stichting
mathematisch
centrum

AFDELING ZUIVERE WISKUNDE
(DEPARTMENT OF PURE MATHEMATICS) ZN 99/80 DECEMBER

J.C.S.P. VAN DER WOUDE

# WEAKLY MIXING REMARKS

kruislaan 413 1098 SJ amsterdam

*Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.*

*The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).*

1980 Mathematics subject classification: 54H20

Weakly mixing remarks

by

Jaap van der Woude

ABSTRACT

We study homomorphisms of minimal transformation groups that admit relatively invariant measures, especially with respect to the equicontinuous structure relation and weak disjointness. In particular we prove that for an open RIM extension the equicontinuous structure relation equals the regionally proximal relation.

KEY WORDS & PHRASES: *Minimal transformation group, ergodicity, Relatively Invariant Measure, equicontinuous structure relation*

## 1. INTRODUCTION

Although we assume basic knowledge about topological dynamics as can be found in $[G_1]$, [B] we will review some basic definitions. A *topological transformation group* (*ttg*) is a tripel $(T,X,\pi)$, where T is a topological group, X a compact $T_2$ space and $\pi\colon T \times X \to X$ is a continuous map such that $\pi(e,x) = x$ and $\pi(s,\pi(t,x)) = \pi(st,x)$ for all $x \in X$, $t,s \in T$. We will fix the group and drop the action symbol. A subset $A \subseteq X$ is called *invariant* if $TA = A$ and X is called *minimal* (*ergodic*) if the only nonempty closed invariant subset of X (with non-empty interior) is X itself.

A continuous surjection $\phi\colon X \to Y$ between two ttg's is called a *homomorphism of ttg's*, or an *extension* if $\phi(tx) = t\phi(x)$ for all $t \in T$, $x \in X$. Any homomorphism induces a closed invariant equivalence relation $R_\phi =$ $= \{(x_1,x_2) \in X \times X \mid \phi(x_1) = \phi(x_2)\}$ on X.

Let $\mathcal{U}_X$ denote the unique uniform structure on X, then we define $P_\phi = \bigcap \{T\alpha \cap R_\phi \mid \alpha \in \mathcal{U}_X\}$, $Q_\phi = \bigcap \{\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathcal{U}_X\}$ the *proximal* and *regionally proximal relation* of $\phi$; and $E_\phi$ the *equicontinuous structure relation* of $\phi$ is defined to be the smallest closed invariant equivalence relation that contains $Q_\phi$. One of the major problems in topological dynamics is to determine $E_\phi$. VEECH $[V_2]$, showed that $E_\phi = Q_\phi$ if the almost periodic points (points with minimal orbitclosure) are dense in $R_\phi$. McMAHON [M] and McMAHON and WU [MW'80] proved related results with totally different methods. We shall use those methods to prove that $E_\phi = Q_\phi$ in case $\phi$ is an open RIM extension. We call $\phi\colon X \to Y$ a RIM *extension* if there exists a homomorphism $\lambda\colon Y \to M(X)$ (into) such that $\hat{\phi}\circ\lambda\colon Y \to M(X) \to M(Y)$ equals $id_Y$, where $M(X)$ is the set of Borel probability measures with the weak star topology and $\hat{\phi}\colon M(X) \to M(Y)$ is the map induced by $\phi$ (for more details see $[G_2]$).

We will also be concerned with the question: when are two homomorphisms $\phi\colon X \to Y$ and $\psi\colon Z \to Y$ of minimal ttg's *weakly disjoint*, i.e. when is $R_{\phi\psi} = \{(x,z) \in X \times Z \mid \phi(x) = \psi(z)\}$ an ergodic ttg. (Notation: $\phi \mathrel{\dot{\perp}} \psi$). We call $\phi$ *weakly mixing* if $\phi \mathrel{\dot{\perp}} \phi$ and a minimal ttg X is *weakly mixing* if $\phi\colon X \to 1$ is weakly mixing.

An interesting result is: if $\phi\colon X \to Y$ is a RIM extension of metric minimal ttg's without non-trivial almost periodic factors, then $\phi$ is weakly disjoint from every open extension of minimal ttg's that values in Y.

## 2. ERGODIC POINTS

Let $\phi\colon X \to Y$ be a homomorphism of minimal ttg's and $n \in \mathbb{N}$, $n \geq 2$. We call $x \in X$ a $\phi$-$n$-*local ergodic point* if for every open subset $W \subseteq X$ there exists a set $U$ open in $\overleftarrow{\phi}\phi(x)$ such that $U \supseteq E_\phi[x]$ and for open (in $\overleftarrow{\phi}\phi(x)$) sets $V_1,\ldots,V_n$ in $U$ we have that $T(V_1\times\ldots\times V_n) \cap \Pi_n W \neq \emptyset$. If $U$ can be chosen to be $\overleftarrow{\phi}\phi(x)$, we call $x$ $\phi$-$n$-*ergodic* and clearly every $x' \in \overleftarrow{\phi}\phi(x)$ is $\phi$-$n$-ergodic iff $x$ is. If $x$ is $\phi$-$n$-(local) ergodic for all $n \in \mathbb{N}$, $n \geq 2$, we call $x$ $\phi$-*(local) ergodic*.

