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The non-renameability of honesty classes*)

P. van Emde Boas

Abstract

An important result in the theory of complexity classes is the naming theorem of E.M. McCreight, which states that the system of complexity classes can be renamed uniformly by a measured set of names. Our investigation of honesty classes shows that for these classes the analogous assertion is false. No measured transformation of programs renames correctly all honesty classes.

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*) This paper is not for review; it is meant for publication in a journal.

1. Introduction and notations

By a function we mean (unless stated otherwise) a *partial recursive function* from the set $\mathbb{N}$ of nonnegative integers into itself. Functions which are defined for all arguments are called *total*. $\mathcal{P}(\mathcal{R})$ denotes the collection of all (total) functions. The set of arguments x for which f(x) is defined, the *domain* of f, is denoted $\mathcal{D}f$. We write $f(x) < \infty$ ($f(x) = \infty$) for $x \in \mathcal{D}f$ ($x \notin \mathcal{D}f$).

The inequality $f \leq g$ means $\mathcal{D}f \supseteq \mathcal{D}g$ and $g(x) \geq f(x)$ for $x \in \mathcal{D}g$. Strict inequality $f < g$ means that $\mathcal{D}f \supseteq \mathcal{D}g$ and $g(x) > f(x)$ for all $x \in \mathcal{D}g$. If $\mathcal{D}f \supseteq \mathcal{D}g$ and $g(x) = f(x)$ for $x \in \mathcal{D}g$ then we write $g \sqsubseteq f$. The range of f is denoted $\mathcal{R}f$.

For finite k the inequalities $k \leq \infty$ and $\infty \leq \infty$ are taken to be true whereas $\infty \leq k$ is false. Beside the inequalities on all arguments we have also inequalities holding "almost everywhere". If P(x) is some predicate we write $\overset{\infty}{\forall}_x[P(x)]$ for "P(x) holds for all except finitely many x " and $\overset{\infty}{\exists}_x[P(x)]$ for "there exist infinitely many x such that P(x)". Using these notations we let $f \underline{\propto} g$ denote $\overset{\infty}{\forall}_x[f(x) \leq g(x)]$. This later notation can be relativised moreover to a subset $A \subseteq \mathbb{N}$: $f \underline{\propto} g$ (A) means $\overset{\infty}{\forall}_x[x \in A \Rightarrow f(x) \leq g(x)]$.

$\mu z[P(z)]$ denotes "the least z such that P(z)".

We use a fixed recursive pairing function $\langle x,y \rangle$ with coordinate projections π_1 and π_2; $\pi_1 \langle x,y \rangle = x$; $\pi_2 \langle x,y \rangle = y$; $\langle \pi_1 x, \pi_2 x \rangle = x$. Moreover, $\langle x,y \rangle$ is increasing in both arguments and consequently $\langle 0,0 \rangle = 0$. ε (<u>zero</u>) denotes the function which is everywhere undefined (zero).

Using this pairing function we can interpret one-variable functions as being many-variable functions; an occasional superindex like for example in $\phi_j^2(x,y) = \phi_j(\langle x,y \rangle)$ indicates the use of this interpretation.

By $(\phi_i)_i$ we denote a fixed *Gödel numbering* of recursive functions [Ro 58]. The function $u(i,x) = \phi_i(x)$ is recursive and there exists a total recursive function s satisfying $\phi_i^2(x,y) = \phi_{s(i,x)}(y)$. Using the convention $\phi_i^2(x,y) = \phi_i(\langle x,y \rangle)$ many-variable functions are included in our enumeration. The functions ϕ_i are called *programs*.

We extend the enumeration $(\phi_i)_i$ to a *complexity measure* by means of a sequence of *step counting functions* $(\Phi_i)_i$; this sequence satisfies the two *Blum axioms* [Bl 67] : for each i, $\mathcal{D}\phi_i = \mathcal{D}\Phi_i$ and the relation $\Phi_i(x) = y$ is decidable. Again we write $\Phi_i^2(x,y)$ for $\Phi_i(\langle x,y\rangle)$.

A *transformation of programs* σ is a total recursive function operating on the indices of programs. In general such a transformation is defined implicitly using the S-n-m axiom and/or the recursion theorem by writing a formula like

$$\phi_{\sigma(i)}(x) \leftarrow P(i,x)$$

where P denotes some expression recursive in i and x.

