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On the frequency of natural numbers m whose prime divisors are all smaller than m^{α}

by

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Introduction

In this report we give a survey of the number theoretical problem of determining the frequency of those natural numbers m whose prime divisors are all smaller than m^α, where α is a fixed real number.

Let g(m) be the largest prime divisor of the natural number m and let $G(n,\alpha)$ be the number of natural numbers m with the properties $m \leq n$ and $g(m) < m^\alpha$. The above mentioned frequency will then be defined by

$$G(\alpha) = \lim_{n\to\infty} \frac{G(n,\alpha)}{n} \quad .$$

It will be shown that $G(\alpha)$ exists and is continuous for all $\alpha \in \mathbb{R}$ and satisfies

$$\begin{cases} G(\alpha) = 0 & (\alpha \leq 0) \\ G(\alpha) = 1 & (\alpha \geq 1) \\ G'(\alpha) = \frac{1}{\alpha} G(\frac{\alpha}{1-\alpha}) & (0 < \alpha < 1). \end{cases}$$

In order to study $G(\alpha)$ in more detail it is convenient to consider the function H(x) defined by

$$\begin{cases} H(x) = 1 & \text{if} \quad 0 \leq x \leq 1 \\ H(x) = G(\frac{1}{x}) & \text{if} \quad x > 1. \end{cases}$$

It is easily verified that H(x) is continuous on $x \geq 0$ and satisfies the equation

$$H'(x) = -\frac{1}{x} H(x-1) \qquad (x > 1).$$

It will be shown that the Laplace transform of H(x)

$$h(s) = \int_0^\infty e^{-sx} H(x)dx$$

is absolutely convergent for each complex number s, which implies that h(s) is an entire function. By virtue of a well-known theorem in the theory of Laplace transforms we have

$$H(x) = \frac{1}{2\pi i} \lim_{A\to\infty} \int_{\sigma-Ai}^{\sigma+Ai} e^{sx}\, h(s)ds.$$

From the equation

$$H'(x) = -\frac{1}{x} H(x-1) \qquad (x > 1)$$

we will deduce that h(s) satisfies the differential equation

$$h'(s) = h(s) \cdot \frac{e^{-s} - 1}{s}.$$

From this it follows that

$$h(s) = c_0 \cdot \exp\left(\int_0^s \frac{e^{-z} - 1}{z}\, dz\right),$$

where

$$c_0 = \int_0^\infty H(x)dx = e^\gamma$$

(γ is Euler's constant).

Hence we may write

$$H(x) = \frac{c_0}{2\pi i} \lim_{A\to\infty} \int_{\sigma-Ai}^{\sigma+Ai} \exp\left(sx + \int_0^s \frac{e^{-z}-1}{z}\, dz\right)ds.$$

By means of this relation we will prove that H(x) is tending very rapidly to 0 as x tends to infinity. More precisely, if we write

$$H(x) = \frac{e^x \cdot \gamma(x)}{(x \log x)^x} \qquad (x > 1),$$

then we have

$$\limsup_{x\to\infty} \gamma(x) \le 1.$$

At the end of the report we will indicate a method to compute H(x) numerically.

For a more detailed description of the numerical computation of H(x) we refer to [16].

The numerical computations were carried out by means of the Electrologica X8 of the Mathematical Centre.

Finally we wish to express our thanks to Prof.dr. A. van Wijngaarden for his useful suggestions concerning the numerical computation of H(x).

1. In this section α will always be a fixed number in the interval $(0,1)$.

Definition 1.1.

$g(m)$ and $G(n,\alpha)$ will be defined as in the introduction whereas $g(1) = 1$.
(1.1)

Definition 1.2.

$$D(x,t) = \text{card}\{m \mid m \leq x,\ g(m) \leq t\}. \tag{1.2}$$

From the last definition it follows immediately that

$$D(x,t) \leq x \text{ for each } x \geq 0 \text{ and each } t.$$

Definition 1.3.

The total number of natural numbers m satisfying $m \leq n$ and $g(m) \geq m^{\alpha}$, will be denoted by $S(n,\alpha)$.

Obviously, $S(n,\alpha)$ is the number of solutions m of the system of inequalities

$$\begin{cases} m \leq n \\ m^{\alpha} \leq g(m). \end{cases}$$

From this it is easily seen that $S(n,\alpha)$ may also be defined as the number of solutions $(\lambda_1, \lambda_2, \ldots, \lambda_v)$ of the system of diophantine inequalities

$$\begin{cases} 2^{\lambda_1} \cdot 3^{\lambda_2} \ldots p_v^{\lambda_v} \leq n \\ (2^{\lambda_1} \cdot 3^{\lambda_2} \ldots p_v^{\lambda_v})^{\alpha} \leq p_v \\ \lambda_v \geq 1. \end{cases} \tag{1.3}$$

Lemma 1.1.

$$G(n,\alpha) = n - S(n,\alpha). \tag{1.4}$$

Proof: This is an immediate consequence of definitions 1.1 and 1.3.

It is easily seen that system (1.3) is equivalent to the following one

$$\begin{cases} 2^{\lambda_1} \cdot 3^{\lambda_1} \cdot \ldots p_v^{\lambda_v} \leq n \\ 2^{\lambda_1} \cdot 3^{\lambda_1} \cdot \ldots p_v^{\lambda_v} \leq p_v^{\frac{1}{\alpha}} \\ \lambda_v \geq 1. \end{cases} \tag{1.5}$$

It is clear that the index v can only take the values 1, 2, 3, ..., $\pi(n)$, and that λ_v can only take the values 1, 2, 3, ..., $[\frac{1}{\alpha}]$. Furthermore, it is easily verified that the number of solutions of (1.5) with $\lambda_v = i$ and $p_v \leq n^{\alpha}$ equals

$$\min\{[\tfrac{1}{\alpha}] - i,\ [{}^2\log n]\} + \sum_{3 \leq p_v \leq n^{\alpha}} D(p_v^{\frac{1}{\alpha} - i}, p_{v-1}) \tag{1.6}$$

and that the number of solutions with $\lambda_v = i$ and $p_v > n^{\alpha}$ is equal to

$$\sum_{n^{\alpha} < p_v \leq n} D(\frac{n}{p_v^i}, p_{v-1}). \tag{1.7}$$

Thus we have

<u>Lemma 1.2.</u>
The total number of solutions of (1.5) is

$$S(n,\alpha) = \sum_{i=1}^{[\frac{1}{\alpha}]} \{\min([\tfrac{1}{\alpha}] - i,\ [{}^2\log n]) + $$

$$+ \sum_{3 \leq p_v \leq n^{\alpha}} D(p_v^{\frac{1}{\alpha} - i}, p_{v-1})$$

$$+ \sum_{n^{\alpha} < p_v \leq n} D(\frac{n}{p_v^i}, p_{v-1})\}. \tag{1.8}$$

Lemma 1.3.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} \min([\tfrac{1}{\alpha}] - i,\ [{}^2\log n]) = 0. \tag{1.9}$$

Proof: $0 \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} \min([\frac{1}{\alpha}] - i,\ [{}^2\log n]) \leq$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} ([\tfrac{1}{\alpha}] - i)$$

$$\leq \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} [\tfrac{1}{\alpha}] = 0.$$

Lemma 1.4.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} \sum_{3\leq p_v\leq n^\alpha} D(p_v^{\frac{1}{\alpha}-i}, p_{v-1}) = 0. \tag{1.10}$$

Proof: $0 \leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} \sum_{3\leq p_v\leq n^\alpha} D(p_v^{\frac{1}{\alpha}-i}, p_{v-1}) \leq$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} \sum_{3\leq p_v\leq n^\alpha} p_v^{\frac{1}{\alpha}-i} \leq$$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} \pi(n^\alpha) \cdot (n^\alpha)^{\frac{1}{\alpha}-i} \leq$$

$$\leq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=1}^{[\frac{1}{\alpha}]} \pi(n^\alpha) \cdot n^{1-\alpha} =$$

$$= \limsup_{n\to\infty} \frac{\pi(n^\alpha)}{n^\alpha} \cdot [\tfrac{1}{\alpha}] = 0.$$

Lemma 1.5.