Obviously every $\phi$-$n$-ergodic point is $\phi$-$n$-local ergodic and if $E_\phi = R_\phi$ the converse is true.

If $\phi\colon X \to 1$ we skip the prefix $\phi$ in the above definitions. Note that if $x$ is $\phi$-$n$-(local) ergodic then $tx$ is for all $t \in T$.

1. <u>PROPOSITION</u>. *Let* $\phi\colon X \to Y$ *be a homomorphism of minimal ttg's.* *If* $x \in X$ *is* $\phi$-2-*local ergodic then* $Q_\phi[x] = E_\phi[x]$. *If* $x \in X$ *is* $\phi$-2-*ergodic then* $Q_\phi[x] = R_\phi[x] = \overleftarrow{\phi}\phi(x)$.

<u>PROOF</u>. Choose $(x,x') \in R_\phi$ $(E_\phi)$ and $\alpha \in \mathcal{U}_X$. Choose $\beta \in \mathcal{U}_X$ with $\beta = \beta^{-1}$ and $\beta\circ\beta \subseteq \alpha$ then $T(\beta(x) \times \beta(x)) \subseteq T\alpha$. Choose $U$ for $W = \beta(x)$ as in the definition. For every neighbourhood $V \times V'$ of $(x,x')$ in $\underline{\overleftarrow{\phi}\phi(x) \times \overleftarrow{\phi}\phi(x)}$ (in $\underline{U\times U}$) we have $V \times V' \cap T(\beta(x) \times \beta(x)) \neq \emptyset$, so $(x,x') \in T\alpha \cap \underline{\overleftarrow{\phi}\phi(x) \times \overleftarrow{\phi}\phi(x)} \subseteq \underline{T\alpha \cap R_\phi}$, i.e. $(x,x') \in Q_\phi$. □

2. <u>COROLLARY</u>. *Let* $\phi\colon X \to Y$ *be a homomorphism of minimal ttg's. If there is a* $\phi$-2-*ergodic point* $x \in X$ *then* $E_\phi = R_\phi$.

<u>PROOF</u>. Since every $x'$ in $\overleftarrow{\phi}\phi(x)$ is a $\phi$-2-ergodic point it follows that $\overleftarrow{\phi}\phi(x) \times \overleftarrow{\phi}\phi(x) = \overleftarrow{\phi}\phi(x) \times \overleftarrow{\phi}\phi(x) \cap Q_\phi \subseteq E_\phi$. But then $\theta\colon X/E_\phi \to Y$ is almost one to one and almost periodic so $\theta$ is a homeomorphism and $X/E_\phi \cong Y$, $E_\phi = R_\phi$. □

For the following we need to remember that $\phi\colon X \to Y$ is open iff for all $y \in Y$, $x \in \overleftarrow{\phi}(y)$ and for any net $y_i \to y$ we can choose $x_i \in \overleftarrow{\phi}(y_i)$ with $x_i \to x$.

3. PROPOSITION. *Let* $\phi: X \to Y$ *be a homomorphism of minimal ttg's such that* $\kappa: X \to X/E_\phi$ *is open. Suppose there exists a* $y \in Y$ *such that* $x$ *is* $\phi$*-2-local ergodic for all* $x \in \overleftarrow{\phi}(y)$. *Then* $E_\phi = Q_\phi$.

PROOF. Since $x$ is $\phi$-2-local ergodic for every $x \in \overleftarrow{\phi}(y)$ it follows from Prop.1 that $Q_\phi \cap \overleftarrow{\phi}(y) \times \overleftarrow{\phi}(y) = E_\phi \cap \overleftarrow{\phi}(y) \times \overleftarrow{\phi}(y)$. Let $z = \kappa(x) \in X/E_\phi$. Choose $(x_1,x_2) \in E_\phi$ and let $z_0 = \kappa(x_1) = \kappa(x_2)$. Choose a net $\{t_i\}$ in $T$ with $t_i z \to z_0$. Since $\kappa$ is open we can choose $x_1^i$ and $x_2^i$ in $\overleftarrow{\kappa}(z)$ such that $t_i(x_1^i,x_2^i) \to (x_1,x_2)$. Now $(x_1^i,x_2^i) \in E_\phi \cap \overleftarrow{\phi}(y) \times \overleftarrow{\phi}(y) \subseteq Q_\phi$ and so $(x_1,x_2) \in \overline{TQ_\phi} = Q_\phi$. □

4. COROLLARY. *Let* $\phi: X \to Y$ *be an open homomorphism of minimal ttg's. If there is an* $x \in X$ *that is* $\phi$*-2-ergodic then* $Q_\phi = R_\phi$.