A *measured set* is a sequence of functions $(\gamma_i)_i$ such that the relation $\gamma_i(x) = y$ is decidable. The sequence of run-times $(\Phi_i)_i$ is an example. A transformation τ such that $(\phi_{\tau(i)})_i$ is measured is called a *measured transformation of programs*.

The above concepts are extended for many-variable functions in a natural way.

For a (partial) function t we define:

$$F_t = \{\phi_i \mid \overset{\infty}{\forall}_x[x \in \mathcal{D}t \Rightarrow \Phi_i(x) \le t(x)]\}$$ (*complexity class of programs*)

$$C_t = \{f \mid \exists_i[f = \phi_i \text{ \underline{and} } \phi_i \in F_t]\}$$ (*complexity class of functions*).

C_t contains all functions computed by some program $\phi_i \in F_t$. Note that in our definitions both C_t and F_t contain partial functions even if t is total. The function t is called a *name* for C_t and F_t.

In the definition of complexity classes functions are discriminated not regarding the computed values. If we account for the results of the computations, this way allowing larger run-times for larger results, we arrive at the concept of a *honest set* of functions, called a honesty class hereafter. These classes have two-variable functions as names.

$$G_R = \{\phi_i \mid \overset{\infty}{\forall}_x[\phi_i(x) < \infty \ \underline{\text{and}}\ \langle x,\phi_i(x)\rangle \in \mathcal{D}R \Rightarrow \Phi_i(x) \le R(x,\phi_i(x))]\}$$

$$H_R = \{\ f \mid \exists_i[f = \phi_i\ \underline{\text{and}}\ \phi_i \in G_R]\}\ .$$

G_R (H_R) is called a *honesty class* of programs (functions). Note that the condition enforced in the definition of G_R holds vacuously if $\phi_i(x)$ diverges; consequently, each honesty class contains all functions with finite domain, whereas it can be shown that no complexity class but the trivial class $C_\varepsilon = \mathcal{P}$ contains any such function.

A further type of classes which we consider are the so-called *weak complexity classes* F_t^W and C_t^W. Their definition reads:

$$F_t^W = \{\phi_i \mid \overset{\infty}{\forall}_x[x \in \mathcal{D}t \ \Rightarrow\ (\Phi_i(x) \le t(x)\ \underline{\text{or}}\ \phi_i(x) = \infty)]\}$$

$$C_t^W = \{\ f \mid \exists_i[f = \phi_i\ \underline{\text{and}}\ \phi_i \in F_t^W]\}\ .$$

These classes are a special type of honesty classes since if we let $R(x,y) = t(x)$ then $F_t^W = G_R$ and $C_t^W = H_R$. Note that C_t^W and C_t contain the same total functions. The weak complexity classes are introduced since they behave like honesty classes without having the additional feature of a two dimensional name.

Another special type of honesty classes results from restricting to names R of the form $R(x,y) = r(\underline{\max}(x,y))$. These classes are called *modified honesty classes* and are denoted by $\hat{H}_r$ and $\hat{G}_r$.

We now are able to formulate the naming theorem of E.M. McCreight [Mc 69] and our main result:

[NAMING THEOREM]. *There exists a measured transformation of programs* σ *such that for each* i *the classes* F_{ϕ_i} *and* $F_{\phi_{\sigma(i)}}$ *are equal* (*and consequently* $C_{\phi_i} = C_{\phi_{\sigma(i)}}$ *also*).

THEOREM 1. For every measured transformation of programs σ there exists an index i such that $H_{\phi_i^2} \neq H_{\phi_{\sigma(i)}^2}$.

This result is strengthened furthermore by showing that i can be chosen to be the index of a total function. Moreover, for the modified complexity classes we can provide an index i such that $H^{\wedge}_{\Phi_i}$ and $H^{\wedge}_{\Phi_{\sigma(i)}}$ contain different total functions.

Before proving theorem 1 and other results in section 3 we investigate the definitions of the above types of classes to clarify why, given the analogy of their definitions, complexity classes and honesty classes behave differently.

2. Honesty classes and weak complexity classes

The concept of honesty classes has been considered frequently in the context of a relation between R-honest and measured sets of functions. A theorem by E.M. McCreight [Mc 69, see also MMo 72, MMo 73], which is frequently referred to, states that for total R the set of R-honest functions is recursively presented by a measured set; conversely each measured set is included in H_R for some total R.

