Automata-based Optimization of Interaction Protocols for Scalable Multicore Platforms (Technical Report)

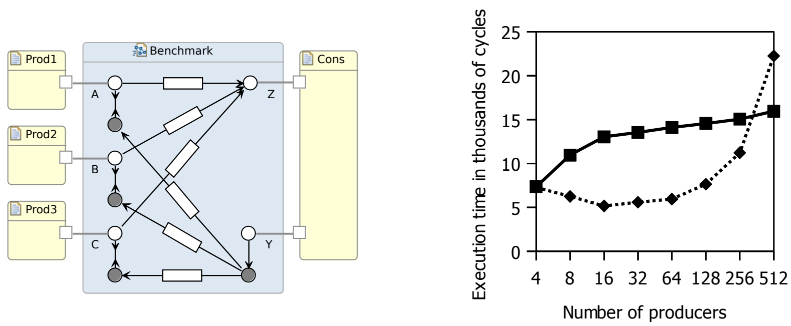
Sung-Shik T.Q. Jongmans, Sean Halle, and Farhad Arbab

Centrum Wiskunde & Informatica, Amsterdam, Netherlands
[jongmans,sean,farhad]@cwi.nl

Abstract. Multicore platforms offer the opportunity for utilizing massively parallel resources. However, programming them is challenging. We need good compilers that optimize commonly occurring synchronization/interaction patterns. To facilitate optimization, a programming language must convey what needs to be done in a form that leaves a considerably large decision space on *how* to do it for the compiler/run-time system.
Reo is a coordination-inspired model of concurrency that allows compositional construction of interaction protocols as declarative specifications. This form of protocol programming specifies only *what* needs to be done and leaves virtually all *how*-decisions involved in obtaining a concrete implementation for the compiler and the run-time system to make, thereby maximizing the potential opportunities for optimization. In contrast, the imperative form of protocol specification in conventional concurrent programming languages, generally, restrict implementation choices (and thereby hamper optimization) due to overspecification.
In this paper, we use the Constraint Automata semantics of Reo protocols as the formal basis for our optimizations. We optimize a generalization of the producer-consumer pattern, by applying CA transformations and prove the correctness of the transforms.

1 Introduction

Context. Coordination languages have emerged for the implementation of protocols among concurrent entities (e.g., threads on multicore hardware). One such language is Reo [1,2], a graphical language for compositional construction of *connectors* (i.e., custom synchronization protocols). Figure 1a shows an example. Briefly, a connector consists of one or more edges (henceforth referred to as *channels*), through which data items flow, and a number of nodes (henceforth referred to as *ports*), on which channel ends coincide. The connector in Figure 1a contains three different channel classes, including standard synchronous channels (normal arrows) and asynchronous channels with a buffer of capacity 1 (arrows decorated with a white rectangle, which represents a buffer). Through connector *composition* (the act of gluing connectors together on their shared ports), programmers can construct arbitrarily complex connectors. As Reo supports both



(a) Connector

(b) Per-interaction overhead for the Pthreads-based implementation (continuous line; squares) and the pre-optimized CA-based implementation (dotted line; diamonds)

Fig. 1: Producers–consumer benchmark

synchronous and asynchronous channels, connector composition enables mixing synchronous and asynchronous communication within the same specification.

Especially when it comes to multicore programming, Reo has a number of advantages over conventional programming languages with a fixed set of low-level synchronization constructs (locks, mutexes, etc.). Programmers using such a conventional language have to translate the synchronization needs of their protocols into the synchronization constructs of that language. Because this translation occurs in the mind of the programmer, invariably some context information either gets irretrievably lost or becomes implicit and difficult to extract in the resulting code. In contrast, Reo allows programmers to compose their own synchronization constructs (i.e., connectors) at a high abstraction level to perfectly fit the protocols of their application. Not only does this reduce the conceptual gap for programmers, which makes it easier to implement and reason about protocols, but by preserving all relevant context information, such user-defined synchronization constructs also offer considerable novel opportunities for compilers to do optimizations on multicore hardware. This report shows one such occasion.

Additionally, Reo has several software engineering advantages as a domain-specific language for protocols [3]. For instance, Reo forces developers to separate their computation code from their protocol code. Such a separation facilitates verbatim reuse, independent modification, and compositional construction of protocol implementations (i.e., connectors) in a straightforward way. Moreover, Reo has a strong mathematical foundation [4], which enables formal connector analyses (e.g., deadlock detection, model checking [5]).

To use connectors in real programs, developers need tools that automatically generate executable code for connectors. In previous work [6], we therefore developed a Reo-to-C compiler, based on Reo's formal semantics of *constraint automata* (CA) [7]. In its simplest form, this tool works roughly as follows. First, it extracts from an input XML representation of a connector a list of its primitive constituents.[1] Second, it consults a database to find for every constituent in the list a "small" CA that formally describes the behavior of that particular constituent. Third, it computes the product of the CA in the constructed collection to obtain one "big" CA describing the behavior of the whole connector. Fourth, it feeds a data structure representing that big CA to a template. Essentially, this template is an incomplete C file with "holes" that need be "filled". The generated code simulates the big CA by repeatedly computing and firing eligible transitions in an event-driven fashion. It runs on top of Proto-Runtime [8,9], an execution environment for C code on multicore hardware. A key feature of Proto-Runtime is that it provides more direct access to processor cores and control over scheduling than threading libraries based on OS threads, such as Pthreads [10].

Problem. Figure 1a shows a connector for a protocol among $k = 3$ producers and one consumer in a producers–consumer benchmark. Every producer loops through the following steps: (i) it produces, (ii) it blocks until the consumer has signaled ready for processing the next batch of productions, and (iii) it sends its production. Meanwhile, the consumer runs the following loop: (i) it signals ready, and (ii) it receives exactly one production from every producer in arbitrary order. We compared the CA-based implementation generated by our tool with a hand-crafted implementation written by a competent C programmer using Pthreads, investigating the time required for communicating a production from a producer to the consumer as a function of the number of producers.

Figure 1b shows our results. On the positive side, for $k \leq 256$, the CA-based implementation outperforms the hand-crafted implementation. For $k = 512$, however, the Pthreads-based implementation outperforms the generated implementation. Moreover, the dotted curve looks disturbing, because it grows more-than-linearly in k: indeed, the CA-based implementation scales poorly. (We skip many details of this benchmark, including those of the Pthreads-based implementation, and the meaning/implications of these experimental results. The reason is that this report is *not* about this benchmark, and its details do not matter. We use this benchmark only as a concrete case to better explain problems of our compilation approach and as a source of inspiration for solutions.)

Contribution. In this report, we report on work at improving the scalability of code generated by our Reo-to-C compiler. First, we identify a cause of poor scalability: briefly, computing eligibility of k transitions in producers–consumer-style protocols (and those generalizations thereof that allow any synchronization involving one party from every one of ℓ groups) takes $\mathcal{O}(k)$ time instead of $\mathcal{O}(1)$, of which the Pthreads-based implementation shows that it is possible. Second, to familiarize the reader with certain essential concepts, we explain a manual

[1] Programmers can use the ECT plugins for Eclipse (`http://reo.project.cwi.nl`) to draw connectors such as the one in Figure 1a, internally represented as XML.

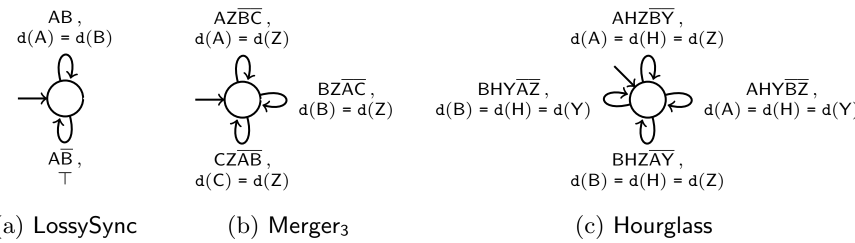


Fig. 2: Example CA, called LossySync, Merger$_3$ and Hourglass

solution (in terms of Reo's CA semantics) that achieves $\mathcal{O}(1)$. Third, we propose an automated, general solution, built upon the same concepts as the manual solution. We formalize this automated solution and prove it correct. Although inspired by our work on Reo and formulated in terms of CA, we make more general contributions beyond Reo and CA, better explained in our conclusion.

We organized the rest of this report as follows. In Section 2, we explain CA. In Section 3, we analyze how the Pthreads-based implementation avoids scalability issues and how we can export that to our setting. In Sections 4–6, we automate the solution proposed in Section 3. Section 7 concludes this report. Definitions and detailed proofs appear in the appendix.

Although inspired by Reo, we can express our main results in a purely automata-theoretic setting. We therefore skip a primer on Reo [1,2].

2 Constraint Automata

Constraint automata are a general formalism for describing systems behavior and have been used to model not only connectors but also, for instance, actors [11]. Figure 2 shows examples.[2,3] In the context of this report, a CA specifies *when* during execution of a connector *which* data items flow *where*. Structurally, every CA consists of finite sets of states, transitions between states, and ports. States represent the internal configurations of a connector, while transitions describe its atomic execution steps. Every transition has a label that consists of two

[2] The LossySync CA models a connector with one input port A and one output port B. It repeatedly chooses between two atomic execution steps (constrained by availability of pending I/O operations): synchronous flow of data from A to B or flow of data only on A (after which the data is lost, before reaching B). The Merger$_3$ CA models a connector with three input ports A, B, and C and one output port Z. It repeatedly chooses between three atomic execution steps: synchronous flow of data from A to Z, from B to Z, or from C to Z. Finally, the Hourglass CA models a connector with two input ports A and B, one internal port H, and two output ports Y and Z. It repeatedly chooses between four atomic execution steps: synchronous flow of data from A via H to Y, from A via H to Z, from B via H to Y, or from B via H to Z.

[3] We show only single state CA for simplicity. Generally, a CA can have any finite number of states, and the results in this report are applicable also to such CA.

p ::= any element from $\mathbb{P}$ORT
Ψ ::= any set of SCs
a ::= 0 | $\underline{1}$ | p
ψ ::= a | $\overline{\psi}$ | $\psi + \psi$ | $\psi \cdot \psi$ | $\bigoplus(\Psi)$

(a) Synchronization constraints

p ::= any element from $\mathbb{P}$ORT
P ::= any subset of $\mathbb{P}$ORT
b ::= $\bot$ | $\top$ | $\mathtt{Eq}(P)$ | $\mathtt{d}(p)$ = $\mathtt{d}(p)$
ϕ ::= b | $\neg\phi$ | $\phi \vee \phi$ | $\phi \wedge \phi$

(b) Data constraints

Fig. 3: Syntax

elements: a *synchronization constraint* (SC) and a *data constraint* (DC). An SC is a propositional formula that specifies which ports synchronize in a firing transition (i.e., where data items flow); a DC is a propositional formula that (under)specifies which particular data items flow where. For instance, in Figure 2a, the DC $\mathtt{d}(\mathsf{A})$ = $\mathtt{d}(\mathsf{B})$ means that the data item on A equals the data item on B; the DC $\top$ means that it does not matter which data items flow. Let $\mathbb{P}$ORT denote the global set of all ports. Formally, an SC is a word ψ generated by the grammar in Figure 3a, while a DC is a word ϕ generated by the grammar in Figure 3b.

Figure 3a generalizes the original definition of SCs as sets of ports interpreted as conjunctions [7] (shortly, we elaborate on the exact correspondence). Operator $\bigoplus$ is a uniqueness quantifier: $\bigoplus(\Psi)$ holds if exactly one SC in Ψ holds. Also, we remark that predicate $\mathtt{Eq}(P)$ is novel. It holds if equal data items are distributed over all ports in P. In many practical cases—but not all—we can replace a DC of the shape $\mathtt{d}(p_1)$ = $\mathtt{d}(p_2)$ with $\mathtt{Eq}(P)$ if $\{p_1 , p_2\} \subseteq P$. In the development of our optimization technique, $\mathtt{Eq}(P)$ plays an important role (see also Section 7).

Let $\mathbb{D}$ATA denote the set of all data items. Formally, we interpret SCs and DCs over *distributions* of data over ports, $\delta : \mathbb{P}\text{ORT} \rightharpoonup \mathbb{D}\text{ATA}$, using relations $\overset{\text{sc}}{\models}$ and $\overset{\text{dc}}{\models}$ and the corresponding equivalence relations $\equiv_{\text{sc}}$ and $\equiv_{\text{dc}}$. Their definition for negation, disjunction, and conjunction is standard; for atoms, we have:

$$\delta \overset{\text{sc}}{\models} p \textbf{ iff } p \in \text{Dom}(\delta) \qquad \begin{array}{lll} \delta \overset{\text{dc}}{\models} \mathtt{Eq}(P) & \textbf{iff} & |\text{Img}(\delta|_P)| = 1 \\ \delta \overset{\text{dc}}{\models} \mathtt{d}(p_1) = \mathtt{d}(p_2) & \textbf{iff} & \delta(p_1) = \delta(p_2) \end{array}$$

Let $\sum(\{\psi_1 , \ldots , \psi_k\})$ and $\prod(\{\psi_1 , \ldots , \psi_k\})$ abbreviate $\psi_1 + \cdots + \psi_k$ and $\psi_1 \cdots \psi_k$, let $\mathbb{SC}$ denote the sets of all SCs, and let $\mathsf{SC}(P)$ and $\mathsf{DC}(P)$ denote the sets of all SCs and all DCs over ports in P.

A constraint automaton is a tuple $(Q , P , \longrightarrow , \imath)$ with Q a set of states, $P \subseteq \mathbb{P}\text{ORT}$ a set of ports, $\longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q$ a transition relation labeled with [SC, DC]-pairs of the form (ψ , ϕ), and $\imath \in Q$ an initial state.

A distribution δ represents a single atomic execution step of a connector in which data item $\delta(p)$ flows on port p (for all ports in the domain of δ). A CA α accepts *streams* (i.e., infinite sequences) of such distributions. Every such a stream represents one possible infinite execution of the connector modeled by α. Intuitively, to see if α accepts a stream σ, starting from the initial state, take the first element $\sigma(0)$ from the stream, check if α has a (ψ , ϕ)-labeled transition from the current state such that $\sigma(0) \overset{\text{sc}}{\models} \psi$ and $\sigma(0) \overset{\text{dc}}{\models} \phi$, and if so, make this transition, remove $\sigma(0)$ from the stream, and repeat.

(a) Current (b) Manual improvement (c) Automated improvement

Fig. 4: Code generation diagrams

Our CA definition generalizes the original definition of CA [7], because Figure 3a generalizes the original definition of SCs. However, CA as originally defined still play a role in the development of our optimization technique: all input CA that this technique operates on are original. Therefore, we make more precise what "originality" means. First, let a *P-complete product* be a product of either a positive or a negative literal for every port in P. Intuitively, a P-complete product specifies not only which ports participate in a transition, but it also makes explicit which ports idle in that transition. Let $\texttt{cp}(P\,,\,P_+)$ denote a P-complete product with positive literals $P_+ \subseteq P$. Then, we call an SC ψ original if a set P_+ exists such that $\texttt{cp}(P\,,\,P_+) \equiv \psi$ (originally, set P_+ would be the SC); we call a CA original if it has only original SCs. All CA in Figure 2 are original.

We adopt *bisimilarity* on CA as behavioral congruence, derived from the definition for original CA of Baier et al. [7]. Roughly, if α and β are bisimilar, denoted as $\alpha \sim \beta$, α can simulate every transition of β in every state and vice versa (see Definition 32 in Appendix B).

3 Enhancing Scalability: Problem and Solution

We study the scalability of code generated by our compiler using Figure 4. We start with Figure 4a, which summarizes the code generation process of our current tool: given an original CA α (computed for the connector to generate code for), it generates a piece of code `alpha` by applying transformation g_1.

Essentially, `alpha` consists of an event-driven *handler*, which simulates α. This handler runs concurrently with the code of its environment (i.e., the code of the entitites under coordination), whose events (i.e., I/O operations performed on ports) it listens and responds to, as follows. Whenever the environment performs an I/O operation on a port p, it assigns a representation of that operation to an `event` variable in a data structure for p (also generated by transformation g_1 and part of `alpha`). This causes the handler to start a new round of simulating α. Based on the state of α that the handler at that point should behave as, the handler knows which transitions of α *may* fire. Which of those transitions actually *can* fire, however, depends also on the pending events that previously occurred (i.e., the pending I/O operations on ports). To investigate this, the handler checks *for every transition* that may fire if the pending events (including the new one) can constitute a distribution δ that satisfies the transition's label. If so, the handler fires the transition: it distributes data over ports according to

δ, and the events involved dissolve. Otherwise, if no transition can fire, all events remain for the next round, and the handler goes dormant.

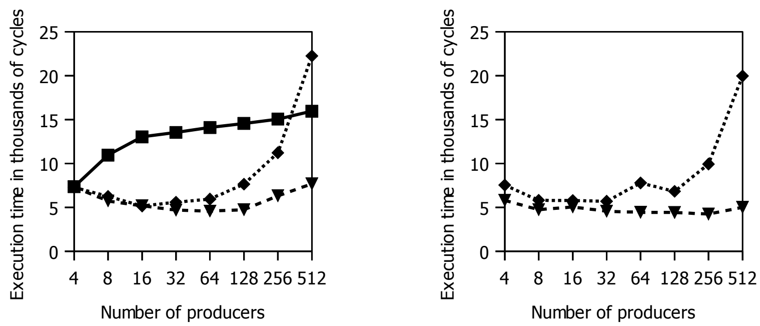
Now, recall our producers–consumer benchmark in Section 1. Figure 2b shows the CA for the connector in Figure 1a.[4] Generally, for an arbitrary number of producers k, the corresponding CA α_k has k transitions. Consequently, in the worst case, the handler in the generated `alpha_k` code performs k checks in every event handling round, which takes $\mathcal{O}(k)$ time. Figure 1b shows this as a more-than-linear increase in execution time for the dotted curve.[5] The Pthreads-based implementation, in contrast, uses a queue for lining up available productions. To receive a production, the consumer simply dequeues, which takes only $\mathcal{O}(1)$ time (ignoring, for simplicity, the overhead of synchronizing queue accesses). Figure 1b shows this as a linear increase in execution time for the continuous curve.

Intuitively, by checking all transitions to make the consumer receive, the generated CA-based implementation performs an exhaustive search for a particular producer that sent a production. In contrast, by using a queue, the Pthreads-based implementation avoids such a search: the queue embodies that in this protocol, it does not matter which *particular* producer sent a production as long as *some* producer has done so (in which case the queue is nonempty). The producers are really *indistinguishable* from the perspective of the consumer. Thus, to improve the scalability of code generated by our tool, we want to export the idea of "using queues to leverage indistinguishability" to our setting.

Figure 4b shows a first attempt at achieving this goal: we introduce a manual transformation h that takes `alpha` as input and hacks together a new piece of code `beta`, which should (i) behave as `alpha`, (ii) demonstrate good scalability, and (iii) use queues. For instance, in our producers–consumer example ($k = 3$), h works roughly as follows. First, h replaces the `event` variable in the data structure for every port $p \in \{\mathsf{A}, \mathsf{B}, \mathsf{C}, \mathsf{Z}\}$ with an `eventQueue` variable that points to a queue of pending events. In this new setup, to perform an I/O operation, the environment enqueues an `eventQueue`, while handler code tests `eventQueue`s for nonemptiness to check SCs, peeks `eventQueue`s to check DCs, and dequeues `eventQueue`s to fire transitions. Subsequently, h adds initialization code to `alpha` to ensure that the `eventQueue` variables of ports A, B, and C all point to the same shared queue, while the `eventQueue` variable for port Z points to a different queue. Here, h effectively exploits the indistinguishability property of producers

[4] To be precise, the CA in Figure 2b describes the behavior of one of the *synchronous regions* of the connector in Figure 1a (i.e., a particular subconnector of the whole). This point is immaterial to our present discussion, however, and ignoring it simplifies our presentation without loss of generality or applicability.

[5] The growth is more-than-linear instead of just linear because of the barrier in the protocol. When producer P is ready to send its $(i+1)$-th production, the consumer may not yet have received the i-th production from all other producers. Then, P must wait until the consumer signals ready (i.e., the barrier). In the worst case, however, the consumer has received an i-th production only from P such that P has to wait $(k-1) \cdot \mathcal{O}(k)$ time. Afterward, it takes another $\mathcal{O}(k)$ time for P to send its $(i+1)$-th production. Consequently, sending the $(i+1)$-th production takes $k \cdot \mathcal{O}(k)$ time, and the complexity of sending a production lies between $\mathcal{O}(k)$ and $\mathcal{O}(k^2)$.



(a) With barrier

(b) Without barrier

Fig. 5: Per-interaction overhead for the Pthreads-based implementation (continuous line; squares), the pre-optimized CA-based implementation (dotted line; diamonds), and the optimized, h-transformed CA-based implementation (dashed line; triangles) of the producers–consumer scenario in Figure 1

by making *the ports* that those producers use indistinguishable in our setting. Finally, h updates the handler code such that it processes the shared queue only once per event handling round instead of thrice (i.e., once for every transition). From an automata-theoretic perspective, h replaces the implementation of the three "physical" transitions with an implementation of one merged "virtual" transition. When the handler fires this virtual transition at run-time, it actually fires one of the three physical transitions.

Property (iii) holds of the piece of code `beta` resulting from applying h to `alpha` as just described. Figure 5 shows that also property (ii) holds. The dashed curve in Figure 5a shows execution times of h-transformed code of the CA-based implementation in the producers–consumer benchmark. The h-transformed code scales much better than the original code. Additionally, Figure 5b shows execution times of the producers–consumer benchmark without a barrier (i.e., producers send productions whenever they want). In this variant, h achieves even better results: it transforms a poorly scalable program into one that scales perfectly.[6]

Establishing property (i), however, is problematic. Although we can informally argue that it holds, proving this—formally showing the equivalence of two concurrent C programs—seems prohibitively complex. That aside, the manual nature of h makes its usage generally impractical, and it seems extremely difficult to automate it: an automated version of h would have to analyze C code

[6] Of course, in many cases and for many applications, a purely asynchronous producers–consumer protocol without a barrier, as in Figure 5b, suffices. The reason that we initially focused on a producers–consumer protocol with a barrier, which is also useful yet in other applications, is that its mix of synchrony and asynchrony makes it a harder, and arguably more interesting, protocol to achieve good scalability for. Comparing the results in Figures 5a and 5b also shows this.

to recover relevant context information about the protocol, which is not only hard but often theoretically impossible. Similarly, it seems infeasible to write an optimizing compiler able to transform, for instance, less scalable Pthreads-based implementations of the producers-consumer scenario (without queues) into the Pthreads-based implementation (with queues) used in our benchmark. The inability of compilers for lower-level languages to do such optimizations seems a significant disadvantage of using such languages for multicore programming.

We therefore pursue an alternative approach, outlined in Figure 4c: we introduce a transformation f that takes CA α as input—instead of the low-level C code generated for it—and transforms it into an equivalent automaton β, a variant of α with *merged transitions* (cf. transformation h, which implicitly replaced the implementation of several physical transitions with one virtual transition). Crucially, α still explicitly contains all relevant context information about the protocol, exactly what makes f eligible to automation. In particular, to merge transitions effectively, f carefully inspects transition labels and takes port indistinguishability into account. The resulting merged transitions have an "obvious" and mechanically obtainable implementation using queues. A subsequent transformation g_2, from β to `beta`, performs this final straightforward step.

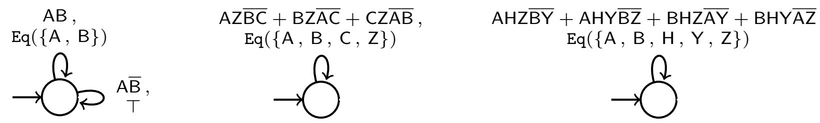
We divide transformation $\alpha \xrightarrow{f} \beta$ into a number of constituent transformations $\alpha \xrightarrow{f_1} \beta' \xrightarrow{f_2} (\beta' , \Gamma) \xrightarrow{f_3} \beta$, discussed in detail in the following sections.

4 Transformation f_1: Preprocessing

Transformation f_1 aims at merging transitions $t_1 , \ldots , t_k$ into one transition $(q , \psi , \mathtt{Eq}(P) , q')$, where $\psi = \sum(\{\psi_1 , \ldots , \psi_k\})$. It consists of two steps.

In the first step, transformation f_1 replaces DCs on transitions of $\alpha = (Q , P , \longrightarrow , \imath)$ with $\mathtt{Eq}(P)$, as follows. Because α is an original CA (our current code generator can handle only original CA), every SC in α is an original SC: for every transition label (ψ , ϕ), a set of ports P_+ exists such that $\mathtt{cp}(P , P_+) \equiv \psi$. Now, for every product in *disjunctive normal form* (DNF) of ϕ, transformation f_1 constructs a graph with vertices P_+ and an edge (p_1 , p_2) for every $\mathtt{d}(p_1) = \mathtt{d}(p_2)$ literal. Because $\mathtt{cp}(P , P_+) \equiv \psi$, if the resulting graph is connected, the product of the $\mathtt{d}(p_1) = \mathtt{d}(p_2)$ literals is equivalent to $\mathtt{Eq}(P)$. Thus, f_1 replaces every transition label (ψ , ϕ) in α with an equivalent label (ψ , ϕ'), where ϕ' denotes the modified DNF of ϕ, with $\mathtt{Eq}(P)$ for every product of $\mathtt{d}(p_1) = \mathtt{d}(p_2)$ literals if those literals induce a connected graph. Let α' denote the resulting CA. We can prove that $\alpha' \sim \alpha$ holds (see Lemma 16 in Appendix B).

In the second step, transformation f_1 merges, for every pair of states (q , q'), all transitions from q to q' labeled by DC ϕ into one new transition. (The individual transitions differ only in their SC.) Every resulting transition has as its SC the sum of the SCs of the individual transitions. Figure 6 shows examples. We denote the resulting CA by $f_1(\alpha)$. The following proposition holds, because choices between individual transitions in α are encoded in $f_1(\alpha)$ by sum-SCs of merged transitions. Consequently, α and $f_1(\alpha)$ can simulate each other's steps.



(a) LossySync (b) Merger_3 (c) Hourglass

Fig. 6: Application of transformations f_1 to the CA in Figure 2

Proposition 1. $f_1(\alpha) \sim \alpha'$

5 Transformation f_2: Constructing Hypergraphs

Every merged transition resulting from the previous preprocessing transformations can *perhaps* be implemented using queues along the same lines as transformation h (see Section 3). In the first place, this depends on the extent to which ports in a merged transition are indistinguishable: no indistinguishable ports means no queues. Second, the SC of a merged transition should make port indistinguishability (i.e., queues), if present, apparent and *mechanically detectable*. The SCs of transitions in $f_1(\alpha)$ fail to do so. For instance, we (hence a computer) cannot directly derive from the syntax of SC $\mathsf{AZ}\overline{\mathsf{BC}} + \mathsf{BZ}\overline{\mathsf{AC}} + \mathsf{CZ}\overline{\mathsf{AB}}$ in Figure 6b that its transition has a scalable implementation with queues. In contrast, the equivalent SC $\bigoplus(\{\mathsf{A}\,,\,\mathsf{B}\,,\,\mathsf{C}\}) \cdot \mathsf{Z}$ makes this much more apparent. From this SC, we can "obviously" (and mechanically by transformation g_2 in Figure 4c) conclude that ports A, B, and C may share the same queue, from which exactly one element is dequeued per firing, because they are indistinguishable indeed: intuitively, if $\delta \overset{\text{sc}}{\models} \bigoplus(\{\mathsf{A}\,,\,\mathsf{B}\,,\,\mathsf{C}\}) \cdot \mathsf{Z}$, we cannot know which one of A, B, or C holds, unless we inspect δ. Thus, beside automatically detecting indistinguishable ports in a transition, to actually reveal them as queues, we additionally need an algorithm for syntactically manipulating that transition's SC. We formulate both these aspects in terms of a per-transition *hypergraph* [12]. Working with hypergraph representations simplifies our reasoning about, and manipulation of, SCs modulo associativity and commutativity. We compute hypergraphs as follows.

Let $\alpha = (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)$ be an original CA as before, and let $(q\,,\,\psi\,,\,\phi\,,\,q')$ be a (merged) transition in $f_1(\alpha)$. Because α is an original CA and by the construction of $f_1(\alpha)$, we know that ψ is a sum of P-complete products of ports (e.g., Figure 6). Because every single port p is equivalent to $\bigoplus(\{p\})$, transformation f_2 can represent ψ as a set $\mathcal{E}$ of sets E of sets V: $\mathcal{E}$ represents the outer sum, every E represents a P-complete product (E includes/excludes every positive/negative port), and every V represents an inner exclusive sum. For instance, $\{\{\{\mathsf{A}\}\,,\,\{\mathsf{Z}\}\}\,,\,\{\{\mathsf{B}\}\,,\,\{\mathsf{Z}\}\}\,,\,\{\{\mathsf{C}\}\,,\,\{\mathsf{Z}\}\}\}$ represents the SC of the transition in Figure 6b. Transformation f_2 considers $\mathcal{E}$ as the set of hyperedges of a hypergraph over the set of vertices $\wp(\mathsf{Port}(\psi))$, where $\mathsf{Port}(\psi)$ denotes the ports occurring in ψ (i.e., every vertex is a set of ports). Formally, f_2 computes a function graph. Let GRAPH denote the set of all hypergraphs with sets of ports as vertices.

Definition 1. $\mathsf{graph} : \mathbb{SC} \rightharpoonup \mathbb{G}\textsc{raph}$ *denotes the partial function from* SCs *to hypergraphs defined as:*[7]

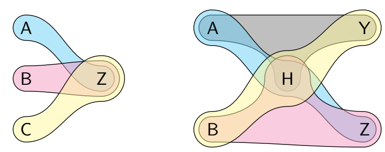
$$\mathsf{graph}(\psi) = (\wp(\mathsf{Port}(\psi))\,,\, \left\{ E \;\middle|\; \begin{array}{l} E = \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\} \\ \textbf{and}\;\; P_+ \subseteq \mathsf{Port}(\psi) \;\;\textbf{and}\;\; P_+ \in \mathcal{P} \end{array} \right\})$$

$$\textbf{if}\;\; \left[\psi = \textstyle\sum(\left\{ \psi' \;\middle|\; \begin{array}{l} \psi' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \;\;\textbf{and} \\ P_+ \subseteq \mathsf{Port}(\psi) \;\;\textbf{and}\;\; P_+ \in \mathcal{P} \end{array} \right\}) \;\;\textbf{for some}\;\; \mathcal{P}\right]$$

(The side condition states just that ψ is a sum of P-complete products of ports.)

Figure 7 shows example hypergraphs (without unconnected vertices).

We define the *meaning* of a hypergraph as a sum of products of exclusive sums, where every product corresponds to a hyperedge. Such a product consists of exclusive sums of positive ports (one for each vertex in the hyperedge), and it consists of negative ports (one for every port outside the vertices in the hyperedge). We can show that graph is an isomorphism (i.e., $\mathsf{graph}(\psi)$ is a sound and complete representation of ψ).



