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POINT DISTAL - AND HPI EXTENSIONS

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Point distal - and HPI extensions *)

by

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ABSTRACT

We present a structure theorem for point distal homomorphisms of minimal transformation groups. Also we prove that a factor of a strictly HPI extension is a highly proximal factor of a strictly-HPI extension.

KEY WORDS & PHRASES: *Highly proximal extension*

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*) This report will be submitted for publication elsewhere

0. INTRODUCTION

Several authors proved structure theorems for point distal transformation groups and homomorphisms. VEECH [7], using and changing the method of Furstenberg for the distal case, proved a structure theorem for metric point distal transformation groups. ELLIS [3] generalized [7] to a relativized version for quasi separable homomorphisms. SHOENFELD [6], using a decent generalization of almost automorphic extensions (the highly proximal extensions) brought the structure theorem in a rather final form (e.g. see [8], 2.1.5). But in every attempt a kind of countability assumption was used: metrizability, σ-compactness and locally compactness of the fase group, quasi-separability. However, noting that a point distal open homomorphism is RIC, we can get rid of any countability assumption (the loss being an extra weakly mixing step). The method is the same as in the previous structure theorems.

Sections 1 and 2 collect notions and notations; section 3 leads to the result, that an open pint distal homomorphism is RIC. In section 4 we prove the structure theorem. The last section uses section 3 to prove that if ψ is strictly-HPI and $\psi = \phi\circ\theta$, then we can find ψ' strictly HPI and θ' highly proximal such that $\psi' = \phi\circ\theta'$.

1. FUNDAMENTALS

In this section we shall briefly mention a few notions, which will be used without explanation in the other sections. We assume basic knowledge about topological dynamics as can be found in [2] and [5].

A *topological transformation group* (ttg for short) is defined to be a triple (T,X,π) with T an arbitrary topological group, $X \neq \emptyset$ a compact T_2 space and $\pi: T\times X \to X$ a continuous action of T on X. In the sequel T will be understood, π surpressed and the action denoted as a left multiplication of elements of X by group elements. If X and Y are ttg's, then a continuous map $\phi: X \to Y$ is called a *homomorphism* if for every $t\in T$ and $x\in X$ $\phi(tx) = t\phi(x)$. If ϕ is surjective, we will often call ϕ an *extension* and Y a *factor* of X. A ttg X is called *minimal* if it does not contain a nonempty, proper, closed, T-invariant subset. There exists a universal minimal ttg M, unique up to isomorphism, of which every minimal ttg is a factor. M is isomorphic to any minimal left ideal in its enveloping semigroup $E(M) (= \overline{T} \subseteq M^M)$, and being a semigroup itself, it acts accordingly on every minimal ttg. Let J be the collection of idempotents in M. Then $\{uM \mid u \in J\}$ is a partition of M and every uM is a subgroup of M. On uM we may define a compact T_1 topology, the so-called τ-topology. Let X be a minimal ttg, $x_0 \in X$ and $u \in J(x_0) = \{v \in J \mid vx_0 = x_0\}$. The *Ellis group* of X relative x_0 in uM is defined to be $\mathcal{G}(X,x_0) = \{\alpha \in uM \mid \alpha x_0 = x_0\}$. It is a τ-closed subgroup of uM, and every τ-closed subgroup $F \subseteq uM$ can be obtained as an Ellis group of some minimal ttg. Let $F_0 = F$ be a τ-closed subgroup of uM, $H(F)$ will be defined as the smallest τ-closed normal subgroup K of F such that F/K is a compact T_2 topological group. For every ordinal β define $F_{\beta+1} := H(F_\beta)$, if β is a limit ordinal $F_\beta := \bigcap\{F_\alpha \mid \alpha < \beta\}$ and F_∞ denotes the inverse limit of the possibly transfinite sequence $\dots F_{\beta+1}, F_\beta, \dots, F_2, F_1, F_0 = F$.

Let X be a ttg and define $2^X := \{A \subseteq X \mid A \neq \emptyset$ and A is closed$\}$. Then 2^X provided with the Vietoris topology is again compact T_2 and a ttg, the action being defined by $tA = \{ta \mid a \in A\}$ for every $t \in T$, $A \in 2^X$. A *quasifactor* of a minimal ttg X is a minimal subttg of 2^X; obviously X is a quasifactor of X.

The action of M on a quasifactor $\mathcal{X}$ of X is deprived of its ambivalency by writing it with a dot, i.e. for $p \in M$, $A \in 2^X$ the element of $\mathcal{X}$ obtained by applying p to A is denoted by $p \circ A$. In general $p \circ A \neq pA = \{pa \mid a \in A\}$, in fact $p \circ A = \{x \in X \mid x = \lim t_i a_i$ for some net $\{t_i\}$ in T converging to p and $a_i \in A\}$. If $A \subseteq X$ is arbitary then $p \circ A := p \circ \overline{A}$. Let $\mathcal{X}$ be a quasifactor of X, then $\mathcal{X}$ can be written as $\mathcal{X} = QF(A,X) := \{p \circ A \mid p \in M\}$ for every $A \in \mathcal{X}$. Let Y be a minimal ttg, $y_0 \in Y$, $u \in J(y_0)$ and $F = \mathcal{G}(Y,y_0) \subseteq uM$. Then $QF(u \circ F, M)$ is the universal proximal extension of Y and the canonical extension $\kappa: QF(u \circ F, M) \to Y$ is defined by $\kappa(p \circ F) = py_0$. Consequently $P = QF(u \circ uM, M)$ is the universal minimal proximal ttg for T.