This relation has led to the feeling that the concepts of measured sets and honest sets are more or less equivalent. This is certainly not the whole truth. For example, deleting the condition that R is total, the above equivalence is lost; the R-honest functions no longer are enumerated by a measured set [EB 74]. The same belief is responsible for the name "honesty procedure" given to the transformation σ which is involved in the naming theorem.

We have considered the honest sets to be a type of classes of recursive functions restricted in some way by the name of the class. Consequently, it is a reasonable question whether the known results for complexity classes remain valid for honesty classes also. Our results have shown that this is the case for the majority of the theorems, the naming theorem being a remarkable exception. However, looking at the proofs themselves the situation looks less trivial since many of the old proofs break down and must be modified essentially.

The difference can be explained by giving a mathematical meaning to the concept "restricted by the name of the class".

Let $(\gamma_i)_i$ be a measured set, and let t be a function. We define the sets $F_S(t)$ and $F_W(t)$ by:

$$F_S(t) = \{ i \mid \overset{\infty}{\forall}_x [x \in \mathcal{D}t \Rightarrow \gamma_i(x) \leq t(x)]\}$$

$$F_W(t) = \{ i \mid \overset{\infty}{\forall}_x [x \in \mathcal{D}t \Rightarrow \gamma_i(x) \leq t(x) \ \underline{\text{or}}\ \gamma_i(x) = \infty]\}.$$

The classes $F_S(t)$ ($F_W(t)$) are called *strongly (weakly) restricted classes*. If we take for $(\gamma_i)_i$ the sequence of run-times $(\Phi_i)_i$ then $F_S(t)$ ($F_W(t)$) precisely becomes the set of indices of programs in F_t (F_t^W). This shows that our model is an abstraction from the complexity classes. To present analogous interpretation of honesty classes we introduce the so-called *honesty run-times* Ψ_i defined by: $\Psi_i(x,y) = \underline{\text{if}}\ \phi_i(x) = y\ \underline{\text{then}}\ \Phi_i(x)\ \underline{\text{else}}\ \infty$. The sequence $(\Psi_i)_i$ is easily seen to be a measured set. Using this measured set the class G_R consists of all programs having indices in $F_W(R)$.

Our model shows that complexity classes are strong classes whereas honesty classes are weak. This explains why the classes behave distinct. For example, the naming theorem holds for strong classes in general, but becomes invalid for weak complexity classes (see theorem 4 in the next section).

We can also explain why the old proofs of theorems which remain valid break down for weak classes. Many of these proofs are based on the concept of a *violation*; i.e. a pair <i,x> such that $\gamma_i(x) > t(x)$. For weak classes a violation means $t(x) < \gamma_i(x) < \infty$; consequently, the collection of violations is no longer recursive but only recursively enumerable, a fact which forces us to reshape many of the known constructions.

As examples of theorems which can be generalized for both weak and strong classes we mention the union theorem [Mc 69] and the gap and operator-gap theorems [Bo 72, Co 72]. The generalizations can be found in [EB 74, 73].

3. The non-renameability proofs

The so-called honesty procedures σ which are constructed in the proof of the naming theorem are known to show irregularities of the following type: if ϕ_i is a program with an extrageously large run-time Φ_i then $\phi_{\sigma(i)}$ becomes a function with unreasonable large values at infinitely many arguments.

This phenomenon was first described by L.J. Bass [Ba 70, BY 73] for the original honesty procedure described in [Mc 69]. More recently R. Moll and A. Meyer [MMo 72] have shown that these irregularities occur for each measured transformation of programs which maps the set of functions with finite domain into itself, a property shared by all honesty procedures, since these functions are precisely the names of the class $C_\varepsilon = \mathcal{P}$. Their proof is a simple application of the recursion theorem.

These irregularities can be explained intuitively as follows. Consider a machine into which is fed the graph of the program ϕ_i and which computes the relation $\phi_{\sigma(i)}(x) = y$. If $\mathcal{D}\phi_i$ is finite, then the answer onto the question is $\phi_{\sigma(i)}(x) = y$ will be almost always negative. Moreover, since σ is measured these answers must be produced in a finite amount of time regardless of the speed at which the graph of ϕ_i is introduced into the machine. Consequently, if we make ϕ_i so expensive that the machine is lured into believing that we are feeding it a function with finite domain it will have decided that $\phi_{\sigma(i)}(x) \neq y$ for the small values of y before receiving a new input.

