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Efficient Updating of Node Importance in Dynamic Real-Life Networks

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Abstract: The analysis of real-life networks, such as the internet, biometrical networks, and social networks, is challenged by the constantly changing structure of these networks. Typically, such networks consist of multiple weakly connected subcomponents and efficiently updating the importance of network nodes, as captured by the ergodic projector of a random walk on these networks, is a challenging task. In this paper, new approximations are introduced that allow to efficiently update the ergodic projector of Markov multi-chains. Properties such as convergence and error bounds for approximations are established. The numerical applicability is illustrated with a real-life social network example.

Keywords: Markov models, Networks, Markov multi-chain, Ergodic projector, Ranking nodes, Resolvent

1. INTRODUCTION

With 3.5 billion searches per day [1], the impact of Google's search engine cannot be underestimated. The underlying mechanism that ensures that the most relevant search results stand out, relies on the inherent information of the internet network consisting of websites (nodes) connected by hyperlinks (directed edges). To distill this information from the network, *Google's PageRank* algorithm essentially performs a random walk on the internet network modeled by a modified Markov chain. At each time step at a current node, the walk randomly chooses an available edge and moves to the corresponding node to continue the random walk. The stationary distribution of the location of this random walk induces the ranking of nodes in terms of importance. Relatively more visits to a node in the long-run means a more important node. For the original work see Brin and Page (1998).

Also in other application domains Markov chain network analysis can lead to important insights, such as in bibliometrics, social network analysis, systems analysis of road networks, as well as in economics, biology, chemistry, neuroscience, and physics. For more details see Gleich (2015).

There are some real-life networks aspects that often complicate network analysis in practice. A feature of many real-life networks is that they continuously change over time. For example, think of a stock market where the network describes the trading dynamics between traders. As a result, one has to recalculate network measures often in order to obtain the latest network insights. As networks can be of gigantic size nowadays, e.g., Facebook's social network reached more than 2 billion users this year according to Zuckerberg (2017), each of these (re-)calculations may pose a computational burden on its own. Another complicating feature is that network nodes are often not strongly connected in practice. A network may consist of multiple weakly or strongly connected components which is reflected as a Markov multi-chain. This lack of connectivity leads to numerical challenges when evaluating network concepts. For example, without making the network artificially strongly connected and thereby distorting the true network dynamics (as in Google's PageRank), the stationary distribution is often not unique. The truly long-term distribution of the location of the random walk is described by the ergodic projector of a Markov multi-chain. For a discussion and an alternative ranking that incorporates the ergodic projector see Berkhout and Heidergott (2018).

The focus of this article is on the efficient updating of the ergodic projector of the random walk on weakly connected networks in a continuously changing network environment in which existing connections are revised. The contribution of this paper is two-fold: (i) a reformulation of the series expansion for the ergodic projector of Markov multi-chains from Schweitzer (1968) is presented that is suitable for numerical updating by providing an error bound, and (ii) a new series expansion is introduced that allows efficient updating of the resolvent after network changes. More specifically, the Markov chain resolvent accurately approximates the ergodic projector Markov multi-chains. Moreover, with little extra effort the resolvent also provides an accurate approximation for the in Markov chain theory fundamental deviation matrix.

The outline of the rest of this article is as follows. In Section 2 the main technical concepts will be introduced. Afterwards, in Section 3 these concepts will be used to develop the series representations of the ergodic projector and the resolvent of Markov multi-chains. The applicabil-

[1] Reported on `http://www.internetlivestats.com/google-search-statistics` (15 December 2017)

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ity of the series expansions will be illustrated in Section 4 via numerical experiments. This article concludes in Section 5.

2. PRELIMINARIES

In the following, it will be shown how to model a random walk on a network in terms of a Markov chain. Afterwards, Markov chain concepts such as the ergodic projector, deviation matrix and resolvent will be introduced.

General networks can typically be modeled as directed graphs with weighted edges. In particular, let $G = (V, E)$ describe a directed graph with *finite* vertex set V and edge set $E \subseteq V \times V$. All edges are weighted by a positive function f based on the underlying network dynamics, i.e., $f : E \to (0, \infty)$. An edge $e = (i, j) \in E$ means that there is a directed relation from i to j and the relation-strength is described by weight $f(e)$. For example, $f(e)$ may be the number of hyperlinks going from website i to website j in an internet network.