Let $\phi: X \to Y$ and $\psi: Z \to Y$ be homomorphisms of minimal ttg's, we call X *and* Z *disjoint relative* Y (ϕ and ψ always assumed to be known) if $R_{\phi\psi} = \{(x,z) \mid \phi(x) = \psi(z)\} \subseteq X \times Z$ is a minimal ttg; notation $(X \perp Z)_Y$. X and Z are *disjoint* $(X \perp Z)$ if $(X \perp Z)_{\{*\}}$, where $\{*\}$ denotes the trivial one point ttg. $\mathcal{P}^\perp$ will be the collection of all minimal ttg's disjoint from every minimal proximal ttg.

We will use several types of homomorphisms of minimal ttg's, the definitions of the distal, proximal and almost periodic ones are assumed to be known. Let ϕ: $X \to Y$ be a homomorphism of minimal ttg's, $\phi(x_0) = y_0$, $u \in J(x_0)$ and $F = \mathcal{J}(Y, y_0)$; ϕ is called a RIC *extension* if for every $p \in M$, $\overset{\leftarrow}{\phi}(py_0) = p \circ F x_0$. Let κ: $QF(u \circ F, M) \to Y$ be the canonical homomorphism, then ϕ is RIC iff $(X \perp QF(u \circ F, M))_Y$. RIC extensions are very useful since they guarantee under certain conditions the existence of an intermediate nontrivial almost periodic extension.

The last fundamental type of extension we will mention in this section is the highly proximal extension ([1],[6]). A homomorphism ϕ: $X \to Y$ of minimal ttg's is called *highly proximal* (h.p.) if for every $y \in Y$, $x \in \overset{\leftarrow}{\phi}(y)$ and $p \in M$: $p \circ \overset{\leftarrow}{\phi}(y) = \{px\}$. Two minimal ttg's are *h.p. equivalent* if there is a common h.p. extension of each. In this way the collection of all minimal ttg's (for T) may be partitioned in highly proximal equivalence classes, each of which has a unique maximal element: a *maximally highly proximal* ttg. If X is a minimal ttg, $x \in X$ and γ: $M \to X$ is an extension, then $X^* = \{p \circ \overset{\leftarrow}{\gamma}(x) \mid p \in M\}$ is the maximal element in the equivalence class to which X belongs. The construction of X^* is independent of the choice of γ, x, and X in the equivalence class. Another characterization of the maximally h.p. ttg's is, that X is maximally h.p. iff every homomorphism ϕ: $Y \to X$ of minimal ttg's is open.

2. TOWERS

A crucial device in structure theorems for minimal ttg's is a tower. Let ϕ: $X \to Y$ be a homomorphism of minimal ttg's, and ν an ordinal. A *tower of height* ν *for* ϕ is a system $\{(W_\alpha, \omega_\alpha), \phi_\alpha \mid \alpha \le \nu\}$ of minimal ttg's with fixed base point, such that:

1° $X = W_\nu$, $Y = W_0$ and $\phi(\omega_\nu) = \omega_0$;

2° for every ordinal α ϕ_α: $W_{\alpha+1} \to W_\alpha$ is a homomorphism with $\phi_\alpha(\omega_{\alpha+1}) = \omega_\alpha$ and $\phi_\nu = id_{W_\nu}$;

3° if α is a limit ordinal $(W_\alpha, \omega_\alpha) := \vee\{(W_\beta, \omega_\beta) \mid \beta < \alpha\}$, i.e. $\omega_\alpha = (\omega_\beta)_{\beta<\alpha}$ and W_α is the minimal orbit closure of ω_α in $\pi\{W_\beta \mid \beta < \alpha\}$;

4° ϕ is the inverse limit of $\{\phi_\alpha \mid \alpha \le \nu\}$.

There are several types of towers depending on the properties of the ϕ_α's. In this paper we will only use three types. A homomorphism $\phi: X \to Y$ of minimal ttg's is a *strictly*-PI (*strictly*-HPI) (*strictly*-HPD) *extension* if there is a tower for ϕ such that the occurring ϕ_α's are proximal or almost periodic (highly proximal or almost periodic) (highly proximal or distal). Let B be one of the symbols PI, HPI and HPD. A homomorphism $\phi: X \to Y$ of minimal ttg's is a B *extension* if there exists a strictly-B homomorphism $\psi: Z \to Y$ of minimal ttg's and a homomorphism $\theta: Z \to X$ such that $\psi = \phi \circ \theta$; in other words ϕ is B if ϕ is a factor of a strictly-B extension.

The next theorem is well-known ([4],[9])

2.1 THEOREM *Let $\phi: X \to Y$ be a homomorphism of minimal ttg's, let $ux_0 = x_0 \in X$ and let H and F be the Ellis groups of X and Y relative to x_0 and $\phi(x_0)$. Then ϕ is* PI *iff $F_\infty \subseteq H$.*

3. POINT DISTAL HOMOMORPHISMS

A homomorphism $\phi: X \to Y$ of minimal ttg's is called *point distal* if there is a $y \in Y$ and $x \in \overleftarrow{\phi}(y)$ such that (x,x') is a distal pair for every $x' \in \overleftarrow{\phi}(y)$. A minimal ttg X is called *point distal* if $\phi: X \to \{*\}$ is point distal. Remark, that $\{x \in X | \ x \text{ distal point in } \overleftarrow{\phi}(\phi(x))\}$ is dense in X, for every point-distal homomorphism ϕ.

