_2 such that $N[x'] \subseteq U$ for all $x' \in V$. Since x_1 has a dense orbit (minimality of X), there is $t \in T$ such that $tx_1 \in V$, hence $N[tx_1] \subseteq U$. Using invariantness of μ and N we obtain

$$\mu(N[x_1]) = \mu(tN[x_1]) = \mu(N[tx_1]) \le \mu(U) < \mu(N[x_2]) + \varepsilon.$$

This holds for every $\varepsilon > 0$, so $\mu(N[x_1]) \le \mu(N[x_2])$. The converse inequality is proved in a similar fashion.
4. For all $x_1,x_2 \in X$, we have

$$\mu(N[x_1]\setminus N[x_2]) = \mu(N[x_2]\setminus N[x_1]) = \tfrac{1}{2}\mu(N[x_1]\Delta N[x_2]),$$

where Δ denotes as usual the symmetric difference. The straightforward proof follows from the observation, that $\mu(N[x_1]\setminus N[x_2]) = \mu(N[x_1]) - \mu(N[x_1]\cap N[x_2])$, in which equality the x_1 and x_2 may be interchanged by 3.
5. In order to show that ρ_N is continuous on X×X, it is sufficient to show that for every $x_1 \in X$ the mapping

$$x \mapsto \rho_N(x_1,x) = \mu(N[x_1]\Delta N[x]) : X \to \mathbb{R}$$

is continuous in the point x_1 of X. So let $\varepsilon > 0$. Let U be an open nbd of $N[x_1]$ in X such that $\mu(U) < \mu(N[x_1]) + \varepsilon/2$ (again, regularity of μ), and let V be a nbd of x_1 such that $N[x] \subseteq U$ for all $x \in V$ (cf. 2). Then for all $x \in V$:

$$\mu(N[x]\setminus N[x_1]) \leq \mu(U \setminus N[x_1]) = \mu(U) - \mu(N[x_1]) < \varepsilon/2.$$

Now it follows from 4 that for all $x \in V$

$$\rho_N(x_1,x) = 2\mu(N[x]\setminus N[x_1]) < \varepsilon. \quad \square$$

The collection S of continuous invariant pseudometrics, referred to above, is

$$S := \{\rho_N \colon N \neq \emptyset \text{ a closed invariant subset of } X\times X\}.$$

We now show that $D_{\underline{X}} := \bigcap_{\rho\in S} D_\rho \subseteq Q_{\underline{X}}$.

Let $(x_1,x_2) \in D_{\underline{X}}$. Then for every closed non-empty invariant subset N of $X\times X$ we have $\mu(N[x_1]\Delta N[x_2]) = 0$. Now the following observation is crucial: for every open subset U of X, $U \neq \emptyset$, one has $\mu(U) > 0$. (Indeed, as μ is invariant, t supp μ = supp $(t\mu)$ = supp μ for every $t \in T$, so supp μ is a non-empty closed invariant subset of X; so supp $\mu = X$, because $\underline{X}$ is minimal.) In particular, if U is a non-empty open subset of X and $U \subseteq N[x_1]$, then also $U \subseteq N[x_2]$ (otherwise $U\setminus N[x_2]$ would be a non-empty open subset of $N[x_1]\Delta N[x_2]$, which has measure zero). This observation will be used below.

We want to show that for arbitrary $\alpha \in \hat{\mathcal{U}}$, $(x_1,x_2) \in \overline{T\alpha}$. To this end, introduce

$$N_\alpha := \overline{T\alpha}.$$

Clearly, N_α is a non-empty closed invariant subset of $X\times X$. Let U be an open nbd of x_2, $U \subseteq \alpha[x_2]$. Then $U \subseteq N_\alpha[x_2]$, hence by the observation above, $U \subseteq N_\alpha[x_1]$. In particular, it follows that $x_2 \in N_\alpha[x_1]$, i.e. $(x_1,x_2) \in N_\alpha = \overline{T\alpha}$. Here $\alpha \in \hat{\mathcal{U}}$ is arbitrary, so $(x_1,x_2) \in \bigcap_{\alpha\in\hat{\mathcal{U}}} \overline{T\alpha} = Q_{\underline{X}}$. So indeed, $D_{\underline{X}} \subseteq Q_{\underline{X}}$. $\square$

<u>REMARK</u>. The proof above originated as follows. Generalising [7], the second author obtained a number of new results (see [12] and [1]). From this, the above proof was extracted as a special case. About the same time, and independently, McMahon found the same proof.

The proof of the second part of the theorem uses the same trick as was used above.

Assume that $E_{\underline{X}} = X\times X$. Since $E_{\underline{X}} \subseteq D_\rho$ for every continuous invariant pseudometric ρ, it follows that $D_{\rho_N} = X\times X$ for every closed invariant non-empty subset N of $X\times X$. Hence $\mu(N[x_1]\Delta N[x_2]) = 0$ for all $x_1, x_2 \in X$.

We want to show that $\underline{X}$ is weakly mixing, that is, that for each open subset O of $X\times X$ the set TO is dense in $X\times X$. So let for i=1,2, U_i and V_i be open in X; we have to show that

$$(V_1\times V_2) \cap T(U_1\times U_2) \neq \emptyset.$$

By minimality of $\underline{X}$, there is $t \in T$ such that

$$W := tU_2 \cap V_2 \neq \emptyset.$$

Next, consider tU_1 and V_1: let $x_1 \in tU_1$ and $x_2 \in V_1$. Then

$$\{x_1\} \times W \subseteq t(U_1\times U_2) \subseteq \overline{T(U_1\times U_2)} =: N.$$

Consequently, $W \subseteq N[x_1]$. Since $W \neq \emptyset$ and W is open, this implies (same trick as above) that $W \subseteq N[x_2]$, that is

$$\{x_2\}\times W \subseteq N = \overline{T(U_1\times U_2)}.$$

Therefore, $V_1\times W$ meets $T(U_1\times U_2)$ and, consequently, $V_1\times V_2$ meets $T(U_1\times U_2)$. □

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