The location of a random walk on the graph-modeled-network will be described by a discrete time Markov chain $(X_t : t = 0, 1, 2, \dots)$ on state space V. Particularly, $X_t \in V$ is the node-location/state of the random surfer at time t. Write P as the transition matrix of the Markov chain of which the (i, j)-th element is given by

$$P(i,j) = \frac{f(i,j)}{\sum_{(i,k)\in E} f(i,k)}, \quad \text{for all } (i,j) \in E,$$

and $P(i, j) = 0$ for all $(i, j) \notin E$. In case $\sum_{j\in V} P(i, j) = 0$, we set $P(i, i) = 1$. In general, a real-life network of C strongly connected components with no outgoing edges leads to a Markov transition matrix of the canonical form (after relabeling the nodes)

$$P = \begin{bmatrix} P_1 & 0 & 0 & \cdots & 0 \\ 0 & P_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_C & 0 \\ \bar{P}_1 & \bar{P}_2 & \cdots & \bar{P}_C & \bar{P}_{C+1} \end{bmatrix} \tag{1}$$

wherein, for $k = 1, 2, \dots, C$, P_k is the transition matrix of the kth ergodic class, and $\bar{P}_k$ contains the transition probabilities from the transient states to ergodic class k. Lastly, $\bar{P}_{C+1}$ describes transition probabilities between transient states. When $C = 1$, P models a Markov *uni*-chain, whereas P with $C > 1$ models a Markov *multi*-chain. For convenience, the described Markov chain with transition probability matrix P will be denoted as 'Markov chain P'.

The probability of finding the random walk at node j after t discrete time steps when starting at node i is given by $(P^t)(i, j) = \Pr(X_t = j \mid X_0 = i)$. In particular, the focus is on the long term distribution of the random walk location as given by the *ergodic projector* Π_P. Using a Cesaro limit, Π_P is defined as

$$\Pi_P = \lim_{T\to\infty} \frac{1}{T} \sum_{t=0}^{T-1} P^t.$$

It holds that Π_P equals $\lim_{t\to\infty} P^t$ when Markov chain P is aperiodic. For more details see Kemeny and Snell (1976). When P is of the canonical form as in (1), ergodic projector Π_P is of the form

$$\Pi_P = \begin{bmatrix} \Pi_1 & 0 & 0 & \cdots & 0 \\ 0 & \Pi_2 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \Pi_C & 0 \\ \bar{\Pi}_1 & \bar{\Pi}_2 & \cdots & \bar{\Pi}_C & 0 \end{bmatrix}.$$

The rows in square matrix Π_k are all equal to the unique stationary distribution of the chain inside the kth ergodic class, i.e., equal to the unique distribution π_{P_k} satisfying $\pi_{P_i}^\top P_i = \pi_{P_i}^\top$. Element (i, j) in Matrix $\bar{\Pi}_k$ gives the equilibrium probability of visiting j (which is part of the kth ergodic class) when starting in transient state i. It holds for $k = 1, 2, \dots, C$, see, e.g., Berkhout and Heidergott (2014), that $\bar{\Pi}_k = (I - \bar{P}_{C+1})^{-1} \bar{P}_k \Pi_{P_k}$, where I is an appropriately sized identity matrix. If the random walk starts according to a random distribution v, the probability of finding the walk at node j in the long term is given by $v^\top \Pi_P(j)$. To exactly calculate Π_P in practice, one has to: i) uncover the multi-chain structure, ii) determine Π_k for $k = 1, 2, \dots, C$, and iii) determine $\bar{\Pi}_k$ for $k = 1, 2, \dots, C$.

Throughout this article, $\|\cdot\|$ denotes the ∞-norm. While any norm can be chosen, this norm choice allows to use simplifying properties such as $\|P\| = 1$.

An alternative way to evaluate Π_P directly (i.e., without having to uncover the multi-chain structure first) by means of an approximation is to determine the so-called *resolvent* of Markov chain P, see Kartashov (1996). The resolvent of Markov chain P with $\alpha \in (0, 1]$ is defined as

$$R_\alpha(P) = \alpha(I - (1 - \alpha)P)^{-1}.$$

It is shown in Theorem 1.5 of Kartashov (1996) that $\lim_{\alpha\to 0} R_\alpha(P) = \Pi_P$. In Berkhout and Heidergott (2016b) this result is elaborated into a numerical approximation framework for Π_P and it is essentially shown [2] that

$$\|R_\alpha(P) - \Pi_P\| \le \alpha c(P),$$

where $c(P)$ is an α-independent constant. This shows that for $\alpha \in (0, 1]$ small, $R_\alpha(P)$ provides an accurate approximation for Π_P.