(a) Merger$_3$ (b) Hourglass

Fig. 7: Hypergraphs for the transitions of the CA in Figure 6

Definition 2. $[\![\cdot]\!] : (\wp(\mathbb{V}\textsc{er}) \times \wp(\mathbb{P}\textsc{ort})) \cup \mathbb{G}\textsc{raph} \to \mathbb{SC}$ *denotes the function from* [*hyperedge, set of ports*]*-pairs and hypergraphs to* SCs *defined as:*

$$\begin{aligned} [\![E]\!]_P &= \textstyle\prod(\{\psi \mid \psi = \bigoplus(V) \;\textbf{and}\; V \in E\} \cup \\ &\qquad \{\psi \mid \psi = \overline{p} \;\textbf{and}\; p \in P \setminus (\bigcup E)\}) \\ [\![(\mathcal{V}\,,\,\mathcal{E})]\!] &= \textstyle\sum(\{\psi \mid \psi = [\![E]\!]_{\bigcup \mathcal{V}} \;\textbf{and}\; E \in \mathcal{E}\}) \end{aligned}$$

Theorem 1. *(Theorem 3 in Appendix B)*

$$\left[\mathsf{graph}(\psi) \textit{ is defined}\right] \;\textbf{implies}\; \psi \equiv [\![\mathsf{graph}(\psi)]\!]$$

In summary, transformation f_2 computes graph for every merged transition in $f_1(\alpha)$ and stores each of those graphs in a set Γ (indexed by transitions).

Hypergraphs as introduced are generic representations of synchronization patterns, isomorphic to but independent of SCs in CA. This reinforces that our optimization approach, transformation f, is not tied CA but a generally applicable technique when relevant context information is available.

6 Transformation f_3: Manipulating SCs

Transformation f_3 aims at making all indistinguishable ports (hence queues) in SCs on (merged) transitions in $f_1(\alpha)$ apparent by analyzing and manipulating the hypergraphs in Γ, computed by transformation f_2. It consists of two steps.

[7] Let $\wp(X)$ denote the power set of X.

In the first step, transformation f_3 computes the indistinguishable ports under every transition $t = (q\,,\,\psi\,,\,\phi\,,\,q')$ in $f_1(\alpha)$. We call the ports in a set I indistinguishable under t if for every distribution δ such that $\delta \overset{\text{sc}}{\models} \psi$ and $|I \cap \mathrm{Dom}(\delta)| = 1$, we cannot deduce from $\delta|_{P \setminus I}$ which particular port in I is satisfied by δ. An example appeared in the first paragraph of Section 5. In an implementation with a queue shared among the ports in I, this means that whenever t fires, we know that exactly one port in I participated in the transition but not which one, even if we know all other participating ports (i.e., those outside I).

By analyzing hypergraph $\gamma_t \in \mathit{\Gamma}$ for the SC ψ of t, transformation f_3 computes maximal sets of indistinguishable ports under t (larger sets of indistinguishable ports means larger queues means better scalability), as follows. Recall from Section 5 that γ_t represents a sum (hyperedge relation) of P-complete products (hyperedges) of singleton exclusive sums (vertices). To understand how port indistinguishability displays in γ_t, suppose that ports $p_1\,,\,p_2 \in P$ are indistinguishable, and let δ be a distribution such that $\delta \overset{\text{sc}}{\models} [\![\gamma_t]\!]$. Because γ_t's hyperedge relation $\mathcal{E}$ represents a sum of P-complete products, exactly one hyperedge $E \in \mathcal{E}$ exists such that δ satisfies $[\![E]\!]_P$. Then, because $|\{p_1\,,\,p_2\} \cap \mathrm{Dom}(\delta)| = 1$, a vertex $V \in E$ exists such that $p_1 \in V$ or $p_2 \in V$.[8] In fact, because every hyperedge consists of singleton vertices, either $\{p_1\} \in E$ or $\{p_2\} \in E$. Now, by inspecting $\delta|_{P \setminus \{p_1, p_2\}}$, we can infer the other vertices in E, beside either $\{p_1\}$ or $\{p_2\}$. Let E' denote this set of vertices, and observe that either $E = E_1 = E' \uplus \{\{p_1\}\}$ or $E = E_2 = E' \uplus \{\{p_2\}\}$. Because both options are possible, $\mathcal{E}$ necessarily includes both E_1 and E_2, and importantly, E_1 and E_2 are equal up to p_1 and p_2.

Generalizing this example from $\{p_1\,,\,p_2\}$ to arbitrarily sized sets I, informally, the ports in I are indistinguishable if every port in I is involved in the same hyperedges as every other port in I *up to occurrences of ports in* I. The following definitions make this generalization formally precise. First, we introduce a function Edge that determines for a port p which hyperedges in $\mathcal{E}$ include p. (In fact, $\mathsf{Edge}(p\,,\,\mathcal{E})$ contains all such hypergedges up to occurrences of vertices with p.) Then, we define a function $\bigstar$ that computes maximal sets of ports with the same set $\mathsf{Edge}(p\,,\,\mathcal{E})$. Importantly, $\bigstar$ yields a partition of the set of ports in vertices connected by $\mathcal{E}$, denoted by $\mathsf{Port}(\mathcal{E})$. Henceforth, we therefore call every maximal set of indistinguishable ports computed by $\bigstar$ a *part*.

Definition 3. $\mathsf{Edge} : \mathbb{P}\text{ORT} \times \wp^2(\mathbb{V}\text{ER}) \rightarrow \wp^2(\mathbb{V}\text{ER})$ *denotes the function from* $\big[$*port, set of hyperedges*$\big]$*-pairs to sets of hyperedges defined as:*

$$\mathsf{Edge}(p\,,\,\mathcal{E}) = \{\mathcal{W} \mid \mathcal{W} = E \setminus \{V\} \textbf{ and } p \in V \in E \in \mathcal{E}\}$$

Definition 4. $\bigstar : \wp^2(\mathbb{V}\text{ER}) \rightarrow \wp^2(\mathbb{P}\text{ORT})$ *denotes the function from sets of hyperedges to sets of sets of ports defined as:*

$$\bigstar(\mathcal{E}) = \{P \mid P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \textbf{ for all } p\big]\}$$

[8] Otherwise, if $p_1\,,\,p_2 \notin V$ for all $V \in E$, the P-complete product represented by E contains $\overline{p}_1$ and $\overline{p}_2$ such that $\delta \overset{\text{sc}}{\not\models} p_1$ and $\delta \overset{\text{sc}}{\not\models} p_2$. This contradicts the assumption $|\{p_1\,,\,p_2\} \cap \mathrm{Dom}(\delta)| = 1$, which implies either $\delta \overset{\text{sc}}{\models} p_1$ or $\delta \overset{\text{sc}}{\models} p_2$.

$$
\begin{aligned}
\mathsf{Edge}(\mathsf{A}\,,\,\mathcal{E}) &= \{\{\{\mathsf{Z}\}\}\}\\
\mathsf{Edge}(\mathsf{B}\,,\,\mathcal{E}) &= \{\{\{\mathsf{Z}\}\}\}\\
\mathsf{Edge}(\mathsf{C}\,,\,\mathcal{E}) &= \{\{\{\mathsf{Z}\}\}\}\\
\mathsf{Edge}(\mathsf{Z}\,,\,\mathcal{E}) &= \{\{\{\mathsf{A}\}\}\,,\,\{\{\mathsf{B}\}\}\,,\,\{\{\mathsf{C}\}\}\}\\
\bigstar(\mathcal{E}) &= \{\{\mathsf{A}\,,\,\mathsf{B}\,,\,\mathsf{C}\}\,,\,\{\mathsf{Z}\}\}
\end{aligned}
$$

(a) Merger_3

$$
\begin{aligned}
\mathsf{Edge}(\mathsf{A}\,,\,\mathcal{E}) &= \{\{\{\mathsf{H}\}\,,\,\{\mathsf{Y}\}\}\,,\,\{\{\mathsf{H}\}\,,\,\{\mathsf{Z}\}\}\}\\
\mathsf{Edge}(\mathsf{B}\,,\,\mathcal{E}) &= \{\{\{\mathsf{H}\}\,,\,\{\mathsf{Y}\}\}\,,\,\{\{\mathsf{H}\}\,,\,\{\mathsf{Z}\}\}\}\\
\mathsf{Edge}(\mathsf{H}\,,\,\mathcal{E}) &= \{\{\{\mathsf{A}\}\,,\,\{\mathsf{Y}\}\}\,,\,\{\{\mathsf{A}\}\,,\,\{\mathsf{Z}\}\}\,,\,\{\{\mathsf{B}\}\,,\,\{\mathsf{Y}\}\}\,,\,\{\{\mathsf{B}\}\,,\,\{\mathsf{Z}\}\}\}\\
\mathsf{Edge}(\mathsf{Y}\,,\,\mathcal{E}) &= \{\{\{\mathsf{A}\}\,,\,\{\mathsf{H}\}\}\,,\,\{\{\mathsf{B}\}\,,\,\{\mathsf{H}\}\}\}\\
\mathsf{Edge}(\mathsf{Z}\,,\,\mathcal{E}) &= \{\{\{\mathsf{A}\}\,,\,\{\mathsf{H}\}\}\,,\,\{\{\mathsf{B}\}\,,\,\{\mathsf{H}\}\}\}\\
\bigstar(\mathcal{E}) &= \{\{\mathsf{A}\,,\,\mathsf{B}\}\,,\,\{\mathsf{H}\}\,,\,\{\mathsf{Y}\,,\,\mathsf{Z}\}\}
\end{aligned}
$$

(b) Hourglass

Fig. 8: Maximal sets of indistinguishable ports of the hypergraphs in Figure 7

Lemma 1. *(Lemma 12 in Appendix B)*

1. $\bigcup \bigstar(\mathcal{E}) = \mathsf{Port}(\mathcal{E})$
2. $\big[P_1 \neq P_2$ **and** $P_1\,,\,P_2 \in \bigstar(\mathcal{E})\big]$ **implies** $P_1 \cap P_2 = \emptyset$

In summary, in the first step, transformation f_3 computes maximal sets of indistinguishable ports in every merged transition $t = (q\,,\,\psi\,,\,\phi\,,\,q')$ by applying $\bigstar$ to hyperedge relation $\mathcal{E}$ in hypergraph γ_t for ψ. Figure 8 shows examples.

In the second step, f_3 manipulates $\mathcal{E}$ of every hypergraph γ_t such that afterward, every vertex in every hyperedge in $\mathcal{E}$ is a part in $\bigstar(\mathcal{E})$. Importantly, every vertex $V \in E \in \mathcal{E}$ such that $V \in \bigstar(\mathcal{E})$ represents not just any $\bigoplus$-formula but one of indistinguishable ports. Consequently, in the meaning of the manipulated γ_t, indistinguishable ports become apparent as inner $\bigoplus$-formulas as in the example in the first paragraph of Section 5.

For manipulating hyperedge relation $\mathcal{E}$, we introduce an operation $\sqcup$ that combines two *combinable* hyperedges into one in a semantics-preserving way. Roughly, we call two distinct hyperedges $E_1\,,\,E_2 \in \mathcal{E}$ combinable if we can select disjoint vertices $V_1\,,\,V_2 \in E_1 \cup E_2$ such that E_1 and E_2 are equal up to inclusion of V_1 and V_2. We denote this property as $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$. Applied to combinable hyperedges E_1 and E_2, operation $\sqcup$ removes E_1 and E_2 from $\mathcal{E}$ and adds their combination $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$ to $\mathcal{E}$. Formally, we have the following. Let $\mathbb{V}\text{ER}$ denote the set of all vertices.

Definition 5. $\curlyvee \subseteq (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times \wp^2(\mathbb{V}\text{ER})$ *denotes the relation on tuples consisting of two sets of* $\big[$*hyperedge, vertex*$\big]$*-pairs and a set of hyperedges defined as:*

$$
(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\;\textbf{iff}\;\; \begin{bmatrix} E_1\,,\,E_2 \in \mathcal{E} \;\textbf{ and }\; E_1 \neq E_2 \;\textbf{ and }\; V_1 \cap V_2 = \emptyset \\ \textbf{and }\; E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \\ \textbf{and }\; E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \end{bmatrix}
$$

Definition 6. $\sqcup : (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times \wp^2(\mathbb{V}\text{ER}) \rightharpoonup \wp^2(\mathbb{V}\text{ER})$ *denotes the partial function from tuples consisting of two* $\big[$*hyperedge, vertex*$\big]$*-pairs and a set of hyperedges to sets of hyperedges defined as:*

$$
\begin{aligned}
(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) &= \mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \\
&\quad \textbf{if }\; (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)
\end{aligned}
$$

$$\textbf{while } \left[\begin{bmatrix}(X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2)\ \textbf{and}\\ V_1 \cup V_2 \subseteq P\ \textbf{and}\ P \in \bigstar(\mathcal{E})\end{bmatrix} \textbf{ for some } X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\right] \textbf{ do}$$

$$\quad \textbf{while } \left[[(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)] \textbf{ for some } E_1\,,\,E_2\right] \textbf{ do}$$

$$\quad\quad \mathcal{E} := (E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)$$

Fig. 9: Algorithm for combining hyperedges

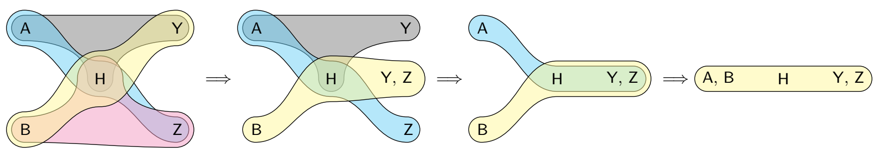


Fig. 10: Evolution of the hypergraphs in Figure 7b

Lemma 2. *(Lemma 8 in Appendix B)*

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\ \textbf{ implies }\ [\![(\mathcal{V}\,,\,\mathcal{E})]\!] \equiv_{\text{sc}} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]$$

Transformation f_3 uses operation $\sqcup$ in the algorithm for combining hyperedges in Figure 9. Essentially, as long as vertices V_1 and V_2 exist such that the ports in $V_1 \cup V_2$ are indistinguishable (as computed by $\bigstar$), the algorithm combines all combinable hyperedges that include V_1 and V_2. For instance, Figure 10 shows the evolution of the hypergraph in Figure 7b during the run of the algorithm in which it first selects Y and Z as V_1 and V_2 and afterward A and B. (In another run, the algorithm may change this order to obtain the same result.)

Let $\mathcal{E}_{\text{in}}$ and $\mathcal{E}_{\text{out}}$ denote the sets of hyperedges before and after running the algorithm. To consider the algorithm correct, $\mathcal{E}_{\text{out}}$ must satisfy two properties: it should represent an SC equivalent to the SC represented by $\mathcal{E}_{\text{in}}$ (i.e., the algorithm is semantics-preserving), and every vertex in every hyperedge in $\mathcal{E}_{\text{out}}$ should be a part in $\bigstar(\mathcal{E}_{\text{in}})$ (i.e., the algorithm effectively reveals indistinguishability). We use *Hoare logic* to prove these properties [13,14]. In particular, we can show that the triple $\{\mathsf{Pre}\}\ A\ \{\mathsf{Post}\}$ holds, where A denotes the algorithm in Figure 9. Precondition Pre states that $\gamma_t = (\mathcal{V}\,,\,\mathcal{E}_{\text{in}})$ is a hypergraph (for the SC of transition t) such that every port in a connected vertex inhabits at most one connected vertex, and such that every connected vertex is nonempty. The definition of graph in Definition 1 implies these conditions. (However, because its precondition is more liberal, the algorithm is more generally applicable.) The postcondition Post states that correctness as previously formulated holds. Formally:

$$[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{out}})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]\ \textbf{ and }\ \left[\begin{bmatrix}E \in \mathcal{E}_{\text{out}}\ \textbf{implies}\\ E \subseteq \bigstar(\mathcal{E}_{\text{in}})\end{bmatrix} \textbf{ for all } E\right]$$

Figure 11 shows the algorithm annotated with assertions for total correctness. By the axioms and rules of Hoare logic, this proof is valid if we can prove that for all six pairs of consecutive assertions, the upper assertion implies the lower one. For brevity, below, we discuss some salient aspects.

$\{\mathsf{Pre}\}$
$\{\mathsf{Inv}_1\}$
while $\left[\begin{bmatrix} (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{ for some } X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\right]$ **do**
  $\{\mathsf{Inv}_1 \textbf{ and } \mathsf{Cond}_1 \textbf{ and } |\mathcal{E}| = z_1\}$
  $\{\mathsf{Inv}_2\}$
  while $[[(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)] \textbf{ for some } E_1\,,\,E_2]$ **do**
    $\{\mathsf{Inv}_2 \textbf{ and } \mathsf{Cond}_2 \textbf{ and } |\mathcal{E}| = z_2\}$
    $\{\mathsf{Inv}_2[\mathcal{E} := (E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)] \textbf{ and } (|\mathcal{E}| < z_2)[\mathcal{E} := (E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)]\}$
    $\mathcal{E} := (E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)$
    $\{\mathsf{Inv}_2 \textbf{ and } |\mathcal{E}| < z_2\}$
  $\{\mathsf{Inv}_2 \textbf{ and } [\textbf{not } \mathsf{Cond}_2]\}$
  $\{\mathsf{Inv}_2 \textbf{ and } \mathsf{Interm} \textbf{ and } |\mathcal{E}| < z_1\}$
  $\{\mathsf{Inv}_1 \textbf{ and } |\mathcal{E}| < z_1\}$
$\{\mathsf{Inv}_1 \textbf{ and } [\textbf{not } \mathsf{Cond}_1]\}$
$\{\mathsf{Post}\}$

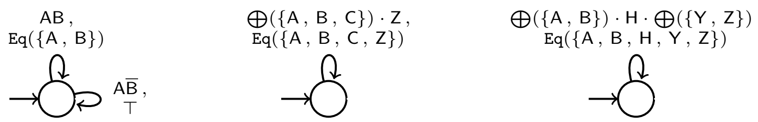
Fig. 11: Algorithm for combining hyperedges with assertions for total correctness

First, the algorithm terminates, because (i) every iteration of the outer loop consists of at least one iteration of the inner loop, for $X = E_1$ and $Y = E_2$, (ii) in every iteration of the inner loop, $\mathcal{E}$ decreases by one, and (iii) $\mathcal{E}$ is finite. Second, the algorithm is semantics-preserving by Lemma 2. The main challenge is proving that the algorithm is also effective. A notable step in this proof is establishing the property labeled Interm from Inv_2 (the invariant of the inner loop) and $[\textbf{not } \mathsf{Cond}_2]$ (the negation of the inner loop's condition). Informally, Interm states that if $\mathcal{F}$ denotes the hyperedge relation *before* running the inner loop, we have $\mathcal{E} = \mathcal{F} \setminus (\mathcal{F}_{1,2}) \cup \mathcal{F}_\dagger$ *after* running the inner loop. Here, $\mathcal{F}_{1,2}$ contains all hyperedges from $\mathcal{F}$ that include V_1 or V_2, while $\mathcal{F}_\dagger$ denotes all new hyperedges added by $\sqcup$ during the loop. This property subsequently enables us to prove Inv_1 (the invariant of the outer loop), which among other properties states $\bigstar(\mathcal{E}_{\text{in}}) = \bigstar(\mathcal{E})$. Consequently, to prove the algorithm's effectiveness, it suffices to show that $E \in \mathcal{E}_{\text{out}}$ implies $E \subseteq \bigstar(\mathcal{E}_{\text{out}})$ (for all E).

Theorem 2. *(Theorem 4 in Appendix B)* $\{\mathsf{Pre}\}\ A\ \{\mathsf{Post}\}$

In summary, in the second step, for every (merged) transition $t = (q\,,\,\psi\,,\,\phi\,,\,q')$ in $f_1(\alpha)$, transformation f_3 manipulates hypergraph γ_t to γ'_t by running the algorithm in Figure 9, given the maximal sets of indistinguishable ports computed in f_3's first step with $\bigstar$. Afterward, f_3 replaces ψ in t with $[\![\gamma'_t]\!]$, which by the correctness of the algorithm is equivalent to $[\![\gamma_t]\!]$ and has made indistinguishable ports (hence queues) apparent. We denote the resulting transition relation by $(f_3 \circ f_1)(\longrightarrow)$ and the resulting CA by $(f_3 \circ f_1)(\alpha)$. Because $\psi \equiv_{\text{sc}} [\![\gamma_t]\!] \equiv_{\text{sc}} [\![\gamma'_t]\!]$ for all transitions t in $f_1(\alpha)$, the following proposition follows from Lemma 16 in Appendix B. Together, Propositions 1 and 2 imply that transformation f is semantics-preserving.

Proposition 2. $(f_3 \circ f_1)(\alpha) \sim f_1(\alpha)$



(a) LossySync (b) Merger_3 (c) Hourglass

Fig. 12: Application of transformation f_3 to the CA in Figure 6

We end with some examples in Figure 12. Transformation f_3 has not had any effect on the LossySync CA, so its implementation does not benefit from queues (no indistinguishable ports), as expected. The Merger_3 and Hourglass CA, in contrast, have changed significantly. In the SC of Merger_3, we can now clearly recognize one queue for ports A, B, and C and one queue for port Z (cf. transformation h in Section 3); similarly, in the SC of Hourglass, we can now clearly recognize one queue for ports A and B and one queue for ports Y and Z.

Applied to Merger_3, transformation f optimizes a multiple-producer-single-consumer protocol. More abstractly, in this case, f optimizes a protocol among two groups of processes, X_1 (producers) and X_2 (consumer), such that $|X_1| = 3$ and $|X_2| = 1$ and all processes in X_1 are indistinguishable to all processes in X_2 and vice versa. Generally, f can optimize protocols among n groups of processes $X_1 , \ldots , X_n$ such that for all $1 \leq i , j \leq n$, all processes in X_i are indistinguishable to all processes in X_j and vice versa. For instance, applied to Hourglass, f optimizes a protocol among three groups of processes such that $|X_1| = |X_3| = 2$ and $|X_2| = 1$.

After having applied transformation f, the automatic generation of actual implementations is straightforward (i.e., transformation g_2 in Figure 4c). The resulting code is, in fact, exactly the same as the code that results from manually applying transformation h as in Section 3 (and consequently, it has the same performance): instead of checking an `event` structure for every port as pre-optimized code does, optimized code checks one `eventQueue` structure for every maximal set of indistinguishable ports, which transformation f has made explicit as $\bigoplus$-formulas in SCs (and are thus easy to detect in the f-transformed CA). As such, optimized code checks the SC of all transitions in the pre-transformation CA that differ only in indistinguishable ports (before applying f) at the same time. For k such transitions, consequently, an unscalable exhaustive $\mathcal{O}(k)$ search is optimized to perfectly scalable $\mathcal{O}(1)$ queue operations. Thus, with respect to Figure 4c, the fully mechanical transformation $g_2 \circ f = g_2 \circ f_3 \circ f_2 \circ f_1$ yields the same code and scalability as the partially manual transformation $h \circ g_1$.

7 Concluding Remarks

In this report, we analyzed scalability issues of the code generated by our Reo-to-C compiler, we explained a manual solution, and we studied the various steps of a mechanical procedure for transforming a CA α to an equivalent CA β, which makes port indistinguishability (hence queues) maximally apparent, using the $\bigoplus$-operator. Our tool can use this mechanical procedure to generate code for α via β with good scalability. In particular, whereas unoptimized code generated for α requires $\mathcal{O}(k)$ time to compute eligibility of k transitions—essentially an exhaustive search—the optimized code generated for β requires only $\mathcal{O}(1)$ time: all maximal sets of indistinguishable ports (explicit in β as a $\bigoplus$-formulas in SCs) in the implementation share the same queue, which optimizes the unscalable $\mathcal{O}(k)$ search to perfectly scalable $\mathcal{O}(1)$ queue operations.

Although inspired by our work on a Reo compiler and formulated generally in terms of CA, we make contributions beyond Reo and CA. The synchronization pattern that we identified and optimized is common and occurs in many classes of protocols and their implementation, regardless of the particular language. Therefore, compilers for other high-level languages may use the same approach as explained in this report to similarly optimize code generated for programs in those languages. In fact, this report led to adding new features to Proto-Runtime to enable our optimization technique, thereby facilitating efficient implementation of our f-transformed CA. Importantly, these new features in Proto-Runtime can now benefit other languages implemented on top of Proto-Runtime as well.

Automatically performing our optimization directly on low-level code such as C (instead of on CA) is extremely complex, if not impossible. This shows that using higher-level languages (that preserve relevant context information about protocols) for multicore programming can indeed be advantageous for performance, a significant general observation in language and compiler design for multicore platforms. Indeed, the work presented in this report serves as evidence that it is possible not only to specify interaction protocols at a higher level of abstraction (than locks, mutex, semaphores, message exchanges, etc.) but also automatically compile and optimize such high-level specifications down to executable code. Such higher-level specifications convey more of the intention behind the protocol, which gives more room for a compiler/optimizer to find and apply efficient implementation alternatives. Lower-level, more imperative, specifications of interaction protocols either lose or obscure the intentions behind protocols and seriously constrict the ability of compilers/optimizers to find efficient implementation alternatives.

Reo is not the only high-level language proposed for programming protocols among threads/processes or implementing multi/manycore applications. For instance, Ng and Yoshida propose to use a parametrized version of the language Scribble [15], called Pabble [16], for implementing protocols in C with MPI. Pabble extends an earlier tool without support for parametrization [17]. It has a theoretical basis in parametrized multiparty session types [18], a type-theoretic approach to specifying protocols that guarantees deadlock-freedom and type-safe communication. The main task of the Pabble tool is to project a *global* parametrized multiparty session type, which specifies the whole protocol, to a number of *local* session types, one for every participant in the protocol. Programmers can use the resulting local session types as a specification for writing new MPI code or for type-checking existing MPI code.

The BIP—behavior, interaction, priority—framework facilitates modeling of application software at three specification levels [19]: behavior of components, interaction between components, and priorities on interactions. Furthermore, it supports validation of a model's functional correctness, code generation, and deployment on manycore platforms [20]. The interaction level, on which components are composed, provides constructs for expressing basic synchronization and communication patterns between components, not unlike the protocols considered in our work. Bonakdarpour et al. developed a tool for automatically generating C++ code from BIP models using TCP sockets, MPI, or Pthreads [21].

Gudenkauff and Hasselbring argue for multicore programming with *tuple spaces* [22]. Tuple spaces, originally introduced in the coordination language Linda [23], are soups in which both processes and data float. Processes can put new data and processes into a tuple space and they can get data from it. Gudenkauff and Hasselbring argue that tuple spaces have appealing properties when it comes to multicore programming and investigate the performance of their extension to the LighTS framework [24], an existing tuple space implementation in Java. From a Mandelbrot benchmark, compared to a reference implementation based on Java threads, Gudenkauff and Hasselbring conclude that their implementation scales reasonably well and that the overhead of their approach is a fair penalty for the ease of programming that tuple spaces provide.

This report makes primarily conceptual and theoretical contributions, and we used performance figures only to motivate and explain the development of our optimization technique. An in-depth study of the use of this technique in practice, including more benchmarks and experiments with different kinds of protocols and contexts, is our next objective, now that we know that the technique is correct. As part of this future work, we will also extend our current, limited proof-of-concept implementation (used in obtaining the data for Figure 5) to a full implementation. We end with the following remarks.

Indistinguishability of data. Transformation f effectively merges transitions with labels of the form $(\psi\,,\,\mathtt{Eq}(P))$. The reason is that the ports in $\mathtt{Eq}(P)$ are indistinguishable from a data perspective. (Whether those ports are also indistinguishable in ψ is exactly what transformation f_3 investigates.) Detecting port indistinguishability in *arbitrary* DCs so as to improve the applicability of f seems an interesting and important future challenge. Perhaps, we can only show that f can transform α to β for arbitrary DCs under a CSP-style refinement relation [25].

Guarded automata. Our SCs, as arbitrary propositional formulas, seem similar to guards on transitions in the guarded automata used by Bonsangue et al. for modeling connector behavior [26]. The intuitive meaning of such guards, however, significantly differs: guards specify a constraint on the environment, while SCs specify a constraint on an execution step. (In fact, transition labels of guarded automata carry both a guard and an SC.)

Model-based testing. We skipped an explanation of the actual code generation process (i.e., transformation g_2 in Figure 4), dismissing it as "straightforward" and "obviously correct". An interesting line of work to better substantiate the latter statement is to have our tool generate not only executable code but also test cases derived from the input CA. Kokash et al. have already worked on such model-based testing for CA in a different context [27].

References

1. Arbab, F.: Reo: a channel-based coordination model for component composition. MSCS **14**(3) (2004) 329–366
2. Arbab, F.: Puff, The Magic Protocol. In: Talcott Festschrift. Volume 7000 of LNCS. Springer (2011) 169–206
3. Jongmans, S.S., Arbab, F.: Modularizing and Specifying Protocols among Threads. In: Proceedings of PLACES 2012. Volume 109 of EPTCS. CoRR (2013) 34–45
4. Jongmans, S.S., Arbab, F.: Overview of Thirty Semantic Formalisms for Reo. SACS **22**(1) (2012) 201–251
5. Kokash, N., Krause, C., de Vink, E.: Reo+mCRL2: A framework for model-checking dataflow in service compositions. FAC **24**(2) (2012) 187–216
6. Jongmans, S.S., Halle, S., Arbab, F.: Reo: A Dataflow Inspired Language for Multicore. In: Proceedings of DFM 2013. (2013)
7. Baier, C., Sirjani, M., Arbab, F., Rutten, J.: Modeling component connectors in Reo by constraint automata. SCP **61**(2) (2006) 75–113
8. Halle, S.: A Study of Frameworks for Collectively Meeting the Productivity, Portability, and Adoptability Goals for Parallel Software. PhD thesis, University of California, Santa Cruz (2011)
9. Halle, S., Cohen, A.: A Mutable Hardware Abstraction to Replace Threads. In: Proceedings of LCPC 2011. Volume 7146 of LNCS. Springer (2013) 185–202
10. Butenhof, D.: Programming with POSIX Threads. Addison-Wesley (1997)
11. Sirjani, M., Jaghoori, M.M., Baier, C., Arbab, F.: Compositional Semantics of an Actor-Based Language Using Constraint Automata. In: Proceedings of COORDINATION 2006. Volume 4038 of LNCS. Springer (2006) 281–297
12. Bretto, A.: Hypergraph Theory: An Introduction. Springer (2013)
13. Hoare, T.: An Axiomatic Basis for Computer Programming. CACM **12**(10) (1969) 576–580
14. Apt, K., de Boer, F., Olderog, E.R.: Verification of Sequential and Concurrent Programs. Springer (2009)
15. Honda, K., Mukhamedov, A., Brown, G., Chen, T.C., Yoshida, N.: Scribbling Interactions with a Formal Foundation. In: Proceedings of ICDCIT 2011. Volume 6536 of LNCS. Springer (2011) 55–75
16. Ng, N., Yoshida, N.: Pabble: Parameterised Scribble for Parallel Programming. In: Proceedings of PDP 2014, IEEE (2014) 707–714
17. Ng, N., Yoshida, N., Honda, K.: Multiparty Session C: Safe Parallel Programming with Message Optimisation. In: Proceedings of TOOLS 2012. Volume 7304 of LNCS. Springer (2012) 202–218
18. Deniélou, P.M., Yoshida, N., Bejleri, A., Hu, R.: Parameterised Multiparty Session Types. LMCS **8**(4) (2012) 1–46
19. Basu, A., Bozga, M., Sifakis, J.: Modeling Heterogeneous Real-time Components in BIP. In: Proceedings of SEFM 2006, IEEE (2006) 3–12
20. Basu, A., Bensalem, S., Bozga, M., Bourgos, P., Maheshwari, M., Sifakis, J.: Component Assemblies in the Context of Manycore. In: Proceedings of FMCO 2011. Volume 7542 of LNCS. Springer (2013) 314–333

21. Bonakdarpour, B., Bozga, M., Jaber, M., Quilbeuf, J., Sifakis, J.: A framework for automated distributed implementation of component-based models. Distributed Computing **25**(5) (2012) 383–409
22. Gudenkauf, S., Hasselbring, W.: Space-Based Multi-Core Programming in Java. In: Proceedings of PPPJ 2011, ACM (2012) 41–50
23. Gelernter, D.: Generative Communication in Linda. TOPLAS **7**(1) (1985) 80–112
24. Balzarotti, D., Costa, P., Picco, G.P.: The LighTS tuple space framework and its customization for context-aware applications. WIAS **5**(2) (2007) 215–231
25. Roscoe, B.: The Theory and Practice of Concurrency. Prentice Hall (1997)
26. Bonsangue, M., Clarke, D., Silva, A.: A model of context-dependent component connectors. SCP **77**(6) (2009) 685–706
27. Kokash, N., Arbab, F., Changizi, B., Makhnist, L.: Input-output Conformance Testing for Channel-based Service Connectors. In: Proceedings of PACO 2011. Volume 60 of EPTCS. CoRR (2011) 19–35

A Definitions and Properties

A.1 Basic Concepts

Definition 7 (Ports). $(\mathbb{P}\text{ORT}, <_{\mathbb{P}\text{ORT}})$ *is a chain, where:*

- $\mathbb{P}\text{ORT}$ *denotes the set of all ports;*
- $<_{\mathbb{P}\text{ORT}} \subseteq \mathbb{P}\text{ORT} \times \mathbb{P}\text{ORT}$ *denotes a strict total order on ports.*

Definition 8 (Data items). $\mathbb{D}\text{ATA}$ *denotes the set of all data items.*

Definition 9 (Distributions).