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=2}^{[\frac{1}{\alpha}]} \sum_{n^\alpha < p_v \le n} D(\frac{n}{i p_v}, p_{v-1}) = 0. \qquad (1.11)$$

Proof: $0 \le \limsup_{n\to\infty} \frac{1}{n} \sum_{i=2}^{[\frac{1}{\alpha}]} \sum_{n^\alpha < p_v \le n} D(\frac{n}{i p_v}, p_{v-1}) \le$

$$\le \limsup_{n\to\infty} \frac{1}{n} \sum_{i=2}^{[\frac{1}{\alpha}]} \sum_{n^\alpha < p_v \le n} \frac{n}{i p_v} \le$$

$$\le \limsup_{n\to\infty} \sum_{i=2}^{[\frac{1}{\alpha}]} \sum_{n^\alpha < p_v \le n} \frac{1}{p_v^2} =$$

$$= \limsup_{n\to\infty} ([\frac{1}{\alpha}] - 1) \sum_{n^\alpha < p_v \le n} \frac{1}{p_v^2} \le$$

$$\le [\frac{1}{\alpha}] \limsup_{n\to\infty} \sum_{k=[n^\alpha]}^{\infty} \frac{1}{k^2} = 0.$$

Lemma 1.6.

$$G(\alpha) = 1 - \lim_{n\to\infty} \frac{S(n,\alpha)}{n} = 1 - \lim_{n\to\infty} \frac{1}{n} \sum_{n^\alpha < p_v \le n} D(\frac{n}{p_v}, p_{v-1}) \qquad (1.12)$$

if these limits exist.

Proof: This is a simple consequence of lemmas 1.1, 1.2, 1.3, 1.4 and 1.5.

Since the number of natural numbers $m \le x$ with the property $g(m) \le p_{v-1}$ is equal to (c.f. Landau [15], §14)

$$D(x,p_{v-1}) = [x] - \sum_{p \geq p_v} \{ [\frac{x}{p}] - \sum_{p<q} [\frac{x}{pq}] + \sum_{q<r} [\frac{x}{pqr}] - + \ldots \}, \quad (1.13)$$

we have by the previous lemma

Lemma 1.7.

$$G(\alpha) = 1 - \lim_{n\to\infty} \frac{1}{n} \sum_{n^{\alpha}<p_v\leq n} \{ [\frac{n}{p_v}] - \sum_{p\geq p_v} ([\frac{n}{pp_v}] - \sum_{p<q} [\frac{n}{pqp_v}] + - \ldots) \}, \quad (1.14)$$

if this limit exists.

In section 3 we will study (1.14) in more detail; in the next section we will prove that $G(\alpha) = 1 + \log \alpha$ if $\frac{1}{2} \leq \alpha < 1$.

2. In this section α will be a fixed number in the interval $[\frac{1}{2}, 1)$.

Since $p_v > n^\alpha$, $p \geq p_v$ and $\alpha \geq \frac{1}{2}$ imply $pp_v > n$, we have

$$\sum_{n^\alpha < p_v \leq n} \sum_{p \geq p_v} \{ [\frac{n}{pp_v}] - \sum_{p<q} [\frac{n}{pqp_v}] + - \ldots \} = 0. \qquad (1.15)$$

Hence
$$G(\alpha) = 1 - \lim_{n\to\infty} \frac{1}{n} \sum_{n^\alpha < p \leq n} [\frac{n}{p}]. \qquad (1.16)$$

If we replace

$$\frac{1}{n} \sum_{n^\alpha < p \leq n} [\frac{n}{p}] \qquad (1.17)$$

by

$$\sum_{n^\alpha < p \leq n} \frac{1}{p} \qquad (1.18)$$

the error will be less than or equal to

$$\frac{1}{n} \sum_{n^\alpha < p \leq n} 1 = \frac{\pi(n) - \pi(n^\alpha)}{n} = o(1) \quad (n \to \infty). \qquad (1.19)$$

Hence, we may write

$$G(\alpha) = 1 - \lim_{n\to\infty} \sum_{n^\alpha < p \leq n} \frac{1}{p}. \qquad (1.20)$$

Since (c.f. Landau [15],p. 201)

$$\sum_{p \leq x} \frac{1}{p} = \log \log x + B + \frac{V(x)}{\log x} \qquad (1.21)$$

where B is a constant and $V(x)$ is a bounded function on $x \geq 2$, it follows that

$$\sum_{n^\alpha < p \leq n} \frac{1}{p} = (\log \log n + B + \frac{V(n)}{\log n}) +$$

$$- (\log \log n^\alpha + B + \frac{V(n^\alpha)}{\alpha \log n}) = \log \frac{1}{\alpha} + O(\frac{1}{\log n}). \qquad (1.22)$$

As an immediate consequence of (1.20) and (1.22) we have

Theorem 2.1.

$$G(\alpha) = 1 + \log \alpha \qquad (\tfrac{1}{2} \leq \alpha < 1).$$

3. In this section α will always be a fixed number in the interval $\left[\frac{1}{k+1}, \frac{1}{k}\right]$.

An immediate consequence of this restriction is that we may write (compare (1.13))

$$\frac{1}{n}\sum_{n^\alpha<p_v\le n} D\left(\frac{n}{p_v}, p_{v-1}\right) = \frac{1}{n}\sum_{n^\alpha<p_v\le n}\left[\frac{n}{p_v}\right] +$$

$$-\frac{1}{n}\sum_{n^\alpha<p_v\le n}\sum_{p\ge p_v}\left[\frac{n}{pp_v}\right] + \frac{1}{n}\sum_{n^\alpha<p_v\le n}\sum_{p\ge p_v}\sum_{p<q}\left[\frac{n}{pqp_v}\right] +$$

$$+ \dots + (-1)^{k+1}\frac{1}{n}\sum_{n^\alpha<p_v\le n}\sum_{p\ge p_v}\sum_{p<q}\dots\sum_{r<s}\left[\frac{n}{pq\dots rsp_v}\right]. \quad (3.1)$$

From this it is clear that, in order to obtain $G(\alpha)$, it is sufficient to study the behaviour of

$$\frac{1}{n}\sum_{n^\alpha<p_v\le n}\sum_{p\ge p_v}\sum_{p<q}\dots\sum_{r<s}\left[\frac{n}{pq\dots rsp_v}\right] \quad (3.2)$$

as n tends to infinity.