Remark 1. It is worth noting that choosing α too small may cause numerical instabilities when approaching the machine precision, see Berkhout and Heidergott (2016b), which implies that $\|R_\alpha(P) - \Pi_P\|$ cannot be made numerically arbitrarily small.

Virtually everything that you would want to know of a Markov chain P can be determined using the so-called *deviation matrix* D_P, see Meyer (1975), where D_P is indicated as the group inverse of $I - P$. It also plays an important role in Markov decision theory and computation, see, e.g., the monograph of Puterman (1994), and serves as a basis for a structural way of ranking nodes in networks described by a Markov multi-chain P, see Berkhout and Heidergott (2018). Matrix D_P is defined as

[2] In Berkhout and Heidergott (2016b), $R_\alpha(P)$ is modified for efficient evaluation. It can be shown that the similar bounds as for the modified version from Berkhout and Heidergott (2016b) also hold true for $R_\alpha(P)$.

$$D_P = \lim_{T\to\infty} \frac{1}{T} \sum_{t_1=0}^{T-1} \sum_{t_2=0}^{t_1-1} (P^{t_2} - \Pi_P) = (I - P + \Pi_P)^{-1} - \Pi_P.$$

Existence of D_P is guaranteed for finite-state Markov chains, see Puterman (1994). A related concept in Markov chain theory is the *fundamental matrix* Z_P which equals

$$Z_P = D_P + \Pi_P = (I - P + \Pi_P)^{-1}.$$

The resolvent can also be used for approximating D_P at the expense of 1 extra matrix multiplication. In particular, defining for $\alpha \in (0,1]$ the *resolvent approximation for* D_P as

$$D_\alpha(P) = \frac{1}{\alpha}(I - R_\alpha(P))R_\alpha(P)$$

it is essentially shown in Berkhout and Heidergott (2016a) that

$$\|D_\alpha(P) - D_P\| \le \alpha \bar{c}(P), \tag{2}$$

where $\bar{c}(P)$ ($> c(P)$) is an α-independent constant. This illustrates that $D_\alpha(P)$ gives an accurate approximation for small $\alpha \in (0,1]$.

3. SERIES REPRESENTATIONS OF THE ERGODIC PROJECT AND THE RESOLVENT

Given the ergodic projector (or resolvent) of a Markov chain P and a Markov chain $Q \ne P$ close to P, it is reasonable to expect that the ergodic projector (or resolvent) of a Markov chain Q is close to that of P. This will be exploited using a series expansion for the ergodic projector in Subsection 3.1 and similarly for the resolvent in Subsection 3.2. It is worth noting that the series developments are inspired by the analysis from Heidergott et al. (2007) in which a series expansion is presented for the ergodic projector. However, this expansion for the ergodic projector is not applicable to Markov multi-chains (as it relies on the property that $Q\Pi_P = \Pi_P$ which in general does not hold).

3.1 Schweitzer Inspired Series Expansion

In Schweitzer (1968) a series expansion is presented that applies to Markov multi-chains in case $\Pi_Q \to \Pi_P$ as $Q \to P$. In particular, it is shown in Schweitzer (1968) that

$$\Pi_Q = \sum_{n=0}^{\infty} B_n$$

where for $n \ge 0$

$$B_n = \sum_{k=0}^{n} Z_P U^k \Pi_P U^{n-k} \quad \text{with} \quad U = (Q-P)Z_P.$$

Differential matrix U measures the distance between P and Q. The series terms can be efficiently evaluated via

$$B_{n+1} = B_n U + Z_P U^{n+1} \Pi_P \quad \text{with} \quad B_0 = \Pi_P.$$

The following lemma allows the development of a similar series as in Schweitzer (1968), the error of which can be meaningfully bounded. The lemma requires that $\Pi_P = \Pi_P \Pi_Q \Pi_P$, which holds when Markov chains P and Q have the same ergodic classes. Furthermore, for a meaningful analysis, P and Q should be not "too different". Specifically, it will be imposed that $||U|| < 1$ so that $(I-U)^{-1} = \sum_{n=0}^{\infty} U^n$ exists. Note that $\|Q-P\| < 1/\|Z_P\|$ implies $||U|| < 1$.