- *A distribution is a partial function* $\delta : \mathbb{P}\text{ORT} \rightharpoonup \mathbb{D}\text{ATA}$ *from ports to data items.*
- $\mathbb{D}\text{IS}$ *denotes the set of all distributions.*

A.2 Synchronization Constraints

Definition 10 (Synchronization constraints).

1. *A synchronization constraint is a word ψ generated by the following grammar:*

$$\begin{aligned} p &::= \textit{any element from } \mathbb{P}\text{ORT} \\ a &::= 0 \mid 1 \mid p \\ \psi &::= a \mid \overline{\psi} \mid \psi + \psi \mid \psi \cdot \psi \mid \bigoplus(\Psi) \end{aligned}$$

2. $(\mathbb{SC}\,,\,<_{\mathbb{SC}})$ *is a chain, where:*
   – $\mathbb{SC}$ *denotes the set of all synchronization constraints;*
   – $<_{\mathbb{SC}} \subseteq \mathbb{SC} \times \mathbb{SC}$ *denotes a strict total order on synchronization constraints such that:*

$$<_{\mathbb{P}\text{ORT}} \subseteq <_{\mathbb{SC}}$$

3. $\overset{\text{sc}}{\models} \subseteq \mathbb{D}\text{IS} \times \mathbb{SC}$ *denotes the relation on* $\big[$*distribution, synchronization constraint*$\big]$*-pairs defined as follows:*

$$\begin{array}{lll} \delta \overset{\text{sc}}{\models} 0 & \textbf{iff} & \textbf{false} \\ \delta \overset{\text{sc}}{\models} 1 & \textbf{iff} & \textbf{true} \\ \delta \overset{\text{sc}}{\models} p & \textbf{iff} & p \in \mathrm{Dom}(\delta) \\ \delta \overset{\text{sc}}{\models} \overline{\psi} & \textbf{iff} & \delta \overset{\text{sc}}{\not\models} \psi \\ \delta \overset{\text{sc}}{\models} \psi_1 + \psi_2 & \textbf{iff} & \big[\delta \overset{\text{sc}}{\models} \psi_1 \ \textbf{or}\ \delta \overset{\text{sc}}{\models} \psi_2\big] \\ \delta \overset{\text{sc}}{\models} \psi_1 \cdot \psi_2 & \textbf{iff} & \big[\delta \overset{\text{sc}}{\models} \psi_1 \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi_2\big] \\ \delta \overset{\text{sc}}{\models} \bigoplus(\Psi) & \textbf{iff} & \left[\left[\psi \in \Psi \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \left[\begin{array}{c} \left[\begin{array}{c} \big[\psi' \in \Psi \ \textbf{and}\ \psi \neq \psi'\big] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'} \end{array}\right] \\ \textbf{for all}\ \psi' \end{array}\right]\right] \textbf{for some}\ \psi\right] \end{array}$$

4. $\equiv_{\text{sc}} \subseteq \mathbb{SC} \times \mathbb{SC}$ *denotes the relation on pairs of synchronization constraints defined as follows:*

$$\psi_1 \equiv_{\text{sc}} \psi_2 \ \textbf{iff}\ \left[\big[\delta \overset{\text{sc}}{\models} \psi_1 \ \textbf{iff}\ \delta \overset{\text{sc}}{\models} \psi_2\big] \ \textbf{for all}\ \delta\right]$$

Definition 11.

1. $\sum : \wp(\mathbb{SC}) \to \mathbb{SC}$ *denotes the function from sets of synchronization constraints to synchronization constraints defined as follows:*

$$\textstyle\sum(\Psi) = \begin{cases} 0 & \textbf{if}\ \Psi = \emptyset \\ \mathrm{least}(\Psi) + \sum(\Psi \setminus \{\mathrm{least}(\Psi)\}) & \textbf{if}\ \Psi \neq \emptyset \end{cases}$$

2. $\prod : \wp(\mathbb{SC}) \to \mathbb{SC}$ *denotes the function from sets of synchronization constraints to synchronization constraints defined as follows:*

$$\textstyle\prod(\Psi) = \begin{cases} 1 & \textbf{if}\ \Psi = \emptyset \\ \mathrm{least}(\Psi) \cdot \prod(\Psi \setminus \{\mathrm{least}(\Psi)\}) & \textbf{if}\ \Psi \neq \emptyset \end{cases}$$

Definition 12 (P-complete product). $\mathsf{cp} : \wp(\mathbb{P}\text{ORT}) \times \wp(\mathbb{P}\text{ORT}) \rightharpoonup \mathbb{SC}$ *denotes the partial function from pairs of sets of ports to synchronization constraints defined as folllows:*

$$\mathsf{cp}(P\,,\,P_+) = \textstyle\prod(\{p \mid p \in P_+\} \cup \{p \mid p = \overline{p'} \ \textbf{and}\ p' \in P \setminus P_+\}) \ \textbf{if}\ P_+ \subseteq P$$

Definition 13. $\mathsf{Port} : \mathbb{SC} \to \wp(\mathbb{P}\text{ORT})$ *denotes the function from synchronization constraints to sets of ports defined as follows:*

$$\begin{array}{ll} \mathsf{Port}(0)\,,\,\mathsf{Port}(1) & = \emptyset \\ \mathsf{Port}(p) & = \{p\} \\ \mathsf{Port}(\overline{\psi}) & = \mathsf{Port}(\psi) \\ \mathsf{Port}(\psi_1 + \psi_2)\,,\,\mathsf{Port}(\psi_1 \cdot \psi_2) & = \mathsf{Port}(\psi_1) \cup \mathsf{Port}(\psi_2) \\ \mathsf{Port}(\bigoplus(\Psi)) & = \bigcup\{P \mid P = \mathsf{Port}(\psi) \ \textbf{and}\ \psi \in \Psi\} \end{array}$$

Definition 14. $\mathsf{SC} : \wp(\mathbb{P}\text{ORT}) \to \wp(\mathbb{SC})$ *denotes the function from sets of ports to sets of synchronization constraints defined as follows:*

$$\mathsf{SC}(P) = \{\psi \mid \mathsf{Port}(\psi) \subseteq P\}$$

Lemma 3 (Property of $\bigoplus$).

$$\begin{gathered}\big[\delta \models \bigoplus(\Psi_1 \cup \Psi_2) \;\textbf{ and }\; \Psi_1 \cap \Psi_2 = \emptyset\big] \;\textbf{ implies}\\ \left[\begin{array}{l} \quad \big[\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_1) \;\textbf{ and }\; \big[\big[\psi' \in \Psi_2 \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \overline{\psi'}\big] \;\textbf{ for all }\; \psi'\big]\big] \\ \textbf{or } \big[\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_2) \;\textbf{ and }\; \big[\big[\psi' \in \Psi_1 \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \overline{\psi'}\big] \;\textbf{ for all }\; \psi'\big]\big] \end{array}\right]\end{gathered}$$

Proof. See page 34.

Lemma 4 (Subsets of $\mathbb{SC}$ have a least element).

1. $\Psi \in \wp(\mathbb{SC})$ **implies** $\text{least}(\Psi) \in \Psi$
2. $\Psi \in \wp(\mathbb{SC})$ **implies** $|\Psi \setminus \{\text{least}(\Psi)\}| < |\Psi|$

Proof. See page 36.

Lemma 5 (Lifting).

1. $\big[\Psi \in \wp(\mathbb{SC}) \;\textbf{ and }\; \big[\big[\psi \in \Psi \;\textbf{ and }\; \delta \overset{\text{sc}}{\models} \psi\big] \;\textbf{ for some }\; \psi\big]\big] \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \sum(\Psi)$
2. $\big[\Psi \in \wp(\mathbb{SC}) \;\textbf{ and }\; \big[\big[\psi \in \Psi \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \psi\big] \;\textbf{ for all }\; \psi\big]\big] \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \prod(\Psi)$
3. $\delta \overset{\text{sc}}{\models} \sum(\Psi) \;\textbf{ implies }\; \big[\big[\psi \in \Psi \;\textbf{ and }\; \delta \overset{\text{sc}}{\models} \psi\big] \;\textbf{ for some }\; \psi\big]$
4. $\delta \overset{\text{sc}}{\models} \prod(\Psi) \;\textbf{ implies }\; \big[\big[\psi \in \Psi \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \psi\big] \;\textbf{ for all }\; \psi\big]$

Proof. See page 37.

A.3 Hypergraphs

Definition 15 (Vertices).

1. *A vertex is a set of ports.*
2. $\mathbb{V}\text{ER}$ *denotes the set of vertices.*

Definition 16 (Hypergraphs).

1. *A hypergraph is a pair* $(\mathcal{V}\,,\,\mathcal{E})$*, where:*
   - $\mathcal{V} \in \wp(\mathbb{V}\text{ER})$ *denotes a set of vertices;*
   - $\mathcal{E} \in \wp^2(\mathcal{V})$ *denotes a set of hyperedges such that:*

$$\big[E \in \mathcal{E} \;\;\textbf{implies}\;\; \bigcap E = \emptyset\big] \;\;\textbf{for all}\;\; E$$

2. $\mathbb{G}\text{RAPH}$ *denotes the set of all hypergraphs.*

Definition 17. $\mathsf{graph} : \mathbb{SC} \rightharpoonup \mathbb{G}\text{RAPH}$ *denotes the partial function from* SC*s to hypergraphs defined as:*

$$\mathsf{graph}(\psi) = (\wp(\mathsf{Port}(\psi))\,,\,\left\{E \;\middle|\; \begin{array}{l} E = \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\} \\ \textbf{and}\;\; P_+ \subseteq \mathsf{Port}(\psi) \;\;\textbf{and}\;\; P_+ \in \mathcal{P} \end{array}\right\})$$
$$\textbf{if}\;\; \left[\psi = \sum(\left\{\psi' \;\middle|\; \begin{array}{c} \psi' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+) \\ \textbf{and}\;\; P_+ \subseteq \mathsf{Port}(\psi) \;\;\textbf{and}\;\; P_+ \in \mathcal{P} \end{array}\right\}) \;\;\textbf{for some}\;\; \mathcal{P}\right]$$

Definition 18. $[\![\cdot]\!] : (\wp(\mathbb{V}\text{ER}) \times \wp(\mathbb{P}\text{ORT})) \cup \mathbb{G}\text{RAPH} \to \mathbb{SC}$ *denotes the function from* [*hyperedge, set of ports*]*-pairs and hypergraphs to* SC*s defined as:*

$$\begin{aligned} [\![E]\!]_P &= \prod(\{\psi \mid \psi = \bigoplus(V) \;\;\textbf{and}\;\; V \in E\} \cup \{\psi \mid \psi = \overline{p} \;\;\textbf{and}\;\; p \in P \setminus (\bigcup E)\}) \\ [\![(\mathcal{V}\,,\,\mathcal{E})]\!] &= \sum(\{\psi \mid \psi = \in E_{\bigcup \mathcal{V}} \;\;\textbf{and}\;\; E \in \mathcal{E}\}) \end{aligned}$$

Definition 19. $\curlyvee \subseteq ((\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER})) \cup ((\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times \wp^2(\mathbb{V}\text{ER}))$ *denotes the relation on*

- *pairs of* [*hyperedge, vertex*]*-pairs;*
- *and tuples consisting of two* [*hyperedge, vertex*]*-pairs and a set of hyperedges*

defined as follows:

$$\begin{aligned} (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2) \;\;&\textbf{iff}\;\; \big[E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \;\;\textbf{and}\;\; E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\}\big] \\ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\;&\textbf{iff}\;\; \big[E_1\,,\,E_2 \in \mathcal{E} \;\;\textbf{and}\;\; E_1 \neq E_2 \;\;\textbf{and}\;\; V_1 \cap V_2 = \emptyset \;\;\textbf{and}\;\; (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2)\big] \end{aligned}$$

Definition 20. $\sqcup : (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times (\wp(\mathbb{V}\text{ER}) \times \mathbb{V}\text{ER}) \times \wp^2(\mathbb{V}\text{ER}) \to \wp^2(\mathbb{V}\text{ER})$ *denotes the function from tuples consisting of two* [*set of verices, vertex*]*-pairs and a set of hyperedges to sets of hyperedges defined as follows:*

$$(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \;\;\textbf{if}\;\; (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Definition 21. $\mathsf{Port} : \wp^2(\mathbb{V}\text{ER}) \to \wp(\mathbb{P}\text{ORT})$ *denotes the function from sets of hyperedges to sets of ports defined as follows:*

$$\mathsf{Port}(\mathcal{E}) = \{p \mid p \in V \in E \in \mathcal{E}\}$$

Lemma 6 (Properties of $\curlyvee$).

1. $(E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2)$ **implies** $\big[V_1 \in E_1$ **and** $V_2 \in E_2\big]$

2. $\left[(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)\ \textbf{ and }\ V_1\neq V_2\right]\ \textbf{implies}\ E_1\neq E_2$

Proof. See page 42.

Lemma 7 (Properties of $\curlyvee_{\mathcal{E}}$).

1. $(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{implies}\ V_1\neq V_2$
2. $(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{implies}\ \left[V_2\notin E_1\ \textbf{ and }\ V_1\notin E_2\right]$

Proof. See page 43.

Lemma 8 ($\sqcup$ is sound, complete, and decreasing).

1. $\left[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{ and }\ \delta\models[\![(\mathcal{V}\,,\,\mathcal{E})]\!]\right]\ \textbf{implies}\ \delta\models[\![(\mathcal{V}\,,\,(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2))]\!]$
2. $\left[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{ and }\ \delta\models[\![(\mathcal{V}\,,\,(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2))]\!]\right]\ \textbf{implies}\ \delta\models[\![(\mathcal{V}\,,\,\mathcal{E})]\!]$
3. $(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{implies}\ [\![(\mathcal{V}\,,\,\mathcal{E})]\!]\equiv_{\mathrm{sc}}[\![(\mathcal{V}\,,\,(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2))]\!]$
4. $(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{implies}\ |(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)|<|\mathcal{E}|$

Proof. See page 44.

Theorem 3 (graph is an isomorphism).

$$\left[\psi=\sum\left(\left\{\psi'\ \middle|\ \begin{array}{l}\psi'\equiv_{\mathrm{sc}}\mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+)\ \textbf{ and}\\ P_+\subseteq\mathsf{Port}(\psi)\ \textbf{ and }\ P_+\in\mathcal{P}\end{array}\right\}\right)\ \textbf{for some}\ \mathcal{P}\right]\ \textbf{implies}\ \psi\equiv_{\mathrm{sc}}[\![\mathsf{graph}(P\,,\,\psi)]\!]$$

Proof. See page 65.

A.4 Algorithm

Definition 22. $\mathsf{Edge} : \mathbb{P}\mathrm{ORT} \times \wp^2(\mathbb{V}\mathrm{ER}) \to \wp^2(\mathbb{V}\mathrm{ER})$ *denotes the function from* $\big[$*port, set of hyperedges*$\big]$*-pairs to sets of hyperedges defined as follows:*

$$\mathsf{Edge}(p\,,\,\mathcal{E}) = \{\mathcal{W} \mid \mathcal{W} = E \setminus \{V\} \ \textbf{ and } \ p \in V \in E \in \mathcal{E}\}$$

Definition 23. $\bigstar : \wp^2(\mathbb{V}\mathrm{ER}) \to \wp^2(\mathbb{P}\mathrm{ORT})$ *denotes the function from sets of hyperedges to sets of sets of ports defined as follows:*

$$\bigstar(\mathcal{E}) = \{P \mid P \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[[p \in P \ \textbf{ iff } \ \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})] \ \textbf{ for all } \ p\big]\}$$

Definition 24. $\checkmark \subseteq \wp^2(\mathbb{V}\mathrm{ER})$ *denotes the relation on hyperedges defined as follows:*

$$\checkmark(\mathcal{E}) \ \textbf{ iff } \ \left[\begin{array}{c} \left[\left[\begin{bmatrix} p \in V_1 \in E_1 \in \mathcal{E} \\ \textbf{and } \ p \in V_2 \in E_2 \in \mathcal{E} \end{bmatrix} \ \textbf{ implies } \ V_1 = V_2\right] \ \textbf{ for all } \ p\,,\,V_1\,,\,V_2\,,\,E_1\,,\,E_2\right] \\ \textbf{and } \ \big[[V \in E \in \mathcal{E} \ \textbf{ implies } \ V \neq \emptyset] \ \textbf{ for all } \ V\,,\,E\big] \ \textbf{ and} \\ \textbf{and } \ \left[\left[V \in E \in \mathcal{E} \ \textbf{ implies } \ \begin{bmatrix} [V \subseteq P \ \textbf{ and } \ P \in \bigstar(\mathcal{E})] \\ \textbf{for some } \ P \end{bmatrix}\right] \ \textbf{ for all } \ V\,,\,E\right] \end{array}\right]$$

Lemma 9 (Properties of Edge).

1. $p \in V \in E \in \mathcal{E} \in \wp^2(\mathbb{V}\mathrm{ER})$ **implies** $E \setminus \{V\} \in \mathsf{Edge}(p\,,\,\mathcal{E})$
2. $p_1\,,\,p_2 \in P \in \bigstar(\mathcal{E})$ **implies** $\mathsf{Edge}(p_1\,,\,\mathcal{E}) = \mathsf{Edge}(p_2\,,\,\mathcal{E})$
3. $\begin{bmatrix} p_1 \in V_1 \in E_1 \in \mathcal{E} \ \textbf{ and} \\ \mathsf{Edge}(p_1\,,\,\mathcal{E}) = \mathsf{Edge}(p_2\,,\,\mathcal{E}) \end{bmatrix} \ \textbf{ implies } \ \left[\begin{bmatrix} p_2 \in V_2 \in E_2 \in \mathcal{E} \ \textbf{ and} \\ (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2) \end{bmatrix} \ \textbf{ for some } \ V_2\,,\,E_2\right]$
4. $\begin{bmatrix} p_2 \in V_2 \in E_2 \in \mathcal{E} \ \textbf{ and} \\ \mathsf{Edge}(p_1\,,\,\mathcal{E}) = \mathsf{Edge}(p_2\,,\,\mathcal{E}) \end{bmatrix} \ \textbf{ implies } \ \left[\begin{bmatrix} p_1 \in V_1 \in E_1 \in \mathcal{E} \ \textbf{ and} \\ (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2) \end{bmatrix} \ \textbf{ for some } \ V_1\,,\,E_1\right]$

Proof. See page 75.

Lemma 10 (More properties of Edge—homomorphisms).

1. $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_1) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_2)$
2. $\big[\checkmark(\mathcal{E}) \ \textbf{ and } \ \mathcal{E}_1\,,\,\mathcal{E}_2 \subseteq \mathcal{E}\big] \ \textbf{ implies } \ \mathsf{Edge}(p\,,\,\mathcal{E}_1 \setminus \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_1) \setminus \mathsf{Edge}(p\,,\,\mathcal{E}_2)$

Proof. See page 77.

Lemma 11 ($\bigstar(\mathcal{E})$ is a partition—groundwork).

$$\begin{bmatrix} k \in \mathbb{N}^+ \ \textbf{ and} \\ p \in \mathsf{Port}(\mathcal{E}) \end{bmatrix} \ \textbf{ implies } \ \left[\begin{array}{c} \left[\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \textbf{and } \ \big[[p' \in P \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})] \ \textbf{ for all } \ p'\big] \ \textbf{ and} \\ \big[|P| = k \ \textbf{ or } \ \big[[\mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E}) \ \textbf{ implies } \ p' \in P] \ \textbf{ for all } \ p'\big]\big] \end{array}\right] \\ \textbf{for some } \ P\,,\,\mathcal{T} \end{array}\right]$$

Proof. See page 82.

Lemma 12 ($\bigstar(\mathcal{E})$ is a partition).

1. $\bigcup \bigstar(\mathcal{E}) = \mathsf{Port}(\mathcal{E})$

$$\textbf{while}\ \big[\big[(X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{E})\big]\ \textbf{for some}\ X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\big]\ \textbf{do}$$
$$\quad\textbf{while}\ \big[\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\big]\ \textbf{for some}\ E_1\,,\,E_2\big]\ \textbf{do}$$
$$\qquad\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)$$

Fig. 13: Algorithm for merging hyperedges

$$\{\mathsf{Pre}\}$$
$$\{\mathsf{Inv}_1\}$$
$$\textbf{while}\ \big[\big[(X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{E})\big]\ \textbf{for some}\ X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\big]\ \textbf{do}$$
$$\quad\{\mathsf{Inv}_1\ \textbf{and}\ \mathsf{Cond}_1\ \textbf{and}\ |\mathcal{E}|=z_1\}$$
$$\quad\{\mathsf{Inv}_2\}$$
$$\quad\textbf{while}\ \big[\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\big]\ \textbf{for some}\ E_1\,,\,E_2\big]\ \textbf{do}$$
$$\qquad\{\mathsf{Inv}_2\ \textbf{and}\ \mathsf{Cond}_2\ \textbf{and}\ |\mathcal{E}|=z_2\}$$
$$\qquad\{\mathsf{Inv}_2[\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)]\ \textbf{and}\ (|\mathcal{E}|<z_2)[\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)]\}$$
$$\qquad\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)$$
$$\qquad\{\mathsf{Inv}_2\ \textbf{and}\ |\mathcal{E}|<z_2\}$$
$$\quad\{\mathsf{Inv}_2\ \textbf{and}\ [\textbf{not}\ \mathsf{Cond}_2]\}$$
$$\quad\{\mathsf{Inv}_2\ \textbf{and}\ \mathsf{Interm}\ \textbf{and}\ |\mathcal{E}|<z_1\}$$
$$\quad\{\mathsf{Inv}_1\ \textbf{and}\ |\mathcal{E}|<z_1\}$$
$$\{\mathsf{Inv}_1\ \textbf{and}\ [\textbf{not}\ \mathsf{Cond}_1]\}$$
$$\{\mathsf{Post}\}$$

Fig. 14: Algorithm for merging hyperedges, annotated with assertions for (total) correctness

$$\mathsf{Pre}\ :\ \big[(\mathcal{V}\,,\,\mathcal{E}_{\mathrm{in}})\in\mathbb{G}\textsc{raph}\ \textbf{and}\ \checkmark(\mathcal{E}_{\mathrm{in}})\ \textbf{and}\ \mathcal{E}=\mathcal{E}_{\mathrm{in}}\big]$$
$$\mathsf{Post}\ :\ \big[\llbracket(\mathcal{V}\,,\,\mathcal{E})\rrbracket=\in(\mathcal{V}\,,\,\mathcal{E}_{\mathrm{in}})\ \textbf{and}\ \big[\big[E\in\mathcal{E}\ \textbf{implies}\ E\subseteq\bigstar(\mathcal{E}_{\mathrm{in}})\big]\ \textbf{for all}\ E\big]\big]$$

$$\mathsf{Cond}_1\ :\ \big[\big[(X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{E})\big]\ \textbf{for some}\ X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\big]$$
$$\mathsf{Cond}_2\ :\ \big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{for some}\ E_1\,,\,E_2\big]$$
$$\mathsf{Inv}_1\ :\ \big[\llbracket(\mathcal{V}\,,\,\mathcal{E})\rrbracket=\in(\mathcal{V}\,,\,\mathcal{E}_{\mathrm{in}})\ \textbf{and}\ \bigstar(\mathcal{E})=\bigstar(\mathcal{E}_{\mathrm{in}})\ \textbf{and}\ \checkmark(\mathcal{E})\big]$$

$$\mathsf{Inv}_2\ :\ \left[\begin{array}{c}\llbracket(\mathcal{V}\,,\,\mathcal{F})\rrbracket=\in(\mathcal{V}\,,\,\mathcal{E}_{\mathrm{in}})\ \textbf{and}\ \bigstar(\mathcal{F})=\bigstar(\mathcal{E}_{\mathrm{in}})\ \textbf{and}\ \checkmark(\mathcal{F})\ \textbf{and}\\ |\mathcal{F}|=z_1\ \textbf{and}\ (X\,,\,V_1)\curlyvee_{\mathcal{F}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{F})\ \textbf{and}\\ \mathcal{F}_1=\{E_1\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\ \mathcal{F}_2=\{E_2\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\\ \textbf{and}\ \mathcal{F}_\dagger=\{E_\dagger\mid E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\\ \textbf{and}\ \llbracket(\mathcal{V}\,,\,\mathcal{E})\rrbracket=\in(\mathcal{V}\,,\,\mathcal{F})\ \textbf{and}\ \big[\big[\mathsf{Cond}_2\ \textbf{and}\ \mathcal{E}=\mathcal{F}\big]\ \textbf{or}\ |\mathcal{E}|<|\mathcal{F}|\big]\ \textbf{and}\\ \left[\left[\begin{array}{c}\mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2\subseteq\mathcal{F}_2\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\\ \textbf{and}\ \left[\left[\begin{array}{c}\big[E_1'\in\mathcal{E}_1\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)\big]\ \textbf{implies}\\ \big[E_2'\in\mathcal{E}_2\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger\big]\end{array}\right]\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \textbf{and}\ \left[\left[\begin{array}{c}\big[E_2'\in\mathcal{E}_2\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)\big]\ \textbf{implies}\\ \big[E_1'\in\mathcal{E}_1\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger\big]\end{array}\right]\ \textbf{for all}\ E_1'\,,\,E_2'\right]\end{array}\right]\ \textbf{for some}\ \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger\right]\end{array}\right]$$

$$\mathsf{Interm}\ :\ \mathcal{E}=(\mathcal{F}\setminus\{\mathcal{F}_1\cup\mathcal{F}_2\})\cup\mathcal{F}_\dagger$$

Fig. 15: Definitions referenced in Figure 14

2. $\left[P_1 \neq P_2 \ \textbf{and}\ P_1\,,\, P_2 \in \bigstar(\mathcal{E})\right] \ \textbf{implies}\ P_1 \cap P_2 = \emptyset$

Proof. See page 87.

Lemma 13 (Properties of Cond_1).

1. $$\left[\begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E})\right] \ \textbf{implies}\ \begin{bmatrix} \{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} = \\ \{E_1 \mid (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2)\} \end{bmatrix}$$

2. $$\left[\begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E})\right] \ \textbf{implies}\ \begin{bmatrix} \{E_2 \mid V_2 \in E_2 \in \mathcal{E}\} = \\ \{E_2 \mid (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2)\} \end{bmatrix}$$

3. $$\left[\begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E})\right] \ \textbf{implies}\ \begin{bmatrix} \left\{E_\dagger \,\middle|\, \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \\ \textbf{and}\ V_1 \in E_1 \in \mathcal{E} \end{matrix}\right\} = \\ \left\{E_\dagger \,\middle|\, \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2) \end{matrix}\right\} \end{bmatrix}$$

4. $$\left[\begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E})\right] \ \textbf{implies}\ \begin{bmatrix} \left\{E_\dagger \,\middle|\, \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_2 \setminus \{V_2\}) \\ \textbf{and}\ V_2 \in E_2 \in \mathcal{E} \end{matrix}\right\} = \\ \left\{E_\dagger \,\middle|\, \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2) \end{matrix}\right\} \end{bmatrix}$$

5. $$\left[\begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E}) \ \textbf{and}\ E \in \mathcal{E}\right] \ \textbf{implies}\ \left[\textbf{not}\ V_1\,,V_2 \in E\right]$$

6. $$\left[\begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E}) \ \textbf{and}\ V_1 \in E_1 \in \mathcal{E}\right] \ \textbf{implies}\ \begin{bmatrix} (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2) \\ \textbf{for some}\ E_2 \end{bmatrix}$$

7. $$\left[\begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E}) \ \textbf{and}\ V_2 \in E_2 \in \mathcal{E}\right] \ \textbf{implies}\ \begin{bmatrix} (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2) \\ \textbf{for some}\ E_1 \end{bmatrix}$$

8. $$\begin{bmatrix} \begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E}) \\ \textbf{and}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\ V \notin \{V_1\,,V_2\} \ \textbf{and} \\ \mathsf{Edge}(p\,,\mathcal{E}) = \mathsf{Edge}(p'\,,\mathcal{E}) \\ \textbf{and}\ \mathcal{E}_1 = \{E_1 \mid (E_1\,,V_1) \curlyvee (E_2\,,V_2)\} \\ \textbf{and}\ \mathcal{E}_2 = \{E_2 \mid (E_1\,,V_1) \curlyvee (E_2\,,V_2)\} \end{bmatrix} \ \textbf{implies}\ \left[\begin{bmatrix} p' \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\ V' \notin \{V_1\,,V_2\} \end{bmatrix} \textbf{for some}\ V'\,,E'\right]$$

9. $$\begin{bmatrix} \begin{bmatrix} (X\,,V_1) \curlyvee_{\mathcal{E}} (Y\,,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{and}\ \checkmark(\mathcal{E}) \\ \textbf{and}\ \mathcal{E}' = (\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and} \\ \mathcal{E}_1 = \{E_1 \mid (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2)\} \ \textbf{and} \\ \mathcal{E}_2 = \{E_2 \mid (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2)\} \ \textbf{and} \\ \mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,V_1) \curlyvee_{\mathcal{E}} (E_2\,,V_2) \end{matrix}\right\} \end{bmatrix} \ \textbf{implies}\ \mathsf{Port}(\mathcal{E}) = \mathsf{Port}(\mathcal{E}')$$

Proof. See page 90.

Lemma 14 (More properties of Cond_1—Edge-equations).