3.1 LEMMA *Let $\phi: X \to Y$ be a homomorphism of minimal ttg's. Then $x \in \overleftarrow{\phi}(y)$ is a distal point in $\overleftarrow{\phi}(y)$ iff $J(x) = J(y)$.*

PROOF Let X be a distal point in $\overleftarrow{\phi}(y)$. For every $w \in J(y)$, $wx \in \overleftarrow{\phi}(y)$. Since (x,wx) is a proximal pair which is by point distality also distal, it follows that $x = wx$ and $J(y) \subseteq J(x)$ so $J(y) = J(x)$. Let $x \in \overleftarrow{\phi}(y)$ be such that $J(x) = J(y)$ and choose $x' \in \overleftarrow{\phi}(y)$. Since $J(x') \subseteq J(y) = J(x)$ and $J(x') \neq \emptyset$ there is a $v \in J$ with $vx = x$ and $vx' = x'$ so (x,x') is a distal pair. □

Two homomorphisms $\phi: X \to Y$ and $\psi: Z \to Y$ are said to satisfy the *generalized Bronstein condition* if $R_{\phi\psi}$ with the relative topology of $X \times Z$ has a dense set of almost periodic points, i.e. points with minimal orbit closure.

3.2 PROPOSITION *Let ϕ: X $\to$ Y and ψ: Z $\to$ Y be homomorphisms of minimal ttg's with ϕ point distal and ψ open. Then the pair (ϕ,ψ) satisfies the generalized Bronstein condition.*

PROOF Let W be an open subset of $R_{\phi\psi}$. Choose $(x,z) \in W$ and open nbhds U_x and U_z of x and z in X and Z such that $U_x \times U_z \cap R_{\phi\psi} \subseteq W$. Since ψ is open $V_x := U_x \cap \overleftarrow{\phi}[\psi[U_z]]$ is an open nhbd of x in X. Choose $x' \in V_x$ with x' is a distal point in $\overleftarrow{\phi}(\phi(x'))$. Then $y' := \phi(x') \in \phi[V_x] \subseteq \phi[U_x] \cap \psi[U_z]$. So there is a $z' \in U_z$ with $\psi(z') = y'$. But $J(z') \subseteq J(y') = J(x')$, hence (x',z') is an almost periodic point in W. Moreover, $(x',z') \in V_x \times U_z \cap R_{\phi\psi} \subseteq U_x \times U_z \cap R_{\phi\psi} \subseteq W$. □

3.3 COROLLARY 1° *Let ϕ: X $\to$ Y be an open and point distal homomorphism of minimal ttg's. Then $R_\phi := R_{\phi\phi}$ has a dense set of almost periodic points (i.e. ϕ satisfies the Bronstein condition (see also [3], 6.4).*

2° *Let X be a point distal minimal ttg. Then $X \in \mathcal{P}^\perp$.*

PROOF 2° Define ϕ: X $\to$ {*} and ψ: P $\to$ {*} (with P the universal minimal proximal ttg for T). Then ϕ is point distal and ψ is open, so $X \times P$ has a dense set of almost periodic points. Since π: $X \times P \to X$, the projection, is proximal and X is minimal, it follows that $X \times P$ has a unique minimal subset. Hence $X \times P$ is minimal and $X \perp P$. □

3.4 PROPOSITION *Let ϕ: X $\to$ Y and ψ: Z $\to$ Y be a homomorphisms of minimal ttg's, with ϕ open and point distal. Then (ϕ,ψ) satisfies the generalized Bronstein condition.*

PROOF Let W be an open subset of $R_{\phi\psi}$. Choose $(x,z) \in W$ and open nbhd's U_x and U_z of x and z in X and Z, with $U_x \times U_z \cap R_{\phi\psi} \subseteq W$. Then $V_z := U_z \cap \overleftarrow{\psi}[\phi[U_x]]$ is an open nhbd of z in Z. Since Z is minimal $\psi[V_z]^\circ \neq \emptyset$ (e.g., see [7] page 215) and $\psi[U_z]^\circ \subseteq \phi[U_x]$, so $V := U_x \cap \overleftarrow{\phi}[\psi[V_z]^\circ]$ is a nonempty open subset of X. Choose $x' \in V$ with x' distal point in $\overleftarrow{\phi}(\phi(x'))$. Then $y' := \phi(x') \in \phi[V] \subseteq \psi[V_z]^\circ \subseteq \psi[U_z]$. Choose $z' \in U_z$ with $\psi(z') = y'$, then $(x',z') \in U_x \times U_z \cap R_{\phi\psi} \subseteq W$ and since $J(z') \subseteq J(y') = J(x')$ it follows that (x',z') is an almost periodic point. □

3.5 THEOREM *Let ϕ: X $\to$ Y be an open and point distal homomorphism of minimal ttg's. Then ϕ is* RIC.

PROOF Choose $x_0 = ux_0 \in X$, and let $F = \mathcal{G}(Y,\phi(x_0))$ be the Ellis group of Y. Let κ: $QF(u\circ F,M) \to Y$ be the canonical map. Then $R_{\phi\kappa}$ has a dense set of almost periodic points. Define ψ: $R_{\phi\kappa} \to Y$ by $\psi(x,z) = \phi(x)$. Then ψ is proximal since κ is, so $R_{\phi\kappa}$ has a unique minimal subset. But now $R_{\phi\kappa}$ is minimal and so ϕ is a RIC extension ([5], X.1.3). □

The next theorem also leads to this result.