Essential in this argumentation and also in the formal proofs is the assumption that σ should work correctly both for total and partial ϕ_i. If this assumption is weakened in the sense that the honesty procedure may run astray on partial functions the irregularities may be suppressed; cf. [Mo 73].

Our negative results are based on an analogous irregularity which is described in the lemma below.

LEMMA 2 [Mirror lemma]. Let σ be a measured transformation of programs. Let t be a total function and let u be a partial function satisfying u > t. Then there exists a program ϕ_e such that

$$\forall_x[(\phi_e(x) = 0 \ \underline{\text{or}} \ \phi_e(x) = u(x)) \ \underline{\text{and}} \ (\phi_e(x) = 0 \Leftrightarrow \phi_{\sigma(e)}(x) > t(x))].$$

The program ϕ_e is "reflected in t" by σ. Moreover, since $\phi_{\sigma(e)}(x) \leq t(x)$ is decidable so is the relation $\phi_e(x) = 0$. Consequently, if u is the empty function, $\mathcal{D}\phi_e$ is recursive.

PROOF. Let the transformation τ be defined by

$$\phi_{\tau(i)}(x) \leftarrow \underline{\text{if}}\ \phi_{\sigma(i)}(x) > t(x)\ \underline{\text{then}}\ 0\ \underline{\text{else}}\ u(x).$$

Since σ is a measured transformation this transformation is well defined. By the recursion theorem there exists a program ϕ_e satisfying $\phi_e = \phi_{\tau(e)}$; this implies:

$$\phi_e(x) = \underline{\text{if}}\ \phi_{\sigma(e)}(x) > t(x)\ \underline{\text{then}}\ 0\ \underline{\text{else}}\ u(x).$$

This program has the properties claimed by the lemma. □

Our proofs need furthermore some diagonalization constructions. Several of these constructions are known in abstract complexity theory and we describe some of them below.

$$\phi_{\delta(i)}(x) \leftarrow \underline{\text{if}}\ \Phi_{\pi_1 x}(x) \leq \phi_i(x)\ \underline{\text{then}}\ 1 \dot{-} \phi_{\pi_1 x}(x)\ \underline{\text{else}}\ 0$$

yields for total ϕ_i a total function $\phi_{\delta(i)}$ which is not contained in C_{ϕ_i}. A modification of this construction yields for partial ϕ_i with infinite domain a function $\phi_{\delta(i)}$ with $\mathcal{D}\phi_{\delta(i)} = \mathcal{D}\phi_i$ and $\phi_{\delta(i)} \notin C_{\phi_i}$; cf. [EB 74].

In his proof of the strong compression theorem M. Blum [Bl 67] describes a "canceling procedure" yielding for total ϕ_i a 0-1 valued function $\phi_{\delta(i)}$ such that for each program $\phi_k = \phi_{\delta(i)}$ $\phi_i \underline{\propto} \Phi_k$. Again this construction can be adapted to partial ϕ_i.

The transformation

$$\phi_{\delta(i)}(x) \leftarrow \mu z[\forall_{k \leq x}[\Phi_k(x) > \phi_i(x)\ \underline{\text{or}}\ z \neq \phi_k(x)]]$$

yields a function $\phi_{\delta(i)}$ such that for $x \geq k$, $\phi_k(x) = \phi_{\delta(i)}(x)$ one has

$\Phi_k(x) > \phi_i(x)$; the individual values of $\phi_{\delta(i)}$ have become expensive.

A fourth method to construct expensive functions is shown below; by deleting a suitable set of values the programs for the remaining values can be made arbitrarily expensive:

$$\phi_{\delta(i,j)}(x) \leftarrow \underline{\text{if}}\ \Phi_{\pi_1 x} \leq \phi_i(x)\ \underline{\text{then}}\ \infty\ \underline{\text{else}}\ \phi_j(x).$$

If ϕ_i and ϕ_j are total, then $\phi_{\delta(i,j)} \notin C^W_{\phi_i}$ and $\phi_{\delta(i,j)} \sqsubseteq \phi_j$. The last construction can be adapted to partial functions ϕ_i and ϕ_j as well.