PROOF. By Cor.2 $E_\phi = R_\phi$ so, with notation as in 3 $\kappa = \phi$ is open. Since $x'$ is $\phi$-2-ergodic for all $x' \in \overleftarrow{\phi}\phi(x)$ it follows from Prop.3 that $Q_\phi = E_\phi = R_\phi$. □

5. COROLLARY. *Let* $X$ *be minimal, then* $Q_X = X \times X$ *iff there exists a 2-ergodic point* ($Q_X = Q_\phi$ *with* $\phi: X \to 1$).

PROOF. The "if" part is just Cor.4. Choose $U$ and $V$ open in $X$ and $(x_1,x_2) \in U \times V$, then $(x_1,x_2) \in Q_X$ so $U \times V \cap T\alpha \neq \emptyset$ for any $\alpha \in \mathcal{U}_X$. Clearly for any $W \subseteq X$ open, there is an $\alpha \in \mathcal{U}_X$ with $\alpha \subseteq T(W\times W)$ and so $T\alpha \subseteq T(W\times W)$. □

Let $\phi: X \to Y$ be a homomorphism of ttg's and $n \in \mathbb{N}$, $n \geq 2$. A point $x_0 \in X$ is called a $P_\phi^n$*-point* if $\{(x_1,\ldots,x_n) \in (\overleftarrow{\phi}\phi(x_0))^n \mid \overline{T(x_1,\ldots,x_n)} \cap \Delta_X^n \neq \emptyset\}$ is dense in $(\overleftarrow{\phi}\phi(x_0))^n$. Clearly, if $\phi$ is proximal then every $x$ is a $P_\phi^n$ point for all $n \in \mathbb{N}$, $n \geq 2$.

7. PROPOSITION. *Let* $\phi: X \to Y$ *be a homomorphism of ttg's and* $n \in \mathbb{N}$, $n \geq 2$.

a. *If* $X$ *is minimal, then every* $P_\phi^n$*-point is a* $\phi$*-n-ergodic point.*

b *If* $x_0$ *has a countable neighbourhood base* $\mathcal{W}_{x_0}$ *for some* $x_0 \in X$, *then every* $\phi$*-n-ergodic point is a* $P_\phi^n$*-point.*

*In particular if* $X$ *is metric and minimal, then the* $\phi$*-n-ergodic points are just the* $P_\phi^n$*-points.*

PROOF. a Let $x \in X$ be a $P^n_\phi$-point. Choose $W \subseteq X$ open and $U_1,\dots,U_n$ open in $\overleftarrow{\phi}\phi(x)$. Then, since $X$ is minimal, $\Delta^n_X \subseteq T(W \times\dots\times W)$ (n-times). Since $U_1 \times\dots\times U_n$ is open in $(\overleftarrow{\phi}\phi(x))^n$ there is a $(x_1,\dots,x_n) \in U_1 \times\dots\times U_n$ with $\Delta^n_X \cap \overline{T(x_1,\dots,x_n)} \neq \emptyset$. So $\Delta^n_X \subseteq \overline{T(U_1\times\dots\times U_n)}$ and $\overline{T(U_1\times\dots\times U_n)} \cap T(W\times\dots\times W) \neq \emptyset$. But then $U_1 \times\dots\times U_n \cap T(W\times\dots\times W) \neq \emptyset$ and $x$ is $\phi$-n-ergodic.

b Let $x \in X$ be $\phi$-n-ergodic, and choose $0 \subseteq (\overleftarrow{\phi}\phi(x))^n$ open in $(\overleftarrow{\phi}\phi(x))^n$. Let $V_1,\dots,V_n$ be open in $\overleftarrow{\phi}\phi(x)$ such that $V_1 \times\dots\times V_n \subseteq 0$ and let $\mathcal{W}_{X_0} = \{W_\alpha \mid \alpha \in \mathbb{N}\}$.