It is easily seen that (3.2) is equal to

$$\frac{1}{n}\sum_{n^\alpha<p_1\le\sqrt[k]{n}}\;\sum_{p_1\le p_2\le\sqrt[k-1]{\frac{n}{p_1}}}\;\sum_{p_2<p_3\le\sqrt[k-2]{\frac{n}{p_1p_2}}}\dots$$

$$\dots\sum_{p_{k-1}<p_k\le\frac{n}{p_1p_2\dots p_{k-1}}}\left[\frac{n}{p_1p_2\dots p_k}\right]. \quad (3.3)$$

If we omit in (3.3) the brackets [ ], the error will be not larger than

$$\frac{1}{n}\sum_{n^\alpha<p_1\le\sqrt[k]{n}}\;\sum_{p_1\le p_2\le\sqrt[k-1]{\frac{n}{p_1}}}\dots\sum_{p_{k-1}<p_k\le\frac{n}{p_1p_2\dots p_k}} 1 \le$$

$$\le\frac{1}{n}\sum_{n^\alpha<p_1\le\sqrt[k]{n}}\dots\sum_{p_{k-2}<p_{k-1}\le\sqrt{\frac{n}{p_1p_2\dots p_{k-2}}}}\pi\left(\frac{n}{p_1p_2\dots p_{k-1}}\right) \le$$

$$\leq \frac{1}{n} \sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \cdots \sum_{p_{k-2}<p_{k-1}\leq\sqrt{\frac{n}{p_1p_2\cdots p_{k-2}}}} \frac{M\cdot\frac{n}{p_1p_2\cdots p_{k-1}}}{\log\frac{n}{p_1p_2\cdots p_{k-1}}} \leq$$ [1)]

$$\leq \frac{M}{\alpha\log n} \sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \cdots \sum_{p_{k-2}<p_{k-1}\leq\sqrt{\frac{n}{p_1p_2\cdots p_{k-2}}}} \frac{1}{p_1p_2\cdots p_{k-1}}$$

(by virtue of $\frac{n}{p_1p_2\cdots p_{k-1}} \geq p_k > p_{k-1} > \cdots > p_1 > n^{\alpha}$)

$$\leq \frac{M}{\alpha\log n} \sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \ \sum_{n^{\alpha}<p_2\leq \sqrt[k-1]{n^{1-\alpha}}} \cdots \sum_{n^{\alpha}<p_{k-1}\leq\sqrt{n^{1-(k-2)\alpha}}} \frac{1}{p_1p_2\cdots p_{k-1}} =$$

$$= \frac{M}{\alpha\log n}\Big(\sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \frac{1}{p_1}\Big)\Big(\sum_{n^{\alpha}<p_2\leq \sqrt[k-1]{n^{1-\alpha}}} \frac{1}{p_2}\Big)\cdots\Big(\sum_{n^{\alpha}<p_{k-1}\leq\sqrt{n^{1-(k-2)\alpha}}} \frac{1}{p_{k-1}}\Big)$$

$$= \frac{M}{\alpha\log n}\Big(\log\frac{1}{k\alpha} + O\big(\frac{1}{\log n}\big)\Big)\cdots\Big(\log\frac{1-(k-2)\alpha}{2\alpha} + O\big(\frac{1}{\log n}\big)\Big) =$$

$$= \text{(compare (1.21))}\ O\Big(\frac{1}{\log n}\Big)\ (n\to\infty). \qquad (3.4)$$

Hence we can confine ourselves to the study of

$$\frac{1}{n} \sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \ \sum_{p_1\leq p_2\leq \sqrt[k-1]{\frac{n}{p_1}}} \cdots \sum_{p_{k-1}<p_k\leq\frac{n}{p_1p_2\cdots p_{k-1}}} \frac{1}{p_1p_2\cdots p_k} =$$

$$= \sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \frac{1}{p_1} \sum_{p_1\leq p_2\leq \sqrt[k-1]{\frac{n}{p_1}}} \frac{1}{p_2} \cdots \sum_{p_{k-1}<p_k\leq\frac{n}{p_1p_2\cdots p_{k-1}}} \frac{1}{p_k} \qquad (3.5)$$

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1) Here we make use of a very weak form of the prime number theorem; M is some constant > 1.

Since

$$\sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \frac{1}{p_1} \sum_{p_1=p_2} \frac{1}{p_2} \cdots \sum_{p_{k-1}<p_k\leq \frac{n}{p_1p_2\cdots p_{k-1}}} \frac{1}{p_k} =$$

$$= \sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \frac{1}{p_1^2} \sum_{p_2<p_3\leq \sqrt[k-2]{\frac{n}{p_1^2}}} \frac{1}{p_3} \cdots \sum_{p_{k-1}<p_k\leq \frac{n}{p_1^2\cdots p_{k-1}}} \frac{1}{p_k} =$$

$$= \sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \frac{O(1)}{p_1^2} = o(1) \ (n \to \infty), \tag{3.6}$$

it is sufficient to study the multiple sum

$$\sum_{n^{\alpha}<p_1\leq \sqrt[k]{n}} \frac{1}{p_1} \sum_{p_1<p_2\leq \sqrt[k-1]{\frac{n}{p_1}}} \frac{1}{p_2} \cdots \sum_{p_{k-1}<p_k\leq \frac{n}{p_1p_2\cdots p_{k-1}}} \frac{1}{p_k}. \tag{3.7}$$

<u>Definition 3.1</u>.

$$\theta(x) = \sum_{p\leq x} \log p, \tag{3.8}$$

$$\varepsilon(x) = \frac{\theta(x) - x}{x}. \tag{3.9}$$

<u>Lemma 3.2</u>.

$$\lim_{x\to\infty} \varepsilon(x) = 0. \tag{3.10}$$

Proof: This lemma is equivalent to the prime number theorem and will be considered as well-known (c.f. Landau [15]).

<u>Definition 3.2</u>.

$$\mu(A) = \sup_{x\geq A} |\varepsilon(x)|. \tag{3.11}$$

In order to obtain the limit of (3.7) as n tends to infinity we prove the following lemma.

Lemma 3.3.

If (i) $1 < A \leq B$, $P = [A] + 1$, $Q = [B]$,

(ii) $f(u,x)$ is a real function on the domain

$$D = \{(u,x) \mid A \leq u \leq B,\ x \geq a\}$$

such that $0 \leq f(u,x) \leq M$ on D,

(iii) for each fixed $x \geq a$, $f(u,x)$ is monotonically nonincreasing on $A \leq u \leq B$,

then
$$\sum_{A<p\leq B} \frac{1}{p} f(p,x) = \int_A^B \frac{f(u,x)}{u \log u}\, du + \Delta \tag{3.12}$$

where

$$|\Delta| \leq O\left(\frac{1}{\log A}\right) + M \cdot \mu(A) \cdot \sum_{r=P}^{Q} \frac{1}{(r+1)\log(r+1)} \,. \tag{3.13}$$