LEMMA 3. Suppose that ϕ_k is a total 1-1 function such that for each x, $\phi_k(x) \in \mathcal{D}\phi_i \cap \mathcal{D}\phi_j$. Let $z(k,x) = \mu t[\phi_k(t) = x]$ and define

$$\phi_{\delta(i,j,k)}(x) \leftarrow \underline{\text{if}}\ \Phi_{\pi_1 z(k,x)}(x) \leq \phi_i(x)\ \underline{\text{then}}\ \infty\ \underline{\text{else}}\ \phi_j(x).$$

Then $$\mathcal{D}\phi_{\delta(i,j,k)} \subseteq \mathcal{R}\phi_k \subseteq \mathcal{D}\phi_i \cap \mathcal{D}\phi_j ,$$

$$\phi_{\delta(i,j,k)} \sqsubseteq \phi_j \text{ and } \phi_{\delta(i,j,k)} \notin C\phi_i$$

The proof is left to the reader. Note that for given i and j the set $\mathcal{D}\phi_i \cap \mathcal{D}\phi_j$ is recursively enumerable. There exists moreover a total function κ such that $k = \kappa(i,j)$ is an index of a program ϕ_k satisfying the assumptions of the lemma provided $\mathcal{D}\phi_i \cap \mathcal{D}\phi_j$ is infinite.

We now turn to our non-renameability results. The first theorem treats the weak complexity classes.

THEOREM 4. Let σ be a measured transformation of programs. Then there exists an index e such that $C^W_{\phi_e} \neq C^W_{\phi_{\sigma(e)}}$.

PROOF. Let t be a total function such that $C^W_t \supsetneq C^W_{\underline{zero}}$. Take $u = \varepsilon$. Application of the mirror lemma yields an index e such that

$$\phi_e(x) = \underline{\text{if}}\ \phi_{\sigma(e)}(x) > t(x)\ \underline{\text{then}}\ 0\ \underline{\text{else}}\ \infty .$$

We claim that $C^W_{\phi_e} \neq C^W_{\phi_{\sigma(e)}}$. We consider three cases:

(1) $\mathcal{D}\phi_e$ is finite. Now $C^W_{\phi_e} = C^W_{\varepsilon} = \mathcal{P}$, whereas $C^W_{\phi_{\sigma(e)}} \subseteq C^W_t \neq \mathcal{P}$.

(2) $\mathbb{N} \setminus \mathcal{D}\phi_e$ is finite. In this case $C^W_{\phi_e} = C^W_{\underline{zero}}$, whereas $C^W_{\underline{zero}} \subsetneqq C^W_t \subseteq C^W_{\phi_{\sigma(e)}}$. Again the classes are distinct.

(3) Both $\mathcal{D}\phi_e$ and $\mathbb{N} \setminus \mathcal{D}\phi_e$ are infinite. Since $\mathcal{D}\phi_e$ is recursive, there exists a total 1-1 ϕ_k such that $\mathcal{R}\phi_k = \mathbb{N} \setminus \mathcal{D}\phi_e$. Let i be an index for t and let ϕ_j be some arbitrary total function. Consider the function $f = \phi_{\delta(i,j,k)}$ as given by lemma 3. Since $\mathcal{D}f \subseteq \mathbb{N} \setminus \mathcal{D}\phi_e$ one has trivially $f \in C^W_{\phi_e}$; at the same time for $x \in \mathcal{D}f$ one has $t(x) \geq \phi_{\sigma(e)}(x)$. Consequently $f \notin C^W_{\phi_i} = C^W_t$ implies $f \notin C^W_{\phi_{\sigma(e)}}$. This proves $C^W_{\phi_e} \neq C^W_{\phi_{\sigma(e)}}$. □

The proof of theorem 1 is analogous, but the diagonalization construction is more complicated.

THEOREM 1. Let σ be a measured transformation operating on two-argument programs. Then there exists an index e such that $H_{\phi^2_e} \neq H_{\phi^2_{\sigma(e)}}$.

PROOF. Let R be total such that $H_R \supsetneqq H_{\underline{zero2}}$ ($\underline{zero2}$ denoting the function with constant value 0). Take $u = \varepsilon 2$ (the everywhere undefined function). The two-dimensional analogue of the mirror lemma yields an index e such that

$$\phi^2_e(x,y) = \underline{if}\ \phi^2_{\sigma(e)}(x,y) \geq R(x,y)\ \underline{then}\ 0\ \underline{else}\ \infty.$$

Let $S = \phi^2_e$, $S' = \phi^2_{\sigma(e)}$. We consider two cases:

(1) $\overset{\infty}{\forall}_x \forall_y\ [S(x,y) = 0]$. This leads to $H_S = H_{\underline{zero2}} \subsetneqq H_R \subseteq H_{S'}$.