For $\alpha \in \mathbb{N}$ define $t_\alpha$ and $V^\alpha_1,\dots,V^\alpha_n$ inductively as follows: let $t_1 \in T$ be such that $t_1(V_1\times\dots\times V_n) \cap W_1 \times\dots\times W_1 \neq \emptyset$ and let $V^1_i := V_i$. Let $t_\alpha$ and $V^\alpha_i$ be defined such that $t_\alpha(V^\alpha_1\times\dots\times V^\alpha_n) \cap W_\alpha \times\dots\times W_\alpha \neq \emptyset$. Then choose $V^{\alpha+1}_i \neq \emptyset$ such that $V^{\alpha+1}_i \subseteq \overline{V^{\alpha+1}_i} \subseteq V^\alpha_i \cap t^{-1}_\alpha W_\alpha$ and let $t_{\alpha+1} \in T$ be such that $t_{\alpha+1}(V^{\alpha+1}_1\times\dots\times V^{\alpha+1}_n) \cap W_{\alpha+1} \times\dots\times W_{\alpha+1} \neq \emptyset$.

Choose $x_i \in \bigcap \{\overline{V^\alpha_i} \mid \alpha \in \mathbb{N}\} \subseteq V_i \cap \{t^{-1}_\alpha W_\alpha \mid \alpha \in \mathbb{N}\}$ for all $i = 1,\dots,n$ then $(x_1,\dots,x_n) \in 0$ and $t_\alpha(x_1,\dots,x_n) \longrightarrow (x_0,\dots,x_0)$, i.e. $x$ is a $P^n_\phi$ point. □

Using an idea of McMAHON we will prove the existence of $\phi$-n-local ergodic points under certain conditions. For that we need the following theorem.

8. THEOREM. *Let $\phi: X \to Y$ be a* RIM *extension of minimal ttg's with section $\lambda(:Y \to M(X))$ and let $x_0 \in X$.*

a *Let* $U$ *be open in* $\overleftarrow{\phi}\phi(x_0)$ *and* $F \subseteq \overleftarrow{\phi}\phi(x_0)$. *Then*

$$E_\phi[F] \times (U \cap \operatorname{supp} \lambda_{\phi(x_0)}) \subseteq T(F\times U)$$

b *Let* $n \in \mathbb{N}$, $n \geq 2$, $U_i$ *open in* $\operatorname{supp} \lambda_{\phi(x_0)}$ *for* $i = 1,\dots,n$ *such that* $E_\phi[U_i]$ *is open in* $\overleftarrow{\phi}\phi(x_0)$ *for* $i = 2,\dots,n$. *Then*
$\underline{E_\phi[U_1] \times (E_\phi[U_2] \cap \operatorname{supp} \lambda_{\phi(x_0)}) \times\dots\times (E_\phi[U_n] \cap \operatorname{supp} \lambda_{\phi(x_0)})} \subseteq$
$\subseteq \overline{T(U_1\times\dots\times U_n)}$.

PROOF. a This is a special case of [M] 1.4.

b This is a special case of [MW'80] 1.1. □

9. PROPOSITION. *Let* $\phi\colon X \to Y$ *be a* RIM *extension of minimal ttg's with section* $\lambda$. *Let* $\kappa\colon X \to X/E_\phi$ *and* $x \in X$. *If* $x$ *has a neighbourhood* $V$ *in* $\overleftarrow{\phi}\phi(x)$ *with* $E_\phi[V] \subseteq \mathrm{supp}\,\lambda_{\phi(x)}$ *and* $\kappa' = \kappa|_{\overleftarrow{\phi}\phi(x)}\colon \overleftarrow{\phi}\phi(x) \to \kappa[\overleftarrow{\phi}\phi(x)]$ *is open in a dense subset of* $V$ *then* $x$ *is* $\phi$*-local ergodic.*

PROOF. Choose $W \subseteq X$ open. Then $\kappa[W]^\circ \neq \emptyset$ ($X$ is minimal!), so there is a neighbourhood $V^*$ of $\kappa(x)$ and a $t \in T$ with $tV^* \subseteq \kappa[W]^\circ$. Define $U := \overleftarrow{\kappa}V^* \cap V$, and choose $n \in \mathbb{N}$, $n \geq 2$. Choose $V_1,\ldots,V_n$ open in $U$. Since the points of openness of $\kappa'$ are dense in $V$ and so in $U$ we can find $V_i' \subseteq V_i$ such that $E_\phi[V_i']$ is open in $\overleftarrow{\phi}\phi(x)$. Obviously $E_\phi[V_i'] \subseteq E_\phi[V_i] \subseteq E_\phi[U] \subseteq E_\phi[V] \subseteq \mathrm{supp}\,\lambda_{\phi(x)}$. Then by 8b we have that

$$E_\phi[V_1'] \times\ldots\times E_\phi[V_n'] \subseteq \overline{T(V_1'\times\ldots\times V_n')} \subseteq \overline{T(V_1\times\ldots\times V_n)}.$$