Lemma 2. When $\Pi_P = \Pi_P \Pi_Q \Pi_P$ and $||U|| < 1$,

$$\Pi_Q = Z_P(I-U)^{-1}\Pi_P + \Pi_Q U.$$

Proof. It holds that

$$\begin{aligned} &\Pi_Q = Z_P(I-U)^{-1}\Pi_P + \Pi_Q U \\ &\Leftrightarrow Z_P^{-1}\Pi_Q = (I-U)^{-1}\Pi_P + Z_P^{-1}\Pi_Q U \\ &\Leftrightarrow \Pi_P = (I-U) Z_P^{-1} \Pi_Q (I-U) \end{aligned}$$

writing out $I-U = I-(Q-P)Z_P$, respectively, and using that $\Pi_Q Q = Q\Pi_Q = \Pi_Q$

$$\begin{aligned} &\Pi_Q = Z_P(I-U)^{-1}\Pi_P + \Pi_Q U \\ &\Leftrightarrow \Pi_P = Z_P^{-1}\Pi_Q(I-U) - (Q-P)\Pi_Q(I-U) \\ &\Leftrightarrow \Pi_P = (Z_P^{-1} - (Q-P))\Pi_Q(I-U) \\ &\Leftrightarrow \Pi_P = (I + \Pi_P - Q)\Pi_Q(I-U) \\ &\Leftrightarrow \Pi_P = \Pi_P\Pi_Q(I-U) \\ &\Leftrightarrow \Pi_P = \Pi_P\Pi_Q(I-(Q-P)Z_P) \\ &\Leftrightarrow \Pi_P = \Pi_P\Pi_Q(I-(I-P)Z_P) \\ &\Leftrightarrow \Pi_P = \Pi_P\Pi_Q(I-(I-\Pi_P)) \\ &\Leftrightarrow \Pi_P = \Pi_P\Pi_Q\Pi_P. \qquad \square \end{aligned}$$

Iterating the result from Lemma 2 gives

$$\begin{aligned} \Pi_Q &= Z_P(I-U)^{-1}\Pi_P + \Pi_Q U \\ &= Z_P(I-U)^{-1}\Pi_P(I+U) + \Pi_Q U^2 \\ &= \ldots \text{ (iterating } N \text{ times in total)} \\ &= Z_P(I-U)^{-1}\Pi_P \sum_{n=0}^{N} U^n + \Pi_Q U^{N+1}. \end{aligned}$$

Utilizing that $\sum_{n=0}^{N} U^n$ is a natural approximation for $(I-U)^{-1} = \sum_{n=0}^{\infty} U^n$ (assuming $||U|| < 1$), gives the following multi-chain series expansion

$$\begin{aligned} \Pi_Q = &\underbrace{Z_P \sum_{n=0}^{N} U^n \Pi_P \sum_{n=0}^{N} U^n}_{:=V(N,P,Q)} \\ &\underbrace{+ \Pi_Q U^{N+1} + Z_P U^{N+1}(I-U)^{-1}\Pi_P \sum_{n=0}^{N} U^n}_{:=W(N,P,Q)}, \end{aligned}$$

where $V(N,P,Q)$ is defined as the *Nth order series expansion approximation* for Π_Q and $W(N,P,Q)$ the corresponding *error*. The following theorem establishes convergence of the series expansion approximation for Π_Q at geometric rate.

Theorem 3. When $\Pi_P = \Pi_P\Pi_Q\Pi_P$ and $||U|| < 1$

$$\|W(N,P,Q)\| \le \left(1 + \frac{2\|Z_P\|}{(1-\|U\|)^2}\right)\|U\|^{N+1}.$$

Proof. First observe that

$$\begin{aligned} \left\|\sum_{n=0}^{N} U^n\right\| &= \left\|(I-U^{N+1})(I-U)^{-1}\right\| \\ &\le \frac{1+\|U\|^{N+1}}{1-\|U\|} < \frac{2}{1-\|U\|}. \end{aligned}$$

Therefore, it holds for the error of $V(N,P,Q)$ that

$$
\begin{aligned}
&\|W(N,P,Q)\| \\
&\le \left\| \Pi_Q U^{N+1} + Z_P U^{N+1}(I-U)^{-1}\Pi_P \sum_{n=0}^{N} U^n \right\| \\
&\le \|U\|^{N+1} + \|U\|^{N+1}\|Z_P\|\|(I-U)^{-1}\| \left\| \sum_{n=0}^{N} U^n \right\| \\
&\le \|U\|^{N+1}(1 + 2\|Z_P\|/(1-\|U\|)^2). \qquad \square
\end{aligned}
$$

Remark 4. In practice, condition $\|U\| < 1$ might be too restrictive. It is topic of further research to relax this requirement.