3.6 THEOREM *Let ψ : X $\to$ Y be a point distal homomorphism of minimal ttg's and let X^* and Y^* be the maximally highly proximal extensions of X and Y. Then ϕ^*: $X^* \to Y^*$ is a RIC extension.*

PROOF Let $x_0 = ux_0$ be a distal point in $\overset{\leftarrow}{\phi}(y_0)$ and let γ: $M \to X$ be defined by $p \mapsto px_0$ and δ: $M \to Y$ by $\delta = \phi\circ\gamma$. Then $X^* = \{p\circ\overset{\leftarrow}{\gamma}(x_0) \mid p \in M\}$ and $Y^* = \{p\circ\overset{\leftarrow}{\delta}(y_0) \mid p \in M\}$, while ϕ^* : $X^* \to Y^*$ is defined by $p\circ\overset{\leftarrow}{\gamma}(x_0) \mapsto p\circ\overset{\leftarrow}{\delta}(y_0)$. Let H and F be the Ellis groups of X and Y with respect to x_0 and y_0. Then H and F are also the Ellis groups of X^* and Y^* with respect to $x_0^* := u\circ\overset{\leftarrow}{\gamma}(x_0)$ and $y_0^* := u\circ\overset{\leftarrow}{\delta}(y_0)$. Clearly $p\circ Fx_0^* \subseteq \overset{*\leftarrow}{\phi}(py_0^*)$ for all $p \in M$. We shall prove now that $\overset{*\leftarrow}{\phi}(py_0^*) \subseteq p\circ Fx_0^*$. Note first that $\overset{\leftarrow}{\gamma}(x_0)=J(x_0)H$, $\overset{\leftarrow}{\delta}(y_0)=J(y_0)F$ and $J(x_0) = J(y_0)$. Let $qx_0^* \in \overset{*\leftarrow}{\phi}(py_0^*)$. Then $qy_0^* = py_0^*$ and $q\circ J(x_0)F = p\circ J(x_0)F$, so $p \in q\circ J(x_0)F$. Let $\{t_i\}$ be a net in T with $t_i \to q$, then there are $v_i \in J(x_0)$ and $f_i \in F$ with $t_iv_if_i \to p$. Since for all $f \in F$ we have $u\circ F = f\circ F$ it follows that $p\circ F = \lim t_iv_if_i \circ F = \lim t_iv_i\circ F$. The compactness of M guarantees the existence of a converging subset $\{t_\alpha v_\alpha\}$, say $t_\alpha v_\alpha \to r$ with $r \in M$. Then $r\circ F = p\circ F$ and $r \in q\circ J(x_0) \subseteq q\circ J(x_0)H = qx_0^*$. But $r\circ F = p\circ F$, so $r \in p\circ F \subseteq \bigcup\{\ell\circ J(x_0)H \mid \ell \in p\circ F\} = \bigcup\{\ell x_0^* \mid \ell \in p\circ F\}$. Let $q' \in p\circ F$ be such that $r \in q'x_0^*$. Then $qx_0^* \cap q'x_0^* \neq \emptyset$. However, $\{px_0^* \mid p \in M\}$ is a partition of M ([1], Thm I.3), so $qx_0^* = q'x_0^* \in p\circ Fx_0^*$. Since we proved that $\overset{*\leftarrow}{\phi}(py_0^*) = p\circ Fx_0^*$ it follows that ϕ^* is RIC. □

3.7 COROLLARY *Let ϕ: X $\to$ Y be an open and point distal homomorphism of minimal ttg's. Then ϕ is RIC.*

PROOF Consider the next diagram. From 3.6 it follows that ϕ^* is RIC. Since ϕ is open and η is highly proximal, we just have to apply [1], Lemma III.5. □

$$\begin{array}{ccc} X^* & \xrightarrow{\theta} & X \\ {\scriptstyle \phi^*}\downarrow & \simeq & \downarrow{\scriptstyle \phi} \\ Y^* & \xrightarrow[\eta]{} & Y \end{array}$$

4. THE VEECH STRUCTURE THEOREM

VEECH [7,8], ELLIS [3] and SHOENFELD [6] constructed structure theorems for point distal homomorphisms of minimal ttg's, all using certain countability assumptions, but the method originating from Veech is also applicable to the general situation.

Let ϕ: $X \to Y$ be a homomorphism of minimal ttg's and let $ux_0 = x_0 \in \overleftarrow{\phi}(y_0)$. Let H and F be the Ellis group of X and Y relative to x_0 and y_0. We can construct two canonical commutative diagrams, consisting of minimal ttg's and homomorphisms as follows ([8]).

EGS(ϕ):

$$\begin{array}{ccc} X_1 & \xrightarrow{\sigma_1} & X \\ {\scriptstyle \phi_1}\downarrow & \simeq & \downarrow{\scriptstyle \phi} \\ Y_1 & \xrightarrow[\tau_1]{} & Y \end{array}$$

with $Y_1 = \{p \circ F x_0 \mid p \in M\}$ and
$X_1 = \{(x,A) \in X \times Y_1 \mid x \in A\}$
Then ϕ_1 is a RIC extension, and σ_1 and τ_1 are proximal extensions.

and

AG(ϕ):

$$\begin{array}{ccc} X_2 & \xrightarrow{\sigma_2} & X \\ {\scriptstyle \phi_2}\downarrow & \simeq & \downarrow{\scriptstyle \phi} \\ Y_2 & \xrightarrow[\tau_2]{} & Y \end{array}$$

with $Y_2 = \{p \circ \overleftarrow{\phi}(y_0) \mid p \in M\}$ and
$X_2 = \{(x,a) \in X \times Y_2 \mid x \in A\}$
Then ϕ_2 is an open homomorphism and σ_2 and τ_2 are highly proximal extensions.

The next proposition generalizes [8], 2.3.6.