(2) $\overset{\infty}{\exists}_x \exists_y\ [S(x,y) = \infty]$.

Since $\mathcal{D}S$ is recursive we can enumerate $\mathbb{N}^2 \setminus \mathcal{D}S$. Consequently, it is possible to enumerate a subset of $\mathbb{N}^2 \setminus \mathcal{D}S$ by a total function ϕ_m such that

(i) $\pi_1 \phi_m(x)$ is increasing in x,

(ii) $\phi_m(x) = \langle\pi_1\phi_m(x),\pi_2\phi_m(x)\rangle \notin \mathcal{D}S$.

Next we define the programs ϕ_i, ϕ_j and ϕ_k and a function z.
The function ϕ_m enumerates the graph of some partial function, which equals the function computed by the program ϕ_j; its domain is enumerated by ϕ_k, and z is an "inverse" of ϕ_k.

$$\phi_k(x) = \pi_1 \phi_m(x) ;$$

$$z(x) = \mu w [\phi_k(w) = x];$$

$$\phi_j(x) = \pi_2 \phi_m(z(x)) ;$$

finally $\phi_i(x) = R(\pi_1\phi_m(z(x)), \pi_2\phi_m(z(x))) = R(x,\phi_j(x))$.

Let $f = \phi_{\delta(i,j,k)}$. Since $f \sqsubseteq \phi_j$ we know that for each $x \in \mathcal{D}f$ one has $S(x,f(x)) = \infty$; consequently, f is a member of H_S.

At the same time we have by construction $f \notin C^W_{\phi_i}$. Hence for each index n of f we have

$$\overset{\infty}{\exists}_x[\phi_i(x) < \Phi_n(x) < \infty].$$

Since for these arguments x we have $\phi_i(x) = R(x,f(x)) \geq S'(x,f(x))$ this proves $f \notin H_{S'}$.

This shows that the classes H_S and $H_{S'}$ are distinct.

4. Further results and open problems

The results in the preceding section are based on the use of partial functions. It is a reasonable question to ask whether partial functions are essential. More particularly one may ask the following questions:

(1) Is it essential that σ is a honesty procedure on $\mathcal{P}$? *)

(2) Do the negative results remain valid if only the total functions in the classes are considered? *)

(3) Does there exist a non-uniform renaming procedure, i.e. does there exist a measured set containing names for all honesty classes?

The first question can be settled completely. We can "uniformize" our proofs in such way that the negative results extend to total honesty procedures as well.

The second question makes no sense for weak complexity classes since $C_t^W \cap \mathcal{R} = C_t \cap \mathcal{R}$. Consequently, the naming theorem itself yields a positive result for the classes $C_t^W \cap R$. Although we have no answer to this problem for general honesty classes we can prove that for the modified honesty classes the negative result remains valid: the classes $H_r^\wedge$ can not be renamed uniformly by a measured set of names.

The third problem is still unsolved.

The results on total honesty procedures are based on a uniformized version of the mirror lemma:

LEMMA 2'. Let σ be a measured transformation of programs; let t be total. Then there exists a transformation ρ such that

$$\phi_{\rho(j)}(x) = \underline{\text{if}}\ \phi_{\sigma(\rho(j))}(x) \leq t(x)\ \underline{\text{then}}\ t(x) + \phi_j(x) + 1\ \underline{\text{else}}\ 0.$$

Moreover, the relation $\phi_{\rho(j)}(x) \neq 0$ is recursive in j and x.

THEOREM 5. Let σ be a measured transformation of programs. Then there exists an index e of a total function such that $C^W_{\Phi_e} \neq C^W_{\Phi_{\sigma(e)}}$.

PROOF. Take a total function t such that $C^W_{\underline{zero}} \subsetneq C^W_t$. Let ρ be the transformation from lemma 2'. Since $\phi_{\rho(j)}(x) \neq 0$ is decidable there exists a transformation κ such that

$$\phi_{\kappa(j)}(x) = \begin{cases} n \text{ if } x \text{ is the n-th element such that } \phi_{\rho(j)}(x) \neq 0\ , \\ \infty \text{ otherwise}\ . \end{cases}$$

*) Questions (1) and (2) were suggested by A. Meyer.