Suppose Π_P and Z_P for Markov chain P are known. When Markov chain P changes to Q (assuming that $\Pi_P = \Pi_P \Pi_Q \Pi_P$ and $||U|| < 1$), an efficient procedure to approximate Π_Q via $V(N,P,Q)$ to a user-defined precision $\epsilon > 0$ is as follows. Matrix multiplication and inverse are denoted by m and i, respectively. Firstly, determine Π_P, Z_P and $\|Z_P\|$ (2 i). Then, for each Q:

(1) Find $U = (Q-P)Z_A$ and $\|U\|$.
(2) Choose $N = \log(\epsilon/\left(1 + \frac{2\|Z_P\|}{(1-\|U\|)^2}\right))/\log(\|U\|)$.
(3) Determine $V(N,P,Q)$. (For each Q: $(N+2)$ m)

Theorem 3 then guarantees that $\|\Pi_Q - V(N,P,Q)\| < \epsilon$.

3.2 Resolvent Series Expansion

In the following, a series expansion for the resolvent will be developed that allows for an accurate approximation of $R_\alpha(Q)$ once $R_\alpha(P)$ is known. Moreover, it follows from (2) that with one extra matrix multiplication this expansion also provides an accurate approximation for deviation matrix D_Q.

The series expansion for the resolvent relies on the following update formula.

Lemma 5. (Resolvent Update Formula). For Markov chains P and Q, the following *update formula* holds for all $\alpha \in (0,1]$

$$R_\alpha(Q) = R_\alpha(P) + R_\alpha(Q)T_\alpha(P,Q),$$

where

$$T_\alpha(P,Q) := \frac{1-\alpha}{\alpha}(Q-P)R_\alpha(P).$$

Proof. Writing out $R_\alpha(Q)$, which is allowed since $\|(1-\alpha)P\| < 1$, and taking the first term out shows that

$$
\begin{aligned}
R_\alpha(Q) &= \alpha \sum_{n=0}^{\infty} ((1-\alpha)Q)^n \\
&= \alpha I + R_\alpha(Q)(1-\alpha)Q.
\end{aligned}
$$

Subtracting $R_\alpha(Q)(1-\alpha)P$ from both sides gives

$$
\begin{aligned}
&R_\alpha(Q)(I-(1-\alpha)P) = \alpha I + R_\alpha(Q)(1-\alpha)(Q-P) \\
&\Leftrightarrow R_\alpha(Q) = R_\alpha(P) + R_\alpha(Q)(1-\alpha)(Q-P) \\
&\qquad\qquad \cdot (I-(1-\alpha)P)^{-1} \\
&\Leftrightarrow R_\alpha(Q) = R_\alpha(P) + R_\alpha(Q)\frac{1-\alpha}{\alpha}(Q-P)R_\alpha(P),
\end{aligned}
$$

using the definition of $T_\alpha(P,Q)$ concludes the proof. $\square$

Note that $T_\alpha(P,Q)$ can be seen as the resolvent version of differential matrix U. Iterating the resolvent update formula to its right hand side gives

$$
\begin{aligned}
R_\alpha(Q) &= R_\alpha(P) + R_\alpha(Q)T_\alpha(P,Q) \\
&= R_\alpha(P)\,(I + T_\alpha(P,Q)) + R_\alpha(P)T_\alpha(P,Q)^2 \\
&= \ldots \text{ (iterating } N \text{ times in total)} \\
&= \underbrace{R_\alpha(P)\sum_{n=0}^{N} T_\alpha(P,Q)^n}_{S_\alpha(N,P,Q)} + \underbrace{R_\alpha(Q)T_\alpha(P,Q)^{N+1}}_{E_\alpha(N,P,Q)},
\end{aligned}
$$

where the *series expansion of order N for $R_\alpha(Q)$* is defined as $S_\alpha(N,P,Q)$ and the corresponding *error term* as $E_\alpha(N,P,Q)$. The claim is that $S_\alpha(N,P,Q)$, for relatively small N (i.e., a few series terms), provides a good approximation for $R_\alpha(Q)$ in case P and Q are not too different, i.e., $\sum_{n=0}^{\infty} T_\alpha(P,Q)^n$ should converge. A sufficient condition for $\sum_{n=0}^{\infty} T_\alpha(P,Q)^n$ to exist is that

$$\exists N \in \mathbb{N} : \|T_\alpha(P,Q)^N\| < 1.$$

The following theorem provides a sharp bound that can be used in practice by multiplying the last term of $S_\alpha(N,P,Q)$ by $\frac{1-\alpha}{\alpha}(Q-P)$ from the right.