1. $$\left[\begin{array}{c} \left[\begin{array}{c} (X\,,V_1)\;\Upsilon_{\mathcal{E}}\;(Y\,,V_2)\ \textbf{and} \\ V_1 \cup V_2 \subseteq P\ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{array}\right]\ \textbf{and}\ \checkmark(\mathcal{E})\ \textbf{and} \\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\ \textbf{and}\ V \notin \{V_1\,,V_2\}\ \textbf{and} \\ \mathcal{E}_1 = \{E_1 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_2 = \{E_2 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{array}{c} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2) \end{array}\right\} \end{array}\right]\ \textbf{implies}\ \left[\begin{array}{l} \mathsf{Edge}(p\,,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\mathcal{E}_\dagger) \\ \textbf{and}\ \mathsf{Edge}(p\,,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\mathcal{E}) \end{array}\right]$$

2. $$\left[\begin{array}{c} \left[\begin{array}{c} (X\,,V_1)\;\Upsilon_{\mathcal{E}}\;(Y\,,V_2)\ \textbf{and} \\ V_1 \cup V_2 \subseteq P\ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{array}\right]\ \textbf{and}\ \checkmark(\mathcal{E})\ \textbf{and} \\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\ \textbf{and}\ V \notin \{V_1\,,V_2\}\ \textbf{and} \\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\ \textbf{and}\ W \notin \{V_1\,,V_2\}\ \textbf{and} \\ \mathsf{Edge}(p\,,\mathcal{E}) = \mathsf{Edge}(q\,,\mathcal{E})\ \textbf{and} \\ \mathcal{E}_1 = \{E_1 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_2 = \{E_2 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{array}{c} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2) \end{array}\right\} \end{array}\right]\ \textbf{implies}\ \left[\begin{array}{c} \mathsf{Edge}(p\,,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(q\,,\mathcal{E}_1 \cup \mathcal{E}_2) \\ \textbf{and}\ \mathsf{Edge}(p\,,\mathcal{E}_\dagger) = \mathsf{Edge}(q\,,\mathcal{E}_\dagger) \end{array}\right]$$

3. $$\left[\begin{array}{c} \left[\begin{array}{c} (X\,,V_1)\;\Upsilon_{\mathcal{E}}\;(Y\,,V_2)\ \textbf{and} \\ V_1 \cup V_2 \subseteq P\ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{array}\right]\ \textbf{and}\ \checkmark(\mathcal{E})\ \textbf{and} \\ \big[[\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2]\ \textbf{for all}\ V\,,E\big] \\ \textbf{and}\ p \in \mathbb{P}\text{ORT}\ \textbf{and} \\ \mathcal{E}_1 = \{E_1 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_2 = \{E_2 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{array}{c} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2) \end{array}\right\} \end{array}\right]\ \textbf{implies}\ \left[\begin{array}{l} \mathsf{Edge}(p\,,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\mathcal{E}_\dagger) \\ \textbf{and}\ \mathsf{Edge}(p\,,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\mathcal{E}) \end{array}\right]$$

4. $$\left[\begin{array}{c} \left[\begin{array}{c} (X\,,V_1)\;\Upsilon_{\mathcal{E}}\;(Y\,,V_2)\ \textbf{and} \\ V_1 \cup V_2 \subseteq P\ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{array}\right]\ \textbf{and}\ \checkmark(\mathcal{E})\ \textbf{and} \\ \mathsf{Edge}(p\,,\mathcal{E}) = \mathsf{Edge}(q\,,\mathcal{E})\ \textbf{and} \\ \mathcal{E}' = (\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger\ \textbf{and} \\ \mathcal{E}_1 = \{E_1 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_2 = \{E_2 \mid (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2)\}\ \textbf{and} \\ \mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{array}{c} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,V_1)\;\Upsilon_{\mathcal{E}}\;(E_2\,,V_2) \end{array}\right\} \end{array}\right]\ \textbf{implies}\ \mathsf{Edge}(p\,,\mathcal{E}') = \mathsf{Edge}(q\,,\mathcal{E}')$$

Proof. See page 109.

Theorem 4 (Figure 14 is correct).

1. $\mathsf{Pre}\ \textbf{implies}\ \mathsf{Inv}_1$
2. $[\mathsf{Inv}_1\ \textbf{and}\ \mathsf{Cond}_1\ \textbf{and}\ |\mathcal{E}| = z_1]\ \textbf{implies}\ \mathsf{Inv}_2$
3. $[\mathsf{Inv}_2\ \textbf{and}\ \mathsf{Cond}_2\ \textbf{and}\ |\mathcal{E}| = z_2]\ \textbf{implies}\ \left[\begin{array}{l} \mathsf{Inv}_2[\mathcal{E} := (E_1\,,V_1) \sqcup_{\mathcal{E}} (E_2\,,V_2)]\ \textbf{and} \\ (|\mathcal{E}| < z_2)[\mathcal{E} := (E_1\,,V_1) \sqcup_{\mathcal{E}} (E_2\,,V_2)] \end{array}\right]$
4. $[\mathsf{Inv}_2\ \textbf{and}\ [\textbf{not}\ \mathsf{Cond}_2]]\ \textbf{implies}\ [\mathsf{Inv}_2\ \textbf{and}\ \mathsf{Interm}\ \textbf{and}\ |\mathcal{E}| < z_1]$
5. $[\mathsf{Inv}_2\ \textbf{and}\ \mathsf{Interm}\ \textbf{and}\ |\mathcal{E}| < z_1]\ \textbf{implies}\ [\mathsf{Inv}_1\ \textbf{and}\ |\mathcal{E}| < z_1]$
6. $[\mathsf{Inv}_1\ \textbf{and}\ [\textbf{not}\ \mathsf{Cond}_1]]\ \textbf{implies}\ \mathsf{Post}$

Proof. See page 141.

A.5 Constraint Automata

Definition 25 (Data constraints).

1. *A data constraint is a word* ψ *generated by the following grammar:*

$$\begin{array}{lcl} p & ::= & \textit{any element from } \mathbb{P}\text{ORT} \\ b & ::= & \bot \mid \top \mid \mathtt{Eq}(P) \mid \mathtt{d}(p) \mathbin{\mathtt{=}} \mathtt{d}(p) \\ \phi & ::= & b \mid \neg\phi \mid \phi \vee \phi \mid \phi \wedge \phi \end{array}$$

2. $\mathbb{DC}$ *denotes the set of all synchronization constraints.*
3. $\overset{\text{dc}}{\models} \subseteq \mathbb{D}\text{IS} \times \mathbb{DC}$ *denotes the relation on* $\big[$*distribution, data constraint*$\big]$*-pairs defined as follows:*

$$\begin{array}{ll} \delta \overset{\text{dc}}{\models} \bot & \textbf{iff false} \\ \delta \overset{\text{dc}}{\models} \top & \textbf{iff true} \\ \delta \overset{\text{dc}}{\models} \mathtt{Eq}(P) & \textbf{iff } |\text{Img}(\delta|_P)| = 1 \\ \delta \overset{\text{dc}}{\models} \mathtt{d}(p_1) \mathbin{\mathtt{=}} \mathtt{d}(p_2) & \textbf{iff } \delta(p_1) = \delta(p_2) \\ \delta \overset{\text{dc}}{\models} \neg\phi & \textbf{iff } \delta \overset{\text{dc}}{\not\models} \phi \\ \delta \overset{\text{dc}}{\models} \phi_1 \vee \phi_2 & \textbf{iff } \big[\delta \overset{\text{dc}}{\models} \phi_1 \textbf{ or } \delta \overset{\text{dc}}{\models} \phi_2\big] \\ \delta \overset{\text{dc}}{\models} \phi_1 \wedge \phi_2 & \textbf{iff } \big[\delta \overset{\text{dc}}{\models} \phi_1 \textbf{ and } \delta \overset{\text{dc}}{\models} \phi_2\big] \end{array}$$

4. $\equiv_{\text{dc}} \subseteq \mathbb{DC} \times \mathbb{DC}$ *denotes the relation on pairs of synchronization constraints defined as follows:*

$$\psi_1 \equiv_{\text{dc}} \psi_2 \textbf{ iff } \Big[\big[\delta \overset{\text{dc}}{\models} \psi_1 \textbf{ iff } \delta \overset{\text{dc}}{\models} \psi_2\big] \textbf{ for all } \delta\Big]$$

Definition 26. $\mathsf{Port} : \mathbb{SC} \to \wp(\mathbb{P}\text{ORT})$ *denotes the function from synchronization constraints to sets of ports defined as follows:*

$$\begin{array}{ll} \mathsf{Port}(\bot)\,,\ \mathsf{Port}(\top) & = \emptyset \\ \mathsf{Port}(\mathtt{Eq}(P)) & = P \\ \mathsf{Port}(\mathtt{d}(p_1) \mathbin{\mathtt{=}} \mathtt{d}(p_2)) & = \{p_1\,,\, p_2\} \\ \mathsf{Port}(\neg\phi) & = \mathsf{Port}(\phi) \\ \mathsf{Port}(\phi_1 \vee \phi_2)\,,\ \mathsf{Port}(\phi_1 \wedge \phi_2) & = \mathsf{Port}(\phi_1) \cup \mathsf{Port}(\phi_2) \end{array}$$

Definition 27. $\mathsf{DC} : \wp(\mathbb{P}\text{ORT}) \to \wp(\mathbb{DC})$ *denotes the function from sets of ports to sets of data constraints defined as follows:*

$$\mathsf{DC}(P) = \{\psi \mid \mathsf{Port}(\psi) \subseteq P\}$$

Definition 28 (States). $\mathbb{S}\text{TATE}$ *denotes the set of all states.*

Definition 29 (Constraint automata).

1. *A constraint automaton is a tuple* $(Q\,,\, P\,,\, \longrightarrow\,,\, \imath)$*, where:*
   - $Q \subseteq \mathbb{S}\text{TATE}$ *(states)*
   - $P \subseteq \mathbb{P}\text{ORT}$ *(ports)*
   - $\longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q$ *(transitions)*
   - $\imath \in Q$ *(initial state)*
2. $\mathbb{CA}$ *denotes the set of all constraint automata.*

Definition 30. $\trianglelefteq \subseteq (\mathbb{SC} \times \mathbb{DC}) \times \wp(\mathbb{SC} \times \mathbb{DC})$ *denotes the relation on* $\big[$*label, set of label*$\big]$*-pairs defined as follows:*

$$(\psi\,,\, \phi) \trianglelefteq X \textbf{ iff } \Bigg[\Big[\big[\delta \overset{\text{sc}}{\models} \psi \textbf{ and } \delta \overset{\text{dc}}{\models} \phi\big] \textbf{ implies } \begin{bmatrix} \begin{bmatrix} (\psi'\,,\, \phi') \in X \textbf{ and} \\ \delta \overset{\text{sc}}{\models} \psi' \textbf{ and } \delta \overset{\text{dc}}{\models} \phi' \end{bmatrix} \\ \textbf{for some } \psi'\,,\, \phi' \end{bmatrix}\Big] \textbf{ for all } \delta\Bigg]$$

Definition 31. $\preceq \subseteq \mathbb{CA} \times \mathbb{CA} \times \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE})$ *denotes the relation on tuples consisting of two* CA *and a binary relation on states defined as follows:*

$$\left[\begin{array}{c} (Q_\alpha\,,\,P_\alpha\,,\,\longrightarrow_\alpha\,,\,\imath_\alpha) \preceq^R (Q_\beta\,,\,P_\beta\,,\,\longrightarrow_\beta\,,\,\imath_\beta) \textbf{ iff} \\ R \subseteq Q_\alpha \times Q_\beta \textbf{ and } P_\alpha = P_\beta \textbf{ and } \imath_\alpha \; R \; \imath_\beta \textbf{ and} \\ \left[\left[\begin{bmatrix} q_\alpha \xrightarrow{\psi_\alpha , \phi_\alpha}_\alpha q'_\alpha \\ \textbf{and } q_\alpha \; R \; q_\beta \end{bmatrix} \textbf{ implies } (\psi_\alpha\,,\,\phi_\alpha) \trianglelefteq \left\{ (\psi_\beta\,,\,\phi_\beta) \;\middle|\; \begin{array}{l} q_\beta \xrightarrow{\psi_\beta , \phi_\beta}_\beta q'_\beta \\ \textbf{and } q'_\alpha \; R \; q'_\beta \end{array} \right\} \right] \textbf{ for all } q_\alpha\,,\,q'_\alpha\,,\,q_\beta\,,\,\psi_\alpha\,,\,\phi_\alpha \right] \end{array}\right]$$

Definition 32. $\sim \subseteq \mathbb{CA} \times \mathbb{CA}$ *denotes the relation on pairs of* CA *defined as follows:*

$$\alpha \sim \beta \textbf{ iff } \left[\left[\alpha \preceq^R \beta \textbf{ and } \beta \preceq^{R^{-1}} \alpha\right] \textbf{ for some } R\right]$$

Definition 33. $\cdot\lceil\cdot/\cdot\rceil : \mathbb{CA} \times (\mathbb{SC} \times \mathbb{DC}) \times (\mathbb{SC} \times \mathbb{DC}) \rightharpoonup \mathbb{CA}$ *denotes the partial function from tuples consisting of a constraint automaton and two labels to constraint automata defined as follows:*

$$(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil = (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \textbf{ if } \left[\mathsf{Port}(\psi_2) \subseteq P \textbf{ and } \mathsf{Port}(\phi_2) \subseteq P\right]$$

for:

$$\longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1 , \phi_1} q'\} \cup \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi , \phi} q'\}$$

Lemma 15 (Property of $\trianglelefteq$).

$$\left[\psi \equiv_{\text{sc}} \psi' \textbf{ and } \phi \equiv_{\text{dc}} \phi' \textbf{ and } (\psi'\,,\,\phi') \in X \in \wp(\mathbb{SC} \times \mathbb{DC})\right] \textbf{ implies } (\psi\,,\,\phi) \trianglelefteq X$$

Proof. See page 182.

Lemma 16 ($\longrightarrow\lceil\cdot/\cdot\rceil\cdot$ is correct).

$$\begin{bmatrix} \psi_1 \equiv_{\text{sc}} \psi_2 \textbf{ and } \phi_1 \equiv_{\text{dc}} \phi_2 \textbf{ and } \alpha \in \mathbb{CA} \\ \textbf{and } \mathsf{Port}(\psi) \subseteq P \textbf{ and } \mathsf{Port}(\phi) \subseteq P \end{bmatrix} \textbf{ implies } \alpha \sim \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil$$

Proof. See page 183.

B Proofs

B.1 Lemma 3

Proof (of Lemma 3). First, assume:

(A1) $\delta \models \bigoplus(\Psi_1 \cup \Psi_2)$

(A2) $\Psi_1 \cap \Psi_2 = \emptyset$

Next, observe:

(Z1) Suppose:

$$\big[\psi \in \Psi_1 \ \textbf{and} \ \psi' \in \Psi_2\big] \ \textbf{for some} \ \psi \, , \, \psi'$$

Then, by introducing (A2), conclude $\big[\psi \in \Psi_1 \ \textbf{and} \ \psi' \in \Psi_2 \ \textbf{and} \ \Psi_1 \cap \Psi_2 = \emptyset\big]$. Then, by applying ZFC, conclude $\big[\psi' \in \Psi_2 \ \textbf{and} \ \psi \neq \psi'\big]$.

(Z2) Suppose:

$$\big[\psi \in \Psi_2 \ \textbf{and} \ \psi' \in \Psi_1\big] \ \textbf{for some} \ \psi \, , \, \psi'$$

Then, by a reduction similar to (Z1), conclude $\big[\psi' \in \Psi_1 \ \textbf{and} \ \psi \neq \psi'\big]$.

(Z3) Suppose:

$$\big[\psi \in \Psi_1 \ \textbf{and} \ \big[\big[\big[\psi' \in \Psi_2 \ \textbf{and} \ \psi \neq \psi'\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\big] \ \textbf{for all} \ \psi'\big]\big] \ \textbf{for some} \ \psi$$

Then, by applying (Z1), conclude $\big[\psi \in \Psi_1 \ \textbf{and} \ \big[\big[\big[\psi \in \Psi_1 \ \textbf{and} \ \psi' \in \Psi_2\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\big] \ \textbf{for all} \ \psi'\big]\big]$. Then, by applying standard inference rules, conclude $\big[\big[\psi' \in \Psi_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\big] \ \textbf{for all} \ \psi'\big]$.

(Z4) Suppose:

$$\big[\psi \in \Psi_2 \ \textbf{and} \ \big[\big[\big[\psi' \in \Psi_1 \ \textbf{and} \ \psi \neq \psi'\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\big] \ \textbf{for all} \ \psi'\big]\big] \ \textbf{for some} \ \psi$$

Then, by a reduction similar to (Z3), conclude $\big[\big[\psi' \in \Psi_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\big] \ \textbf{for all} \ \psi'\big]$.

(Z5) Recall $\delta \models \bigoplus(\Psi_1 \cup \Psi_2)$ from (A1). Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\Psi_1 \cup \Psi_2 \in \wp(\mathbb{SC})$. Then, by applying ZFC, conclude $\Psi_1 \, , \, \Psi_2 \in \wp(\mathbb{SC})$.

(Z6) Suppose:

$$\big[\psi \in \Psi_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and} \ \Big[\begin{bmatrix}\big[\psi' \in \Psi_1 \ \textbf{and} \ \psi \neq \psi'\big] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{bmatrix} \ \textbf{for all} \ \psi'\Big]\big] \ \textbf{for some} \ \psi$$

Then, by introducing (Z5), conclude:

$$\psi \in \Psi_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and} \ \Big[\begin{bmatrix}\big[\psi' \in \Psi_1 \ \textbf{and} \ \psi \neq \psi'\big] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{bmatrix} \ \textbf{for all} \ \psi'\Big] \ \textbf{and} \ \Psi_1 \in \wp(\mathbb{SC})$$

Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude $\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_1)$.

(Z7) Suppose:

$$\big[\psi \in \Psi_2 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and} \ \Big[\begin{bmatrix}\big[\psi' \in \Psi_2 \ \textbf{and} \ \psi \neq \psi'\big] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{bmatrix} \ \textbf{for all} \ \psi'\Big]\big] \ \textbf{for some} \ \psi$$

Then, by a reduction similar to (Z6), conclude $\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_2)$.

(Z8) Suppose:

$$\big[\psi \in \Psi_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and} \ \Big[\begin{bmatrix}\big[\psi' \in \Psi_1 \cup \Psi_2 \ \textbf{and} \ \psi \neq \psi'\big] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{bmatrix} \ \textbf{for all} \ \psi'\Big]\big] \ \textbf{for some} \ \psi$$

Then, by applying ZFC, conclude:

$$\psi \in \Psi_1 \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \left[\begin{array}{c}\left[\begin{array}{c}[\psi' \in \Psi_1 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \\ \textbf{for all}\ \psi'\end{array}\right] \ \textbf{and}\ \left[\begin{array}{c}\left[\begin{array}{c}[\psi' \in \Psi_2 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \\ \textbf{for all}\ \psi'\end{array}\right]$$

Then, by applying (Z3), conclude:

$$\psi \in \Psi_1 \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \left[\begin{array}{c}\left[\begin{array}{c}[\psi' \in \Psi_1 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \\ \textbf{for all}\ \psi'\end{array}\right] \ \textbf{and}\ \big[[\psi' \in \Psi_2 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}] \ \textbf{for all}\ \psi'\big]$$

Then, by applying (Z6), conclude $\big[\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_1) \ \textbf{and}\ \big[[\psi' \in \Psi_2 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}] \ \textbf{for all}\ \psi'\big]\big]$.

(Z9) Suppose:

$$\Big[\psi \in \Psi_2 \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \Big[\left[\begin{array}{c}[\psi' \in \Psi_1 \cup \Psi_2 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \ \textbf{for all}\ \psi'\Big]\Big] \ \textbf{for some}\ \psi$$

Then, by a reduction similar to (Z8), conclude:

$$\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_2) \ \textbf{and}\ \big[[\psi' \in \Psi_1 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}] \ \textbf{for all}\ \psi'\big]$$

Now, prove the lemma by the following reduction. Recall $\delta \models \bigoplus(\Psi_1 \cup \Psi_2)$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude:

$$\Big[\psi \in \Psi_1 \cup \Psi_2 \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \Big[\left[\begin{array}{c}[\psi' \in \Psi_1 \cup \Psi_2 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \ \textbf{for all}\ \psi'\Big]\Big] \ \textbf{for some}\ \psi$$

Then, by applying ZFC, conclude:

$$\left[\begin{array}{l}\quad\ \psi \in \Psi_1 \\ \textbf{or}\ \ \psi \in \Psi_2\end{array}\right] \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \Big[\left[\begin{array}{c}[\psi' \in \Psi_1 \cup \Psi_2 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \ \textbf{for all}\ \psi'\Big]$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{rl} & \Big[\psi \in \Psi_1 \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \Big[\left[\begin{array}{c}[\psi' \in \Psi_1 \cup \Psi_2 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \ \textbf{for all}\ \psi'\Big]\Big] \\ \textbf{or} & \Big[\psi \in \Psi_2 \ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi \ \textbf{and}\ \Big[\left[\begin{array}{c}[\psi' \in \Psi_1 \cup \Psi_2 \ \textbf{and}\ \psi \neq \psi'] \\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}\end{array}\right] \ \textbf{for all}\ \psi'\Big]\Big]\end{array}$$

Then, by applying (Z8)(Z9), conclude:

$$\begin{array}{rl} & \big[\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_1) \ \textbf{and}\ \big[[\psi' \in \Psi_2 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}] \ \textbf{for all}\ \psi'\big]\big] \\ \textbf{or} & \big[\delta \overset{\text{sc}}{\models} \bigoplus(\Psi_2) \ \textbf{and}\ \big[[\psi' \in \Psi_1 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{\psi'}] \ \textbf{for all}\ \psi'\big]\big]\end{array}$$

(| QED. |)

B.2 Lemma 4

Proof (of Lemma 4).

1. First, assume:

   (A1) $\Psi \in \wp(\mathbb{SC})$

   Now, prove the lemma by the following reduction. Recall $\Psi \in \wp(\mathbb{SC})$ from (A1). Then, by applying ZFC, conclude $\Psi \subseteq \mathbb{SC}$. Then, by introducing Definition 10 of $\mathbb{SC}$, conclude:

   $$\big[(\mathbb{SC}\,,\,<_{\mathbb{SC}}) \text{ is a chain}\big] \;\; \textbf{and} \;\; \Psi \subseteq \mathbb{SC}$$

   Then, by applying ZFC, conclude $\big[(\Psi\,,\,<_{\mathbb{SC}}) \text{ is a chain}\big]$. Then, by applying ZFC, conclude:

   $$\psi = \text{least}(\Psi) \;\; \textbf{for some} \;\; \psi$$

   Then, by applying ZFC, conclude $\text{least}(\Psi) \in \Psi$.

   ⦇ QED. ⦈

2. First, assume:

   (B1) $\Psi \in \wp(\mathbb{SC})$

   Now, prove the lemma by the following reduction. Recall $\Psi \in \wp(\mathbb{SC})$ from (B1). Then, by applying Lemma 4:1, conclude $\text{least}(\Psi) \in \Psi$. Then, by applying ZFC, conclude $|\Psi \setminus \{\text{least}(\Psi)\}| = |\Psi| - 1$. Then, by applying ZFC, conclude $|\Psi \setminus \{\text{least}(\Psi)\}| < |\Psi|$.

   ⦇ QED. ⦈

B.3 Lemma 5

Proof (of Lemma 5).

1. First, assume:

   (A1) $\Psi \in \wp(\mathbb{SC})$

   (A2) $\big[\psi \in \Psi \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\big]$ **for some** ψ

   Now, prove the lemma by induction on $|\Psi|$.

   – **Base:** $|\Psi| = 0$
   Prove the base case by the following reduction. Recall $|\Psi| = 0$ from **Base**. Then, by applying ZFC, conclude $\Psi = \emptyset$. Then, by introducing (A2), conclude $\big[\psi \in \Psi \ \textbf{and}\ \Psi = \emptyset\big]$. Then, by applying substitution, conclude $\psi \in \emptyset$. Then, by applying ZFC, conclude **false**.

   – **IH:**

   $$\Big[\big[|\hat{\Psi}| < |\Psi| \ \textbf{and}\ \hat{\Psi} \in \wp(\mathbb{SC}) \ \textbf{and}\ \begin{bmatrix}\big[\hat{\psi} \in \hat{\Psi} \ \textbf{and}\ \hat{\delta} \stackrel{\text{sc}}{\models} \hat{\psi}\big] \\ \textbf{for some}\ \hat{\psi}\end{bmatrix}\big] \ \textbf{implies}\ \hat{\delta} \stackrel{\text{sc}}{\models} \textstyle\sum(\hat{\Psi})\Big] \ \textbf{for all}\ \hat{\delta}, \hat{\Psi}$$

   – **Step:** $|\Psi| > 0$
   First, observe:

   (Z1) Recall $\big[\psi \in \Psi \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\big]$ from (A2). Then, by applying standard inference rules, conclude:

   $$\textbf{true}\ \textbf{and}\ \psi \in \Psi \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi$$

   Then, by applying standard inference rules, conclude:

   $$\big[\psi = \text{least}(\Psi) \ \textbf{or}\ \psi \neq \text{least}(\Psi)\big] \ \textbf{and}\ \psi \in \Psi \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi$$

   Then, by applying standard inference rules, conclude:

   $$\big[\psi = \text{least}(\Psi) \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\big] \ \textbf{or}\ \big[\psi \neq \text{least}(\Psi) \ \textbf{and}\ \psi \in \Psi \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\big]$$

   Then, by applying substitution, conclude:

   $$\delta \stackrel{\text{sc}}{\models} \text{least}(\Psi) \ \textbf{or}\ \big[\psi \neq \text{least}(\Psi) \ \textbf{and}\ \psi \in \Psi \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\big]$$

   Then, by applying ZFC, conclude $\big[\delta \stackrel{\text{sc}}{\models} \text{least}(\Psi) \ \textbf{or}\ \big[\psi \in \Psi \setminus \{\text{least}(\Psi)\} \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\big]\big]$. Then, by introducing (A1), conclude:

   $$\delta \stackrel{\text{sc}}{\models} \text{least}(\Psi) \ \textbf{or}\ \big[\Psi \in \wp(\mathbb{SC}) \ \textbf{and}\ \psi \in \Psi \setminus \{\text{least}(\Psi)\} \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\big]$$

   Then, by applying ZFC, conclude:

   $$\delta \stackrel{\text{sc}}{\models} \text{least}(\Psi) \ \textbf{or}\ \begin{bmatrix}\Psi \in \wp(\mathbb{SC}) \ \textbf{and}\ \Psi \setminus \{\text{least}(\Psi)\} \in \wp(\mathbb{SC}) \\ \textbf{and}\ \psi \in \Psi \setminus \{\text{least}(\Psi)\} \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\end{bmatrix}$$

   Then, by applying Lemma 4:2, conclude:

   $$\delta \stackrel{\text{sc}}{\models} \text{least}(\Psi) \ \textbf{or}\ \begin{bmatrix}|\Psi \setminus \{\text{least}(\Psi)\}| < |\Psi| \ \textbf{and}\ \Psi \setminus \{\text{least}(\Psi)\} \in \wp(\mathbb{SC}) \\ \textbf{and}\ \psi \in \Psi \setminus \{\text{least}(\Psi)\} \ \textbf{and}\ \delta \stackrel{\text{sc}}{\models} \psi\end{bmatrix}$$

   Then, by applying **IH**, conclude $\big[\delta \stackrel{\text{sc}}{\models} \text{least}(\Psi) \ \textbf{or}\ \delta \stackrel{\text{sc}}{\models} \sum(\Psi \setminus \{\text{least}(\Psi)\})\big]$. Then, by applying Definition 10 of $\stackrel{\text{sc}}{\models}$, conclude $\delta \stackrel{\text{sc}}{\models} \text{least}(\Psi) + \sum(\Psi \setminus \{\text{least}(\Psi)\})$.

(Z2) Recall $|\varPsi| > 0$ from **Step**. Then, by applying ZFC, conclude $\varPsi \neq \emptyset$. Then, by introducing (A1), conclude $\big[\varPsi \in \wp(\mathbb{SC})$ **and** $\varPsi \neq \emptyset\big]$. Then, by applying Definition 11 of $\sum$, conclude:

$$\textstyle\sum(\varPsi) = \mathrm{least}(\varPsi) + \sum(\varPsi \setminus \{\mathrm{least}(\varPsi)\})$$

Now, prove the inductive step by the following reduction. Recall $\delta \overset{\text{sc}}{\models} \mathrm{least}(\varPsi) + \sum(\varPsi \setminus \{\mathrm{least}(\varPsi)\})$ from (Z1). Then, by applying (Z2), conclude $\delta \overset{\text{sc}}{\models} \sum(\varPsi)$.

(| QED. |)

2. First, assume:

(B1) $\varPsi \in \wp(\mathbb{SC})$

(B2) $\big[\psi \in \varPsi$ **implies** $\delta \overset{\text{sc}}{\models} \psi\big]$ **for all** ψ

Now, prove the lemma by induction on $|\mathbb{SC}|$.

– **Base:** $|\varPsi| = 0$
Prove the base case by the following reduction. Recall $|\varPsi| = 0$ from **Base**. Then, by applying ZFC, conclude $\varPsi = \emptyset$. Then, by applying ZFC, conclude $\big[\emptyset \in \wp(\mathbb{SC})$ **and** $\varPsi = \emptyset\big]$. Then, by applying substitution, conclude $\big[\varPsi \in \wp(\mathbb{SC})$ **and** $\varPsi = \emptyset\big]$. Then, by applying Definition 11 of $\prod$, conclude $\prod(\varPsi) = 1$. Then, by applying standard inference rules, conclude $\big[$**true and** $\prod(\varPsi) = 1\big]$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude $\big[\delta \overset{\text{sc}}{\models} 1$ **and** $\prod(\varPsi) = 1\big]$. Then, by applying substitution, conclude $\delta \overset{\text{sc}}{\models} \prod(\varPsi)$.

– **IH:**

$$\left[\left[\begin{array}{c} |\hat{\varPsi}| < |\varPsi| \ \textbf{and} \ \hat{\varPsi} \in \wp(\mathbb{SC}) \ \textbf{and} \\ \big[\big[\hat{\psi} \in \hat{\varPsi} \ \textbf{implies} \ \hat{\delta} \overset{\text{sc}}{\models} \hat{\psi}\big] \ \textbf{for all} \ \hat{\psi}\big] \end{array}\right] \ \textbf{implies} \ \hat{\delta} \overset{\text{sc}}{\models} \textstyle\sum(\hat{\varPsi})\right] \ \textbf{for all} \ \hat{\delta}, \hat{\varPsi}$$

– **Step:** $|\varPsi| > 0$
First, observe:

(Y1) Recall $|\varPsi| > 0$ from **Step**. Then, by applying ZFC, conclude $\varPsi \neq \emptyset$. Then, by introducing (B1), conclude $\big[\varPsi \in \wp(\mathbb{SC})$ **and** $\varPsi \neq \emptyset\big]$. Then, by applying Definition 11 of $\prod$, conclude:

$$\textstyle\prod(\varPsi) = \mathrm{least}(\varPsi) \cdot \prod(\varPsi \setminus \{\mathrm{least}(\varPsi)\})$$

(Y2) Recall $\varPsi \in \wp(\mathbb{SC})$ from (B1). Then, by applying Lemma 4:1, conclude $\mathrm{least}(\varPsi) \in \varPsi$. Then, by applying (B2), conclude $\delta \overset{\text{sc}}{\models} \mathrm{least}(\varPsi)$.

(Y3) Recall $\big[\psi \in \varPsi$ **implies** $\delta \overset{\text{sc}}{\models} \psi\big]$ **for all** ψ from (B2). Then, by applying ZFC, conclude:

$$\big[\psi \in \varPsi \setminus \{\mathrm{least}(\varPsi)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all} \ \psi$$

Then, by introducing (B1), conclude:

$$\varPsi \in \wp(\mathbb{SC}) \ \textbf{and} \ \big[\big[\psi \in \varPsi \setminus \{\mathrm{least}(\varPsi)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all} \ \psi\big]$$

Then, by applying ZFC, conclude:

$$\begin{array}{c} \varPsi \in \wp(\mathbb{SC}) \ \textbf{and} \ \varPsi \setminus \{\mathrm{least}(\varPsi)\} \in \wp(\mathbb{SC}) \ \textbf{and} \\ \big[\big[\psi \in \varPsi \setminus \{\mathrm{least}(\varPsi)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all} \ \psi\big] \end{array}$$

Then, by applying Lemma 4:2, conclude:

$$\begin{array}{c} |\varPsi \setminus \mathrm{least}(\varPsi)| < |\varPsi| \ \textbf{and} \ \varPsi \setminus \{\mathrm{least}(\varPsi)\} \in \wp(\mathbb{SC}) \ \textbf{and} \\ \big[\big[\psi \in \varPsi \setminus \{\mathrm{least}(\varPsi)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all} \ \psi\big] \end{array}$$

Then, by applying **IH**, conclude $\delta \overset{\text{sc}}{\models} \wp(\varPsi \setminus \{\mathrm{least}(\varPsi)\})$.