Proof: $\sum_{A<p\leq B} \frac{1}{p} f(p,x) =$

$$= \sum_{r=P}^{Q} \frac{f(r,x)}{r} \cdot \frac{\theta(r) - \theta(r-1)}{\log r} =$$

$$= \sum_{r=P}^{Q} \frac{f(r,x)}{r \log r} + \sum_{r=P}^{Q} \frac{\theta(r) - \theta(r-1) - 1}{r \log r} \cdot f(r,x) =$$

$$= \sum_{r=P}^{Q} \frac{f(r,x)}{r \log r} + \sum_{r=P}^{Q} \frac{r + r\cdot\varepsilon(r) - (r-1) - (r-1)\varepsilon(r-1) - 1}{r \log r} \cdot f(r,x) =$$

$$= \sum_{r=P}^{Q} \frac{f(r,x)}{r \log r} + \sum_{r=P}^{Q} \frac{r \cdot \varepsilon(r) - (r-1) \cdot \varepsilon(r-1)}{r \log r} f(r,x) =$$

$$= \sum_{r=P}^{Q} \frac{f(r,x)}{r \log r} + \sum_{r=P}^{Q} r\cdot \varepsilon(r) \left\{\frac{f(r,x)}{r \log r} - \frac{f(r+1,x)}{(r+1)\log(r+1)}\right\} +$$

$$- \frac{(P-1) \cdot \varepsilon(P-1) \cdot f(P,x)}{P \log P} + \frac{Q \cdot \varepsilon(Q) \cdot f(Q+1,x)}{(Q+1)\log(Q+1)} \,. \tag{3.14}$$

It is easy to show that

$$\sum_{r=P}^{Q} \frac{f(r,x)}{r \log r} = \int_P^Q \frac{f(u,x)}{u \log u}\, du + O\left(\frac{1}{P \log P}\right), \qquad (3.15)$$

$$\frac{(P-1) \cdot \varepsilon(P-1) \cdot f(P,x)}{P \log P} = O\left(\frac{1}{\log P}\right) \quad \text{and} \qquad (3.16)$$

$$\frac{Q \cdot \varepsilon(Q) \cdot f(Q+1,x)}{(Q+1)\log(Q+1)} = O\left(\frac{1}{\log P}\right). \qquad (3.17)$$

Furthermore we have

$$\left|\sum_{r=P}^{Q} r \cdot \varepsilon(r) \cdot \left\{\frac{f(r,x)}{r \log r} - \frac{f(r+1,x)}{(r+1)\log(r+1)}\right\}\right| \leq$$

$$\leq \sum_{r=P}^{Q} r \cdot |\varepsilon(r)| \left\{\frac{f(r,x)}{r \log r} - \frac{f(r+1,x)}{(r+1)\log(r+1)}\right\} =$$

$$= \sum_{r=P}^{Q} |\varepsilon(r)| \cdot \left\{\frac{f(r,x)}{\log r} - \frac{f(r+1,x)}{\log(r+1)} + \frac{f(r+1,x)}{(r+1)\log(r+1)}\right\}$$

$$\leq \max_{P \leq r \leq Q} |\varepsilon(r)| \cdot \left\{\frac{f(P,x)}{\log P} + \sum_{r=P}^{Q} \frac{f(r+1,x)}{(r+1)\log(r+1)}\right\}$$

$$\leq O\left(\frac{1}{\log P}\right) + M \cdot \mu(P) \cdot \sum_{r=P}^{Q} \frac{1}{(r+1)\log(r+1)}. \qquad (3.18)$$

Since

$$O\left(\frac{1}{\log P}\right) = O\left(\frac{1}{\log A}\right)$$

and

$$\mu(P) \leq \mu(A),$$

the lemma follows.

As an application we prove

<u>Lemma 3.4.</u>
If $\frac{1}{k+1} \leq \alpha \leq \frac{1}{k}$, $(k = 1, 2, 3, \ldots)$ then

$$\sum_{n^\alpha<p_1\le \sqrt[k]{n}} \frac{1}{p_1} \sum_{p_1<p_2\le \sqrt[k-1]{\frac{n}{p_1}}} \frac{1}{p_2} \cdots \sum_{p_{k-1}<p_k\le \frac{n}{p_1p_2\cdots p_{k-1}}} \frac{1}{p_k} =$$

$$= \int_{n^\alpha}^{\sqrt[k]{n}} \frac{du_1}{u_1 \log u_1} \int_{u_1}^{\sqrt[k-1]{\frac{n}{u_1}}} \frac{du_2}{u_2 \log u_2} \cdots \int_{u_{k-2}}^{\sqrt{\frac{n}{u_1u_2\cdots u_{k-2}}}} \frac{du_{k-1}}{u_{k-1} \log u_{k-1}}$$

$$\int_{u_{k-1}}^{\frac{n}{u_1u_2\cdots u_{k-1}}} \frac{du_k}{u_k \log u_k} + \delta_k(n,\alpha) \qquad (3.19)$$

where (uniformly) $\lim_{n\to\infty} \delta_k(n,\alpha) = 0$ (k fixed, $\frac{1}{k+1} \le \alpha \le \frac{1}{k}$).

Proof: From (1.22) it follows that the lemma is correct for $k = 1$.

Assume the lemma is correct for $k = 1, 2, 3, \ldots, q$.
For $\frac{1}{q+2} \le \alpha \le \frac{1}{q+1}$ we then may write

$$\sum_{n^\alpha<p_1\le \sqrt[q+1]{n}} \frac{1}{p_1} \sum_{p_1<p_2\le \sqrt[q]{\frac{n}{p_1}}} \frac{1}{p_2} \cdots \sum_{p_q<p_{q+1}\le \frac{n}{p_1p_2\cdots p_q}} \frac{1}{p_{q+1}} =$$

$$= \sum_{n^\alpha<p_1\le \sqrt[q+1]{n}} \frac{1}{p_1} \{ \int_{p_1}^{\sqrt[q]{\frac{n}{p_1}}} \frac{du_2}{u_2 \log u_2} \int_{u_2}^{\sqrt[q-1]{\frac{n}{p_1u_2}}} \frac{du_3}{u_3 \log u_3} \cdots$$

$$\cdots \int_{u_q}^{\frac{n}{p_1u_2\cdots u_q}} \frac{du_{q+1}}{u_{q+1} \log u_{q+1}} + \delta_q(\frac{n}{p_1}, \frac{\log p_1}{\log n - \log p_1}) \}$$

$$= \sum_{n^\alpha<p_1\le \sqrt[q+1]{n}} \frac{1}{p_1} \int_{p_1}^{\sqrt[q]{\frac{n}{p_1}}} \frac{du_2}{u_2 \log u_2} \cdots \int_{u_q}^{\frac{n}{p_1u_2\cdots u_q}} \frac{du_{q+1}}{u_{q+1} \log u_{q+1}} + o(1)$$

$(n \to \infty)$, (3.20)

because of $\frac{n}{p_1} \to \infty$ as $n \to \infty$

and $$\frac{1}{q+1} \leq \frac{\log p_1}{\log n - \log p_1} \leq \frac{1}{q}.$$

Now we let

$$f(u,x) = \int_u^{\sqrt[q]{\frac{x}{u}}} \frac{du_2}{u_2 \log u_2} \int_{u_2}^{\sqrt[q-1]{\frac{x}{uu_2}}} \frac{du_3}{u_3 \log u_3} \cdots \int_{u_q}^{\frac{x}{uu_2 \cdots u_q}} \frac{du_{q+1}}{u_{q+1} \log u_{q+1}} \tag{3.21}$$

on the domain

$$D = \{(u,x) \,|\, x^{\alpha} \leq u \leq x^{\frac{1}{q+1}},\ x \geq 2^{\frac{1}{\alpha}}\}.$$