Define the transformation τ by

$$\phi_{\tau(j)}(x) = \underline{\text{if}}\ \phi_{\rho(j)}(x) = 0\ \underline{\text{then}}\ \infty$$

$$\underline{\text{else}}\ \underline{\text{if}}\ \Phi_{\pi_1\phi_{\kappa(j)}(x)}(x) \leq t(x)\ \underline{\text{then}}\ 1 \dot{-} \phi_{\pi_1\phi_{\kappa(j)}(x)}(x)\ \underline{\text{else}}\ 0.$$

Since $\mathcal{D}\phi_{\tau(j)} = \{x | \phi_{\rho(j)}(x) \neq 0\}$ is recursive the following function m is total:

$$m(x) = \max_{j \leq x} \{\Phi_{\tau(j)}(x) \mid x \in \mathcal{D}\phi_{\tau(j)}\}.$$

The definition of τ ensures that whenever $\mathcal{D}\phi_{\tau(j)}$ is infinite $\phi_{\tau(j)} \notin C_t^W$. By the definition of $m, \phi_{\tau(j)} \in C_m^W$ for each j. Let j_0 be an index of m and let $e = \rho(j_0)$. If we take $u = \phi_e$, $u' = \phi_{\sigma(e)}$ then we have

$$u(x) = \underline{\text{if}}\ u'(x) \leq t(x)\ \underline{\text{then}}\ t(x) + m(x) + 1\ \underline{\text{else}}\ 0$$

and

$$\mathcal{D}\phi_{\tau(j_0)} = \{x | u'(x) \leq t(x)\}.$$

Since m is total so is u. If $u(x) = 0$ almost everywhere, then $u'(x) > t(x)$ almost everywhere and then

$$C_u^W = C_{zero}^W \subsetneqq C_t^W \subseteq C_{u'}^W .$$

If $u(x) \neq 0$ infinitely often then $\phi_{\tau(j_0)} \notin C_t^W$ and hence $\phi_{\tau(j_0)} \notin C_{u'}^W$, whereas $\phi_{\tau(j_0)} \in C_u^W$ by construction.

This proves that $C_{\phi_e}^W \neq C_{\phi_{\sigma(e)}}^W$. □

THEOREM 6. Let σ be a measured transformation of programs. Then there exists an index e of a total function such that $H_{\phi_e^2} = H_{\phi_{\sigma(e)}^2}$.

PROOF. The proof combines ideas from the proofs of theorems 1 and 5. Let $\underline{H_{zero2}} \subsetneq H_R$, R total. By lemma 2' there exists ρ satisfying:

$$\phi^2_{\rho(j)}(x,y) = \underline{\text{if}}\ \phi^2_{\sigma(\rho(j))}(x,y) \le R(x,y)\ \underline{\text{then}}\ R(x,y) + \phi^2_j(x,y) + 1\ \underline{\text{else}}\ 0.$$

Hence $\phi^2_{\rho(j)}(x,y) = 0$ is recursive. From this we derive the existence of transformations κ and δ' such that

$\phi_{\kappa(j)}(n) = \langle x_n, y_n \rangle$ where x_n is increasing in n and

$\phi_{\rho(j)}(x_n,y_n) \neq 0$.

$\mathcal{D}\phi_{\delta'(j)} \subseteq \pi_1\ \mathcal{R}\phi_{\kappa(j)}$ and

if $x = x_n$ and $\phi_{\delta'(j)}(x) < \infty$ then $\phi_{\delta'(j)}(x) = y_n$.

Careful inspection of the construction form the proof of theorem 1 learns that it is decidable whether $\langle x,y \rangle \in \mathcal{R}\phi_{\kappa(j)}$ and whether $y = \phi_{\delta'(j)}(x)$. This means that $(\phi_{\delta'(j)})_j$ is a measured set. Consequently $(\phi_{\delta'(j)})_j \subseteq H_S$ for some total S, by the equivalence of measured and honest sets. Let $S = \phi^2_{j_0}$. By the construction $\phi_{\delta'(j_0)} \in H_{\phi^2_{\rho(j_0)}} \setminus H_{\phi^2_{\sigma(\rho(j_0))}}$ whenever $\mathcal{R}\phi_{\kappa(j_0)}$ is infinite.

Moreover, the case that $\mathcal{R}\phi_{\kappa(j_0)}$ is finite leads to the inequality $H_{\phi^2_{\rho(j_0)}} = \underline{H_{zero2}} \subsetneq H_R \subseteq H_{\phi_{\sigma(\rho(j_0))}}$. Hence $e = \rho(j_0)$ satisfies the condition of the theorem. □

Theorems 5 and 6 yield a satisfying answer to question (1): there exist no total honesty procedures for weak complexity classes or honesty classes. We next consider the second question.