Now, prove the inductive step by the following reduction. Recall $\delta \overset{\text{sc}}{\models} \wp(\Psi \setminus \{\text{least}(\Psi)\})$ from (Y3). Then, by introducing (Y2), conclude $\left[\delta \overset{\text{sc}}{\models} \text{least}(\Psi) \textbf{ and } \delta \overset{\text{sc}}{\models} \prod(\Psi \setminus \{\text{least}(\Psi)\})\right]$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude $\delta \overset{\text{sc}}{\models} \text{least}(\Psi) \cdot \prod(\Psi \setminus \{\text{least}(\Psi)\})$. Then, by introducing (Y1), conclude $\prod(\Psi) = \text{least}(\Psi) \cdot \prod(\Psi \setminus \{\text{least}(\Psi)\}) \textbf{ and } \delta \overset{\text{sc}}{\models} \text{least}(\Psi) \cdot \prod(\Psi \setminus \{\text{least}(\Psi)\})$. Then, by applying substitution, conclude $\delta \overset{\text{sc}}{\models} \prod(\Psi)$.

(| QED. |)

3. First, assume:

(C1) $\delta \overset{\text{sc}}{\models} \sum(\Psi)$

Now, prove the lemma by induction on $|\Psi|$.

- **Base:** $|\Psi| = 0$
  Prove the base case by the following reduction. Recall $\delta \overset{\text{sc}}{\models} \sum(\Psi)$ from (C1). Then, by introducing **Base**, conclude $\left[|\Psi| = 0 \textbf{ and } \delta \overset{\text{sc}}{\models} \sum(\Psi)\right]$. Then, by applying ZFC, conclude:

  $$\Psi = \emptyset \textbf{ and } \delta \overset{\text{sc}}{\models} \sum(\Psi)$$

  Then, by applying Definition 11 of $\sum$, conclude $\delta \overset{\text{sc}}{\models} 0$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude **false**.

- **IH:**

  $$\left[\left[|\hat{\Psi}| < |\Psi| \textbf{ and } \hat{\delta} \overset{\text{sc}}{\models} \sum(\hat{\Psi})\right] \textbf{ implies } \left[\left[\hat{\psi} \in \hat{\Psi} \textbf{ and } \hat{\delta} \overset{\text{sc}}{\models} \hat{\psi}\right] \textbf{ for some } \hat{\psi}\right]\right] \textbf{ for all } \hat{P}, \hat{\Psi}$$

- **Step:** $|\Psi| > 0$
  First, observe:

  (X1) Recall $|\Psi| > 0$ from **Step**. Then, by applying ZFC, conclude $\Psi \neq \emptyset$. Then, by introducing (C1), conclude $\left[\delta \overset{\text{sc}}{\models} \sum(\Psi) \textbf{ and } \Psi \neq \emptyset\right]$. Then, by applying Definition 11 of $\sum$, conclude $\delta \overset{\text{sc}}{\models} \text{least}(\Psi) + \sum(\Psi \setminus \{\text{least}(\Psi)\})$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude:

  $$\delta \overset{\text{sc}}{\models} \text{least}(\Psi) \textbf{ or } \delta \overset{\text{sc}}{\models} \sum(\Psi \setminus \{\text{least}(\Psi)\})$$

  (X2) Recall $\delta \overset{\text{sc}}{\models} \sum(\Psi)$ from (C1). Then, by applying Definition 11 of $\sum$, conclude $\Psi \in \wp(\mathbb{SC})$.

  (X3) Suppose $\delta \overset{\text{sc}}{\models} \text{least}(\Psi)$. Then, by introducing (X2), conclude:

  $$\Psi \in \wp(\mathbb{SC}) \textbf{ and } \delta \overset{\text{sc}}{\models} \text{least}(\Psi)$$

  Then, by applying Lemma 4:1, conclude $\left[\text{least}(\Psi) \in \Psi \textbf{ and } \delta \overset{\text{sc}}{\models} \text{least}(\Psi)\right]$. Then, by applying ZFC, conclude $\left[\left[\left[\psi = \text{least}(\Psi) \textbf{ and } \psi \in \Psi\right] \textbf{ for some } \psi\right] \textbf{ and } \delta \overset{\text{sc}}{\models} \text{least}(\Psi)\right]$. Then, by applying standard inference rules, conclude:

  $$\left[\psi = \text{least}(\Psi) \textbf{ and } \psi \in \Psi \textbf{ and } \delta \overset{\text{sc}}{\models} \text{least}(\Psi)\right] \textbf{ for some } \psi$$

  Then, by applying substitution, conclude $\left[\psi \in \Psi \textbf{ and } \delta \overset{\text{sc}}{\models} \psi\right]$.

  (X4) Suppose $\delta \overset{\text{sc}}{\models} \sum(\Psi \setminus \{\text{least}(\Psi)\})$. Then, by introducing (X2), conclude:

  $$\Psi \in \wp(\mathbb{SC}) \textbf{ and } \delta \overset{\text{sc}}{\models} \sum(\Psi \setminus \{\text{least}(\Psi)\})$$

  Then, by applying Lemma 4:2, conclude $\left[|\Psi \setminus \{\text{least}(\Psi)\}| < |\Psi| \textbf{ and } \delta \overset{\text{sc}}{\models} \sum(\Psi \setminus \{\text{least}(\Psi)\})\right]$. Then, by applying **IH**, conclude:

  $$\left[\psi \in \Psi \setminus \{\text{least}(\Psi)\} \textbf{ and } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for some } \psi$$

  Then, by applying ZFC, conclude $\left[\psi \in \Psi \textbf{ and } \delta \overset{\text{sc}}{\models} \psi\right]$.

Now, prove the inductive step by the following reduction. Recall from (X1):

$$\delta \stackrel{sc}{\models} \text{least}(\Psi) \ \textbf{or} \ \delta \stackrel{sc}{\models} \sum(\Psi \setminus \{\text{least}(\Psi)\})$$

Then, by applying (X3), conclude:

$$\big[\big[\psi \in \Psi \ \textbf{and} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for some} \ \psi\big] \ \textbf{or} \ \delta \stackrel{sc}{\models} \sum(\Psi \setminus \{\text{least}(\Psi)\})$$

Then, by applying (X4), conclude:

$$\big[\big[\psi \in \Psi \ \textbf{and} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for some} \ \psi\big] \ \textbf{or} \ \big[\big[\psi \in \Psi \ \textbf{and} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for some} \ \psi\big]$$

Then, by applying standard inference rules, conclude:

$$\big[\psi \in \Psi \ \textbf{and} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for some} \ \psi$$

⦇ QED. ⦈

4. First, assume:

(D1) $\delta \stackrel{sc}{\models} \wp(\Psi)$

Now, prove the lemma by induction on $|\Psi|$.

- **Base:** $|\Psi| = 0$
  Prove the base case by the following reduction. Recall $|\Psi| = 0$ from **Base**. Then, by applying ZFC, conclude $\Psi = \emptyset$. Then, by applying standard inference rules, conclude $\big[\textbf{true} \ \textbf{and} \ \Psi = \emptyset\big]$ Then, by applying standard inference rules, conclude:

  $$\big[\big[\big[\textbf{false} \ \textbf{implies} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for all} \ \psi\big] \ \textbf{and} \ \Psi = \emptyset\big]$$

  Then, by applying ZFC, conclude $\big[\big[\big[\psi \in \emptyset \ \textbf{implies} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for all} \ \psi\big] \ \textbf{and} \ \Psi = \emptyset\big]$. Then, by applying subsitution, conclude $\big[\big[\psi \in \Psi \ \textbf{implies} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for all} \ \psi\big]$.

- **IH:**

  $$\big[\big[|\hat{\Psi}| < |\Psi| \ \textbf{and} \ \hat{\delta} \stackrel{sc}{\models} \prod(\hat{\Psi})\big] \ \textbf{implies} \ \big[\big[\hat{\psi} \in \hat{\Psi} \ \textbf{implies} \ \hat{\delta} \stackrel{sc}{\models} \hat{\psi}\big] \ \textbf{for all} \ \hat{\psi}\big]\big] \ \textbf{for all} \ \hat{P}, \hat{\Psi}$$

- **Step:** $|\Psi| > 0$
  First, observe:

  (W1) Recall $|\Psi| > 0$ from **Step**. Then, by applying ZFC, conclude $\Psi \neq \emptyset$. Then, by introducing (D1), conclude $\big[\delta \stackrel{sc}{\models} \wp(\Psi) \ \textbf{and} \ \Psi \neq \emptyset\big]$. Then, by applying Definition 11 of $\prod$, conclude $\delta \stackrel{sc}{\models} \text{least}(\Psi) \cdot \prod(\Psi \setminus \{\text{least}(\Psi)\})$. Then, by applying Definition 10 of $\stackrel{sc}{\models}$, conclude:

  $$\delta \stackrel{sc}{\models} \text{least}(\Psi) \ \textbf{and} \ \delta \stackrel{sc}{\models} \prod(\Psi \setminus \{\text{least}(\Psi)\})$$

  (W2) Suppose $\delta \stackrel{sc}{\models} \text{least}(\Psi)$. Then, by applying standard inference rules, conclude:

  $$\big[\psi = \text{least}(\Psi) \ \textbf{implies} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for all} \ \psi$$

  Then, by applying ZFC, conclude $\big[\big[\psi \in \{\text{least}(\Psi)\} \ \textbf{implies} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for all} \ \psi\big]$.

  (W3) Suppose $\delta \stackrel{sc}{\models} \prod(\Psi \setminus \{\text{least}(\Psi)\})$. Then, by introducing (D1), conclude:

  $$\delta \stackrel{sc}{\models} \prod(\Psi) \ \textbf{and} \ \delta \stackrel{sc}{\models} \prod(\Psi \setminus \{\text{least}(\Psi)\})$$

  Then, by applying Definition 11 of $\prod$, conclude $\big[\Psi \in \wp(\mathbb{SC}) \ \textbf{and} \ \delta \stackrel{sc}{\models} \prod(\Psi \setminus \{\text{least}(\Psi)\})\big]$. Then, by applying Lemma 4:2, conclude:

  $$|\Psi \setminus \{\text{least}(\Psi)\}| < |\Psi| \ \textbf{and} \ \delta \stackrel{sc}{\models} \prod(\Psi \setminus \{\text{least}(\Psi)\})$$

  Then, by applying **IH**, conclude $\big[\big[\psi \in \Psi \setminus \{\text{least}(\Psi)\} \ \textbf{implies} \ \delta \stackrel{sc}{\models} \psi\big] \ \textbf{for all} \ \psi\big]$.

(W4) Recall $\delta \overset{sc}{\models} \prod(\Psi)$ from (D1). Then, by applying Definition 11 of $\prod$, conclude $\Psi \in \wp(\mathbb{SC})$. Then, by applying Lemma 4:1, conclude $\mathrm{least}(\Psi) \in \Psi$. Then, by applying ZFC, conclude:

$$\Psi = \Psi \setminus \{\mathrm{least}(\Psi) \cup (\Psi \setminus \{\mathrm{least}(\Psi)\})\}$$

Now, prove the lemma by the following reduction. Recall from (W1):

$$\delta \overset{sc}{\models} \mathrm{least}(\Psi) \;\; \textbf{and} \;\; \delta \overset{sc}{\models} \prod(\Psi \setminus \{\mathrm{least}(\Psi)\})$$

Then, by applying (W2), conclude:

$$\big[\big[\psi \in \{\mathrm{least}(\Psi)\} \;\; \textbf{implies} \;\; \delta \overset{sc}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi\big] \;\; \textbf{and} \;\; \delta \overset{sc}{\models} \prod(\Psi \setminus \{\mathrm{least}(\Psi)\})$$

Then, by applying (W3), conclude:

$$\begin{aligned} &\big[\big[\psi \in \{\mathrm{least}(\Psi)\} \;\; \textbf{implies} \;\; \delta \overset{sc}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi\big] \\ \textbf{and} \;\; &\big[\big[\psi \in \Psi \setminus \{\mathrm{least}(\Psi)\} \;\; \textbf{implies} \;\; \delta \overset{sc}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi\big] \end{aligned}$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\psi \in \{\mathrm{least}(\Psi)\} \;\; \textbf{or} \;\; \psi \in \Psi \setminus \{\mathrm{least}(\Psi)\}\big] \;\; \textbf{implies} \;\; \delta \overset{sc}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying ZFC, conclude

$$\big[\psi \in \{\mathrm{least}(\Psi)\} \cup (\Psi \setminus \{\mathrm{least}(\Psi))\} \;\; \textbf{implies} \;\; \delta \overset{sc}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying (W4), conclude $\big[\big[\psi \in \Psi \;\; \textbf{implies} \;\; \delta \overset{sc}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi\big]$.

(| QED. |)

B.4 Lemma 6

Proof (of Lemma 6).

1. First, assume:

   (A1) $(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$

   Now, prove the lemma by the following reduction. Recall $(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$ from (A1). Then, by applying Definition 19 of $\curlyvee$, conclude $\left[E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \textbf{ and } E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\}\right]$. Then, by applying ZFC, conclude $\left[V_1 \in E_1 \textbf{ and } V_2 \in E_2\right]$.

   ⦇ QED. ⦈

2. First, assume:

   (B1) $(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$

   (B2) $V_1 \neq V_2$

   Next, observe:

   (Z1) Recall $(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$ from (B1). Then, by applying Definition 19 of $\curlyvee$, conclude:

   $$E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\}$$

   Then, by introducing (B2), conclude $\left[E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \textbf{ and } V_1 \neq V_2\right]$. Then, by applying ZFC, conclude $E_1 = (E_2 \cup \{V_1\}) \setminus \{V_2\}$.

   (Z2) Recall $(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$ from (B1). Then, by applying Lemma 6:1, conclude:

   $$V_1 \in E_1 \textbf{ and } V_2 \in E_2$$

   Now, prove the lemma by the following reduction. Suppose $E_1 = E_2$. Then, by introducing (Z1)(Z2), conclude $\left[E_1 = E_2 \textbf{ and } E_1 = (E_2 \cup \{V_1\}) \setminus \{V_2\} \textbf{ and } V_1 \in E_1 \textbf{ and } V_2 \in E_2\right]$. Then, by applying substitution, conclude $\left[E_1 = (E_1 \cup \{V_1\}) \setminus \{V_2\} \textbf{ and } V_1 \in E_1 \textbf{ and } V_2 \in E_1\right]$. Then, by applying ZFC, conclude $\left[E_1 = E_1 \setminus \{V_2\} \textbf{ and } V_2 \in E_1\right]$. Then, by applying ZFC, conclude:

   $$E_1 = E_1 \setminus \{V_2\} \textbf{ and } E_1 \neq E_1 \setminus \{V_2\}$$

   Then, by applying standard inference rules, conclude **false**.

   ⦇ QED. ⦈

B.5 Lemma 7

Proof (of Lemma 7).

1. First, assume:

   (A1) $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$

   Next, observe:

   (Z1) Recall $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$ from (A1). Then, by applying Definition 19 of $\curlyvee$, conclude:

   $$E_1 \neq E_2 \;\;\textbf{and}\;\; (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2)$$

   (Z2) Recall $(E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2)$ from (Z1). Then, by applying Definition 19 of $\curlyvee$, conclude:

   $$E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\}$$

   (Z3) Recall $(E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2)$ from (Z2). Then, by applying Lemma 6:1, conclude $V_2 \in E_2$.

   Now, prove the lemma by the following reduction. Suppose $V_1 = V_2$. Then, by introducing (Z2), conclude:

   $$E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \;\;\textbf{and}\;\; V_1 = V_2$$

   Then, by applying substitution, conclude $E_1 = (E_2 \setminus \{V_2\}) \cup \{V_2\}$. Then, by applying ZFC, conclude $E_1 = E_2 \cup \{V_2\}$. Then, by introducing (Z3), conclude $\big[E_1 = E_2 \cup \{V_2\} \;\;\textbf{and}\;\; V_2 \in E_2\big]$. Then, by applying ZFC, conclude $E_1 = E_2$. Then, by introducing (Z1), conclude $\big[E_1 = E_2 \;\;\textbf{and}\;\; E_1 \neq E_2\big]$. Then, by applying standard inference rules, conclude **false**.

   ⦇ QED. ⦈

2. First, assume:

   (B1) $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$

   Next, observe:

   (Y1) Recall $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$ from (B1). Then, by applying Lemma 7:1, conclude $V_1 \neq V_2$.

   Now, prove the lemma by the following reduction. Recall $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$ from (B1). Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2)$. Then, by applying Definition 19 of $\curlyvee$, conclude $\big[E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \;\;\textbf{and}\;\; E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\}\big]$. Then, by introducing (Y1), conclude:

   $$E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \;\;\textbf{and}\;\; E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \;\;\textbf{and}\;\; V_1 \neq V_2$$

   Then, by applying ZFC, conclude $\big[E_1 = (E_2 \cup \{V_1\}) \setminus \{V_2\} \;\;\textbf{and}\;\; E_2 = (E_1 \cup \{V_2\}) \setminus \{V_1\}\big]$. Then, by applying ZFC, conclude $\big[V_2 \notin E_1 \;\;\textbf{and}\;\; V_1 \notin E_2\big]$.

   ⦇ QED. ⦈

B.6 Lemma 8

Proof (of Lemma 8).

1. First, assume:

   (A1) $(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)$

   (A2) $\delta \overset{\text{sc}}{\models} [\![\mathcal{V}\,,\,\mathcal{E}]\!]$

   (A3) $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$

   Next, observe:

   (Z1) Recall $(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)$ from (A1). Then, by applying Definition 19 of $\curlyvee$, conclude:

   $$E_1\,,\,E_2 \in \wp(\mathbb{V}\text{ER}) \ \textbf{and}\ V_1\,,\,V_2 \in \mathbb{V}\text{ER} \ \textbf{and}\ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}\ E_1\,,\,E_2 \in \mathcal{E} \ \textbf{and}\ (E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$$

   (Z2) Suppose:

   $$E \in \mathcal{E} \ \textbf{for some}\ E$$

   Then, by introducing (Z1), conclude $E_1\,,\,E_2\,,\,E \in \mathcal{E}$. Then, by applying ZFC, conclude:

   $$\{E_1\,,\,E_2\} \subseteq \mathcal{E} \ \textbf{and}\ E \in \mathcal{E}$$

   Then, by applying ZFC, conclude $\big[E \in \{E_1\,,\,E_2\} \ \textbf{or}\ E \in \mathcal{E}\setminus\{E_1\,,\,E_2\}\big]$. Then, by applying ZFC, conclude $\big[E \in \{E_1\} \ \textbf{or}\ E \in \{E_2\} \ \textbf{or}\ E \in \mathcal{E}\setminus\{E_1\,,\,E_2\}\big]$.

   (Z3) Suppose:

   $$\big[E \in \mathcal{E}\setminus\{E_1\,,\,E_2\} \ \textbf{and}\ \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup\mathcal{V}}\big] \ \textbf{for some}\ E$$

   Then, by applying ZFC, conclude $\big[E \in (\mathcal{E}\setminus\{E_1\,,\,E_2\})\cup\{E_\dagger\} \ \textbf{and}\ \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup\mathcal{V}}\big]$. Then, by applying standard inference rules, conclude:

   $$\big[E'' \in (\mathcal{E}\setminus\{E_1\,,\,E_2\})\cup\{E_\dagger\} \ \textbf{and}\ \delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup\mathcal{V}}\big] \ \textbf{for some}\ E''$$

   (Z4) Suppose $\big[[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \ \textbf{and}\ V \in E_1\} \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{for all}\ \psi\big]$. Then, by applying ZFC, conclude $\big[\big[\big[[\psi = \bigoplus(V) \ \textbf{and}\ V \in E_1] \ \textbf{for some}\ V\big] \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all}\ \psi\big]$. Then, by applying standard inference rules, conclude:

   $$\big[\big[[\psi = \textstyle\bigoplus(V) \ \textbf{and}\ V \in E_1] \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for some}\ V\big] \ \textbf{for all}\ \psi$$

   Then, by applying substitution, conclude:

   $$\big[[V \in E_1 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \textstyle\bigoplus(V)] \ \textbf{for some}\ V\big] \ \textbf{for all}\ \psi$$

   Then, by applying standard inference rules, conclude $\big[[V \in E_1 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)] \ \textbf{for all}\ V\big]$.

   (Z5) Suppose:

   $$\big[[\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{and}\ V \in E_2\} \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{for all}\ \psi\big] \ \textbf{for some}\ \delta$$

   Then, by a reduction similar to (Z4), conclude $\big[[V \in E_2 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)] \ \textbf{for all}\ V\big]$.

   (Z6) Suppose $\big[[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and}\ p \in (\bigcup\mathcal{V})\setminus\bigcup E_1\} \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{for all}\ \psi\big]$. Then, by applying ZFC, conclude $\big[\big[\big[[\psi = \overline{p} \ \textbf{and}\ p \in (\bigcup\mathcal{V})\setminus\bigcup E_1] \ \textbf{for some}\ p\big] \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all}\ \psi\big]$. Then, by applying standard inference rules, conclude:

   $$\big[\big[[\psi = \overline{p} \ \textbf{and}\ p \in (\textstyle\bigcup\mathcal{V})\setminus\bigcup E_1] \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for some}\ p\big] \ \textbf{for all}\ \psi$$

   Then, by applying substitution, conclude:

   $$\big[[p \in (\textstyle\bigcup\mathcal{V})\setminus\bigcup E_1 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{p}] \ \textbf{for some}\ p\big] \ \textbf{for all}\ \psi$$

   Then, by applying standard inference rules, conclude:

   $$\big[p \in (\textstyle\bigcup\mathcal{V})\setminus\bigcup E_1 \ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \overline{p}\big] \ \textbf{for all}\ p$$

(Z7) Recall $V_1, V_2 \in \mathbb{V}\text{ER}$ from (Z1). Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1, V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})$.

(Z8) Recall $E_1, E_2 \in \wp(\mathbb{V}\text{ER})$ from (Z1). Then, by applying ZFC, conclude $E_1 \cap E_2 \in \wp(\mathbb{V}\text{ER})$.

(Z9) Suppose:

$$\big[p \in V \ \textbf{ and } \ V = V_1\big] \ \textbf{for some} \ p, V$$

Then, by applying substitution, conclude $p \in V_1$. Then, by introducing (Z7), conclude:

$$p \in V_1 \ \textbf{ and } \ V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})$$

Then, by applying ZFC, conclude $p \in V_1 \cup V_2$. Then, by applying ZFC, conclude $p \in \bigcup\{V_1 \cup V_2\}$. Then, by introducing (Z8), conclude $\big[p \in \bigcup\{V_1 \cup V_2\} \ \textbf{and} \ E_1 \cap E_2 \in \wp(\mathbb{V}\text{ER})\big]$. Then, by applying ZFC, conclude $p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))$.

(Z0) Suppose:

$$\big[p \in V \ \textbf{ and } \ V = V_2\big] \ \textbf{for some} \ p, V$$

Then, by a reduction similar to (Z9), conclude $p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))$.

(Y1) Recall $(E_1, V_1) \curlyvee (E_2, V_2)$ from (Z1). Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \ \textbf{ and } \ E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\}$$

(Y2) Recall $(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$ from (A1). Then, by applying Lemma 7:2, conclude:

$$V_2 \notin E_1 \ \textbf{ and } \ V_1 \notin E_2$$

(Y3) Recall $E_1, E_2 \in \mathcal{E}$ from (Z1). Then, by applying ZFC, conclude $E_1 \cap E_2 \in \mathcal{E}$. Then, by applying standard inference rules, conclude $E_1 \cap E_2 = E_1 \cap E_2$. Then, by applying (Y1), conclude:

$$E_1 \cap E_2 = E_1 \cup ((E_1 \setminus \{V_1\}) \cup \{V_2\})$$

Then, by introducing (Y2), conclude $\big[E_1 \cap E_2 = E_1 \cup ((E_1 \setminus \{V_1\}) \cup \{V_2\}) \ \textbf{and} \ V_2 \notin E_1\big]$. Then, by applying ZFC, conclude $E_1 \cap E_2 = E_1 \cup (E_1 \setminus \{V_1\})$. Then, by applying ZFC, conclude:

$$E_1 \cap E_2 = E_1 \setminus \{V_1\}$$

(Y4) By a reduction similar to (Y3), conclude $E_1 \cap E_2 = E_2 \setminus \{V_2\}$.

(Y5) Recall $E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\}$ from (Y1). Then, by applying (Y4), conclude $E_1 = (E_1 \cap E_2) \cup \{V_1\}$.

(Y6) By a reduction similar to (Y5), conclude $E_2 = (E_1 \cap E_2) \cup \{V_2\}$.

(Y7) Suppose:

$$\big[p \in V \in E_1 \ \textbf{ and } \ V \neq V_1\big] \ \textbf{for some} \ p, V$$

Then, by applying (Y5), conclude $\big[p \in V \in (E_1 \cap E_2) \cup \{V_1\} \ \textbf{and} \ V \neq V_1\big]$. Then, by applying ZFC, conclude $\big[p \in V \in (E_1 \cap E_2) \cup \{V_1\} \ \textbf{and} \ V \notin \{V_1\}\big]$. Then, by applying ZFC, conclude $p \in V \in E_1 \cap E_2$. Then, by introducing (Z7), conclude $\big[p \in V \in E_1 \cap E_2 \ \textbf{and} \ V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})\big]$. Then, by applying ZFC, conclude $p \in V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$. Then, by applying ZFC, conclude $p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))$.

(Y8) Suppose:

$$\big[p \in V \in E_2 \ \textbf{ and } \ V \neq V_2\big] \ \textbf{for some} \ p, V$$

Then, by a reduction similar to (Y7), conclude $p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))$.

(Y9) Suppose:

$$p \in \bigcup E_1 \ \textbf{for some} \ p$$

Then, by applying ZFC, conclude:

$$p \in V \in E_1 \ \textbf{for some} \ V$$

Then, by applying standard inference rules, conclude $\left[p \in V \in E_1 \ \textbf{and} \ \textbf{true}\right]$. Then, by applying standard inference rules, conclude $\left[p \in V \in E_1 \ \textbf{and} \ \left[V = V_1 \ \textbf{or} \ V \neq V_1\right]\right]$. Then, by applying standard inference rules, conclude $\left[\left[p \in V \ \textbf{and} \ V = V_1\right] \ \textbf{or} \ \left[p \in V \in E_1 \ \textbf{and} \ V \neq V_1\right]\right]$. Then, by applying (Z9), conclude $\left[p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{or} \ \left[p \in V \in E_1 \ \textbf{and} \ V \neq V_1\right]\right]$. Then, by applying (Y7), conclude $\left[p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{or} \ p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\right]$. Then, by applying standard inference rules, conclude $p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))$.

(Y0) Suppose:

$$p \in \bigcup E_2 \ \textbf{for some} \ p$$

Then, by applying a reduction similar to (Y9), conclude $p \in \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))$.

(X1) Suppose:

$$p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{for some} \ p$$

Then, by applying ZFC, conclude $\left[p \in (\bigcup \mathcal{V}) \ \textbf{and} \ p \notin \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\right]$. Then, by applying (Y9), conclude $\left[p \in (\bigcup \mathcal{V}) \ \textbf{and} \ p \notin \bigcup E_1\right]$. Then, by applying ZFC, conclude $p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1$.

(X2) Suppose:

$$p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{for some} \ p$$

Then, by a reduction similar to (X1), conclude $p \in (\bigcup \mathcal{V}) \setminus \bigcup E_2$.

(X3) Recall $(E_1 , V_1) \curlyvee (E_2 , V_2)$ from (Z1). Then, by applying Lemma 6:1, conclude:

$$V_1 \in E_1 \ \textbf{and} \ V_2 \in E_2$$

(X4) Suppose $\left[\left[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{for all} \ V\right]$. Then, by introducing (X3), conclude:

$$\left[\left[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{for all} \ V\right] \ \textbf{and} \ V_1 \in E_1$$

Then, by applying standard inference rules, conclude $\delta \overset{\text{sc}}{\models} \bigoplus(V_1)$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude:

$$\left[p \in V_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} p \ \textbf{and} \ \left[\begin{array}{c}\left[\begin{array}{l}\left[p' \in V_1 \ \textbf{and} \ p \neq p'\right] \\ \quad \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{array}\right] \\ \textbf{for all} \ p'\end{array}\right]\right] \ \textbf{for some} \ p$$

(X5) Recall $\delta \overset{\text{sc}}{\models} [\![(\mathcal{V} , \mathcal{E})]\!]$ from (A2). Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $(\mathcal{V} , \mathcal{E}) \in \mathbb{G}\text{RAPH}$. Then, by applying Definition 16 of $\mathbb{G}\text{RAPH}$, conclude:

$$\mathcal{V} \in \wp(\mathbb{V}\text{ER}) \ \textbf{and} \ \mathcal{E} \in \wp^2(\mathcal{V}) \ \textbf{and} \ \left[\left[E \in \mathcal{E} \ \textbf{implies} \ \bigcap E = \emptyset\right] \ \textbf{for all} \ E\right]$$

(X6) Recall $E_1 \in \mathcal{E}$ from (Z1). Then, by introducing (X3), conclude $V_1 \in E_1 \in \mathcal{E}$. Then, by introducing (X5), conclude $V_1 \in E_1 \in \mathcal{E} \in \wp^2(\mathcal{V})$. Then, by applying ZFC, conclude $V_1 \in E_1 \in \wp(\mathcal{V})$. Then, by applying ZFC, conclude $V_1 \in \mathcal{V}$. Then, by applying ZFC, conclude $V_1 \subseteq \bigcup \mathcal{V}$.

(X7) By a reduction similar to (X6), conclude $V_2 \subseteq \bigcup \mathcal{V}$.

(X8) Recall $E_1 , E_2 \in \mathcal{E}$ from (Z1). Then, by applying (X5), conclude $\left[\bigcap E_1 = \emptyset \ \textbf{and} \ \bigcap E_2 = \emptyset\right]$.

(X9) Suppose:

$$p \in V_2 \setminus V_1 \ \textbf{for some} \ p$$

Then, by applying ZFC, conclude $\big[p \in V_2 \ \textbf{and} \ p \notin V_1\big]$. Then, by applying ZFC, conclude:

$$p \in ((\bigcup \mathcal{V}) \setminus ((\bigcup E_2) \cup V_1)) \cup V_2 \ \textbf{and} \ p \notin V_1$$

Then, by applying ZFC, conclude $\big[p \in ((\bigcup \mathcal{V}) \setminus ((\bigcup E_2) \cup V_1)) \cup V_2 \ \textbf{and} \ p \notin V_1 \cap V_2\big]$. Then, by applying ZFC, conclude $p \in (((\bigcup \mathcal{V}) \setminus ((\bigcup E_2) \cup V_1)) \cup V_2) \setminus (V_1 \cap V_2)$. Then, by introducing (X7), conclude $\big[p \in (((\bigcup \mathcal{V}) \setminus ((\bigcup E_2) \cup V_1)) \cup V_2) \setminus (V_1 \cap V_2) \ \textbf{and} \ V_2 \subseteq \bigcup \mathcal{V}\big]$. Then, by applying ZFC, conclude $p \in ((\bigcup \mathcal{V}) \setminus (((\bigcup E_2) \cup V_1) \setminus V_2)) \setminus (V_1 \cap V_2)$. Then, by applying ZFC, conclude:

$$p \in (\bigcup \mathcal{V}) \setminus ((((\bigcup E_2) \cup V_1) \setminus V_2) \cup (V_1 \cap V_2))$$

Then, by applying ZFC, conclude $p \in (\bigcup \mathcal{V}) \setminus (((\bigcup E_2) \setminus V_2) \cup (V_1 \setminus V_2) \cup (V_1 \cap V_2))$. Then, by applying ZFC, conclude $p \in (\bigcup \mathcal{V}) \setminus (((\bigcup E_2) \setminus V_2) \cup V_1)$. Then, by introducing (X3), conclude:

$$p \in (\bigcup \mathcal{V}) \setminus (((\bigcup E_2) \setminus V_2) \cup V_1) \ \textbf{and} \ V_2 \in E_2$$

Then, by introducing (X8), conclude $\big[p \in (\bigcup \mathcal{V}) \setminus (((\bigcup E_2) \setminus V_2) \cup V_1) \ \textbf{and} \ V_2 \in E_2 \ \textbf{and} \ \bigcap E_2 = \emptyset\big]$. Then, by applying ZFC, conclude $p \in (\bigcup \mathcal{V}) \setminus ((\bigcup (E_2 \setminus \{V_2\})) \cup V_1)$. Then, by applying ZFC, conclude $p \in (\bigcup \mathcal{V}) \setminus \bigcup ((E_2 \setminus \{V_2\}) \cup \{V_1\})$. Then, by applying (Y1), conclude $p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1$.