4.1 PROPOSITION *Let* ϕ: $X \to Y$ *be a point distal homomorphism of minimal ttg's. Then* EGS(ϕ) *and* AG(ϕ) *coincide.*

PROOF It is easy to check that ϕ_2 is a point distal homomorphism too. Since ϕ_2 is open it is RIC, so for every $p \in M$ we have

$$\overleftarrow{\phi_2}(p\circ\overleftarrow{\phi}(y_0)) = \{(x,p\circ\overleftarrow{\phi}(y_0)) \mid x \in p\circ\overleftarrow{\phi}(y_0)\} = p\circ F(x_0,u\circ\overleftarrow{\phi}(y_0)).$$

But then $p\circ Fx_0 = p\circ\overleftarrow{\phi}(y_0)$, so $Y_1 = Y_2$. □

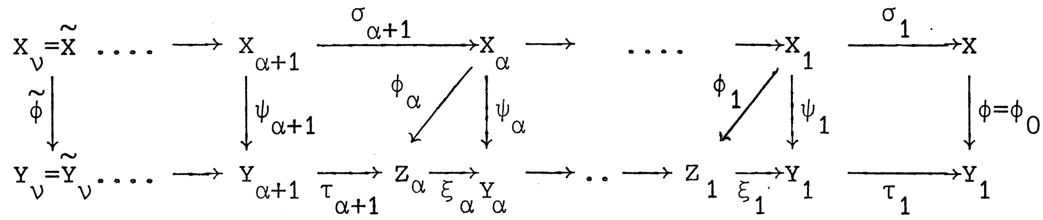
4.2 THEOREM *Let ϕ: $X \to Y$ be a point distal homomorphism of minimal ttg's. Then there is a commutative diagram consisting of minimal ttg's and homomorphisms:*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\sigma} & X \\ {\scriptstyle\tilde{\phi}}\downarrow & \simeq & \downarrow{\scriptstyle\phi} \\ \tilde{T} & \xrightarrow[\tau]{} & Y \end{array}$$

*with σ is a highly proximal extension, τ is a strictly-*HPI *extension and $\tilde{\phi}$ is open, point distal* (RIC) *and weakly mixing.*

Moreover $\tilde{\phi}$ is a homeomorphism (and so ϕ is HPI) *iff ϕ is a* PI *extension.*

PROOF Let



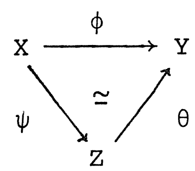
be the canonical EGS tower for ϕ (cf. [4], [5] or [8]). In that tower, for every ordinal $\beta < \nu$ the next diagram is just EGS(ϕ_β). In addition $\xi_{\beta+1}: Z_{\beta+1} \to Y_{\beta+1}$ is

$$\begin{array}{ccc} X_{\beta+1} & \xrightarrow{\sigma_{\beta+1}} & X_\beta \\ {\scriptstyle\psi_{\beta+1}}\downarrow & & \downarrow{\scriptstyle\phi_\beta} \\ Y_{\beta+1} & \longrightarrow & Z_\beta \end{array}$$

the maximal almost periodic extension of $X_{\beta+1}$ and $Y_{\beta+1}$ (so $\psi_{\beta+1} = \xi_{\beta+1} \circ \phi_{\beta+1}$). Since $\phi = \phi_0$ is point distal EGS(ϕ_0) and AG(ϕ_0) coincide, so σ_1 and τ_1 are highly proximal and ψ_1 is point distal. But then also ϕ_1 is point distal and EGS(ϕ_1) coincides with AG(ϕ_1). Transfinite induction shows that the EGS - tower for ϕ is just an AG-tower. Let σ be the inverse limit of all σ_α's and define τ to be the inverse limit of the $\tau_\alpha\circ\xi_\alpha$'s. Then σ is highly proximal, τ is strictly-HPI and $\tilde{\phi}$ is open and point distal. Moreover $\tilde{\phi}$ is a homeomorphism if ϕ is a PI-extension. If ϕ is not a PI extension, it follows from [8], Thm 2.1.3, that $\tilde{\phi}$ is weakly mixing (i.e. $R_{\tilde{\phi}}$ does not contain a proper closed invariant subset with nonempty interior in $R_{\tilde{\phi}}$). □

5. HPI EXTENSIONS

5.1 LEMMA *Consider the commutative diagram*



of minimal ttg's and homomorphisms, with ϕ *strictly-HPD and* ψ *proximal. Then* ψ *is highly proximal.*

PROOF Let X^* and Z^* be the maximally highly proximal extensions of X and Z. Then $\psi^*: X^* \to Z^*$ (cf. Thm 3.6) is proximal and $\phi': X^* \to Y$ is still strictly-HPD. Applying the maximally highly proximality of Z^* to the highly proximal steps in the tower of ϕ' and [1] lemma II.2 to the distal one's, it becomes clear that Z^* can be mapped (in a "commutative way") onto every minimal ttg in the tower of ϕ'. Then we may conclude (paying the necessary attention to the base points) that $X^* = Z^*$ and so that ψ is highly proximal. □

5.2 PROPOSITION *Let* $\phi: X \to Y$ *be an almost periodic homomorphism of minimal ttg's. Then* $\phi^*: X^* \to Y^*$ *is strictly*-HPI. *In fact* $\phi^* = X^* \to Z \to Y^*$ *with* $X^* \to Z$ *is highly proximal and* $Z \to Y^*$ *is the maximal almost periodic extension of* Y^* *between* X^* *and* Y^*.

PROOF From 3.6 it follows that ϕ^* is a RIC extension. Let H and F be the Ellis groups of X and Y, then $\mathsf{H}(\mathsf{F}) \subseteq \mathsf{H}$ ([5], IX.2.1.(4)). Let Z be the maximal almost periodic extension of Y^* between X^* and Y^*, then $\mathsf{H}(\mathsf{F}).\mathsf{H} = \mathsf{H}$ is the Ellis group of Z ([5], X.2.1), and Z is a factor of X^* under a proximal map. Since $X^* \to X \to Y$ is strictly-HPI it follows from 5.1 that $X^* \to Z$ is highly proximal, and so that ϕ^* is strictly-HPI. □

5.3 COROLLARY *Let* $\phi: X \to Y$ *be a homomorphism of minimal ttg's. Then*

1° ϕ^* *is strictly-HPI if* ϕ *is strictly-HPI*

2° ϕ^* *is HPI iff* ϕ *is HPI.*

REMARK In 5.8 more will be shown, namely, that ϕ^* is even strictly-HPI if (and only if) ϕ is HPI.