All our diagonalization procedures used up to this point constructed expensive functions by deleting values from some given function. This way we produce partial diagonalization functions. If we want a total function we must provide also finite "escape values". Our aim was to define $\phi_{\delta(j)}$ in such a way that $\phi_{\delta(j)}(x) = y$ only when $\phi_{\rho(j)}(x,y)$ was large (and consequently $\phi_{\sigma(\rho(j))}(x,y)$ was small). Therefore we need for each x at least two values of y such that $\phi_{\rho(j)}(x,y) \neq 0$.

Up to now we are unable to solve this difficulty for ordinary honesty classes. Using modified honesty classes the problem however disappears; if $r(x) \neq 0$ for infinitely many x and if $R(x,y) = r(\underline{\max}(x,y))$ then $R(x,y) \neq 0$ whenever $x \leq y$ and $r(y) \neq 0$.

THEOREM 7. For each measured transformation σ there exists an index e (of a total function) such that $\hat{H}_{\phi_e} \cap \mathcal{R} \neq \hat{H}_{\phi_{\sigma(e)}} \cap \mathcal{R}$.

PROOF. We describe the diagonalization procedure for the case that ϕ_e is obtained by application of the mirror lemma using $u = \varepsilon$, leaving the modification yielding a total ϕ_e to the reader.

Suppose $\hat{H}_{\underline{zero}} \neq \hat{H}_t$ and let ϕ_e satisfy the relation

$$\phi_e(x) = \underline{\text{if}}\ \phi_{\sigma(e)}(x) \leq t(x)\ \underline{\text{then}}\ \infty\ \underline{\text{else}}\ 0$$

Let $R(x,y) = \phi_e(\max(x,y))$, $R'(x,y) = \phi_{\sigma(e)}(\underline{\max}(x,y))$. The case that $\mathcal{D}\phi_e$ is cofinite leads to the inclusions $\hat{H}_{\phi_e} = \hat{H}_{\underline{zero}} \subsetneq \hat{H}_t \subseteq \hat{H}_{\phi_{\sigma(e)}}$.

Otherwise there exist infinitely many x such that $\phi_e(x) = \infty$; these x form a recursive set. Let

$$y_1(x) = \mu z[z \geq x\ \underline{\text{and}}\ \phi_e(z) = \infty]\ \text{and}$$

$$y_2(x) = \mu z[z > y_1(x)\ \underline{\text{and}}\ \phi_e(z) = \infty].$$

Then y_1 and y_2 are total and for each x, $R(x,y_1(x)) = R(x,y_2(x)) = \infty$. Define f by

$$f(x) = \underline{\text{if}}\ \Phi_{\pi_1 x}(x) \leq t(y_1(x))\ \underline{\text{and}}\ \phi_{\pi_1 x}(x) = y_1(x)$$
$$\underline{\text{then}}\ y_2(x)\ \underline{\text{else}}\ y_1(x).$$

Then as before $f \notin H_{R'}$, and $f \in H_R$; moreover, f is total. This proves that $\hat{H}_{\phi_e} \cap \mathcal{R} \neq \hat{H}_{\phi_{\sigma(e)}} \cap \mathcal{R}$. □

The third question on the existence of non-uniform honesty procedures remains unsolved. An idea which is used as a short cut in the proof of theorem 6 suggests a way to approach the solution. Let $(R_i)_i$ be a measured set of names of honesty classes. Then there exists a transformation δ such that $\phi_{\delta(i)} \notin H_{R_i}$ (unless $H_{R_i} = P$). It is not very difficult to construct such a transformation.

However, if we are able to define δ in such a way that δ becomes a measured transformation of programs then we are done. In this case $(\phi_{\delta(i)})_i$ is a measured set which is contained in H_R for some total R. This honesty class H_R clearly has no name in the sequence $(R_i)_i$. We therefore specialize our third problem to:

UNSOLVED PROBLEM. Let $(R_i)_i$ be a measured sequence of functions in two variables. Does there exist a measured transformation δ such that, for $H_{R_i} \neq P$, $\phi_{\delta(i)} \notin H_{R_i}$?

One may weaken the condition by asking $\phi_{\delta(i)} \notin H_{R_i}$ only for total functions R_i. A positive answer to this question should show the non-existence of a measured set containing total names for all honesty classes named by a total function.

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