Since $tE_\phi[V_i'] = E_\phi[tV_i'] = \overleftarrow{\kappa}\kappa[tV_i']$ and $\kappa[tV_i'] \subseteq tV^* \subseteq \kappa[W]^\circ$ we have that $W \cap tE_\phi[V_i'] \neq \emptyset$ and so

$$\emptyset \neq W \times\ldots\times W \cap T(E_\phi[V_1'] \times\ldots\times E_\phi[V_n']) \subseteq W \times\ldots\times W \cap \overline{T(V_1\times\ldots\times V_n)}$$

but then $W \times\ldots\times W \cap T(V_1\times\ldots\times V_n) \neq \emptyset$. □

10. PROPOSITION. *If* $X$ *is minimal and has an invariant measure* $\mu$ *then every* $x \in X$ *is local ergodic. In particular* $Q_X = E_X$.

PROOF. Note that $X = \mathrm{supp}\,\mu$ ($=\mathrm{supp}\,\lambda_1 = \overleftarrow{\phi}\phi(x)$ with $\phi\colon X \to 1$). Since in the proof of Prop. 9 the openness of $\kappa'$ in some points was only used to make sure that any $V$ open in $U$ contained a $V'$ with $E_\phi[V']$ open in $\overleftarrow{\phi}\phi(x)$, it is enough to prove that any open $V$ in $X$ contains an open $V'$ in $X$ with $E_\phi[V']$ open. The proposition follows then as in 9.
Let $\kappa\colon X \to X/E_X$ and let $V' := \overleftarrow{\kappa}(\kappa[V]^\circ) \cap V$. Then $E_\phi[V'] = \overleftarrow{\kappa}\kappa[V'] = \overleftarrow{\kappa}(\kappa[V]^\circ)$ is open and non empty. From Prop. 1 it follows that $Q_X[x] = E_X[x]$ for all $x \in X$ and so $Q_X = E_X$ (which is a special case of [M]1.5). □

11. COROLLARY. a *If* $\phi\colon X \to Y$ *is a* RIM *extension of metric minimal ttg's then there is a residual subset of* $\phi$*-local ergodic points.*
b *If* $\phi\colon X \to Y$ *is a* RIM *extension of minimal ttg's with* $R_\phi = E_\phi$ *then every*

$x \in X$ *with* supp $\lambda_{\phi(x)} = \overleftarrow{\phi}\phi(x)$ *is* $\phi$*-ergodic.*

PROOF. a Let $\kappa: X \to X/E_\phi$. Since X is metric and minimal, there is a residual set $X_1 \subseteq X$ in each point of which $\kappa$ is open and, by $[G_2]$ 3.3, there is a residual set $X_2 \subseteq X$ with supp $\lambda_{\phi(x)} = \overleftarrow{\phi}\phi(x)$ for all $x \in X_2$. Let $X_0 = X_1 \cap X_2$ then $X_0$ is residual. By $[V_1]$ prop.3.1 there is a residual subset $Y_0$ of Y such that $X_0 \cap \overleftarrow{\phi}(y)$ is residual in $\overleftarrow{\phi}(y)$ for all $y \in Y_0$. Choose $x \in \overleftarrow{\phi}[Y_0]$, then $\kappa$ is open in a dense subset of $\overleftarrow{\phi}\phi(x)$ and so $\kappa'$ is open in a dense subset of $\overleftarrow{\phi}\phi(x)$, also supp $\lambda_{\phi(x)} = \overleftarrow{\phi}\phi(x)$. So from Prop.9 it follows that x is $\phi$-local ergodic. Clearly, $\overleftarrow{\phi}[Y_0]$ is residual in X.
b In this case $\kappa = \phi$. If supp $\lambda_{\phi(x)} = \overleftarrow{\phi}\phi(x)$ then $\kappa = \phi$ is open in x and so $\kappa'$ is open in all $x' \in \overleftarrow{\phi}\phi(x)(=\overleftarrow{\phi}\phi(x'))$. From Prop.9 and the observation that $E_\phi[x] = \overleftarrow{\phi}\phi(x)$ it follows that x is $\phi$-ergodic. □

The following theorem as well as its proof is a generalization of $[G_1]$ II.2.1.