5.4 PROPOSITION *Let* Y^* *be a maximally highly proximal minimal ttg and let* $\phi: X \to Y^*$ *be a* HPD *homomorphism of minimal ttg's. Then* ϕ *is a* RIC *extension.*

PROOF Suppose ϕ is strictly-HPD. Let F be the Ellis group of Y^*, then the canonical homomorphism κ: $QF(u\circ F, M) \to Y^*$ is proximal and open. Apply [1] lemma III,1 and III,2 to the highly proximal and distal steps in the tower for ϕ respectively. It then turns out that X and $QF(u\circ F, M)$ are disjoint over Y^* and so that ϕ is RIC ([5], X 1.3). If ϕ is an HPD extension, there exist a strictly-HPD extension $\psi: Z \to Y^*$ and a homomorphism of minimal ttg's $\theta: Z \to X$ such that $\phi\circ\theta = \psi$. Since X is a factor of Z between Z and Y^* and Z and $QF(u\circ F, M)$ are disjoint over Y^* it follows that also X and $QF(u\circ F, M)$ are disjoint over Y^*, so ϕ is RIC. □

5.5 THEOREM *Let* $\phi: X \to Y$ *be an open HPI homomorphism of minimal ttg's. Then* ϕ *is RIC.*

PROOF From 5.3 it follows that $\phi^*: X^* \to Y^*$ is HPI and so by 5.4 ϕ^* is RIC. Since ϕ is open [1] lemma III.5 gives that ϕ is RIC. □

5.6 PROPOSITION *Let* $\phi^*: X^* \to Y^*$ *be a strictly*-HPI *homomorphism of maximally highly proximal minimal ttg's. Then there is a commutative diagram*

$$\begin{array}{ccc} X^* & \xrightarrow{\phi^*} & Y^* \\ {\scriptstyle\theta}\downarrow & \simeq & \downarrow \\ U^* & \longrightarrow & U \end{array}$$

where U *is the maximal almost periodic extension of* Y^* *and* $\theta: X^* \to U^*$ *is strictly*-HPI.

PROOF Let

$$V_\nu^* = X^* \to \ldots \to V'_{\alpha+1} \to V_{\alpha+1} \to V'_\alpha \to V_\alpha \to \ldots \to V'_2 \to V_2 \to V'_1 \to V_1 \to Y^* = V_0^*$$

be an HPI tower for ϕ^* with $V'_\beta \to V_\beta$ h.p. and $V_{\beta+1}\ V'_\beta$ almost periodic. Then by 5.2

$$V_\nu^* = X^* \to \dots \to \tilde{V}_{\alpha+1} \to V_\alpha^* \to \dots \to V_2^* \to \tilde{V}_2 \to V_1^* \to \tilde{V}_1 \to Y^* = V_0^*$$

is an HPI tower for ϕ^* with $\tilde{V}_{\beta+1}^* = V_{\beta+1}^*$ and $\tilde{V}_{\beta+1} \to V_\beta^*$ almost periodic, as in 5.2. For every ordinal β denote the Ellis group of V_β by $L(\beta)$ and $L(\nu) := H$. Since $X^* \to V_\beta^*$ is strictly-HPI, it is RIC by 5.4. Let $W_{\beta+1}$ be the maximal almost periodic extension of V_β^* between X^* and V_β^*. Note that $H(L(\beta)).H$ is the Ellis group of $W_{\beta+1}$ ([5], X.2.1) and so that $W_{\beta+1} = V_\beta^*$ iff $X^* = V_\beta^*$. We claim that

$$X^* \to \dots \to W_{\alpha+2}^* \to W_{\alpha+1}^* \to W_\alpha^* \to \dots \to W_2^* \to W_1^* \to Y^*$$

is an HPI tower for ϕ^*, where for every limit ordinal γ $W_\gamma^* = (W_\gamma')^*$ with $W_\gamma' = \vee\{W_\beta^* | \ \beta > \gamma\}$ is the inverse limit of $\{W_\beta^* | \ \beta < \gamma\}$. Let α be an ordinal. Since $W_{\alpha+1}$ is the maximal almost periodic extension of V_α^* between X^* and V_α^* and $\tilde{V}_{\alpha+1} \to V_\alpha^*$ is an almost periodic extension of V_α^* between X^* and V_α^*, there is an almost periodic $\eta_{\alpha+1}: W_{\alpha+1} \to \tilde{V}_{\alpha+1}$. By 5.2 there is an almost periodic extension $Q_{\alpha+1} \to V_{\alpha+1}^*$ such that $Q_{\alpha+1}^* = W_{\alpha+1}^*$. Since $W_{\alpha+2}$ is the maximal almost periodic extension of $V_{\alpha+2}^*$ between X^* and $V_{\alpha+1}^*$ and $Q_{\alpha+1} \to V_{\alpha+1}^*$ and $\tilde{V}_{\alpha+2} \to V_{\alpha+1}^*$ are almost periodic extensions of $V_{\alpha+1}^*$ between X^* and $V_{\alpha+1}^*$, there are almost periodic extensions $\xi_{\alpha+2}: W_{\alpha+2} \to Q_{\alpha+1}$ and $\eta_{\alpha+2}: W_{\alpha+2} \to \tilde{V}_{\alpha+2}$. By 5.2, $\xi_{\alpha+2}^*: W_{\alpha+2}^* \to Q_{\alpha+1}^* = W_{\alpha+1}^*$ is strictly-HPI. Continuation proves our claim until we meet the first limit ordinal.