Theorem 6. For Markov chains P and Q and $\alpha \in (0,1]$, when $\sum_{n=0}^{\infty} T_\alpha(P,Q)^n < \infty$

$$\|E_\alpha(N,P,Q)\| \le \left\| R_\alpha(P)T_\alpha(P,Q)^N \frac{1-\alpha}{\alpha}(Q-P) \right\|.$$

Proof. The following resolvent property will be used in this proof

$$I = \frac{1}{\alpha}(I-(1-\alpha)P)R_\alpha(P) = R_\alpha(P)\frac{1}{\alpha}(I-(1-\alpha)P). \quad (3)$$

A necessary condition for $\sum_{n=0}^{\infty} T_\alpha(P,Q)^n$ to converge is that $\lim_{N\to\infty} T_\alpha(P,Q)^N = 0$. This implies that

$$\lim_{N\to\infty} \|E_\alpha(N,P,Q)\| \le \lim_{N\to\infty} \|T_\alpha(P,Q)^{N+1}\| = 0,$$

therefore,

$$\lim_{N\to\infty} S_\alpha(N,P,Q) = R_\alpha(Q). \quad (4)$$

This means that

$$
\begin{aligned}
E_\alpha(N,P,Q) &= R_\alpha(P) \sum_{n=N+1}^{\infty} T_\alpha(P,Q)^n \\
&= R_\alpha(P)T_\alpha(P,Q)^{N+1}\sum_{n=0}^{\infty} T_\alpha(P,Q)^n,
\end{aligned}
$$

for which, using (3) and (4),

$$\sum_{n=0}^{\infty} T_\alpha(P,Q)^n = \frac{1}{\alpha}(I-(1-\alpha)P)\underbrace{R_\alpha(P)\sum_{n=0}^{\infty} T_\alpha(P,Q)^n}_{=R_\alpha(Q)}$$

so that

$$
\begin{aligned}
E_\alpha(N,P,Q) &= \frac{1}{\alpha}R_\alpha(P)T_\alpha(P,Q)^{N+1}(I-(1-\alpha)P)R_\alpha(Q) \\
&= \frac{1}{\alpha}R_\alpha(P)T_\alpha(P,Q)^N(1-\alpha)(Q-P)R_\alpha(Q),
\end{aligned}
$$

where the last equality uses that

$$T_\alpha(P,Q)(I-(1-\alpha)P) = (1-\alpha)(Q-P),$$

because of (3).

Taking the norm of the last found expression of $E_\alpha(N,P,Q)$ leads to

$$\|E_\alpha(N,P,Q)\| \leq \left\| R_\alpha(P)T_\alpha(P,Q)^N \frac{1-\alpha}{\alpha}(Q-P) \right\| \|R_\alpha(Q)\|,$$

which proves the result since $\|R_\alpha(Q)\| = 1$. □

Suppose $R_\alpha(P)$ for Markov chain P is known. When Markov chain P changes to Q, an efficient procedure to approximate $R_\alpha(Q)$ via $S_\alpha(N,P,Q)$ to a user-defined precision $\epsilon > 0$ is as follows, where again, a matrix multiplication and inverse are denoted by m and i, respectively. Determine $R_\alpha(P)$ and $T_\alpha(P,Q)$ $(1\ i + 1\ m)$, and initialize $N = 0$ and $S_\alpha(0,P,Q) = R_\alpha(P)$. For each Q:

WHILE $\|R_\alpha(P)T_\alpha(P,Q)^N(Q-P)\| \geq \frac{\alpha\epsilon}{1-\alpha}$:

(1) CALCULATE
$$R_\alpha(P)T_\alpha(P,Q)^{N+1} = R_\alpha(P)T_\alpha(P,Q)^N T_\alpha(P,Q).$$
(2) SAVE
$$S_\alpha(N+1,P,Q) = S_\alpha(N,P,Q) + R_\alpha(P)T_\alpha(P,Q)^{N+1}.$$
(3) SET $N = N + 1$.

RETURN $S_\alpha(N,P,Q)$. (For each Q: $(2(N+1)-1)\ m$)

In practice, one immediately notices when the error bound drifts away for increasing N. In that case, P and Q differ too much. It follows from Theorem 6 that this procedure leads to an approximation $S_\alpha(N,P,Q)$ which satisfies $\|R_\alpha(Q) - S_\alpha(N,P,Q)\| < \epsilon$.