(X0) Suppose:

$$p \in V_1 \setminus V_2 \ \textbf{for some} \ p$$

Then, by applying a reduction similar to (X9), conclude $p \in (\bigcup \mathcal{V}) \setminus \bigcup E_2$.

(W1) Suppose:

$$\Big[\Big[\begin{bmatrix}[p' \in V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p'\Big] \ \textbf{and} \ \Big[\begin{bmatrix}(\bigcup \mathcal{V}) \setminus \bigcup E_1 \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p'\Big]\Big] \ \textbf{for some} \ p$$

Then, by applying (X9), conclude:

$$\begin{bmatrix}[p' \in V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p' \ \textbf{and} \ \Big[\begin{bmatrix}p' \in V_2 \setminus V_1 \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p'\Big]$$

Then, by applying ZFC, conclude:

$$\begin{bmatrix}[p' \in V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p' \ \textbf{and} \ \Big[\begin{bmatrix}[p' \in V_2 \setminus V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p'\Big]$$

Then, by applying standard inference rules, conclude:

$$\Big[\begin{bmatrix}[p' \in V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{and} \ \begin{bmatrix}[p' \in V_2 \setminus V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix}\Big] \ \textbf{for all} \ p'$$

Then, by applying standard inference rules, conclude:

$$\big[\big[[p' \in V_1 \ \textbf{and} \ p \neq p'] \ \textbf{or} \ [p' \in V_2 \setminus V_1 \ \textbf{and} \ p \neq p']\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'$$

Then, by applying standard inference rules, conclude:

$$\big[\big[[p' \in V_1 \ \textbf{or} \ p' \in V_2 \setminus V_1] \ \textbf{and} \ p \neq p'\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'$$

Then, by applying ZFC, conclude $\big[\big[[p' \in V_1 \cup (V_2 \setminus V_1) \ \textbf{and} \ p \neq p'] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]$. Then, by applying ZFC, conclude $\big[\big[[p' \in V_1 \cup V_2 \ \textbf{and} \ p \neq p'] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]$.

(W2) Suppose:

$$\Big[\Big[\begin{bmatrix}[p' \in V_2 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p'\Big] \ \textbf{and} \ \Big[\begin{bmatrix}(\bigcup \mathcal{V}) \setminus \bigcup E_2 \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \ \textbf{for all} \ p'\Big]\Big] \ \textbf{for some} \ p \,, \delta$$

Then, by a reduction similar to (W1), conclude:

$$\big[\big[p' \in V_1 \cup V_2 \ \textbf{and} \ p \neq p'\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'$$

(W3) Suppose:

$$\big[\big[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big] \ \textbf{and} \ \big[\big[p' \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]$$

Then, by applying (X4), conclude:

$$\begin{array}{c}\Big[\Big[p \in V_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} p \ \textbf{and} \ \begin{bmatrix}\begin{bmatrix}[p' \in V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \\ \textbf{for all} \ p'\end{bmatrix}\Big] \ \textbf{for some} \ p\Big] \\ \textbf{and} \ \big[\big[p' \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in V_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} p \ \textbf{and} \ \begin{bmatrix}\begin{bmatrix}[p' \in V_1 \ \textbf{and} \ p \neq p'] \\ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\end{bmatrix} \\ \textbf{for all} \ p'\end{bmatrix} \\ \textbf{and} \ \big[\big[p' \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]\end{bmatrix} \ \textbf{for some} \ p$$

Then, by applying (W1), conclude:

$$p \in V_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} p \ \textbf{and} \ \big[\big[\big[p' \in V_1 \cup V_2 \ \textbf{and} \ p \neq p'\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]$$

Then, by applying ZFC, conclude:

$$p \in V_1 \cup V_2 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} p \ \textbf{and} \ \big[\big[\big[p' \in V_1 \cup V_2 \ \textbf{and} \ p \neq p'\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]$$

Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude $\delta \overset{\text{sc}}{\models} \bigoplus(V_1 \cup V_2)$. Then, by applying standard inference rules, conclude $\big[\big[V = V_1 \cup V_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big]$. Then, by applying ZFC, conclude $\big[\big[V \in \{V_1 \cup V_2\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big]$.

(W4) Suppose:

$$\big[\big[V \in E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big] \ \textbf{and} \ \big[\big[p' \in (\bigcup \mathcal{V}) \setminus \bigcup E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]$$

Then, by a reduction similar to (W3), conclude $\big[\big[V \in \{V_1 \cup V_2\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big]$.

(W5) Suppose:

$$\big[\big[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big] \ \textbf{and} \ \big[\big[p' \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p'}\big] \ \textbf{for all} \ p'\big]$$

Then, by applying (W3), conclude:

$$\begin{array}{r}\big[\big[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big] \\ \textbf{and} \ \big[\big[V \in \{V_1 \cup V_2\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \ \textbf{for all} \ V\big]\end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{r}\big[\big[V \in E_1 \cap E_2 \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for all} \;\; V\big] \\ \textbf{and} \;\; \big[\big[V \in \{V_1 \cup V_2\} \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for all} \;\; V\big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{r}\big[V \in E_1 \cap E_2 \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \\ \textbf{and} \;\; \big[V \in \{V_1 \cup V_2\} \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\end{array}\right] \;\; \textbf{for all} \;\; V$$

Then, by applying standard inference rules, conclude:

$$\Big[\left[\begin{array}{r} V \in E_1 \cap E_2 \\ \textbf{or} \;\; V \in \{V_1 \cup V_2\}\end{array}\right] \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\Big] \;\; \textbf{for all} \;\; V$$

Then, by applying ZFC, conclude $\big[\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for all} \;\; V\big]$.

(W6) Suppose:

$$\big[\big[V \in E_2 \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for all} \;\; V\big] \;\; \textbf{and} \;\; \big[\big[p' \in (\bigcup \mathcal{V}) \setminus \bigcup E_2 \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \overline{p'}\big] \;\; \textbf{for all} \;\; p'\big]$$

Then, by a reduction similar to (W6), conclude:

$$\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for all} \;\; V$$

(W7) Suppose $\big[\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for all} \;\; V\big]$. Then, by applying standard inference rules, conclude:

$$\big[\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for some} \;\; V\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \;\; \psi = \bigoplus(V)\big] \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\; \textbf{for some} \;\; V\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying substitution, conclude:

$$\big[\big[\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \;\; \psi = \bigoplus(V)\big] \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \psi\big] \;\; \textbf{for some} \;\; V\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \;\; \psi = \bigoplus(V)\big] \;\; \textbf{for some} \;\; V\big] \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying ZFC, conclude:

$$\big[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \;\; \textbf{and} \;\; V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi$$

(W8) Suppose $\big[\big[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \overline{p}\big] \;\; \textbf{for all} \;\; p\big]$. Then, by applying standard inference rules, conclude:

$$\big[\big[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \overline{p}\big] \;\; \textbf{for some} \;\; p\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\big[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \;\; \textbf{and} \;\; \psi = \overline{p}\big] \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \overline{p}\big] \;\; \textbf{for some} \;\; p\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying substitution, conclude:

$$\big[\big[\big[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \;\; \textbf{and} \;\; \psi = \overline{p}\big] \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \psi\big] \;\; \textbf{for some} \;\; p\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\big[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \;\; \textbf{and} \;\; \psi = \overline{p}\big] \;\; \textbf{for some} \;\; p\big] \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi$$

Then, by applying ZFC, conclude:

$$\big[\psi \in \{\psi' \mid \psi' = \overline{p} \;\; \textbf{and} \;\; p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \;\; \textbf{implies} \;\; \delta \overset{\text{sc}}{\models} \psi\big] \;\; \textbf{for all} \;\; \psi$$

(W9) Recall $V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})$ from (Z7). Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1 \cup V_2 \in \mathbb{V}\text{ER}$. Then, by applying ZFC, conclude $\{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})$. Then, by introducing (Z8), conclude:

$$\{V_1 \cup V_2\}\,,\, E_1 \cap E_2 \in \wp(\mathbb{V}\text{ER})$$

Then, by applying ZFC, conclude $\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})$.

(W0) Suppose:

$$V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{for some} \ V$$

Then, by introducing (W9), conclude $V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $V \in \mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V \in \wp(\mathbb{P}\text{ORT})$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $V \in \wp(\mathbb{SC})$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\bigoplus(V) \in \mathbb{SC}$.

(V1) Suppose:

$$\big[\psi = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big] \ \textbf{for some} \ \psi\,,\, V$$

Then, by applying (W0), conclude $\big[\psi = \bigoplus(V) \ \textbf{and} \ \bigoplus(V) \in \mathbb{SC}\big]$. Then, by applying substitution, conclude $\psi \in \mathbb{SC}$.

(V2) Suppose **true**. Then, by applying ZFC, conclude $\{\psi \mid \psi = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \in \Omega$. Then, by applying (V1), conclude $\{\psi \mid \psi = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and} \ \psi \in \mathbb{SC}\} \in \Omega$. Then, by applying ZFC, conclude:

$$\{\psi \mid \psi = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and} \ \psi \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying (V1), conclude $\{\psi \mid \psi = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \in \wp(\mathbb{SC})$.

(V3) Recall $\mathcal{V} \in \wp(\mathbb{V}\text{ER})$ from (X5). Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $\mathcal{V} \in \wp^2(\mathbb{P}\text{ORT})$.

(V4) Suppose:

$$p \in (\bigcup \mathcal{V}) \setminus \bigcup E \ \textbf{for some} \ p\,,\, E$$

Then, by applying ZFC, conclude $p \in \bigcup \mathcal{V}$. Then, by applying ZFC, conclude:

$$p \in V \in \mathcal{V} \ \textbf{for some} \ V$$

Then, by introducing (V3), conclude $p \in V \in \mathcal{V} \in \wp^2(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude:

$$p \in V \in \wp(\mathbb{P}\text{ORT})$$

Then, by applying ZFC, conclude $p \in \mathbb{P}\text{ORT}$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $p \in \mathbb{SC}$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\overline{p} \in \mathbb{SC}$.

(V5) Suppose:

$$\big[\psi = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E\big] \ \textbf{for some} \ \psi\,,\, p\,,\, E$$

Then, by applying (V4), conclude $\big[\psi = \overline{p} \ \textbf{and} \ \overline{p} \in \mathbb{SC}\big]$. Then, by applying substitution, conclude $\psi \in \mathbb{SC}$.

(V6) Suppose **true**. Then, by applying ZFC, conclude:

$$\{\psi \mid \psi = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \in \Omega$$

Then, by applying (V5), conclude:

$$\{\psi \mid \psi = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{and} \ \psi \in \mathbb{SC}\} \in \Omega$$

Then, by applying ZFC, conclude:

$$\{\psi \mid \psi = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{and} \ \psi \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying (V5), conclude $\{\psi \mid \psi = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \in \wp(\mathbb{SC})$.

(V7) Recall from (V2)(V6):

$$\begin{array}{l}\{\psi \mid \psi = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\,, \\ \{\psi \mid \psi = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \in \wp(\mathbb{SC})\end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{l}\{\psi \mid \psi = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \cup \\ \{\psi \mid \psi = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \in \wp(\mathbb{SC})\end{array}$$

(V8) Suppose $\delta \overset{\text{sc}}{\models} [\![E_1]\!]_{\bigcup \mathcal{V}}$. Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\delta \overset{\text{sc}}{\models} \prod(\{\psi' \mid \psi' = \bigoplus(V) \ \textbf{and} \ V \in E_1\} \cup \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1\})$$

Then, by applying Lemma 5:4, conclude:

$$\left[\begin{bmatrix}\psi \in \{\psi' \mid \psi' = \bigoplus(V) \ \textbf{and} \ V \in E_1\} \cup \\ \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1\}\end{bmatrix} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi$$

Then, by applying ZFC, conclude:

$$\left[\begin{bmatrix}\psi \in \{\psi' \mid \psi' = \bigoplus(V) \ \textbf{and} \ V \in E_1\} \\ \textbf{or} \ \psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1\}\end{bmatrix} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix}\left[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \ \textbf{and} \ V \in E_1\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \\ \textbf{and} \ \left[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right]\end{bmatrix} \ \textbf{for all} \ \psi$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{r}\left[\left[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \ \textbf{and} \ V \in E_1\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi\right] \\ \textbf{and} \ \left[\left[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi\right]\end{array}$$

Then, by applying (Z4), conclude:

$$\begin{array}{c}\left[\left[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{for all} \ V\right] \ \textbf{and} \\ \left[\left[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi\right]\end{array}$$

Then, by applying (Z6), conclude:

$$\begin{array}{c}\left[\left[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{for all} \ V\right] \ \textbf{and} \\ \left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{for all} \ p\right]\end{array}$$

Then, by applying (X1), conclude:

$$\begin{array}{c}\left[\left[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{for all} \ V\right] \ \textbf{and} \\ \left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{for all} \ p\right] \ \textbf{and} \\ \left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{for all} \ p\right]\end{array}$$

Then, by applying (W5), conclude:

$$\begin{array}{c}\left[\left[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{for all} \ V\right] \ \textbf{and} \\ \left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{for all} \ p\right]\end{array}$$

Then, by applying (W7), conclude:

$$\begin{array}{c}\left[\left[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi\right] \\ \textbf{and} \ \left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{for all} \ p\right]\end{array}$$

Then, by applying (W8), conclude:

$$\begin{aligned} &\big[\big[\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all} \ \psi\big] \\ \textbf{and} \ &\big[\big[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all} \ \psi\big] \end{aligned}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{aligned} &\big[\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \\ \textbf{and} \ &\big[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \end{aligned}\right] \ \textbf{for all} \ \psi$$

Then, by applying standard inference rules, conclude:

$$\left[\left[\begin{aligned} &\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \\ \textbf{or} \ &\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \end{aligned}\right] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi$$

Then, by applying ZFC, conclude:

$$\left[\left[\begin{aligned} \psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \cup \\ \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \end{aligned}\right] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi$$

Then, by introducing (V7), conclude:

$$\begin{aligned} &\left[\left[\left[\begin{aligned} \psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \cup \\ \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \end{aligned}\right] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi\right] \\ &\qquad \textbf{and} \ \left[\begin{aligned} &\{\psi \mid \psi = \textstyle\bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \cup \\ &\{\psi \mid \psi = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \in \wp(\mathbb{SC}) \end{aligned}\right] \end{aligned}$$

Then, by applying Lemma 5:2, conclude:

$$\begin{aligned} \delta \overset{\text{sc}}{\models} \textstyle\prod(\{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{and} \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \cup \\ \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}) \end{aligned}$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\delta \overset{\text{sc}}{\models} [\![\{V_1 \cup V_2\} \cup (E_1 \cap E_2)]\!]_{\bigcup \mathcal{V}}$. Then, by applying (A3), conclude $\delta \overset{\text{sc}}{\models} [\![E_\dagger]\!]_{\bigcup \mathcal{V}}$.

(V9) Suppose $\delta \overset{\text{sc}}{\models} [\![E_2]\!]_{\bigcup \mathcal{V}}$. Then, by a reduction similar to (V8), conclude $\delta \overset{\text{sc}}{\models} [\![E_\dagger]\!]_{\bigcup \mathcal{V}}$.

(V0) Recall $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$ from (Z1). Then, by applying ZFC, conclude $\mathcal{E} \setminus \{E_1 , E_2\} \in \wp^2(\mathbb{V}\text{ER})$

(U1) Suppose:

$$E = E_\dagger \ \textbf{for some} \ E$$

Then, by applying ZFC, conclude $E \in \{E_\dagger\}$. Then, by introducing (V0), conclude:

$$E \in \{E_\dagger\} \ \textbf{and} \ \mathcal{E} \setminus \{E_1 , E_2\} \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying ZFC, conclude $E \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\}$.

(U2) Suppose:

$$\big[E \in \{E_1\} \ \textbf{and} \ \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big] \ \textbf{for some} \ E$$

Then, by applying ZFC, conclude $\big[E = E_1 \ \textbf{and} \ \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big]$. Then, by applying substitution, conclude $\delta \overset{\text{sc}}{\models} [\![E_1]\!]_{\bigcup \mathcal{V}}$. Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}} \ \textbf{and} \ E'' = E_\dagger\big] \ \textbf{for some} \ E''$$

Then, by applying (U1), conclude $\big[\delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}} \ \textbf{and} \ E'' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\}\big]$.

(U3) Suppose:

$$\left[E \in \{E_2\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\right] \textbf{ for some } E$$

Then, by a reduction similar to (U2), conclude $\left[\delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}} \textbf{ and } E'' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\}\right]$.

(U4) Suppose:

$$\delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ for some } E$$

Then, by applying standard inference rules, conclude:

$$\delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } \left[\psi = [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ for some } \psi\right]$$

Then, by applying standard inference rules, conclude:

$$\left[\delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } \psi = [\![E]\!]_{\bigcup \mathcal{V}}\right] \textbf{ for some } \psi$$

Then, by applying substitution, conclude $\left[\delta \overset{\text{sc}}{\models} \psi \textbf{ and } \psi = [\![E]\!]_{\bigcup \mathcal{V}}\right]$.

(U5) Suppose **true**. Then, by applying ZFC, conclude:

$$\{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\}\} \in \Omega$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\} \textbf{ and } [\![E']\!]_{\bigcup \mathcal{V}} \in \mathbb{SC}\} \in \Omega$$

Then, by applying substitution, conclude:

$$\{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\} \textbf{ and } \psi' \in \mathbb{SC}\} \in \Omega$$

Then, by applying ZFC, conclude:

$$\{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\} \textbf{ and } \psi' \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying substitution, conclude:

$$\{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\} \textbf{ and } [\![E']\!]_{\bigcup \mathcal{V}} \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\}\} \in \wp(\mathbb{SC})$$

(U6) Recall $(E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)$ from (A1). Then, by introducing (Z1), conclude:

$$(E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2) \textbf{ and } E_1 , E_2 \in \wp(\mathbb{V}\text{ER}) \textbf{ and } V_1 , V_2 \in \mathbb{V}\text{ER}$$

Then, by applying Definition 20 of $\sqcup$, conclude:

$$(E_1 , V_1) \sqcup_{\mathcal{E}} (E_2 , V_2) = (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying (A3), conclude $(E_1 , V_1) \sqcup_{\mathcal{E}} (E_2 , V_2) = (\mathcal{E} \setminus \{E_1 , E_2\}) \cup \{E_\dagger\}$.

Now, prove the lemma by the following reduction. Recall $\delta \overset{\text{sc}}{\models} [\![(\mathcal{V} , \mathcal{E})]\!]$ from (A2). Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\delta \overset{\text{sc}}{\models} \sum(\{\psi'' \mid \psi'' = [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } E \in \mathcal{E}\})$. Then, by applying Lemma 5:3, conclude:

$$\left[\psi \in \{\psi'' \mid \psi'' = [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } E \in \mathcal{E}\} \textbf{ and } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for some } \psi$$

Then, by applying ZFC, conclude $\left[\left[\left[\psi = [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } E \in \mathcal{E}\right] \textbf{ for some } E\right] \textbf{ and } \delta \overset{\text{sc}}{\models} \psi\right]$. Then, by applying standard inference rules, conclude:

$$\left[\psi = [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } E \in \mathcal{E} \textbf{ and } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for some } E$$

Then, by applying substitution, conclude $\big[E \in \mathcal{E}$ **and** $\delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big]$. Then, by applying (Z2), conclude:

$$\begin{bmatrix} E \in \mathcal{E} \setminus \{E_1\,,\,E_2\} \\ \textbf{or}\;\; E \in \{E_1\} \\ \textbf{or}\;\; E \in \{E_2\} \end{bmatrix} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{l} \big[E \in \mathcal{E} \setminus \{E_1\,,\,E_2\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big] \\ \quad \textbf{or } \big[E \in \{E_1\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big] \\ \quad \textbf{or } \big[E \in \{E_2\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big] \end{array}$$

Then, by applying (Z3), conclude:

$$\begin{array}{rl} & \big[\big[E'' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}}\big] \textbf{ for some } E''\big] \\ \textbf{or} & \big[E \in \{E_1\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } \Big[\begin{bmatrix} [E' \in \mathcal{E} \textbf{ and } E \neq E'] \\ \textbf{implies } \delta \overset{\text{sc}}{\models} [\![E']\!]_{\bigcup \mathcal{V}} \end{bmatrix} \textbf{ for all } E'\Big]\big] \\ \textbf{or} & \big[E \in \{E_2\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \textbf{ and } \Big[\begin{bmatrix} [E' \in \mathcal{E} \textbf{ and } E \neq E'] \\ \textbf{implies } \delta \overset{\text{sc}}{\models} [\![E']\!]_{\bigcup \mathcal{V}} \end{bmatrix} \textbf{ for all } E'\Big]\big] \end{array}$$

Then, by applying (U2)(U3), conclude:

$$\begin{array}{rl} & \big[\big[E'' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\} \textbf{ and } \delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}}\big] \textbf{ for some } E''\big] \\ \textbf{or} & \big[\big[\delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}} \textbf{ and } E'' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\big] \textbf{ for some } E''\big] \\ \textbf{or} & \big[\big[\delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}} \textbf{ and } E'' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\big] \textbf{ for some } E''\big] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{\text{sc}}{\models} [\![E'']\!]_{\bigcup \mathcal{V}} \textbf{ and } E'' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\big] \textbf{ for some } E''$$

Then, by applying (U4), conclude:

$$\big[\big[\delta \overset{\text{sc}}{\models} \psi'' \textbf{ and } \psi'' = [\![E'']\!]_{\bigcup \mathcal{V}}\big] \textbf{ for some } \psi''\big] \textbf{ and } E'' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}$$

Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{\text{sc}}{\models} \psi'' \textbf{ and } \psi'' = [\![E'']\!]_{\bigcup \mathcal{V}} \textbf{ and } E'' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\big] \textbf{ for some } \psi''$$

Then, by applying ZFC, conclude:

$$\delta \overset{\text{sc}}{\models} \psi'' \textbf{ and } \psi'' \in \{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\}$$

Then, by introducing (U5), conclude:

$$\begin{array}{l} \delta \overset{\text{sc}}{\models} \psi'' \textbf{ and } \psi'' \in \{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\} \\ \quad \textbf{and } \{\psi' \mid \psi' = [\![E']\!]_{\bigcup \mathcal{V}} \textbf{ and } E' \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\} \in \wp(\mathbb{SC}) \end{array}$$

Then, by applying Lemma 5:1, conclude $\delta \overset{\text{sc}}{\models} \sum(\{\psi' \mid \psi' = [\![E]\!]_{\bigcup \mathcal{V}}$ **and** $E \in (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{E_\dagger\}\})$. Then, by applying (U6), conclude $\delta \overset{\text{sc}}{\models} \sum(\{\psi' \mid \psi' = [\![E]\!]_{\bigcup \mathcal{V}}$ **and** $E \in (E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)\})$. Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]$.

(| QED. |)

2. First, assume:

(B1) $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$

(B2) $\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]$

(B3) $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$

Next, observe:

(T1) Recall $(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2)$ from (B1). Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_1\,,\, E_2 \in \wp(\mathbb{V}\text{ER}) \;\textbf{ and }\; V_1\,,\, V_2 \in \mathbb{V}\text{ER} \;\textbf{ and }\; \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$$
$$\textbf{and }\; E_1\,,\, E_2 \in \mathcal{E} \;\textbf{ and }\; V_1 \cap V_2 = \emptyset \;\textbf{ and }\; (E_1\,,\, V_1) \curlyvee (E_2\,,\, V_2)$$

(T2) Recall $(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2)$ from (B1). Then, by introducing (T1), conclude:

$$(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \;\textbf{ and }\; E_1\,,\, E_2 \in \wp(\mathbb{V}\text{ER}) \;\textbf{ and }\; V_1\,,\, V_2 \in \mathbb{V}\text{ER}$$

Then, by applying Definition 20 of $\sqcup$, conclude:

$$(E_1\,,\, V_1) \sqcup_{\mathcal{E}} (E_2\,,\, V_2) = (\mathcal{E} \setminus \{E_1\,,\, E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying (B3), conclude $(E_1\,,\, V_1) \sqcup_{\mathcal{E}} (E_2\,,\, V_2) = (\mathcal{E} \setminus \{E_1\,,\, E_2\}) \cup \{E_\dagger\}$.

(T3) Suppose:

$$\big[E \in \mathcal{E} \setminus \{E_1\,,\, E_2\} \;\textbf{ and }\; \delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big] \;\textbf{ for some }\; E$$

Then, by applying ZFC, conclude $\big[E \in \mathcal{E}$ **and** $\delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\big]$. Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{\text{sc}}{\models} [\![E']\!]_{\bigcup \mathcal{V}} \;\textbf{ and }\; E' \in \mathcal{E}\big] \;\textbf{ for some }\; E'$$

(T4) Suppose $\delta \overset{\text{sc}}{\models} [\![E_1]\!]_{\bigcup \mathcal{V}}$. Then, by introducing (T1), conclude $\big[\delta \overset{\text{sc}}{\models} [\![E_1]\!]_{\bigcup \mathcal{V}}$ **and** $E_1 \in \mathcal{E}\big]$. Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \;\textbf{ and }\; E \in \mathcal{E}\big] \;\textbf{ for some }\; E$$

(T5) Suppose $\delta \overset{\text{sc}}{\models} [\![E_2]\!]_{\bigcup \mathcal{V}}$. Then, by a reduction similar to (T4), conclude:

$$\big[\delta \overset{\text{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \;\textbf{ and }\; E \in \mathcal{E}\big] \;\textbf{ for some }\; E$$

(T6) Suppose $\big[\big[V \in \{V_1 \cup V_2\}$ **implies** $\delta \overset{\text{sc}}{\models} \bigoplus(V)\big]$ **for all** $V\big]$ Then, by applying ZFC, conclude:

$$\big[V = V_1 \cup V_2 \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \;\textbf{ for all }\; V$$

Then, by applying substitution, conclude $\big[\delta \overset{\text{sc}}{\models} \bigoplus(V_1 \cup V_2)$ **for all** $V\big]$. Then, by applying standard inference rules, conclude $\delta \overset{\text{sc}}{\models} \bigoplus(V_1 \cup V_2)$. Then, by introducing (T1), conclude:

$$\delta \overset{\text{sc}}{\models} \bigoplus(V_1 \cup V_2) \;\textbf{ and }\; V_1 \cap V_2 = \emptyset$$

Then, by applying Lemma 3, conclude:

$$\delta \overset{\text{sc}}{\models} \bigoplus(V_1) \;\textbf{ and }\; \big[\big[p \in V_2 \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \overline{p}\big] \;\textbf{ for all }\; p\big]$$
$$\textbf{or }\; \delta \overset{\text{sc}}{\models} \bigoplus(V_2) \;\textbf{ and }\; \big[\big[p \in V_1 \;\textbf{ implies }\; \delta \overset{\text{sc}}{\models} \overline{p}\big] \;\textbf{ for all }\; p\big]$$

(T7) Suppose:

$$\delta \overset{\text{sc}}{\models} \bigoplus(V') \;\textbf{ for some }\; V'$$

Then, by applying substitution, conclude $\big[\big[V = V'$ **implies** $\delta \overset{\text{sc}}{\models} \bigoplus(V)\big]$ **for all** $V\big]$. Then, by applying ZFC, conclude $\big[\big[V \in \{V'\}$ **implies** $\delta \overset{\text{sc}}{\models} \bigoplus(V)\big]$ **for all** $V\big]$.

(T8) Recall $(E_1\,,\, V_1) \curlyvee (E_2\,,\, V_2)$ from (Z1). Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \;\textbf{ and }\; E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\}$$

(T9) Recall $(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$ from (B1). Then, by applying Lemma 7:2, conclude:

$$V_2 \notin E_1 \ \textbf{and}\ V_1 \notin E_2$$

(T0) Recall $E_1, E_2 \in \mathcal{E}$ from (Z1). Then, by applying ZFC, conclude $E_1 \cap E_2 \in \mathcal{E}$. Then, by applying standard inference rules, conclude $E_1 \cap E_2 = E_1 \cap E_2$. Then, by applying (T8), conclude:

$$E_1 \cap E_2 = E_1 \cup ((E_1 \setminus \{V_1\}) \cup \{V_2\})$$

Then, by introducing (T9), conclude $\big[E_1 \cap E_2 = E_1 \cup ((E_1 \setminus \{V_1\}) \cup \{V_2\})\ \textbf{and}\ V_2 \notin E_1\big]$. Then, by applying ZFC, conclude $E_1 \cap E_2 = E_1 \cup (E_1 \setminus \{V_1\})$. Then, by applying ZFC, conclude:

$$E_1 \cap E_2 = E_1 \setminus \{V_1\}$$

(S1) By a reduction similar to (T0), conclude $E_1 \cap E_2 = E_2 \setminus \{V_2\}$.

(S2) Recall $E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\}$ from (T8). Then, by applying (S1), conclude $E_1 = (E_1 \cap E_2) \cup \{V_1\}$.

(S3) By a reduction similar to (S2), conclude $E_2 = (E_1 \cap E_2) \cup \{V_2\}$.

(S4) Suppose $\big[\delta \overset{\text{sc}}{\models} \bigoplus(V_1)\ \textbf{and}\ \big[\big[V \in E_1 \cap E_2\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V\big]\big]$. Then, by applying (T7), conclude:

$$\begin{aligned} &\big[\big[V \in \{V_1\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V\big] \\ \textbf{and}\ &\big[\big[V \in E_1 \cap E_2\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V\big] \end{aligned}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{aligned} &\big[V \in \{V_1\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \\ \textbf{and}\ &\big[V \in E_1 \cap E_2\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big] \end{aligned}\right]\ \textbf{for all}\ V$$

Then, by applying standard inference rules, conclude:

$$\big[\big[V \in \{V_1\}\ \textbf{or}\ V \in E_1 \cap E_2\big]\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V$$

Then, by applying ZFC, conclude $\big[\big[V \in \{V_1\} \cup (E_1 \cap E_2)\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V\big]$. Then, by applying (S2), conclude $\big[\big[V \in E_1\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V\big]$.

(S5) Suppose $\big[\delta \overset{\text{sc}}{\models} \bigoplus(V_2)\ \textbf{and}\ \big[\big[V \in E_1 \cap E_2\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V\big]\big]$. Then, by a reduction similar to (S4), conclude $\big[\big[V \in E_2\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V\big]$.