By the induction hypothesis, $f(u,x)$ is bounded on D (compare (3.4) and (3.20)); moreover, it is easily verified that $f(u,x)$ satisfies all other conditions of lemma 3.3 ($A = x^{\alpha}$, $B = x^{1/q+1}$, $a = 2^{1/\alpha}$).
Hence

$$\sum_{n^{\alpha} < p \leq \sqrt[q+1]{n}} \frac{1}{p_1} f(p_1,n) =$$

$$= \int_{n^{\alpha}}^{\sqrt[q+1]{n}} \frac{du_1}{u_1 \log u_1} \int_{u_1}^{\sqrt[q]{\frac{n}{u_1}}} \frac{du_2}{u_2 \log u_2} \cdots \int_{u_q}^{\frac{n}{u_1 u_2 \cdots u_q}} \frac{du_{q+1}}{u_{q+1} \log u_{q+1}} + \Delta, \tag{3.22}$$

where

$$|\Delta| \leq O(\frac{1}{\log n^{\alpha}}) + M \cdot \mu(n^{\alpha}) \cdot \sum_{r=P}^{Q} \frac{1}{(r+1)\log(r+1)}$$

$$= O(\frac{1}{\log n}) + M \cdot \mu(n^{\alpha}) \cdot O(\int_{n^{\alpha}}^{\sqrt[q+1]{n}} \frac{du}{u \log u}) =$$

$$= O(\frac{1}{\log n}) + \mu(n^{\alpha}) \cdot O(1) = o(1) \quad (n \to \infty). \tag{3.23}$$

Together with (3.20) this proves the lemma by induction.

Lemma 3.5.
If $x > 1$, $0 < a < \frac{1}{k}$, $0 < b_i < \frac{1}{k-i}$ $(i = 0, 1, \ldots, k-1)$ and $b_k > 0$ then we have

$$\int_{x^a}^{x^{b_0}} \frac{du_1}{u_1 \log u_1} \int_{u_1}^{\left(\frac{x}{u_1}\right)^{b_1}} \frac{du_2}{u_2 \log u_2} \int_{u_2}^{\left(\frac{x}{u_1 u_2}\right)^{b_2}} \frac{du_3}{u_3 \log u_3} \cdots$$

$$\cdots \int_{u_k}^{\left(\frac{x}{u_1 u_2 \cdots u_k}\right)^{b_k}} \frac{du_{k+1}}{u_{k+1} \log u_{k+1}} =$$

$$= \int_a^{b_0} \frac{dv_1}{v_1} \int_{\frac{v_1}{1-v_1}}^{b_1} \frac{dv_2}{v_2} \int_{\frac{v_2}{1-v_2}}^{b_2} \frac{dv_3}{v_3} \cdots \int_{\frac{v_k}{1-v_k}}^{b_k} \frac{dv_{k+1}}{v_{k+1}} . \qquad (3.24)$$

Proof: If $x > 1$, $a > 0$, $b_0 > 0$ we have

$$\int_{x^a}^{x^{b_0}} \frac{du_1}{u_1 \log u_1} = \int_{a \log x}^{b_0 \log x} \frac{dv_1}{v_1} = \int_a^{b_0} \frac{dv_1}{v_1},$$

which shows that the lemma is correct if $k = 0$. Now assume that the lemma is correct for $k = 0, 1, \ldots, r-1$.
Then

$$\int_{x^a}^{x^{b_0}} \frac{du_1}{u_1 \log u_1} \int_{u_1}^{\left(\frac{x}{u_1}\right)^{b_1}} \frac{du_2}{u_2 \log u_2} \cdots \int_{u_r}^{\left(\frac{x}{u_1 u_2 \cdots u_r}\right)^{b_r}} \frac{du_{r+1}}{u_{r+1} \log u_{r+1}} =$$

$$= \int_{x^a}^{x^{b_0}} \frac{du_1}{u_1 \log u_1} \int_{\frac{\log u_1}{\log x - \log u_1}}^{b_1} \frac{dv_2}{v_2} \int_{\frac{u_2}{1-u_2}}^{b_2} \frac{dv_3}{v_3} \cdots \int_{\frac{u_r}{1-u_r}}^{b_r} \frac{dv_{r+1}}{v_{r+1}} =$$

$$= \int_{a \log x}^{b_0 \log x} \frac{dw}{w} \int_{\frac{w}{\log x - w}}^{b_1} \frac{dv_2}{v_2} \int_{\frac{u_2}{1-u_2}}^{b_2} \frac{dv_3}{v_3} \cdots \int_{\frac{u_r}{1-u_r}}^{b_r} \frac{dv_{r+1}}{v_{r+1}} =$$

$$= \int_{a}^{b_0} \frac{dv_1}{v_1} \int_{\frac{v_1}{1-v_1}}^{b_1} \frac{dv_2}{v_2} \int_{\frac{v_2}{1-v_2}}^{b_2} \frac{dv_3}{v_3} \cdots \int_{\frac{u_r}{1-u_r}}^{b_r} \frac{dv_{r+1}}{v_{r+1}}, \qquad (3.25)$$

which proves the lemma by induction.

It is clear that this lemma is applicable to (3.19) and consequently we find

Lemma 3.6.

$$\lim_{n\to\infty} \sum_{n^\alpha < p_1 \le \sqrt[k]{n}} \frac{1}{p_1} \sum_{p_1 < p_2 \le \sqrt[k-1]{\frac{n}{p_1}}} \frac{1}{p_2} \cdots \sum_{p_{k-1} < p_k \le \frac{n}{p_1 p_2 \cdots p_{k-1}}} \frac{1}{p_k} =$$

$$= \int_{\alpha}^{1/k} \frac{dv_1}{v_1} \int_{\frac{v_1}{1-v_1}}^{1/k-1} \frac{dv_2}{v_2} \int_{\frac{v_2}{1-v_2}}^{1/k-2} \frac{dv_3}{v_3} \cdots \int_{\frac{v_{k-1}}{1-v_{k-1}}}^{1} \frac{dv_k}{v_k}. \qquad (3.26)$$

It will be clear that we can now state without proof

Theorem 3.1.

$G(\alpha)$ exists for all real values of α and moreover

$G(\alpha) = 1$ if $\alpha \geq 1$,

$G(\alpha) = 1 + \log \alpha$ if $\frac{1}{2} \leq \alpha < 1$,

$$G(\alpha) = 1 + \log \alpha + \sum_{r=2}^{\left[\frac{1}{\alpha}\right]} (-1)^r \int_{\alpha}^{\frac{1}{r}} \frac{dv_1}{v_1} \int_{\frac{v_1}{1-v_1}}^{\frac{1}{r-1}} \frac{dv_2}{v_2} \int_{\frac{v_2}{1-v_2}}^{\frac{1}{r-2}} \frac{dv_3}{v_3} \cdots \tag{3.27}$$

$$\cdots \int_{\frac{v_{r-1}}{1-v_{r-1}}}^{1} \frac{dv_r}{v_r} \quad \text{if } 0 < \alpha < \frac{1}{2},$$

$G(\alpha) = 0$ if $\alpha \leq 0$.

4. From (3.27) it is easily deduced that $G(\alpha)$ is continuous on $\alpha > 0$. Moreover, it is even easy to show that $G(\alpha)$ is differentiable on $0 < \alpha < 1$ and satisfies the equation

$$G'(\alpha) = \frac{1}{\alpha} G\left(\frac{\alpha}{1 - \alpha}\right) \qquad (0 < \alpha < 1). \tag{4.1}$$

It is obvious that $G(\alpha)$ is completely determined on $\alpha > 0$ by

(i) $G(\alpha) = 1$ if $\alpha \geq 1$

(ii) $G(\alpha)$ is continuous on $\alpha > 0$

(iii) $G'(\alpha) = \frac{1}{\alpha} G\left(\frac{\alpha}{1 - \alpha}\right)$ if $0 < \alpha < 1$.