In the following, a connection will be established between $S_\alpha(N,P,Q)$, $V(N,P,Q)$ and the series expansion for the ergodic projector from Heidergott et al. (2007), respectively. The connection requires that $Q\Pi_P = \Pi_P$ which in general does not hold for Markov multi-chains.

Proposition 7. When $Q\Pi_P = \Pi_P$, it holds that

$$\lim_{\alpha\downarrow 0} S_\alpha(N,P,Q) = V(N,P,Q) = \Pi_P \sum_{n=0}^{N}((Q-P)D_P)^n$$

and also the errors coincide, i.e.,

$$\lim_{\alpha\downarrow 0} E_\alpha(N,P,Q) = W(N,P,Q) = \Pi_Q((Q-P)D_P)^{N+1}.$$

Proof. Since $Q\Pi_P = \Pi_P$ (as prerequisite) and $P\Pi_P = \Pi_P$, it holds that

$$T_\alpha(N,P,Q) = \frac{1-\alpha}{\alpha}(Q-P)R_\alpha(P) = (1-\alpha)(Q-P)\sum_{i=0}^{\infty}(1-\alpha)^i(P^i - \Pi_P),$$

this expression allows us to let α tend to 0 which leads to

$$\lim_{\alpha\downarrow 0} T_\alpha(N,P,Q) = (Q-P)D_P.$$

Furthermore, noting that $\lim_{\alpha\downarrow 0} R_\alpha(P) = \Pi_P$ and $\lim_{\alpha\downarrow 0} R_\alpha(Q) = \Pi_Q$, proves the connection between the resolvent series expansion and the series expansion from Heidergott et al. (2007).

Because $Z_P\Pi_P = \Pi_P$ and $Q\Pi_P = \Pi_P$, it holds that

$$Z_P \sum_{n=0}^{N} U^n \Pi_P = Z_P \sum_{n=0}^{N}((Q-P)Z_P)^n \Pi_P = \Pi_P.$$

Furthermore, U may be written as

$$U = (Q-P)Z_P = (Q-P)(Z_P + \Pi_P) = (Q-P)D_P.$$

Together this shows that $V(N,P,Q)$ coincides with the series expansion approximation of Heidergott et al. (2007). The same holds true for the error by observing that $W(N,P,Q) = \Pi_Q U^{N+1} = \Pi_Q((Q-P)D_P)^{N+1}$ (where it is tacitly used that $U^{N+1}\Pi_P = 0$ for $N \geq 0$). □

4. NUMERICAL EXPERIMENTS

This section illustrates the practical applicability of the presented series expansions using a numerical experiment. To that end, a social network from Moody (2001) is used, which is the result of an in-school questionnaire. The specific dataset can be found in KONECT (2017). Each student of the school was asked to list his 5 best female and his 5 male friends. Each node represents a student and when student i chose student j as a friend this is represented by an edge $(i,j) \in E$. In total the dataset consists of 2,539 nodes and 12,969 edges.

All edges are weighted by a function $f : E \to \{1, 2, \ldots, 6\}$. A larger edge weight indicates more interaction. In particular for $(i,j) \in E$, $f(i,j) = 1 + x_{ij}$ where x_{ij} is the number of the following activities students i and j participated in. These activities were:

- you went to (his/her) house in the last seven days,
- you met (him/her) after school to hang out or go somewhere in the last seven days,
- you spent time with (him/her) last weekend,
- you talked with (him/her) about a problem in the last seven days,
- you talked with (him/her) on the telephone in the last seven days.

Value $f(i,j) = 1$ means that student i nominated j as friend but reported no activities, whereas $f(i,j) = 6$ means that student i nominated j as friend and reported participating in all five activities with the friend. The network is converted to Markov chain P as described in Section 2. It leads to a Markov multi-chain of 228 ergodic states and 2,311 transient states. The rows of Π_P give insight into the popularity of the students, for more details see Berkhout and Heidergott (2018). This is useful information for, e.g., social support and marketing purposes. In the following, the network edges will be revised and it will be demonstrated that the presented series expansions efficiently lead to accurate approximations for the new ergodic projector, i.e., the popularity of students in the revised network.