(S6) Suppose $\big[\big[\psi \in \{\psi' \mid \psi' = \bigoplus(V)\ \textbf{and}\ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi\big]\ \textbf{for all}\ \psi\big]$. Then, by applying ZFC, conclude:

$$\big[\big[\big[\psi = \bigoplus(V)\ \textbf{and}\ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big]\ \textbf{for some}\ V\big]\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi\big]\ \textbf{for all}\ \psi$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\big[\psi = \bigoplus(V)\ \textbf{and}\ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big]\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi\big]\ \textbf{for some}\ V\big]\ \textbf{for all}\ \psi$$

Then, by applying substitution, conclude:

$$\big[\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for some}\ V\big]\ \textbf{for all}\ \psi$$

Then, by applying standard inference rules, conclude:

$$\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V$$

Then, by applying ZFC, conclude:

$$\big[\big[V \in \{V_1 \cup V_2\}\ \textbf{or}\ V \in E_1 \cap E_2\big]\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(V)\big]\ \textbf{for all}\ V$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{r}\big[V \in \{V_1 \cup V_2\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \\ \textbf{and} \ \big[V \in E_1 \cap E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big]\end{array}\right] \ \textbf{for all} \ V$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{r}\big[\big[V \in \{V_1 \cup V_2\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \ \textbf{for all} \ V\big] \\ \textbf{and} \ \big[\big[V \in E_1 \cap E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \ \textbf{for all} \ V\big]\end{array}$$

Then, by applying (T6), conclude:

$$\begin{array}{r}\left[\begin{array}{r}\delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V_1) \ \textbf{and} \ \big[\big[p \in V_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \ \textbf{for all} \ p\big] \\ \textbf{or} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V_2) \ \textbf{and} \ \big[\big[p \in V_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \ \textbf{for all} \ p\big]\end{array}\right] \\ \textbf{and} \ \big[\big[V \in E_1 \cap E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \ \textbf{for all} \ V\big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{r}\Big[\delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V_1) \ \textbf{and} \ \left[\begin{array}{c}\big[p \in V_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[\begin{array}{c}\big[V \in E_1 \cap E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \\ \textbf{for all} \ V\end{array}\right]\Big] \\ \textbf{or} \ \Big[\delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V_2) \ \textbf{and} \ \left[\begin{array}{c}\big[p \in V_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[\begin{array}{c}\big[V \in E_1 \cap E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \\ \textbf{for all} \ V\end{array}\right]\Big]\end{array}$$

Then, by applying (S4)(S5), conclude:

$$\begin{array}{r}\Big[\left[\begin{array}{c}\big[p \in V_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \big[\big[V \in E_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \ \textbf{for all} \ V\big]\Big] \\ \textbf{or} \ \Big[\left[\begin{array}{c}\big[p \in V_1 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \big[\big[V \in E_2 \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \textstyle\bigoplus(V)\big] \ \textbf{for all} \ V\big]\Big]\end{array}$$

(S7) Suppose:

$$\big[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \psi\big] \ \textbf{for all} \ \psi$$

Then, by applying ZFC, conclude:

$$\big[\big[\big[\psi = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\big] \ \textbf{for some} \ p\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \psi\big] \ \textbf{for all} \ \psi$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\big[\psi = \overline{p} \ \textbf{and} \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\big] \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \psi\big] \ \textbf{for some} \ p\big] \ \textbf{for all} \ \psi$$

Then, by applying substitution, conclude:

$$\big[\big[p \in (\textstyle\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \ \textbf{for some} \ p\big] \ \textbf{for all} \ \psi$$

Then, by applying standard inference rules, conclude:

$$\big[p \in (\textstyle\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \ \textbf{for all} \ p$$

Then, by applying ZFC, conclude $\big[\big[p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\text{sc}}{\vDash} \overline{p}\big] \ \textbf{for all} \ p\big]$.

(S8) Recall V_1, $V_2 \in \mathbb{V}\text{ER}$ from (T1). Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude V_1, $V_2 \in \wp(\mathbb{P}\text{ORT})$.

(S9) Suppose:

$$p \in (\textstyle\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{for some} \ p$$

Then, by introducing (S8), conclude $\big[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{and} \ V_2 \in \wp(\mathbb{P}\text{ORT})\big]$. Then, by applying ZFC, conclude $p \in ((\bigcup \mathcal{V}) \setminus \bigcup E_1) \cup V_2$. Then, by applying ZFC, conclude $p \in (((\bigcup \mathcal{V}) \setminus \bigcup E_1) \setminus V_2) \cup V_2$. Then, by applying ZFC, conclude $p \in ((\bigcup \mathcal{V}) \setminus (\bigcup E_1 \cup V_2)) \cup V_2$. Then, by applying (S2), conclude:

$$p \in ((\textstyle\bigcup \mathcal{V}) \setminus (\bigcup((E_1 \cap E_2) \cup \{V_1\}) \cup V_2)) \cup V_2$$

Then, by applying ZFC, conclude $p \in ((\bigcup \mathcal{V}) \setminus (\bigcup(E_1 \cap E_2) \cup V_1 \cup V_2)) \cup V_2$. Then, by applying ZFC, conclude $\big[p \in (\bigcup \mathcal{V}) \setminus (\bigcup(E_1 \cap E_2) \cup V_1 \cup V_2) \ \textbf{or} \ p \in V_2\big]$.

(S0) Suppose:

$$p \in (\bigcup \mathcal{V}) \setminus \bigcup E_2 \ \textbf{ for some } \ p$$

Then, by a reduction similar to (S9), conclude $\left[p \in (\bigcup \mathcal{V}) \setminus (\bigcup(E_1 \cap E_2) \cup V_1 \cup V_2) \ \textbf{ or } \ p \in V_1\right]$.

(R1) Suppose:

$$\left[\left[p \in V_2 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for all } \ p\right]$$
$$\textbf{and } \left[\left[p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2)) \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for all } \ p\right]$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{matrix}\left[p \in V_2 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \\ \left[p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2)) \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right]\end{matrix}\right] \ \textbf{ for all } \ p$$

Then, by applying standard inference rules, conclude:

$$\left[\left[p \in V_2 \ \textbf{ or } \ p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2))\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for all } \ p$$

Then, by applying (S9), conclude $\left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for all } \ p\right]$. Then, by applying standard inference rules, conclude:

$$\left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for some } \ p\right] \ \textbf{ for all } \ \psi$$

Then, by applying standard inference rules, conclude:

$$\left[\left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{ and } \ \psi = \overline{p}\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for some } \ p\right] \ \textbf{ for all } \ \psi$$

Then, by applying substitution, conclude:

$$\left[\left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{ and } \ \psi = \overline{p}\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for some } \ p\right] \ \textbf{ for all } \ \psi$$

Then, by applying standard inference rules, conclude:

$$\left[\left[\left[p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1 \ \textbf{ and } \ \psi = \overline{p}\right] \ \textbf{ for some } \ p\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for all } \ \psi$$

Then, by applying ZFC, conclude:

$$\left[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_1\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for all } \ \psi$$

(R2) Suppose:

$$\left[\left[p \in V_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for all } \ p\right]$$
$$\textbf{and } \left[\left[p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2)) \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}\right] \ \textbf{ for all } \ p\right]$$

Then, by a reduction similar to (R1), conclude:

$$\left[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\bigcup \mathcal{V}) \setminus \bigcup E_2\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for all } \ \psi$$

(R3) Suppose $\left[\left[V \in E_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{ for all } \ V\right]$. Then, by applying standard inference rules, conclude $\left[\left[\left[V \in E_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{ for some } \ V\right] \ \textbf{ for all } \ \psi\right]$. Then, by applying standard inference rules, conclude:

$$\left[\left[\left[V \in E_1 \ \textbf{ and } \ \psi = \bigoplus(V)\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \bigoplus(V)\right] \ \textbf{ for some } \ V\right] \ \textbf{ for all } \ \psi$$

Then, by applying substitution, conclude:

$$\left[\left[\left[V \in E_1 \ \textbf{ and } \ \psi = \bigoplus(V)\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for some } \ V\right] \ \textbf{ for all } \ \psi$$

Then, by applying standard inference rules, conclude:

$$\left[\left[\left[V \in E_1 \ \textbf{ and } \ \psi = \bigoplus(V)\right] \ \textbf{ for some } \ V\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for all } \ \psi$$

Then, by applying ZFC, conclude:

$$\left[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \ \textbf{ and } \ V \in E_1\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for all } \ \psi$$

(R4) Suppose $\big[[V \in E_2$ **implies** $\delta \overset{\text{sc}}{\vDash} \bigoplus(V)]$ **for all** $V\big]$. Then, by a reduction similar to (R3), conclude:

$$\big[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \text{ and } V \in E_2\} \text{ implies } \delta \overset{\text{sc}}{\vDash} \psi\big] \text{ for all } \psi$$

(R5) Suppose:

$$V \in E_1 \text{ for some } V$$

Then, by introducing (T1), conclude $V \in E_1 \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $V \in \mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V \in \wp(\mathbb{P}\text{ORT})$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $V \in \wp(\mathbb{SC})$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\bigoplus(V) \in \mathbb{SC}$.

(R6) Suppose:

$$V \in E_2 \text{ for some } V$$

Then, by a reduction similar to (R5), conclude $\bigoplus(V) \in \mathbb{SC}$

(R7) Suppose:

$$\big[\psi = \bigoplus(V) \text{ and } V \in E_1\big] \text{ for some } \psi\,,\, V$$

Then, by applying (R5), conclude $\big[\psi = \bigoplus(V)$ **and** $\bigoplus(V) \in \mathbb{SC}\big]$. Then, by applying substitution, conclude $\psi \in \mathbb{SC}$.

(R8) Suppose:

$$\big[\psi = \bigoplus(V) \text{ and } V \in E_1\big] \text{ for some } \psi\,,\, V$$

Then, by a reduction similar to (R7), conclude $\psi \in \mathbb{SC}$

(R9) Suppose **true**. Then, by applying ZFC, conclude $\{\psi \mid \psi = \bigoplus(V)$ **and** $V \in E_1\} \in \varOmega$. Then, by applying (R7), conclude $\{\psi \mid \psi = \bigoplus(V)$ **and** $V \in E_1$ **and** $\psi \in \mathbb{SC}\} \in \varOmega$. Then, by applying ZFC, conclude $\{\psi \mid \psi = \bigoplus(V)$ **and** $V \in E_1$ **and** $\psi \in \mathbb{SC}\} \in \wp(\mathbb{SC})$. Then, by applying (R7), conclude:

$$\{\psi \mid \psi = \bigoplus(V) \text{ and } V \in E_1\} \in \wp(\mathbb{SC})$$

(R0) Suppose **true**. Then, by a reduction similar to (R9), conclude:

$$\{\psi \mid \psi = \bigoplus(V) \text{ and } V \in E_2\} \in \wp(\mathbb{SC})$$

(Q1) Recall $\delta \overset{\text{sc}}{\vDash} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]$ from (B2). Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)) \in \mathbb{G}\text{RAPH}$. Then, by applying Definition 16 of $\mathbb{G}\text{RAPH}$, conclude:

$$\mathcal{V} \in \wp(\mathbb{V}\text{ER})$$

Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $\mathcal{V} \in \wp^2(\mathbb{P}\text{ORT})$.

(Q2) Suppose:

$$p \in (\textstyle\bigcup \mathcal{V}) \setminus \bigcup E \text{ for some } p\,,\, E$$

Then, by applying ZFC, conclude $p \in \bigcup \mathcal{V}$. Then, by applying ZFC, conclude:

$$p \in V \in \mathcal{V} \text{ for some } V$$

Then, by introducing (Q1), conclude $p \in V \in \mathcal{V} \in \wp^2(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude:

$$p \in V \in \wp(\mathbb{P}\text{ORT})$$

Then, by applying ZFC, conclude $p \in \mathbb{P}\text{ORT}$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $p \in \mathbb{SC}$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\overline{p} \in \mathbb{SC}$.

(Q3) Suppose:

$$\left[\psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E\right] \textbf{ for some } \psi, p, E$$

Then, by applying (Q2), conclude $\left[\psi = \overline{p} \textbf{ and } \overline{p} \in \mathbb{SC}\right]$. Then, by applying substitution, conclude $\psi \in \mathbb{SC}$.

(Q4) Suppose **true**. Then, by applying ZFC, conclude $\{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \in \Omega$. Then, by applying (V5), conclude $\{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1 \textbf{ and } \psi \in \mathbb{SC}\} \in \Omega$. Then, by applying ZFC, conclude $\{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1 \textbf{ and } \psi \in \mathbb{SC}\} \in \wp(\mathbb{SC})$. Then, by applying (V5), conclude $\{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \in \wp(\mathbb{SC})$.

(Q5) Suppose **true**. Then, by a reduction similar to (Q4), conclude:

$$\{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_2\} \in \wp(\mathbb{SC})$$

(Q6) Recall $\{\psi \mid \psi = \bigoplus(V) \textbf{ and } V \in E_1\}, \{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \in \wp(\mathbb{SC})$ from (R9)(Q4). Then, by applying ZFC, conclude:

$$\{\psi \mid \psi = \bigoplus(V) \textbf{ and } V \in E_1\} \cup \{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \in \wp(\mathbb{SC})$$

(Q7) By a reduction similar to (Q6), conclude:

$$\{\psi \mid \psi = \bigoplus(V) \textbf{ and } V \in E_2\} \cup \{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_2\} \in \wp(\mathbb{SC})$$

(Q8) Suppose:

$$\begin{array}{r} \left[\left[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \textbf{ and } V \in E_1\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for all } \psi\right] \\ \textbf{and } \left[\left[\psi \in \{\psi' \mid \psi' = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for all } \psi\right] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{r} \left[\psi \in \{\psi' \mid \psi' = \bigoplus(V) \textbf{ and } V \in E_1\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi\right] \\ \textbf{and } \left[\psi \in \{\psi' \mid \psi' = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi\right] \end{array}\right] \textbf{ for all } \psi$$

Then, by applying standard inference rules, conclude:

$$\left[\left[\begin{array}{r} \psi \in \{\psi' \mid \psi' = \bigoplus(V) \textbf{ and } V \in E_1\} \\ \textbf{or } \psi \in \{\psi' \mid \psi' = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \end{array}\right] \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for all } \psi$$

Then, by applying ZFC, conclude:

$$\left[\left[\begin{array}{l} \psi \in \{\psi' \mid \psi' = \bigoplus(V) \textbf{ and } V \in E_1\} \cup \\ \qquad \{\psi' \mid \psi' = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \end{array}\right] \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for all } \psi$$

Then, by introducing (Q6), conclude:

$$\begin{array}{c} \left[\left[\left[\begin{array}{l} \psi \in \{\psi' \mid \psi' = \bigoplus(V) \textbf{ and } V \in E_1\} \cup \\ \qquad \{\psi' \mid \psi' = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \end{array}\right] \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi\right] \textbf{ for all } \psi\right] \textbf{ and} \\ \{\psi \mid \psi = \bigoplus(V) \textbf{ and } V \in E_1\} \cup \{\psi \mid \psi = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\} \in \wp(\mathbb{SC}) \end{array}$$

Then, by applying Lemma 5:2, conclude:

$$\delta \overset{\text{sc}}{\models} \prod(\{\psi' \mid \psi' = \bigoplus(V) \textbf{ and } V \in E_1\} \cup \{\psi' \mid \psi' = \overline{p} \textbf{ and } p \in (\bigcup\mathcal{V}) \setminus \bigcup E_1\})$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\delta \overset{\text{sc}}{\models} [\![E_1]\!]_{\bigcup\mathcal{V}}$.

(Q9) Suppose:

$$\begin{aligned}&\big[[\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{ and } \ V \in E_2\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{ for all } \ \psi\big]\\ \textbf{and } \ &\big[[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup E_2\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{ for all } \ \psi\big]\end{aligned}$$

Then, by a reduction similar to (Q8), conclude $\delta \overset{\text{sc}}{\models} [\![E_2]\!]_{\bigcup \mathcal{V}}$.

(Q0) Suppose:

$$\left[\begin{matrix}[p \in V_2 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}]\\ \textbf{for all } \ p\end{matrix}\right] \ \textbf{ and } \ \big[[V \in E_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \textstyle\bigoplus(V)] \ \textbf{ for all } \ V\big]$$
$$\textbf{and } \ \big[[p \in (\textstyle\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \textstyle\bigcup(E_1 \cap E_2)) \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}] \ \textbf{ for all } \ p\big]$$

Then, by applying (R1), conclude:

$$\begin{aligned}&\big[[V \in E_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \textstyle\bigoplus(V)] \ \textbf{ for all } \ V\big] \ \textbf{ and}\\ &\big[[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup E_1\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{ for all } \ \psi\big]\end{aligned}$$

Then, by applying (R3), conclude:

$$\begin{aligned}&\big[[\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{ and } \ V \in E_1\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{ for all } \ \psi\big]\\ \textbf{and } \ &\big[[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup E_1\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{ for all } \ \psi\big]\end{aligned}$$

Then, by applying (Q8), conclude $\delta \overset{\text{sc}}{\models} [\![E_1]\!]_{\bigcup \mathcal{V}}$.

(P1) Suppose:

$$\left[\begin{matrix}[p \in V_1 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}]\\ \textbf{for all } \ p\end{matrix}\right] \ \textbf{ and } \ \big[[V \in E_2 \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \textstyle\bigoplus(V)] \ \textbf{ for all } \ V\big]$$
$$\textbf{and } \ \big[[p \in (\textstyle\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \textstyle\bigcup(E_1 \cap E_2)) \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \overline{p}] \ \textbf{ for all } \ p\big]$$

Then, by a reduction similar to (Q0), conclude $\delta \overset{\text{sc}}{\models} [\![E_2]\!]_{\bigcup \mathcal{V}}$.

(P2) Suppose $\delta \overset{\text{sc}}{\models} [\![E_\dagger]\!]_{\bigcup \mathcal{V}}$. Then, by applying (B3), conclude $\delta \overset{\text{sc}}{\models} [\![\{V_1 \cup V_2\} \cup (E_1 \cap E_2)]\!]_{\bigcup \mathcal{V}}$. Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\begin{aligned}\delta \overset{\text{sc}}{\models} \textstyle\prod(\{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{ and } \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \cup\\ \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\})\end{aligned}$$

Then, by applying Lemma 5:4, conclude:

$$\left[\left[\begin{matrix}\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{ and } \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \cup\\ \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\}\end{matrix}\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for all } \ \psi$$

Then, by applying ZFC, conclude:

$$\left[\left[\begin{matrix}\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{ and } \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\\ \textbf{or } \ \psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\}\end{matrix}\right] \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi\right] \ \textbf{ for all } \ \psi$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{matrix}&[\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{ and } \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi]\\ \textbf{and } &[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi]\end{matrix}\right] \ \textbf{ for all } \ \psi$$

Then, by applying standard inference rules, conclude:

$$\begin{aligned}&\big[[\psi \in \{\psi' \mid \psi' = \textstyle\bigoplus(V) \ \textbf{ and } \ V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{ for all } \ \psi\big]\\ \textbf{and } \ &\big[[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{ and } \ p \in (\textstyle\bigcup \mathcal{V}) \setminus \textstyle\bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \ \textbf{ implies } \ \delta \overset{\text{sc}}{\models} \psi] \ \textbf{ for all } \ \psi\big]\end{aligned}$$

Then, by applying (S6), conclude:

$$\left[\begin{array}{l} \quad \left[\left[\begin{array}{c}[p \in V_2 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[[V \in E_1 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \bigoplus(V)] \ \textbf{for all} \ V\right]\right] \\ \textbf{or} \ \left[\left[\begin{array}{c}[p \in V_1 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[[V \in E_2 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \bigoplus(V)] \ \textbf{for all} \ V\right]\right] \end{array}\right] \ \textbf{and}$$
$$\left[\left[\psi \in \{\psi' \mid \psi' = \overline{p} \ \textbf{and} \ p \in (\bigcup \mathcal{V}) \setminus \bigcup(\{V_1 \cup V_2\} \cup (E_1 \cap E_2))\} \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \psi\right] \ \textbf{for all} \ \psi\right]$$

Then, by applying (S7), conclude:

$$\left[\begin{array}{l} \quad \left[\left[\begin{array}{c}[p \in V_2 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[[V \in E_1 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \bigoplus(V)] \ \textbf{for all} \ V\right]\right] \\ \textbf{or} \ \left[\left[\begin{array}{c}[p \in V_1 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[[V \in E_2 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \bigoplus(V)] \ \textbf{for all} \ V\right]\right] \end{array}\right]$$
$$\textbf{and} \ \left[[p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \ \textbf{for all} \ p\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{l} \left[\begin{array}{l} \left[\begin{array}{c}[p \in V_2 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[[V \in E_1 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \bigoplus(V)] \ \textbf{for all} \ V\right] \\ \quad \textbf{and} \ \left[[p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \ \textbf{for all} \ p\right] \end{array}\right] \\ \textbf{or} \ \left[\begin{array}{l} \left[\begin{array}{c}[p \in V_1 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \\ \textbf{for all} \ p\end{array}\right] \ \textbf{and} \ \left[[V \in E_2 \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \bigoplus(V)] \ \textbf{for all} \ V\right] \\ \quad \textbf{and} \ \left[[p \in (\bigcup \mathcal{V}) \setminus (V_1 \cup V_2 \cup \bigcup(E_1 \cap E_2)) \ \textbf{implies} \ \delta \overset{\mathrm{sc}}{\models} \overline{p}] \ \textbf{for all} \ p\right] \end{array}\right] \end{array}$$

Then, by applying (Q0)(P1), conclude $\left[\delta \overset{\mathrm{sc}}{\models} [\![E_1]\!]_{\bigcup \mathcal{V}} \ \textbf{or} \ \delta \overset{\mathrm{sc}}{\models} [\![E_2]\!]_{\bigcup \mathcal{V}}\right]$.

(P3) Suppose:

$$\left[E \in \{E_\dagger\} \ \textbf{and} \ \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\right] \ \textbf{for some} \ E$$

Then, by applying ZFC, conclude $\left[E = E_\dagger \ \textbf{and} \ \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}}\right]$. Then, by applying substitution, conclude $\delta \overset{\mathrm{sc}}{\models} [\![E_\dagger]\!]_{\bigcup \mathcal{V}}$. Then, by applying (P2), conclude $\left[\delta \overset{\mathrm{sc}}{\models} [\![E_1]\!]_{\bigcup \mathcal{V}} \ \textbf{or} \ \delta \overset{\mathrm{sc}}{\models} [\![E_2]\!]_{\bigcup \mathcal{V}}\right]$. Then, by applying (T4)(T5), conclude:

$$\left[\left[\delta \overset{\mathrm{sc}}{\models} [\![E']\!]_{\bigcup \mathcal{V}} \ \textbf{and} \ E' \in \mathcal{E}\right] \ \textbf{for some} \ E'\right] \ \textbf{or} \ \left[\left[\delta \overset{\mathrm{sc}}{\models} [\![E']\!]_{\bigcup \mathcal{V}} \ \textbf{and} \ E' \in \mathcal{E}\right] \ \textbf{for some} \ E'\right]$$

Then, by applying standard inference rules, conclude:

$$\left[\delta \overset{\mathrm{sc}}{\models} [\![E']\!]_{\bigcup \mathcal{V}} \ \textbf{and} \ E' \in \mathcal{E}\right] \ \textbf{for some} \ E'$$

(P4) Suppose:

$$\delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \ \textbf{for some} \ E$$

Then, by applying standard inference rules, conclude:

$$\delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \ \textbf{and} \ \left[\psi = [\![E]\!]_{\bigcup \mathcal{V}} \ \textbf{for some} \ \psi\right]$$

Then, by applying standard inference rules, conclude:

$$\left[\delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\bigcup \mathcal{V}} \ \textbf{and} \ \psi = [\![E]\!]_{\bigcup \mathcal{V}}\right] \ \textbf{for some} \ \psi$$

Then, by applying substitution, conclude $\left[\delta \overset{\mathrm{sc}}{\models} \psi \ \textbf{and} \ \psi = [\![E]\!]_{\bigcup \mathcal{V}}\right]$.

(P5) Suppose **true**. Then, by applying ZFC, conclude $\{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}}$ **and** $E' \in \mathcal{E}\} \in \Omega$. Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}}$ **and** $E' \in \mathcal{E}$ **and** $[\![E']\!]_{\bigcup\mathcal{V}} \in \mathbb{SC}\} \in \Omega$. Then, by applying substitution, conclude $\{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}}$ **and** $E' \in \mathcal{E}$ **and** $\psi' \in \mathbb{SC}\} \in \Omega$. Then, by applying ZFC, conclude $\{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}}$ **and** $E' \in \mathcal{E}$ **and** $\psi' \in \mathbb{SC}\} \in \wp(\mathbb{SC})$. Then, by applying substitution, conclude $\{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}}$ **and** $E' \in \mathcal{E}$ **and** $[\![E']\!]_{\bigcup\mathcal{V}} \in \mathbb{SC}\} \in \wp(\mathbb{SC})$. Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}}$ **and** $E' \in \mathcal{E}\} \in \wp(\mathbb{SC})$.

Now, prove the lemma by the following reduction. Recall $\delta \overset{sc}{\models} [\![(\mathcal{V}, (E_1, V_1) \sqcup_{\mathcal{E}} (E_2, V_2))]\!]$ from (B2). Then, by applying (T2), conclude $\delta \overset{sc}{\models} [\![(\mathcal{V}, (\mathcal{E} \setminus \{E_1, E_2\}) \cup \{E_\dagger\})]\!]$. Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\delta \overset{sc}{\models} \sum(\{\psi'' \mid \psi'' = [\![E]\!]_{\bigcup\mathcal{V}}$ **and** $E \in (\mathcal{E} \setminus \{E_1, E_2\}) \cup \{E_\dagger\}\})$. Then, by applying Lemma 5:3, conclude:

$$\big[\psi \in \{\psi'' \mid \psi'' = [\![E]\!]_{\bigcup\mathcal{V}} \textbf{ and } E \in (\mathcal{E} \setminus \{E_1, E_2\}) \cup \{E_\dagger\}\} \textbf{ and } \delta \overset{sc}{\models} \psi\big] \textbf{ for some } \psi$$

Then, by applying ZFC, conclude:

$$\big[\big[\psi = [\![E]\!]_{\bigcup\mathcal{V}} \textbf{ and } E \in (\mathcal{E} \setminus \{E_1, E_2\}) \cup \{E_\dagger\}\big] \textbf{ for some } E\big] \textbf{ and } \delta \overset{sc}{\models} \psi$$

Then, by applying standard inference rules, conclude:

$$\big[\psi = [\![E]\!]_{\bigcup\mathcal{V}} \textbf{ and } E \in (\mathcal{E} \setminus \{E_1, E_2\}) \cup \{E_\dagger\} \textbf{ and } \delta \overset{sc}{\models} \psi\big] \textbf{ for some } E$$

Then, by applying substitution, conclude $\big[E \in (\mathcal{E} \setminus \{E_1, E_2\}) \cup \{E_\dagger\}$ **and** $\delta \overset{sc}{\models} [\![E]\!]_{\bigcup\mathcal{V}}\big]$. Then, by applying ZFC, conclude $\big[\big[E \in \mathcal{E} \setminus \{E_1, E_2\}$ **or** $E \in \{E_\dagger\}\big]$ **and** $\delta \overset{sc}{\models} [\![E]\!]_{\bigcup\mathcal{V}}\big]$. Then, by applying standard inference rules, conclude:

$$\big[E \in \mathcal{E} \setminus \{E_1, E_2\} \textbf{ and } \delta \overset{sc}{\models} [\![E]\!]_{\bigcup\mathcal{V}}\big] \textbf{ or } \big[E \in \{E_\dagger\} \textbf{ and } \delta \overset{sc}{\models} [\![E]\!]_{\bigcup\mathcal{V}}\big]$$

Then, by applying (T3), conclude:

$$\big[\big[\delta \overset{sc}{\models} [\![E'']\!]_{\bigcup\mathcal{V}} \textbf{ and } E'' \in \mathcal{E}\big] \textbf{ for some } E''\big] \textbf{ or } \big[E \in \{E_\dagger\} \textbf{ and } \delta \overset{sc}{\models} [\![E]\!]_{\bigcup\mathcal{V}}\big]$$

Then, by applying (P3), conclude:

$$\big[\big[\delta \overset{sc}{\models} [\![E'']\!]_{\bigcup\mathcal{V}} \textbf{ and } E'' \in \mathcal{E}\big] \textbf{ for some } E''\big] \textbf{ or } \big[\big[\delta \overset{sc}{\models} [\![E'']\!]_{\bigcup\mathcal{V}} \textbf{ and } E'' \in \mathcal{E}\big] \textbf{ for some } E''\big]$$

Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{sc}{\models} [\![E'']\!]_{\bigcup\mathcal{V}} \textbf{ and } E'' \in \mathcal{E}\big] \textbf{ for some } E''$$

Then, by applying (P4), conclude $\big[\big[\big[\delta \overset{sc}{\models} \psi''$ **and** $\psi'' = [\![E'']\!]_{\bigcup\mathcal{V}}\big]$ **for some** $\psi''\big]$ **and** $E'' \in \mathcal{E}\big]$. Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{sc}{\models} \psi'' \textbf{ and } \psi'' = [\![E'']\!]_{\bigcup\mathcal{V}} \textbf{ and } E'' \in \mathcal{E}\big] \textbf{ for some } \psi''$$

Then, by applying ZFC, conclude $\big[\delta \overset{sc}{\models} \psi''$ **and** $\psi'' \in \{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}}$ **and** $E' \in \mathcal{E}\}\big]$. Then, by introducing (P5), conclude:

$$\delta \overset{sc}{\models} \psi'' \textbf{ and } \psi'' \in \{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}} \textbf{ and } E' \in \mathcal{E}\} \textbf{ and } \{\psi' \mid \psi' = [\![E']\!]_{\bigcup\mathcal{V}} \textbf{ and } E' \in \mathcal{E}\} \in \wp(\mathbb{SC})$$

Then, by applying Lemma 5:1, conclude $\delta \overset{sc}{\models} \sum(\{\psi' \mid \psi' = [\![E]\!]_{\bigcup\mathcal{V}}$ **and** $E \in \mathcal{E}\})$. Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\delta \overset{sc}{\models} [\![(\mathcal{V}, \mathcal{E})]\!]$.

(| QED. |)

3. First, assume:

(C1) $(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$

Next, observe:

(O1) Suppose:

$$\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!] \;\; \textbf{for some} \;\; \delta$$

Then, by introducing (C1), conclude $\big[\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!] \;\textbf{and}\; (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big]$. Then, by applying Lemma 8:1, conclude $\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]$.

(O2) Suppose:

$$[\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!] \;\; \textbf{for some} \;\; \delta$$

Then, by introducing (C1), conclude $\big[[\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!] \;\textbf{and}\; (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big]$. Then, by applying Lemma 8:2, conclude $\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!]$.

Now, prove the lemma by the following reduction. Recall from (O1):

$$\big[\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]\big] \;\textbf{for all}\; \delta$$

Then, by introducing (O2), conclude:

$$\begin{array}{r l} & \Big[\big[\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]\big] \;\textbf{for all}\; \delta\Big] \\ \textbf{and} & \Big[\big[\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!]\big] \;\textbf{for all}\; \delta\Big] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{r l} & \big[\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]\big] \\ \textbf{and} & \big[\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!]\big] \end{array}\right] \;\textbf{for all}\; \delta$$

Then, by applying standard inference rules, conclude:

$$\big[\delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,\mathcal{E})]\!] \;\textbf{iff}\; \delta \overset{\text{sc}}{\models} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]\big] \;\textbf{for all}\; \delta$$

Then, by applying Definition 10 of $\equiv_{\text{sc}}$, conclude $[\![(\mathcal{V}\,,\,\mathcal{E})]\!] \equiv_{\text{sc}} [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]$.

(| QED. |)

4. First, assume:

(D1) $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$

Next, observe:

(N1) Recall $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$. Then, by applying Definition 19 of $\curlyvee$, conclude $E_1 \neq E_2$. Then, by applying ZFC, conclude $|\{E_1\,,\,E_2\}| = 2$.

(N2) $\{E_1\,,\,E_2\} \subseteq \mathcal{E}$. Then, by applying ZFC, conclude $|\mathcal{E} \setminus \{E_1\,,\,E_2\}| = |\mathcal{E}| - |\{E_1\,,\,E_2\}|$. Then, by applying (N1), conclude $|\mathcal{E} \setminus \{E_1\,,\,E_2\}| = |\mathcal{E}| - 2$.