Definition 4.1.

$$\begin{cases} H(x) = 1 & \text{if } 0 \leq x \leq 1 \\ H(x) = G\left(\frac{1}{x}\right) & \text{if } x > 1. \end{cases} \tag{4.2}$$

Lemma 4.1.

$$H'(x) = -\frac{1}{x} \cdot H(x - 1) \quad (x > 1). \tag{4.3}$$

Proof:
$$H'(x) = \frac{d}{dx} G\left(\frac{1}{x}\right) =$$

$$= xG\left(\frac{\frac{1}{x}}{1 - \frac{1}{x}}\right) \cdot \frac{-1}{x^2} = -\frac{1}{x} \cdot G\left(\frac{1}{x - 1}\right) = -\frac{1}{x} H(x - 1).$$

Lemma 4.2.

$$\int_{t-1}^{t} H(x)dx = t \cdot H(t) \quad (t \geq 1)^{1)}. \tag{4.4}$$

---

1) This formula is due to L.E. Fleischhacker.

Proof: Obviously the lemma is correct for $t = 1$. For $t > 1$ we put

$$\phi(t) = \int_{t-1}^{t} H(x)dx - t \cdot H(t); \tag{4.5}$$

differentiating $\phi(t)$ we find

$$\phi'(t) = H(t) - H(t-1) - H(t) - t \cdot H'(t) = 0.$$

Hence $\phi(t)$ is constant on $t > 1$.
Since

$$\lim_{t \to 1} \phi(t) = \int_0^1 H(x)dx - H(1) = 0$$

we have

$$\phi(t) = 0 \quad (t > 1)$$

which proves the lemma.

Lemma 4.3.

$$H(x) > 0 \quad \text{if} \quad x \geq 0. \tag{4.6}$$

Proof: Suppose x_0 is the smallest zero of $H(x)$. Since $H(x)$ is positive for $0 \leq x \leq 1$ we have $H(x) > 0$ on $0 \leq x < x_0$ so that

$$\int_{x_0-1}^{x_0} H(x)dx > 0.$$

But this is a contradiction because

$$\int_{x_0-1}^{x_0} H(x)dx = x_0 \cdot H(x_0) = 0.$$

Lemma 4.4.
$H(x)$ is strictly decreasing on $x \geq 1$.

Proof: This is a simple consequence of lemmas 4.1 and 4.3.

Lemma 4.5.

H(x) is concave on $x \geq 1$.

Proof: If $1 \leq x \leq 2$ we have

$$H''(x) = \frac{d^2}{dx^2}(1 - \log x) = \frac{1}{x^2} > 0.$$

If $x \geq 2$ we have

$$H''(x) = \frac{d}{dx}\left(-\frac{1}{x} H(x-1)\right) = \frac{1}{x^2} H(x-1) + \frac{1}{x}\,\frac{H(x-2)}{x-1} > 0.$$

Since H(x) is differentiable in x = 2, the lemma has been proved.

Lemma 4.6.

$$H(x+1) < \frac{1}{2x+1} \quad H(x) \qquad (x \geq 1). \tag{4.7}$$

Proof: Since H(x) is concave on $x \geq 1$ we have

$$\frac{1}{2}\{H(x) + H(x+1)\} > \int_x^{x+1} H(t)dt = (x+1)H(x+1). \tag{4.8}$$

From this inequality it is easily deduced that

$$H(x+1) < \frac{1}{2x+1} \quad H(x) \qquad (x \geq 1).$$

As an immediate consequence we have

Lemma 4.7.

$$H(n) < \frac{1}{3\cdot5\cdot7\cdot\ldots(2n-1)} = \frac{2^n \cdot n!}{(2n)!} \qquad (n=2,3,4,\ldots). \tag{4.9}$$

By means of Stirling's formula we see that

$$H(n) < \frac{2^n \cdot n!}{(2n)!} \approx \frac{2^n \cdot n^n \cdot e^{-n}\sqrt{2\pi n}}{(2n)^{2n} \cdot e^{-2n}\sqrt{4\pi n}} =$$

$$= \frac{e^n}{2^{n+\frac{1}{2}} \cdot n^n} \qquad (n = 2, 3, 4, \ldots). \tag{4.10}$$

Remark: By the concavety of H(x) on $x \geq 1$ we can also state that

$$(t+1)H(t+1) = \int_t^{t+1} H(x)dx > H(t + \tfrac{1}{2}) \quad (t \geq 2).$$

Hence

$$H(t+1) > \frac{1}{t+1} H(t + \tfrac{1}{2}) > \frac{1}{(t+1)(t+\frac{1}{2})} H(t) \quad (t \geq 2\tfrac{1}{2})$$

and from this inequality it is easily seen that

$$H(n) > \frac{2^{2n}}{(2n)!} \cdot H(2\tfrac{1}{2}) \qquad (n = 4, 5, 6, \ldots).$$

The upper bound of H(x), together with the fact that H(x) is decreasing on $x \geq 1$, shows that the Laplace transform of H(x)

$$h(s) = \int_0^\infty e^{-sx} \cdot H(x)dx \tag{4.11}$$

converges absolutely for each complex number s. Hence h(s) will be an entire function.

Lemma 4.8.

$$h(s) = \exp(\gamma + \int_0^s \frac{e^{-z} - 1}{z} dz) \tag{4.12}$$

where γ is Euler's constant.

Proof: $\int_0^\infty e^{-sx} x \, dH(x) = - \frac{d}{ds} \int_0^\infty e^{-sx} \, dH(x) =$

$$= - \frac{d}{ds} \{e^{-sx} \cdot H(x)\Big|_0^\infty + s \int_0^\infty H(x) \cdot e^{-sx} dx\} =$$

$$= - \frac{d}{ds} \{H(0) + s \cdot h(s)\} = - sh'(s) - h(s). \tag{4.13}$$

Furthermore we have

$$\int_0^\infty e^{-sx} \cdot xdH(x) = \int_1^\infty e^{-sx} \cdot xdH(x) = \int_1^\infty e^{-sx} \cdot x \cdot \frac{-1}{x} H(x-1)dx$$

$$= - \int_1^\infty e^{-sx} H(x-1)dx = - \int_0^\infty e^{-s(x+1)}H(x)dx = - e^{-s} \cdot h(s). \qquad (4.14)$$

Hence

$$sh'(s) + h(s) = e^{-s} \cdot h(s) \qquad (4.15)$$

and it follows that

$$h'(s) = h(s) \cdot \frac{e^{-s} - 1}{s} \qquad (4.16)$$

and so we have

$$h(s) = c_0 \cdot \exp(\int_0^s \frac{e^{-z} - 1}{z} dz). \qquad (4.17)$$

We may also write

$$h(s) = c_0^* \cdot \exp(- \int_s^\infty \frac{e^{-z}}{z} dz - \log s). \qquad (4.18)$$

From the theory of Laplace transforms we know that

$$\lim_{\sigma \to \infty} \sigma \cdot h(\sigma) = H(0);$$

from (4.18) it follows that

$$\lim_{\sigma \to \infty} \sigma \cdot h(\sigma) = c_0^*.$$

Hence,

$$c_0^* = H(0) = 1,$$

and

$$h(s) = \exp(- \int_s^\infty \frac{e^{-z}}{z} dz - \log s). \qquad (4.19)$$

It is well-known that

$$\lim_{\sigma\to 0} \left(-\int_\sigma^\infty \frac{e^{-z}}{z}\,dz - \log\sigma\right) = \gamma$$

so that

$$h(0) = \int_0^\infty H(x)dx = e^\gamma = 1.781{,}072{,}417{,}99.$$

This proves that

$$h(s) = \exp\left(\gamma + \int_0^s \frac{e^{-z}-1}{z}\,dz\right).$$

Theorem 4.1.

$$H(x) = \frac{c_0}{2\pi i} \lim_{A\to\infty} \int_{\sigma-Ai}^{\sigma+Ai} e^{sx} \cdot \exp\left(\int_0^s \frac{e^{-z}-1}{z}\,dz\right)ds.$$

Proof: This is an immediate consequence of the previous lemma and a well-known theorem in the theory of Laplace transforms.