For the numerical experiment a random matrix Y is drawn with elements

$$Y(i,j) = \begin{cases} 0 & \text{if } (i,j) \notin E \\ X & \text{if } (i,j) \in E \end{cases},$$

where X is a uniform random variable on interval $(0, 10)$. Subsequently, the rows of Y are normalized so that it becomes a stochastic matrix. The objective is to approximate Π_{Q_θ}, where

$$Q_\theta = Q = (1-\theta)P + \theta Y.$$

The results for a typical random Q_θ with $\theta = 0.01$ can be found in Figure 1 (for which $\|P - Q_\theta\| \approx 0.0163$). To obtain a sharper bound, $\|U^{N+1}\|$ instead of $\|U\|^{N+1}$ is used in the bound for $\|W(N,P,Q_\theta)\|$ as given in Theorem 3. The bound for $\|E_\alpha(N,P,Q_\theta)\|$ is as in Theorem 6. The value $\alpha = 10^{-8}$ is used for the resolvent.

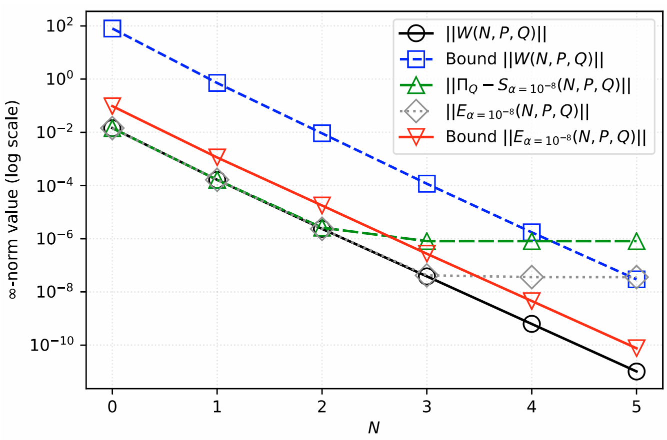

Fig. 1. The series expansion approximation errors for $\theta = 0.01$ ($\|P - Q_\theta\| \approx 0.0163$). A few series terms is enough for an accurate approximation.

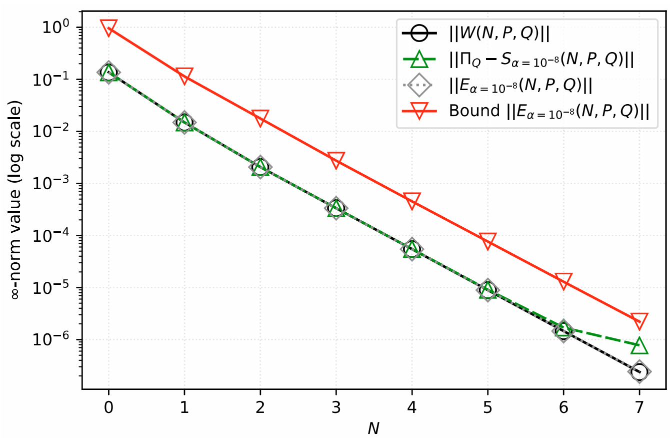

Fig. 2. The series expansion approximation errors for $\theta = 0.1$ ($\|P - Q_\theta\| \approx 0.163$).

It follows from Figure 1 that 2 series terms leads to a norm error below 10^{-4}. In other words, $V(2, P, Q_\theta)$ and $S_{\alpha=10^{-8}}(2, P, Q_\theta)$ already provide accurate approximations for Π_{Q_θ}. For $N \geq 4$, the bound for $\|E_\alpha(N, P, Q_\theta)\|$ becomes smaller than the actual value of $\|E_\alpha(N, P, Q_\theta)\|$. This is due to similar numerical issues as mentioned in Remark 1.

The value of θ is small in this initial experiment to ensure that $\|U\| < 1$ so that the bound for $\|W(N, P, Q_\theta)\|$ (from Theorem 3) holds. In this context see also Remark 4. However, the development of $V(N, P, Q_\theta)$ does not rely on this restriction so that θ may be increased for testing purposes. The results for approximating Π_{Q_θ} with $\theta = 0.1$ can be found in Figure 2.

When $\theta = 0.1$, P and Q_θ differ more so that relatively more series terms are required to obtain a precision of 10^{-4}, i.e., $N = 4$ terms. Furthermore, the bound for $\|E_\alpha(N, P, Q_\theta)\|$ is sharp enough to be used in practice to guarantee a certain precision.

5. CONCLUSION

The introduced series expansions in this paper allow efficient updating of the ergodic projector of Markov multi-chains. Numerical examples illustrate the usefulness of the presented approximations and their error bounds. Future research will be on the analysis of local perturbations, i.e., perturbations restricted to a subnetwork, extensions to perturbations of the structure of the network, and tightening the error bound for the ergodic projector series expansion.

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