12. THEOREM. *Let* $\phi: X \to Y$ *and* $\psi: Z \to Y$ *be homomorphisms of minimal ttg's with* $\psi$ *open. If for every* $n \in \mathbb{N}$, $n \geq 2$ *there exists a* $\phi$*-n-ergodic point* $x \in X$, *then* $\phi \perp \psi$ *(i.e.* $R_{\phi\psi} = \{(x,z) \in X \times Z \mid \phi(x) = \psi(z)\}$ *is ergodic.*

PROOF. Choose $W = \overline{TW} \subseteq R_{\phi\psi}$ with $\mathrm{int}_{R_{\phi\psi}} W \neq \emptyset$. Since $\psi$ is open and X minimal we may choose open sets U and V in X and Z such that $U \times V \cap R_{\phi\psi} \subseteq W^\circ$ and for every $v \in V$ there is a $u \in U$ with $(u,v) \in R_{\phi\psi}$.
Since Z is minimal there are $t_1,\ldots,t_m$ in T with $Z = \bigcup_{i=1}^m t_i V$. Choose $(\bar{x},\bar{z}) \in R_{\phi\psi}$ and a neighbourhood O of $(\bar{x},\bar{z})$ in $R_{\phi\psi}$. Since $\psi$ is open there are open neighbourhoods $O_{\bar{x}}$ and $O_{\bar{z}}$ of $\bar{x}$ and $\bar{z}$ in X and Z such that $O_{\bar{x}} \times O_{\bar{z}} \cap R_{\phi\psi} \subseteq O$ and for every $o_1 \in O_{\bar{x}}$ there is an $o_2 \in O_{\bar{z}}$ with $(o_1,o_2) \in R_{\phi\psi}$.
Choose a $\phi$-m-ergodic point $x \in O_{\bar{x}}$ [Note that x is $\phi$-n-ergodic for all n with $2 \leq n \leq m$], and choose $z \in O_{\bar{z}}$ with $\phi(x) = \psi(z)$. Let $y := \phi(x) = \psi(z)$ and $U_x := O_{\bar{x}}$, $V_z := O_{\bar{z}}$. Without loss of generality choose $\{t_1,\ldots,t_n\} \subseteq$ $\subseteq \{t_1,\ldots,t_m\}$ such that $\overleftarrow{\psi}(y) \subseteq \bigcup_{i=1}^n t_i V$ and $t_i V \cap \overleftarrow{\psi}(y) \neq \emptyset$ for $i = 1,\ldots,n$. Then $L := t_1 U \times \ldots \times t_n U \cap (\overleftarrow{\phi}(y))^n$ is open in $(\overleftarrow{\phi}(y))^n$ and non-empty, for choose $z_i \in V \cap t_i^{-1}\overleftarrow{\psi}(y)$ and $x_i \in U$ such that $\phi(x_i) = \psi(z_i)$ then $(t_1 x_1,\ldots,t_n x_n) \in L$. Since $U_x$ is open in X and x is $\phi$-n-ergodic it follows

that there exists a $t \in T$ with

$$t(t_1U\times\ldots\times t_nU \cap (\phi^{\leftarrow}(y))^n) \cap \prod_n U_x \neq \emptyset.$$

Choose $\bar{x}_i \in U$ with $tt_i\bar{x}_i \in U_x \cap \phi^{\leftarrow}(ty)$ for $i = 1,\ldots,n$.
Choose $z' \in V_z$ such that $\phi(tt_i\bar{x}_i) = \psi(z') = ty$, then $t^{-1}z' \in t_{i_0}V \cap \psi^{\leftarrow}(y)$ for some $t_{i_0} \in \{t_1,\ldots,t_n\}$. But now

$$(tt_{i_0}\bar{x}_{i_0}, z') \in tt_{i_0}(U\times V) \cap R_{\phi\psi} \cap U_x \times V_z \subseteq TW^\circ \cap O \neq \emptyset.$$

Since $O$ was an arbitrary neighbourhood of $(\bar{x},\bar{z})$ in $R_{\phi\psi}$ it follows that $(\bar{x},\bar{z}) \in \overline{TW} = W$ and so $R_{\phi\psi} = W$; i.e. $R_{\phi\psi}$ is ergodic. □

13. <u>COROLLARY</u>. *Let $\phi: X \to Y$ be an open RIM extension of minimal ttg's with section $\lambda$. Suppose there is an $x \in X$ with $\phi^{\leftarrow}\phi(x) = \mathrm{supp}\,\lambda_{\phi(x)}$ (e.g. X is metric). Equivalent are*

<u>a</u> $Q_\phi = R_\phi$ \
<u>b</u> $E_\phi = R_\phi$ } (*cf.* [MW'80] Prop.2.2)

<u>c</u> *$\phi$ is weakly mixing*

<u>d</u> *$\phi \perp \psi$ for every open homomorphism $\psi: Z \to Y$ of minimal ttg's.*

<u>PROOF</u>. d ⇒ c ⇒ a ⇒ b is obvious

b ⇒ d follows from 11 and 12 and [M]2.2.. □

14. <u>COROLLARY</u>. *If X is minimal and has an invariant measure, then X is weakly mixing iff X is weakly disjoint from every minimal ttg (iff $E_X = Q_X = X \times X$).*