From this theorem it is easily deduced that

$$H(x) = \frac{c_0}{2\pi i} \lim_{A\to\infty} \int_{\sigma-Ai}^{\sigma+Ai} e^{-sx} \exp\left(\int_0^s \frac{e^z-1}{z}\,dz\right)ds. \qquad (4.20)$$

Integration by parts yields

$$H(x) = \frac{c_0}{2\pi i} \lim_{A\to\infty} \left\{\frac{e^{-sx}}{-x} \exp\left(\int_0^s \frac{e^z-1}{z}\,dz\right)\Bigg|_{\sigma-Ai}^{\sigma+Ai} + \int_{\sigma-Ai}^{\sigma+Ai} \frac{e^{-sx}}{x} \cdot \frac{e^s-1}{s} \cdot \exp\left(\int_0^s \frac{e^z-1}{z}\,dz\right)ds\right\} =$$

$$= \frac{c_0}{2\pi i} \exp\left(\int_0^1 \frac{e^z-1}{z}\,dz\right) \lim_{A\to\infty} \left\{\frac{e^{-sx}}{-x} \exp\left(\int_1^s \frac{e^z-1}{z}\,dz\right)\Bigg|_{\sigma-Ai}^{\sigma+Ai} + \int_{\sigma-Ai}^{\sigma+Ai} \frac{e^{-sx}}{x} \cdot \frac{e^s-1}{s} \exp\left(\int_1^s \frac{e^z-1}{z}\,dz\right)ds\right\} =$$

$$= \frac{c_0 \cdot c_1}{2\pi i} \lim_{A\to\infty} \{\frac{e^{-sx}}{-x} \cdot \frac{1}{s} \exp(\int_1^s \frac{e^z}{z}\, dz)\Big|_{\sigma-Ai}^{\sigma+Ai} +$$

$$+ \int_{\sigma-Ai}^{\sigma+Ai} \frac{e^{-sx}}{x} \cdot \frac{e^s - 1}{s} \cdot \frac{1}{s} \cdot \exp(\int_1^s \frac{e^z}{z}\, dz) ds\}, \qquad (4.21)$$

where $c_1 = \exp(\int_0^1 \frac{e^z - 1}{z}\, dz)$.

It is easily verified that $\int_1^s \frac{e^z}{z}\, dz$ is bounded for $Re(s) = \sigma > 0$ (σ fixed); hence we obtain from (4.21) that

$$H(x) = \frac{c_0 \cdot c_1}{2\pi i} \lim_{A\to\infty} \int_{\sigma-Ai}^{\sigma+Ai} \frac{e^{-sx}}{x} \cdot \frac{e^s - 1}{s^2} \cdot \exp(\int_1^s \frac{e^z}{z}\, dz) ds. \qquad (4.22)$$

However, this integral is absolutely convergent, so that we may write

$$H(x) = \frac{c_0}{2\pi i} \int_{\sigma-\infty i}^{\sigma+\infty i} \frac{e^{-sx}}{x} \cdot \frac{e^s - 1}{s} \cdot \exp(\int_0^s \frac{e^z - 1}{z}\, dz) ds. \qquad (4.23)$$

From this relation we will deduce

Theorem 4.2.

If one writes

$$H(x) = \frac{e^x \cdot \gamma(x)}{(x \log x)^x} \quad (x > 1) \qquad (4.24)$$

then

$$\limsup_{x\to\infty} \gamma(x) \leq 1. \qquad (4.25)$$

It is well known that we may take $\sigma < 0$ in (4.19); hence, in (4.23) σ will be > 0. If we stipulate that each integration path will be a straight line or segment, we may write

$$H(x) = \frac{c_0 \cdot c_1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-(\sigma+it)x}}{x} \cdot \frac{e^{\sigma+it}-1}{(\sigma+it)^2} \exp\left(\int_1^{\sigma+it} \frac{e^z}{z}\, dz\right)dt =$$

$$= \frac{c_0 \cdot c_1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-(\sigma+it)x}}{x} \cdot \frac{e^{\sigma+it} - 1}{(\sigma+it)^2} \exp\left(\int_1^{1+it} \frac{e^z}{z}\, dz + \int_{1+it}^{\sigma+it} \frac{e^z}{z}\, dz\right)dt$$

$$\leq \frac{c_0 \cdot c_1 \cdot c_2}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-\sigma x}}{x} \cdot \frac{e^{\sigma} + 1}{\sigma^2 + t^2} \exp\left(\left|\int_{1+it}^{\sigma+it} \frac{e^z}{z}\, dz\right|\right)dt$$

(where $c_2 = \sup\limits_{-\infty<t<+\infty} \exp\left(\left|\int_1^{1+it} \frac{e^z}{z}\, dz\right|\right)$)

$$= \frac{c_0 \cdot c_1 \cdot c_2}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-\sigma x}}{x} \cdot \frac{e^{\sigma} + 1}{\sigma^2 + t^2} \exp\left(\left|\int_1^{\sigma} \frac{e^{u+it}}{u+it} du\right|\right)dt \leq$$

$$\leq \frac{c_0 \cdot c_1 \cdot c_2}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-\sigma x}}{x} \cdot \frac{e^{\sigma} + 1}{\sigma^2 + t^2} \cdot \exp\left(\int_1^{\sigma} \frac{e^u}{u}\, du\right)dt =$$

$$= \frac{c_0 \cdot c_1 \cdot c_2}{2\pi} \cdot \frac{e^{-\sigma x}}{x} \cdot (e^{\sigma} + 1)\exp\left(\int_1^{\sigma} \frac{e^u}{u}\, du\right) \cdot \int_{-\infty}^{+\infty} \frac{dt}{\sigma^2 + t^2} =$$

$$= \frac{c_0 \cdot c_1 \cdot c_2}{2\pi} \cdot \frac{e^{-\sigma x}}{x} \cdot (e^{\sigma} + 1)\exp\left(\int_1^{\sigma} \frac{e^u}{u}\, du\right) \cdot \frac{\pi}{\sigma}. \tag{4.26}$$