Let γ be a limit ordinal and let Q_β be defined as above for all $\beta < \gamma$. Then $W_\gamma^* \to W_\gamma' (=\vee\{W_\beta^* \ |\beta < \gamma\}) \longrightarrow \Pi((Q_\beta \ |\beta < \gamma\}$. Define Q_γ as the image of W_γ^*. Clearly $Q_\gamma^* = W_\gamma^*$ and Q_γ maps onto $V_\gamma' := \vee\{V_\beta^* | \ \beta < \gamma\}$. We shall prove that $Q_\gamma \to V_\gamma'$ is almost periodic, so $W_\gamma^* \to V_\gamma'^* (=V_\gamma^*)$ can be written as $W_\gamma^* \to Q_\gamma' \to V_\gamma^*$ with $W_\gamma^* = Q_\gamma'^*$ and $Q_\gamma' \to V_\gamma^*$ is almost periodic (5.2). It then follows that $W_{\gamma+1} \to Q_\gamma'$ is almost periodic by definition of $W_{\gamma+1}$ and $W_{\gamma+1}^* \to Q_\gamma'^* = W_\gamma^*$ is strictly-HPI (5.2).

Since $Q_\gamma \to V_\gamma'$ is almost periodic in all the coordinates, it is distal. The Ellis group of Q_γ is $\cap\{H(L(\beta)) \ |\beta < \gamma\}.H$, for $H(L(\beta)).H$ was the Ellis group of W_β and so of W_β^* and Q_β, and $L(\gamma) = \cap\{L(\beta) | \ \beta < \gamma\}$ is the Ellis group of V'_γ. Since $H(L(\gamma)) = H(\cap\{L(\beta) \mid \beta < \gamma\}) \subseteq \cap\{H(L(\beta)) \mid \beta < \gamma\}$ it follows from IX.2.1.(4) that $Q_\gamma \to V'_\gamma$ is almost periodic. This proves our claim, and the proposition if we define $U := W_1$. □

REMARK In the construction of the tower consisting of W^*_α's in the proof above, every $W^*_{\alpha+1} \to W^*_\alpha$ is strictly-HPI, and by 5.2 in such a way that $\tilde{W}_{\alpha+1}$ exists with $\tilde{W}^*_{\alpha+1} = W^*_{\alpha+1}$ and $\tilde{W}_{\alpha+1} \to W^*_\alpha$ is almost periodic.

5.7 PROPOSITION *Let $\phi^*: X^* \to Y^*$ be a strictly-*HPI *homomorphism of maximally highly proximal minimal ttg's. Then*

$$X^* \to \dots \to U^*_{\alpha+1} \to U_{\alpha+1} \to U^*_\alpha \to \dots \to U^*_2 \to U_2 \to U^*_1 \to U_1 \to Y^* = U^*_0$$

is an HPI *tower for ϕ^**, with $U_{\alpha+1}$ *is the maximal almost periodic extension of U^*_α between X^** and *U^*_α, and for a limit ordinal γ, $U^*_\gamma = (\vee\{U^*_\beta \mid \beta > \gamma\})^*$.*

PROOF Without loss of generality let

$$X^* \to \dots \to V^*_{\alpha+1} \to \tilde{V}_{\alpha+1} \to V^*_\alpha \to \dots \to V^*_2 \to \tilde{V}_2 \to V^*_1 \to \tilde{V}_1 \to Y^* = V^*_0$$

be an HPI tower for ϕ^*, with $V^*_{\alpha+1} = \tilde{V}^*_{\alpha+1}$ and $\tilde{V}_{\alpha+1} \to V^*_\alpha$ almost periodic. Let $W'_{\alpha+1}$ be the maximal almost periodic extension of V^*_α between X^* and V^*_α and for a limit ordinal γ, $W^*_\gamma = (\vee\{W^*_\beta \mid \beta < \gamma\})^*$.

By the preceding remark and the proof of 5.6:

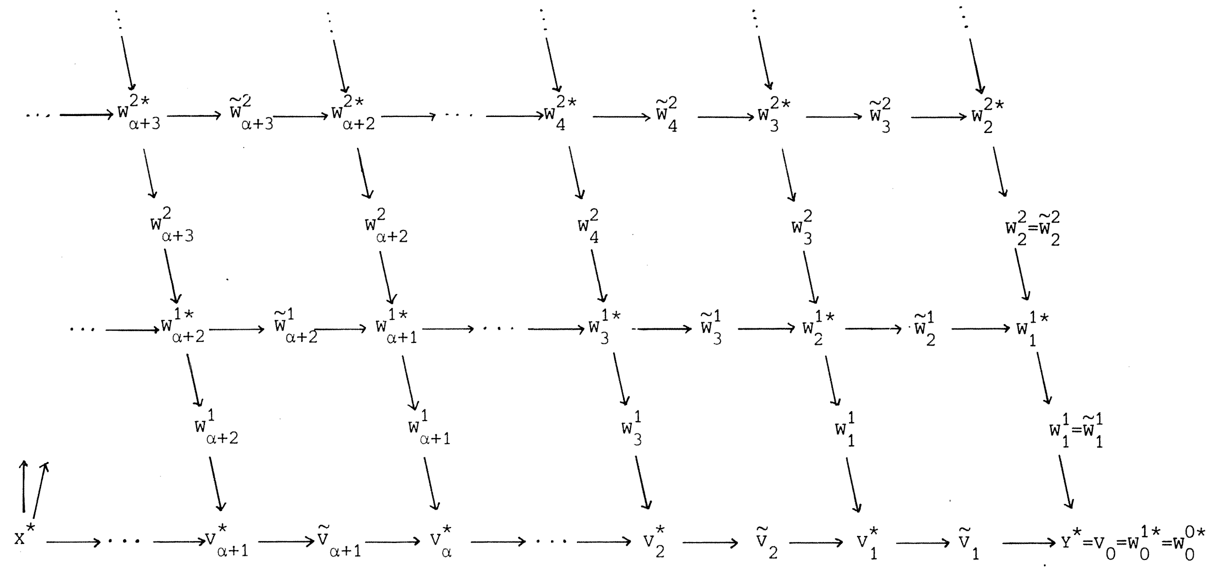
$$X^* \to \dots \to W^{1*}_{\alpha+1} \to \tilde{W}^1_{\alpha+1} \to W^{1*}_\alpha \to \dots \to W^{1*}_2 \to \tilde{W}^1_2 \to W^{1*}_2 \to \tilde{W}^1_1 = W^1_1 \to Y^* = W^{1*}_0$$