15. <u>COROLLARY</u>. *Let $\phi: X \to Y$ and $\psi: Z \to Y$ be homomorphisms of minimal ttg's with $\psi$ open.*

<u>a</u> *If $\phi$ is a RIM extension with $E_\phi = R_\phi$ and X metric then $\phi \perp \psi$.*

<u>b</u> *If $\phi$ is proximal then $\phi \perp \psi$.*

*In particular any open proximal map is weakly mixing.*

## 3. $E_\phi = Q_\phi$ AND "JOE's CONJECTURE"

The next theorem gives a partial solution to a question raised by J. AUSLANDER. If $x_1$ and $x_2$ are regionally proximal and if we have a net $(x_i^1, x_i^2) \to (x_1, x_2)$, is it possible to find a net $(\bar{x}_i^1, \bar{x}_i^2)$, suitably close to the first one, that converges to $(x_1, x_2)$ such that for some net $\{t_i\}$ in T $t_i(\bar{x}_i^1, \bar{x}_i^2) \to \Delta$? It is not difficult to see that we can state that question as follows: When do we have that $Q_\phi = \bigcap \{\mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi} \mid \alpha \in \mathcal{U}_X\}$. In the absolute case McMAHON proved the equality for minimal ttg's with invariant measure.

16. LEMMA. *Let* $\phi: X \to Y$ *be an open* RIM *extension of minimal ttg's and* $W \subseteq X$ *open. Then there is an open set* $U = E_\phi[U]$ *in* X *such that* $R_\phi \cap U_1 \times U_2 \cap \cap T(W{\times}W) \neq \emptyset$ *for all* $U_1$ *and* $U_2$ *open in* U *with* $R_\phi \cap U_1 \times U_2 \neq \emptyset$.

PROOF. Let $\kappa: X \to X/E_\phi$ *and define* $U := \overleftarrow{\kappa}(\kappa[W]^\circ)$. Clearly $U = E_\phi[U] = \overleftarrow{\kappa}\kappa[U]$ is open in X.
Choose $U_1$ and $U_2$ open in U with $\phi[U_1] \cap \phi[U_2] =: V^* \neq \emptyset$. Since $U_1 \cap \overleftarrow{\phi}(V^*) \neq \emptyset$ and open we can choose $V_1 \subseteq U_1 \cap \overleftarrow{\phi}(V^*)$ open with $E_\phi[V_1]$ open in X. Then $\phi[V_1] \subseteq \phi[U_2]$ and $U_2 \cap \overleftarrow{\phi}(\phi[V_1]) \neq \emptyset$ and open. Choose $V_2 \subseteq U_2 \cap \cap \overleftarrow{\phi}[\phi[V_1]]$ open with $E_\phi[V_2]$ open in X and remark that $\phi[V_2] \subseteq \phi[V_1]$. As $W \cap E_\phi[V_2]$ $(\neq \emptyset)$ is open we can choose $x_0 \in W \cap E_\phi[V_2]$ with $x_0 \in \operatorname{supp} \lambda_{\phi(x_0)}$, say $y_0 = \phi(x_0)$.
Clearly $y_0 \in \phi[V_2] \subseteq \phi[V_1]$ so $\tilde{V}_i := V_i \cap \overleftarrow{\phi}(y_0) \neq \emptyset$ and open in $\overleftarrow{\phi}(y_0)$ and so $E_\phi[\tilde{V}_i] = E_\phi[V_i \cap \overleftarrow{\phi}(y_0)] = E_\phi[V_i] \cap \overleftarrow{\phi}(y_0)$ is open in $\overleftarrow{\phi}(y_0)$. By Theorem 8b we have that

$$E_\phi[\tilde{V}_1] \times (E_\phi[\tilde{V}_2] \cap \operatorname{supp} \lambda_{y_0}) \subseteq \overline{T(\tilde{V}_1{\times}\tilde{V}_2)} \subseteq$$
$$\subseteq \overline{T(V_1{\times}V_2 \cap \overleftarrow{\phi}y_0{\times}\overleftarrow{\phi}y_0)} \subseteq \overline{T(U_1{\times}U_2 \cap R_\phi)} \quad .$$

But $x_0 \in E_\phi[\tilde{V}_2] \cap \operatorname{supp} \lambda_{y_0} \cap W$ and obviously $W \cap E_\phi[\tilde{V}_1] \neq \emptyset$ so $W \times W \cap E_\phi[\tilde{V}_1] \times (E_\phi[\tilde{V}_2] \cap \operatorname{supp} \lambda_{y_0}) \neq \emptyset$ and $W \times W \cap T(U_1{\times}U_2 \cap R_\phi) \neq \emptyset$. As $W \times W$ is open we have $W \times W \cap T(U_1{\times}U_2 \cap R_\phi) = W \times W \cap T(U_1{\times}U_2) \cap R_\phi \neq \emptyset$. □

17. <u>THEOREM</u>. *Let* $\phi: X \to Y$ *be an open* RIM *extension of minimal ttg's. Then* $E_\phi = Q_\phi = \bigcap \{\mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi} \mid \alpha \in \mathcal{U}_X\}$.