We define

$$c_3 = c_0 \cdot c_1 \cdot c_2$$

and

$$\int_1^{\sigma} \frac{e^u}{u}\, du = (1 + \hat{\varepsilon}(\sigma)) \cdot \frac{e^{\sigma}}{\sigma}; \tag{4.27}$$

it is easily shown that

$$\lim_{\sigma\to\infty} \hat{\varepsilon}(\sigma) = 0. \tag{4.28}$$

Hence

$$\frac{c_3}{2} \cdot \frac{e^{-\sigma x}}{x} \left(\frac{e^{\sigma} + 1}{\sigma}\right) \exp\{(1 + \hat{\varepsilon}(\sigma)) \frac{e^{\sigma}}{\sigma}\} \leq$$

$$\leq c_3 \cdot \frac{e^{-\sigma x}}{x} \cdot \frac{e^{\sigma}}{\sigma} \cdot \exp\{(1 + \hat{\varepsilon}(\sigma)) \cdot \frac{e^{\sigma}}{\sigma}\} =$$

$$= c_3 \cdot \exp\{-\sigma x - \log x + \sigma - \log \sigma + (1 + \hat{\varepsilon}(\sigma)) \cdot \frac{e^{\sigma}}{\sigma}\} =$$

$$= c_3 \exp\{-x(\sigma - \frac{e^{\sigma}}{\sigma x}) - \log x + \sigma - \log \sigma + \hat{\varepsilon}(\sigma) \cdot \frac{e^{\sigma}}{\sigma}\}. \qquad (4.29)$$

Now we take σ so that

$$\psi(\sigma) = \sigma - \frac{e^{\sigma}}{\sigma x} \quad (x \text{ fixed}) \qquad (4.30)$$

is maximal.

From this we see that σ is the only solution of

$$1 = \frac{1}{x} \cdot \frac{(\sigma - 1) \cdot e^{\sigma}}{\sigma^2} \quad . \qquad (4.31)$$

If this solution is denoted by $\sigma(x)$ then it is easily verified that

$$\sigma(x) > \log x + \log \log x \qquad (4.32)$$

$$\frac{e^{\sigma(x)}}{\sigma(x) \cdot x} = \frac{\sigma(x)}{\sigma(x) - 1} \qquad (4.33)$$

and

$$\frac{\sigma(x) - \log \sigma(x)}{x} = o(1) \qquad (x \to \infty). \qquad (4.34)$$

For $\sigma = \sigma(x)$ (4.29) equals

$$c_3 \cdot \exp\{-x(\sigma(x) - \frac{\sigma(x)}{\sigma(x)-1} + \frac{\log x}{x} - \frac{\sigma(x)}{x} + \frac{\log \sigma(x)}{x} - \hat{\varepsilon}(\sigma(x)) \cdot \frac{\sigma(x)}{\sigma(x)-1})\}$$

$$\leq c_3 \cdot \exp\{-x(\log x + \log \log x - (1 + \hat{\varepsilon}(\sigma(x))) \frac{\sigma(x)}{\sigma(x)-1} + o(1))\} =$$

$$= \frac{1}{(x \log x)^x} \exp \{x(1 + \hat{\varepsilon}(\sigma(x))) \frac{\sigma(x)}{\sigma(x)-1} + o(1))\}. \qquad (4.35)$$

We thus find that

$$\gamma(x) \leq (1 + \hat{\varepsilon}(\sigma(x))) \cdot \frac{\sigma(x)}{\sigma(x)-1} + o(1) \tag{4.36}$$

which proves the theorem.

5. In this section we will discuss briefly a method to compute $H(x)$ numerically.
Our starting point is

$$\begin{cases} H(x) = 1 & (0 \le x \le 1) \\ (x+1)H(x+1) = \int_x^{x+1} H(t)dt & (x \ge 0). \end{cases} \tag{5.1}$$

If we approximate the integral

$$\int_{x_0}^{x_0+1} H(t)dt \qquad (x \ge 1) \tag{5.2}$$

by means of the trapezoidal formula

$$\frac{1}{2n}\left\{H(x_0) + 2\sum_{k=1}^{n-1} H(x_0 + \frac{k}{n}) + H(x_0 + 1)\right\}$$

we obtain, because of the concavity of $H(x)$ on $x \ge 1$, that

$$(x_0+1)H(x_0+1) = \int_{x_0}^{x_0+1} H(t)dt < \frac{1}{2n}\left\{H(x_0) + 2\sum_{k=1}^{n-1} H(x_0 + \frac{k}{n}) + H(x_0+1)\right\}.$$

It follows that

$$H(x_0+1) < \frac{1}{2n(x_0+1) - 1}\left\{H(x_0) + 2\sum_{k=1}^{n-1} H(x_0 + \frac{k}{n})\right\}.$$

Thus, if one has upperbounds for $H(x)$ at the points

$$x_0 + \frac{k}{n}, \quad (k = 0, 1, 2, \ldots, n-1)$$

one can compute an upperbound for $H(x_0+1)$.
Continuing in this way one can compute upper bounds for $H(x)$ at the points $x_0 + 1 + \frac{v}{n}$, $(v = 1, 2, 3, \ldots)$. On the other hand, approximating the integral (5.2) by

$$\frac{1}{n}\sum_{k=1}^{n} H(x_0 + \frac{2k-1}{2n})$$

one finds, also because of the concavity of $H(x)$ on $x \geq 1$, that

$$H(x_0+1) > \frac{1}{n(x_0+1)} \sum_{k=1}^{n} H(x_0 + \frac{2k-1}{2n}).$$

Hence, as soon as one has lower bounds for $H(x)$ at the points $x_0 + \frac{2k-1}{2n}$, $(k = 1, 2, 3, \ldots, n)$ one can compute a lower bound for $H(x_0+1)$. If one also knows lower bounds for $H(x)$ at the points $x_0 + \frac{k}{n}$, $(k = 1, 2, \ldots, n-1)$ one can apply the same method to compute a lower bound for $H(x_0 + 1 + \frac{1}{2n})$. Repeating this process one finds lower bounds for $H(x)$ at the points $x_0 + 1 + \frac{k}{2n}$, $(k = 2, 3, 4, \ldots)$.
As a starting point for the computations one may take of course $x_0 = 1$.
If one chooses the grid sizes in the above integral-approximating procedures small enough, one may expect that the corresponding upper and lower bounds for $H(x)$ will not differ very much.
Actual computations show that this is indeed the case. Performing the computations on the Electrologica-X8 of the Mathematical Centre in Amsterdam, using an ALGOL 60 program (with $n = 200$), we found that the corresponding upper and lower bounds for $H(x)$ were equal up to at least the first significant digit for all $x < 100$.
Using more refined integral-approximating formulae and smaller grid sizes we were able to compute $H(x)$ for values of x up to at least $x = 1000$.
Below we include a table for $H(x)$ with a five or more significant figure accuracy.

Table. $H(x) = a(x) \cdot 10^{-b(x)}$.

x	a(x)	b(x)
2	0.306 852	0
3	0.486 083	1
4	0.491 092	2
5	0.354 724	3
6	0.196 496	4
7	0.874 566	6
8	0.323 206	7
9	0.101 624	8
10	0.277 017	10
20	0.246 178	28
30	0.326 904	49
40	0.682 549	72
50	0.671 533	96
60	0.589 802	121
70	0.702 809	147
80	0.152 686	173
90	0.753 402	201
100	0.100 059	228
120	0.576 171	286
140	0.659 516	345
160	0.267 213	405
180	0.588 780	467
200	0.983 383	530
300	0.477 838	857
400	0.279 185	1201
500	0.505 734	1558
1000	0.458 767	3463

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