is an HPI tower for ϕ^*. We can repeat the procedure for $X^* \to W^{1*}_1$. So let $W^2_{\alpha+1}$ be the maximal almost periodic extension of W^{1*}_α between X^* and W^{1*}_α. Then

$$X^* \to \dots \to W^{2*}_{\alpha+1} \to \tilde{W}^2_{\alpha+1} \to W^{2*}_\alpha \to \dots \to W^{2*}_3 \to \tilde{W}^2_3 \to W^{2*}_2 \to \tilde{W}^2_2 = W^2_2 \to W^{1*}_1$$

is an HPI tower for $X^* \to W^{1*}_1$.

Repeating this procedure results in the following picture



Let γ be the first limit ordinal. Define

$$U^{\gamma *}_{\alpha} := (V\{W^{\beta *}_{\beta+\alpha} \mid \beta < \gamma\})^{*}$$

for non limit ordinals α and for a limit ordinal δ

$$U^{\gamma *}_{\delta} = (V\{W^{\beta *}_{\delta} \mid \beta < \gamma\})^{*} = (V\{U^{\gamma *}_{\alpha} \mid \alpha < \delta\})^{*}.$$

Now

$$X^{*} \to \ldots \to U^{\gamma *}_{\alpha+1} \to U^{\gamma *}_{\alpha} \to \ldots U^{\gamma *}_{2} \to U^{\gamma\alpha}_{1} \to U^{\gamma *}_{0}$$

is an HPI tower for $X^{*} \to U^{\gamma *}_{0}$ as follows:

Define $Q^{\gamma}_{\alpha+1}$ to be the image of $U^{\gamma *}_{\alpha+1}$ in $\Pi\{\tilde{W}^{\beta}_{\beta+\alpha+1} \mid \beta < \gamma\}$. Then $Q^{\gamma *}_{\alpha+1} =_{*} U^{\gamma *}_{\alpha+1}$ and in the same way as in the proof of 5.6 we can prove that $Q^{\gamma}_{\alpha+1} \to V\{W^{\beta}_{\beta+\alpha} | \beta > \gamma\}$ is almost periodic, so $U^{\gamma *}_{\alpha+1} \to U^{\gamma *}_{\alpha}$ is strictly-HPI.

Since $U^{\gamma *}_{\alpha}$ is the inverse limit of the first γ steps in the tower we ask for in this proposition, it will be clear that the obtained observation, that $X^{*} \to U^{\gamma *}_{0}$ is strictly-HPI proves our proposition. □

The next theorem is a relativized version of [1] cor. III.1, and it shows that our definition of an HPI extension, as just a factor of a strictly-HPI extension, coincides with the one of AUSLANDER and GLASNER where it had to be a highly proximal factor.

5.8 THEOREM *Let* $\psi: Z \to Y$ *be an HPI homomorphism of minimal ttg's. Then* $\psi^*: Z^* \to Y^*$ *is strictly-HPI. In particular it follows that* $Z^* \to Y$ *is strictly-HPI.*

PROOF Let $\phi: X \to Y$ and $\theta: X \to Z$ be homomorphism of minimal ttg's, such that ϕ is strictly-HPI and $\psi \circ \phi = \phi$. Then $\phi^*: X^* \to Y^*$ is strictly-HPI and $\psi^*: Z^* \to Y^*$ is HPI. Let

$$X^* \to \ldots \to U^*_{\alpha+1} \to U^*_{\alpha} \to \ldots \to U^*_2 \to U^*_1 \to Y^* = U^*_0$$

be the HPI tower for ϕ^* we found in proposition 5.7. Note that ϕ^* is strictly-PI and so ψ^* is PI. Since by 5.4 ψ^* is RIC, there is a maximal almost periodic extension W_1 of Y^* between Z^* and Y^*, and $W_1 = Y^*$ iff $Z^* = Y^*$. As U_1 is the maximal almost periodic extension of Y^* between X^* and Y^*, there is an almost periodic $\eta_1: U_1 \to W_1$ and $\eta_1^*: U_1^* \to W_1^*$ is strictly-HPI by 5.2.. Consequently $X^* \to W_1^*$ is strictly-HPI and $Z^* \to W_1^*$ is HPI.

Construct a tower as in 5.7 for $X^* \to W_1^*$, and let W_2 be the maximal almost periodic extension of W_1^* between Z^* and W_1^*. Since ψ^* is PI there exists an ordinal ν such that $W_\nu^* = Z^*$ and so $Z^* = W_\nu^* \to \ldots W_2^* \to W_1^* \to Y^*$ is a strictly-HPI tower for ψ^*. □

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