<u>PROOF</u>. Choose $\alpha \in \mathcal{U}_X$ and $\beta \in \mathcal{U}_X$ such that $\beta = \beta^{-1}$ and $\beta\circ\beta \subseteq \alpha$. For $W := \beta(x_0)$ for some fixed $x_0 \in X$ we have

$$\overline{T(W\times W) \cap R_\phi} \subseteq \overline{T\alpha \cap R_\phi}.$$

Choose $U = E_\phi[U]$ open in $X$ as in lemma <u>16</u>.
Choose $(x_1,x_2) \in U \times U \cap R_\phi$ and open neighbourhoods $V_1$ and $V_2$ of $x_1$ and $x_2$ in $U$ then $V_1 \times V_2 \cap R_\phi \neq \emptyset$ so by lemma <u>16</u>

$$T(W\times W) \cap R_\phi \cap V_1 \times V_2 \neq \emptyset.$$

But then $(x_1,x_2) \in \overline{T(W\times W) \cap R_\phi} \subseteq \overline{T\alpha \cap R_\phi}$ and so $U \times U \cap R_\phi \subseteq \overline{T\alpha \cap R_\phi}$, even $U \times U \cap R_\phi \subseteq \mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$. Choose $(x_1,x_2) \in E_\phi$ and $t \in T$ with $tx_1 \in U$. Then $(tx_1,tx_2) \in E_\phi[tx_1] \times E_\phi[tx_1] \subseteq E_\phi[U] \times E_\phi[U] \cap R_\phi = U \times U \cap R_\phi \subseteq$ $\subseteq \mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$. As $\mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$ is $T$ invariant we have $(x_1,x_2) \in \mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$ and consequently $E_\phi \subseteq \mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi}$.
So $E_\phi \subseteq \bigcap \{\mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi} \mid \alpha \in \mathcal{U}_X\} \subseteq \bigcap \{\overline{T\alpha \cap R_\phi} \mid \alpha \in \mathcal{U}_X\} = Q_\phi$. □

18. <u>REMARK</u>. We used the openness of $\phi$ to conclude that $(\phi[V_1] \cap \phi[V_2])^\circ \neq \emptyset$ for open $V_1$ and $V_2$ in $X$ with $\phi[V_1] \cap \phi[V_2] \neq \emptyset$.
Now suppose $\phi$ satisfies B.c, i.e. the almost periodic points are dense in $R_\phi$. Then $V_1 \times V_2 \cap R_\phi$ contains an almost periodic point $(x_1,x_2)$ and $V_1 \times V_2 \cap \overline{T(x_1,x_2)}$ in an open subset of the minimal ttg $\overline{T(x_1,x_2)}$. So $\phi[V_1] \times \phi[V_2] \cap \Delta_Y$ has a non-empty interior in $\Delta_Y$, or what is the same $(\phi[V_1] \cap \phi[V_2])^\circ \neq \emptyset$.

19. <u>COROLLARY</u>. *If* $\phi: X \to Y$ *is a* RIM *extension of minimal ttg's that satisfies the Bronstein condition, then*

$$E_\phi = Q_\phi = \bigcap \{\mathrm{int}_{R_\phi} \overline{T\alpha \cap R_\phi} \mid \alpha \in \mathcal{U}_X\}.$$

REFERENCES

[B] BRONŠTEǏN, I.U., *Extensions of minimal transformation groups*, Sythoff & Noordhoff, Alphen aan den Rijn, 1979.

[$G_1$] GLASNER, S., *Proximal flows*, Lecture Notes in Math. vol. 517, Springer Verlag, Berlin and New York 1976.

[$G_2$] GLASNER, S., *Relatively invariant measures*, Pacific J. Math. 58 (1975) 393-410.

[M] McMAHON, D.C., *Relativized weak disjointness and relatively invariant measures*, Trans. Amer. Math. Soc. 236 (1978), 225-237.

[MW'80] McMAHON, D.C. & T.S. WU, *Homomorphisms of minimal flows and generalizations of weak mixing*, preprint, (1980).

[$V_1$] VEECH, W.A., *Point-distal flows*, Amer. J. Math. 92 (1970) 205-242.

[$V_2$] VEECH, W.A., *Topological dynamics*, Bull. Amer. Math. Soc. 83 (1977), 775-830.