Now, prove the lemma by the following reduction. Recall $(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$ from (D1). Then, by applying Definition 19 of $\curlyvee$, conclude:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\textbf{and}\; E_1\,,\,E_2 \in \wp(\mathbb{V}\text{ER}) \;\textbf{and}\; V_1\,,\,V_2 \in \mathbb{V}\text{ER} \;\textbf{and}\; \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying Definition 20 of $\sqcup$, conclude:

$$(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = (\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying ZFC, conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| = |(\mathcal{E} \setminus \{E_1\,,\,E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}|$. Then, by applying ZFC, conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| \leq |(\mathcal{E} \setminus \{E_1\,,\,E_2\})| + |\{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}|$. Then, by applying (N2), conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| \leq (|\mathcal{E}| - 2) + |\{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}|$. Then, by applying ZFC, conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| \leq (|\mathcal{E}| - 2) + 1$. Then, by applying PA, conclude:

$$|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| \leq |\mathcal{E}| - 1$$

Then, by applying PA, conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{E}|$.

(| QED. |)

B.7 Theorem 3

Proof (of Theorem 3). First, assume:

(A1) $\psi = \sum(\left\{\psi' \;\middle|\; \begin{array}{l}\psi' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+)\ \textbf{and}\\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P}\end{array}\right\})$ **for some** $\mathcal{P}$

Next, observe:

(Z1) Suppose:

$$\big[[p'' \in \{p \mid p \in P_+\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} p''\big]\ \textbf{for all}\ p''\big]\ \textbf{for some}\ P_+\,,\,\delta$$

Then, by applying ZFC, conclude $\big[[p'' \in P_+\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} p''\big]\ \textbf{for all}\ p''\big]$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude $\big[[p'' \in P_+\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(\{p''\})\big]\ \textbf{for all}\ p''\big]$. Then, by applying standard inference rules, conclude $\big[\big[[p'' \in P_+\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(\{p''\})\big]\ \textbf{for some}\ p''\big]\ \textbf{for all}\ \psi''\big]$. Then, by applying standard inference rules, conclude:

$$\big[\big[[p'' \in P_+\ \textbf{and}\ \psi'' = \bigoplus(\{p''\})\big]\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \bigoplus(\{p''\})\big]\ \textbf{for some}\ p''\big]\ \textbf{for all}\ \psi''$$

Then, by applying substitution, conclude:

$$\big[\big[[p'' \in P_+\ \textbf{and}\ \psi'' = \bigoplus(\{p''\})\big]\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi''\big]\ \textbf{for some}\ p''\big]\ \textbf{for all}\ \psi''$$

Then, by applying standard inference rules, conclude:

$$\big[\big[[p'' \in P_+\ \textbf{and}\ \psi'' = \bigoplus(\{p''\})\big]\ \textbf{for some}\ p''\big]\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi''\big]\ \textbf{for all}\ \psi''$$

Then, by applying ZFC, conclude:

$$\big[\psi'' \in \{\psi' \mid \psi' = \bigoplus(\{p''\})\ \textbf{and}\ p'' \in P_+\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi''\big]\ \textbf{for all}\ \psi''$$

Then, by applying ZFC, conclude:

$$\big[\psi'' \in \{\psi' \mid \psi' = \bigoplus(\{p''\})\ \textbf{and}\ \{p''\} \in \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\}\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi''\big]\ \textbf{for all}\ \psi''$$

Then, by applying standard inference rules, conclude:

$$\big[\psi'' \in \{\psi' \mid \psi' = \bigoplus(V')\ \textbf{and}\ V' \in \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\}\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi''\big]\ \textbf{for all}\ \psi''$$

(Z2) Suppose:

$$\big[\psi'' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+)\ \textbf{and}\ \delta \overset{\text{sc}}{\models} \psi''\big]\ \textbf{for some}\ \psi''\,,\,P_+\,,\,\delta$$

Then, by applying Definition 10 of $\equiv_{\text{sc}}$, conclude $\delta \overset{\text{sc}}{\models} \mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+)$. Then, by applying Definition 12 of cp, conclude $\delta \overset{\text{sc}}{\models} \prod(\{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'}\ \textbf{and}\ p' \in \mathsf{Port}(\psi) \setminus P_+\})$. Then, by applying Lemma 5:4, conclude:

$$\big[\psi''' \in \{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'}\ \textbf{and}\ p' \in \mathsf{Port}(\psi) \setminus P_+\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi'''\big]\ \textbf{for all}\ \psi'''$$

Then, by applying ZFC, conclude:

$$\left[\begin{array}{c}[\psi''' \in \{p''' \mid p''' \in P_+\}\ \textbf{or}\ \psi''' \in \{p''' \mid p''' = \overline{p'}\ \textbf{and}\ p' \in \mathsf{Port}(\psi) \setminus P_+\}]\\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi'''\end{array}\right]\ \textbf{for all}\ \psi'''$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{r}\big[\big[\psi''' \in \{p''' \mid p''' \in P_+\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi'''\big]\ \textbf{for all}\ \psi'''\big]\\ \textbf{and}\ \big[\big[\psi''' \in \{p''' \mid p''' = \overline{p'}\ \textbf{and}\ p' \in \mathsf{Port}(\psi) \setminus P_+\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi'''\big]\ \textbf{for all}\ \psi'''\big]\end{array}$$

Then, by applying (Z1), conclude:

$$\begin{array}{r}\big[\big[\psi''' \in \{\psi' \mid \psi' = \bigoplus(V')\ \textbf{and}\ V' \in \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\}\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi'''\big]\ \textbf{for all}\ \psi'''\big]\\ \textbf{and}\ \big[\big[\psi''' \in \{p''' \mid p''' = \overline{p'}\ \textbf{and}\ p' \in \mathsf{Port}(\psi) \setminus P_+\}\ \textbf{implies}\ \delta \overset{\text{sc}}{\models} \psi'''\big]\ \textbf{for all}\ \psi'''\big]\end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{l} \left[\begin{array}{c}\left[\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \ \textbf{and} \ V' \in \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\}\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi'''\right] \\ \textbf{for all} \ \psi''' \end{array}\right] \\ \textbf{and} \ \left[\begin{array}{c}\left[\psi''' \in \{p''' \mid p''' = \overline{p'} \ \textbf{and} \ p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\}\} \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi'''\right] \\ \textbf{for all} \ \psi''' \end{array}\right] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c}\left[\left[\begin{array}{r}\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \ \textbf{and} \ V' \in \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\}\} \\ \textbf{or} \ \psi''' \in \{p''' \mid p''' = \overline{p'} \ \textbf{and} \ p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\}\}\end{array}\right] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi'''\right] \\ \textbf{for all} \ \psi''' \end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c}\left[\left[\begin{array}{r}\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \ \textbf{and} \ V' \in \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\}\} \cup \\ \{p''' \mid p''' = \overline{p'} \ \textbf{and} \ p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\}\}\end{array}\right] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi'''\right] \\ \textbf{for all} \ \psi''' \end{array}$$

(Z3) Suppose:

$$P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{for some} \ P_+$$

Then, by applying Definition 13 of Port, conclude $\left[P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ \mathsf{Port}(\psi) \in \wp(\mathbb{P}\text{ORT})\right]$. Then, by applying ZFC, conclude $\left[P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ \mathsf{Port}(\psi) \subseteq \mathbb{P}\text{ORT}\right]$. Then, by applying ZFC, conclude $P_+ \subseteq \mathbb{P}\text{ORT}$. Then, by applying ZFC, conclude $P_+ \in \wp(\mathbb{P}\text{ORT})$.

(Z4) Suppose:

$$\left[p \in P_+ \ \textbf{and} \ P_+ \subseteq \mathsf{Port}(\psi)\right] \ \textbf{for some} \ p\,,\, P_+$$

Then, by applying (Z3), conclude $p \in P_+ \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $p \in \mathbb{P}\text{ORT}$.

(Z5) Suppose:

$$\left[p \in P_+ \ \textbf{and} \ P_+ \subseteq \mathsf{Port}(\psi)\right] \ \textbf{for some} \ p\,,\, P_+$$

Then, by applying (Z4), conclude $p \in \mathbb{P}\text{ORT}$. Then, by applying ZFC, conclude $\{p\} \in \wp(\mathbb{P}\text{ORT})$.

(Z6) Suppose:

$$\left[V = \{p\} \ \textbf{and} \ p \in P_+ \ \textbf{and} \ P_+ \subseteq \mathsf{Port}(\psi)\right] \ \textbf{for some} \ V\,,\, p\,,\, P_+$$

Then, by applying (Z5), conclude $\left[V = \{p\} \ \textbf{and} \ \{p\} \in \wp(\mathbb{P}\text{ORT})\right]$. Then, by applying substitution, conclude $V \in \wp(\mathbb{P}\text{ORT})$.

(Z7) Suppose:

$$P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{for some} \ P_+$$

Then, by applying standard inference rules, conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\} \in \Omega$$

Then, by applying (Z6), conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+ \ \textbf{and} \ V \in \wp(\mathbb{P}\text{ORT})\} \in \Omega$$

Then, by applying ZFC, conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ \{V \mid V = \{p\} \ \textbf{and} \ p \in P_+ \ \textbf{and} \ V \in \wp(\mathbb{P}\text{ORT})\} \in \wp^2(\mathbb{P}\text{ORT})$$

Then, by applying (Z6), conclude $\{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\} \in \wp^2(\mathbb{P}\text{ORT})$.

(Z8) Suppose:

$$\big[P_+ \subseteq \mathsf{Port}(\psi) \text{ and } \psi' = \bigoplus(V') \text{ and } V' \in \{V \mid V = \{p\} \text{ and } p \in P_+\}\big] \text{ for some } P_+\,,\, \psi', V'$$

Then, by applying (Z7), conclude $\big[\psi' = \bigoplus(V') \text{ and } V' \in \{V \mid V = \{p\} \text{ and } p \in P_+\} \in \wp^2(\mathbb{P}\text{ORT})\big]$. Then, by applying ZFC, conclude $\big[\psi' = \bigoplus(V') \text{ and } V' \in \wp(\mathbb{P}\text{ORT})\big]$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\big[\psi' = \bigoplus(V') \text{ and } \bigoplus(V') \in \mathbb{SC}\big]$ Then, by applying substitution, conclude $\psi' \in \mathbb{SC}$.

(Z9) Suppose:

$$P_+ \subseteq \mathsf{Port}(\psi) \text{ for some } P_+$$

Then, by applying standard inference rules, conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \text{ and } \{\psi' \mid \psi' = \bigoplus(V') \text{ and } V' \in \{V \mid V = \{p\} \text{ and } p \in P_+\}\} \in \Omega$$

Then, by applying (Z8), conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \text{ and } \{\psi' \mid \psi' = \bigoplus(V') \text{ and } V' \in \{V \mid V = \{p\} \text{ and } p \in P_+\} \text{ and } \psi' \in \mathbb{SC}\} \in \Omega$$

Then, by applying ZFC, conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \text{ and } \{\psi' \mid \psi' = \bigoplus(V') \text{ and } V' \in \{V \mid V = \{p\} \text{ and } p \in P_+\} \text{ and } \psi' \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying (Z8), conclude $\{\psi' \mid \psi' = \bigoplus(V') \text{ and } V' \in \{V \mid V = \{p\} \text{ and } p \in P_+\}\} \in \wp(\mathbb{SC})$.

$$\{\psi' \mid \psi' = \bigoplus(V') \text{ and } V' \in \{V \mid V = \{p\} \text{ and } p \in P_+\}\} \in \wp(\mathbb{SC})$$

(Z0) Suppose:

$$p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \text{ and } p \in P_+\} \text{ for some } p\,,\, p'\,,\, P_+$$

Then, by applying ZFC, conclude $p' \in \mathsf{Port}(\psi)$. Then, by applying Definition 13 of Port, conclude $p' \in \mathsf{Port}(\psi) \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $p' \in \mathbb{P}\text{ORT}$.

(Y1) Suppose:

$$\big[p''' = \overline{p'} \text{ and } p' \in \mathsf{Port}(\psi) \setminus X\big] \text{ for some } p\,,\, p'\,,\, p'''\,,\, X$$

Then, by applying (Z0), conclude $\big[p''' = \overline{p'} \text{ and } p' \in \mathbb{P}\text{ORT}\big]$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\big[p''' = \overline{p'} \text{ and } \overline{p'} \in \mathbb{SC}\big]$. Then, by applying substitution, conclude $p''' \in \mathbb{SC}$.

(Y2) Suppose **true**. Then, by applying standard inference rules, conclude:

$$\{p''' \mid p''' = \overline{p'} \text{ and } p' \in \mathsf{Port}(\psi) \setminus X\} \in \Omega$$

Then, by applying (Y1), conclude:

$$\{p''' \mid p''' = \overline{p'} \text{ and } p' \in \mathsf{Port}(\psi) \setminus X \text{ and } p''' \in \mathbb{SC}\} \in \Omega$$

Then, by applying ZFC, conclude:

$$\{p''' \mid p''' = \overline{p'} \text{ and } p' \in \mathsf{Port}(\psi) \setminus X \text{ and } p''' \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying (Y1), conclude

$$\{p''' \mid p''' = \overline{p'} \text{ and } p' \in \mathsf{Port}(\psi) \setminus X\} \in \wp(\mathbb{SC})$$

(Y3) Suppose:

$$P_+ \subseteq \mathsf{Port}(\psi) \textbf{ for some } P_+$$

Then, by applying (Z9), conclude $\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC})$. Then, by introducing (Y2), conclude:

$$\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\}\,,\\ \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC})$$

Then, by applying ZFC, conclude:

$$\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \\ \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC})$$

(Y4) Suppose:

$$\left[\psi'' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \textbf{ and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \delta \overset{\mathrm{sc}}{\models} \psi''\right] \textbf{ for some } \psi''\,,\, P_+\,,\, \delta$$

Then, by applying (Z2), conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and }$$
$$\left[\begin{array}{c}\left[\left[\begin{array}{l}\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \\ \quad \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\}\end{array}\right] \textbf{ implies } \delta \overset{\mathrm{sc}}{\models} \psi'''\right] \\ \textbf{for all } \psi'''\end{array}\right]$$

Then, by applying (Y3), conclude:

$$\left[\begin{array}{c}\left[\left[\begin{array}{l}\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \\ \quad \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\}\end{array}\right] \textbf{ implies } \delta \overset{\mathrm{sc}}{\models} \psi'''\right] \\ \textbf{for all } \psi'''\end{array}\right]$$
$$\textbf{and } \left[\begin{array}{l}\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \\ \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC})\end{array}\right]$$

Then, by applying Lemma 5:2, conclude:

$$\delta \overset{\mathrm{sc}}{\models} \prod(\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \\ \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC}))$$

(Y5) Suppose:

$$P_+ \subseteq \mathsf{Port}(\psi) \textbf{ for some } P_+$$

Then, by applying (Y5), conclude $\{V \mid V = \{p\} \textbf{ and } p \in P_+\} \in \wp^2(\mathbb{P}\textsc{ort})$. Then, by applying Definition 15 of $\mathbb{V}\textsc{er}$, conclude $\{V \mid V = \{p\} \textbf{ and } p \in P_+\} \in \wp(\mathbb{V}\textsc{er})$.

(Y6) Suppose:

$$\left[\psi'' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \textbf{ and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \delta \overset{\mathrm{sc}}{\models} \psi''\right] \textbf{ for some } \psi''\,,\, P_+\,,\, \delta$$

Then, by applying (Y4), conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and }$$
$$\delta \overset{\mathrm{sc}}{\models} \prod(\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \\ \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC}))$$

Then, by applying (Y5), conclude:

$$\delta \overset{\text{sc}}{\models} \prod(\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC})) \textbf{ and } \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \in \wp(\mathbb{V}\text{ER})$$

Then, by applying Definition 13 of Port, conclude:

$$\delta \overset{\text{sc}}{\models} \prod(\{\psi' \mid \psi' = \bigoplus(V') \textbf{ and } V' \in \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \textbf{ and } p \in P_+\}\} \in \wp(\mathbb{SC})) \textbf{ and } \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \in \wp(\mathbb{V}\text{ER}) \textbf{ and } \mathsf{Port}(\psi) \in \wp(\mathbb{P}\text{ORT})$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude $\delta \overset{\text{sc}}{\models} [\![\{V \mid V = \{p\} \textbf{ and } p \in P_+\}]\!]_{\mathsf{Port}(\psi)}$.

(Y7) Suppose **true**. Then, by applying standard inference rules, conclude:

$$\{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \textbf{ and } E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \\ \textbf{and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P} \end{array}\right\}\} \in \Omega$$

Then,by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \textbf{ and } E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \\ \textbf{and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P} \end{array}\right\} \textbf{ and } [\![E]\!]_{\mathsf{Port}(\psi)} \in \mathbb{SC}\} \in \Omega$$

Then, by applying substitution, conclude:

$$\{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \textbf{ and } E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \\ \textbf{and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P} \end{array}\right\} \textbf{ and } \psi' \in \mathbb{SC}\} \in \Omega$$

Then, by applying ZFC, conclude:

$$\{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \textbf{ and } E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \\ \textbf{and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P} \end{array}\right\} \textbf{ and } \psi' \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying substitution, conclude:

$$\left\{\psi' \,\middle|\, \begin{array}{c} \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \textbf{ and} \\ E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \\ \textbf{and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P} \end{array}\right\} \\ \textbf{and } [\![E]\!]_{\mathsf{Port}(\psi)} \in \mathbb{SC} \end{array}\right\} \in \wp(\mathbb{SC})$$

Then,by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \textbf{ and } E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \textbf{ and } p \in P_+\} \\ \textbf{and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P} \end{array}\right\}\} \in \wp(\mathbb{SC})$$

(Y8) Suppose:

$$\delta \overset{\text{sc}}{\models} \psi \textbf{ for some } \delta$$

Then, by applying (A1), conclude:

$$\delta \overset{\text{sc}}{\models} \sum(\{\psi' \mid \psi' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \textbf{ and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P}\})$$

Then, by applying Lemma 5:3, conclude:

$$\psi'' \in \{\psi' \mid \psi' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \textbf{ and } P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } P_+ \in \mathcal{P}\} \textbf{ and } \delta \overset{\text{sc}}{\models} \psi'' \textbf{ for some } \psi''$$

Then, by applying ZFC, conclude:

$$\big[\big[\psi'' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+) \ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P}\big]\ \textbf{for some}\ P_+\big]\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} \psi''$$

Then, by applying standard inference rules, conclude:

$$\big[\psi'' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+) \ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} \psi''\big]\ \textbf{for some}\ P_+$$

Then, by applying (Y6), conclude

$$P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} [\![\{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\}]\!]_{\mathsf{Port}(\psi)}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered} P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P}\ \textbf{and} \\ \big[\big[E = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\mathsf{Port}(\psi)}\big]\ \textbf{for some}\ E\big] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\big[P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P}\ \textbf{and}\ E = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\mathsf{Port}(\psi)}\big]\ \textbf{for some}\ E$$

Then, by applying ZFC, conclude:

$$E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\mathsf{Port}(\psi)}$$

Then, by applying standard inference rules, conclude:

$$E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}\ \textbf{and}\ \big[\big[\psi''' = [\![E]\!]_{\mathsf{Port}(\psi)}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} \psi'''\big]\ \textbf{for some}\ \psi'''\big]$$

Then, by applying standard inference rules, conclude:

$$\Big[E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}\ \textbf{and}\ \psi''' = [\![E]\!]_{\mathsf{Port}(\psi)}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} \psi'''\Big]\ \textbf{for some}\ \psi'''$$

Then, by applying ZFC, conclude:

$$\psi''' \in \{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)}\ \textbf{and}\ E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}\}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} \psi'''$$

Then, by applying (Y7), conclude:

$$\begin{gathered} \psi''' \in \{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)}\ \textbf{and}\ E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}\}\ \textbf{and}\ \delta \overset{\mathrm{sc}}{\models} \psi''' \\ \textbf{and}\ \{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)}\ \textbf{and}\ E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}\} \in \wp(\mathbb{SC}) \end{gathered}$$

Then, by applying Lemma 5:1, conclude:

$$\delta \models \textstyle\sum(\{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)}\ \textbf{and}\ E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}\})$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\delta \models [\![\wp(\mathsf{Port}(\psi))\,,\,\left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}]\!]$$

Then, by introducing (A1), conclude:

$$\begin{gathered} \delta \models [\![\wp(\mathsf{Port}(\psi))\,,\,\left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\}\ \textbf{and}\ p \in P_+\} \\ \textbf{and}\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\}]\!] \\ \textbf{and}\ \psi = \textstyle\sum(\left\{\psi' \,\middle|\, \begin{array}{l} \psi' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\,P_+)\ \textbf{and} \\ P_+ \subseteq \mathsf{Port}(\psi)\ \textbf{and}\ P_+ \in \mathcal{P} \end{array}\right\} \end{gathered}$$

Then, by applying Definition 17 of graph, conclude $\delta \models [\![\mathsf{graph}(\psi)]\!]$.

(Y9) Suppose:

$$\big[\big[\psi'' \in \{\psi' \mid \psi' = \bigoplus(V') \;\textbf{and}\; V' \in \{V \mid V = \{p''\} \;\textbf{and}\; p'' \in P_+\}\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi''\big] \;\textbf{for all}\; \psi''\big] \;\textbf{for some}\; P_+\,,\,\delta$$

Then, by applying ZFC, conclude:

$$\big[\psi'' \in \{\psi' \mid \psi' = \bigoplus(V') \;\textbf{and}\; V' = \{p''\} \;\textbf{and}\; p'' \in P_+\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi''\big] \;\textbf{for all}\; \psi''$$

Then, by applying substitution, conclude:

$$\big[\psi'' \in \{\psi' \mid \psi' = \bigoplus(\{p''\}) \;\textbf{and}\; p'' \in P_+\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi''\big] \;\textbf{for all}\; \psi''$$

Then, by applying ZFC, conclude:

$$\big[\big[\big[\psi'' = \bigoplus(\{p''\}) \;\textbf{and}\; p'' \in P_+\big] \;\textbf{for some}\; p''\big] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi''\big] \;\textbf{for all}\; \psi''$$

Then, by applying standard inference rules, conclude:

$$\big[\big[\big[\psi'' = \bigoplus(\{p''\}) \;\textbf{and}\; p'' \in P_+\big] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi''\big] \;\textbf{for some}\; p''\big] \;\textbf{for all}\; \psi''$$

Then, by applying substitution, conclude:

$$\big[\big[p'' \in P_+ \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \bigoplus(\{p''\})\big] \;\textbf{for some}\; p''\big] \;\textbf{for all}\; \psi''$$

Then, by applying standard inference rules, conclude $\big[\big[p'' \in P_+ \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \bigoplus(\{p''\})\big] \;\textbf{for all}\; p''\big]$. Then, by applying Definition 10 of $\overset{\text{sc}}{\models}$, conclude $\big[\big[p'' \in P_+ \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} p''\big] \;\textbf{for all}\; p''\big]$. Then, by applying ZFC, conclude $\big[\big[p'' \in \{p \mid p \in P_+\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} p''\big] \;\textbf{for all}\; p''\big]$.

(Y0) Suppose:

$$\delta \overset{\text{sc}}{\models} [\![\{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}]\!]_{\mathsf{Port}(\psi)} \;\textbf{for some}\; \delta\,,\,P_+$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\delta \overset{\text{sc}}{\models} \prod(\{\psi' \mid \psi' = \bigoplus(V') \;\textbf{and}\; V' \in \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\} \cup \{p''' \mid p''' = \overline{p'} \;\textbf{and}\; p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\} \in \wp(\mathbb{SC}))$$

Then, by applying Lemma 5:4, conclude:

$$\left[\left[\begin{array}{l}\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \;\textbf{and}\; V' \in \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\} \cup \\ \quad \{p''' \mid p''' = \overline{p'} \;\textbf{and}\; p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\}\end{array}\right] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi'''\right] \;\textbf{for all}\; \psi'''$$

Then, by applying ZFC, conclude:

$$\left[\left[\begin{array}{l}\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \;\textbf{and}\; V' \in \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\} \\ \textbf{or}\; \psi''' \in \{p''' \mid p''' = \overline{p'} \;\textbf{and}\; p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\}\end{array}\right] \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi'''\right] \;\textbf{for all}\; \psi'''$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{l}\big[\big[\psi''' \in \{\psi' \mid \psi' = \bigoplus(V') \;\textbf{and}\; V' \in \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi'''\big] \;\textbf{for all}\; \psi'''\big] \\ \textbf{and}\; \big[\big[\psi''' \in \{p''' \mid p''' = \overline{p'} \;\textbf{and}\; p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi'''\big] \;\textbf{for all}\; \psi'''\big]\end{array}$$

Then, by applying (Y9), conclude:

$$\begin{array}{l}\big[\big[\psi''' \in \{p''' \mid p''' \in P_+\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi'''\big] \;\textbf{for all}\; \psi'''\big] \\ \textbf{and}\; \big[\big[\psi''' \in \{p''' \mid p''' = \overline{p'} \;\textbf{and}\; p' \in \mathsf{Port}(\psi) \setminus \{V \mid V = \{p\} \;\textbf{and}\; p \in P_+\}\} \;\textbf{implies}\; \delta \overset{\text{sc}}{\models} \psi'''\big] \;\textbf{for all}\; \psi'''\big]\end{array}$$

Then, by applying ZFC, conclude:

$$\begin{aligned}&\big[\big[\psi''' \in \{p''' \mid p''' \in P_+\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi'''\big] \textbf{ for all } \psi'''\big]\\ \textbf{and } &\big[\big[\psi''' \in \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi'''\big] \textbf{ for all } \psi'''\big]\end{aligned}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c}\big[\psi''' \in \{p''' \mid p''' \in P_+\} \textbf{ or } \psi''' \in \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\}\big]\\ \textbf{implies } \delta \overset{\text{sc}}{\models} \psi'''\end{array}\right] \textbf{ for all } \psi'''$$

Then, by applying ZFC, conclude:

$$\big[\psi''' \in \{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi'''\big] \textbf{ for all } \psi'''$$

(X1) Suppose:

$$P_+ \subseteq \mathsf{Port}(\psi) \textbf{ for some } P_+$$

Then, by applying standard inference rules, conclude $\big[P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \{p \mid p \in P_+\} \in \Omega\big]$. Then, by applying (Z4), conclude $\big[P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \{p \mid p \in P_+ \textbf{ and } p \in \mathbb{P}\text{ORT}\} \in \Omega\big]$. Then, by applying ZFC, conclude $\big[P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \{p \mid p \in P_+ \textbf{ and } p \in \mathbb{P}\text{ORT}\} \in \wp(\mathbb{P}\text{ORT})\big]$. Then, by applying (Z4), conclude $\{p \mid p \in P_+\} \in \wp(\mathbb{P}\text{ORT})$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude:

$$\{p \mid p \in P_+\} \in \wp(\mathbb{SC})$$

(X2) Suppose:

$$P_+ \subseteq \mathsf{Port}(\psi) \textbf{ for some } P_+$$

Then, by applying (X1), conclude $\{p''' \mid p''' \in P_+\} \in \wp(\mathbb{SC})$. Then, by introducing (Y2), conclude:

$$\{p''' \mid p''' \in P_+\}\,,\, \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\} \in \wp(\mathbb{SC})$$

Then, by applying ZFC, conclude $\{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\} \in \wp(\mathbb{SC})$.

(X3) Suppose:

$$\big[P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \delta \overset{\text{sc}}{\models} [\![\{V \mid V = \{p\} \textbf{ and } p \in P_+\}]\!]_{\mathsf{Port}(\psi)}\big] \textbf{ for some } \delta\,,\, P_+$$

Then, by applying (Y0), conclude:

$$\begin{array}{c}P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and}\\ \big[\big[\psi''' \in \{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi'''\big] \textbf{ for all } \psi'''\big]\end{array}$$

Then, by applying (X2), conclude:

$$\begin{array}{c}\big[\big[\psi''' \in \{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\} \textbf{ implies } \delta \overset{\text{sc}}{\models} \psi'''\big] \textbf{ for all } \psi'''\big]\\ \textbf{and } \{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\} \in \wp(\mathbb{SC})\end{array}$$

Then, by applying Lemma 5:2, conclude $\delta \overset{\text{sc}}{\models} \prod(\{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\})$.

(X4) Suppose:

$$\big[P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \delta \overset{\text{sc}}{\models} [\![\{V \mid V = \{p\} \textbf{ and } p \in P_+\}]\!]_{\mathsf{Port}(\psi)}\big] \textbf{ for some } \delta\,,\, P_+$$

Then, by applying (X3), conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \textbf{ and } \delta \overset{\text{sc}}{\models} \prod(\{p''' \mid p''' \in P_+\} \cup \{p''' \mid p''' = \overline{p'} \textbf{ and } p' \in \mathsf{Port}(\psi) \setminus P_+\})$$

Then, by applying Definition 12 of cp, conclude $\delta \overset{\text{sc}}{\models} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+)$. Then, by applying standard inference rules, conclude:

$$\big[\psi'' = \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \textbf{ and } \delta \overset{\text{sc}}{\models} \psi''\big] \textbf{ for some } \psi''$$

Then, by applying Definition 10 of $\equiv_{\text{sc}}$, conclude $\big[\psi'' \equiv \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \textbf{ and } \delta \overset{\text{sc}}{\models} \psi''\big]$.

(X5) Suppose **true**. Then, by applying standard inference rules, conclude:

$$\{\psi' \mid \psi' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \;\textbf{ and }\; P'_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P'_+ \in \mathcal{P}\} \in \Omega$$

Then, by applying Definition 10 of $\equiv_{\mathrm{sc}}$, conclude:

$$\{\psi' \mid \psi' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \;\textbf{ and }\; P'_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P'_+ \in \mathcal{P} \;\textbf{ and }\; \psi' \in \mathbb{SC}\} \in \Omega$$

Then, by applying ZFC, conclude:

$$\{\psi' \mid \psi' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \;\textbf{ and }\; P'_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P'_+ \in \mathcal{P} \;\textbf{ and }\; \psi' \in \mathbb{SC}\} \in \wp(\mathbb{SC})$$

Then, by applying Definition 10 of $\equiv_{\mathrm{sc}}$, conclude:

$$\{\psi' \mid \psi' \equiv_{\mathrm{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \;\textbf{ and }\; P'_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P'_+ \in \mathcal{P}\} \in \wp(\mathbb{SC})$$

(X6) Suppose:

$$\delta \models [\![\mathsf{graph}(\psi)]\!] \;\textbf{ for some }\; \delta$$

Then, by applying Definition 17 of graph, conclude:

$$\delta \models [\![\wp(\mathsf{Port}(\psi))\,,\, \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \end{array}\right\}]\!]$$

Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

$$\delta \models \textstyle\sum(\{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \;\textbf{ and }\; E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \end{array}\right\}\})$$

Then, by applying Lemma 5:3, conclude:

$$[\psi''' \in \{\psi' \mid \psi' = [\![E]\!]_{\mathsf{Port}(\psi)} \;\textbf{ and }\; E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \end{array}\right\}\} \;\textbf{ and }\; \delta \overset{\mathrm{sc}}{\models} \psi'''] \;\textbf{ for some }\; \psi'''$$

Then, by applying ZFC, conclude:

$$[[\psi''' = [\![E]\!]_{\mathsf{Port}(\psi)} \;\textbf{ and }\; E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \end{array}\right\}] \;\textbf{ for some }\; E] \;\textbf{ and }\; \delta \overset{\mathrm{sc}}{\models} \psi'''$$

Then, by applying standard inference rules, conclude:

$$[\psi''' = [\![E]\!]_{\mathsf{Port}(\psi)} \;\textbf{ and }\; E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \end{array}\right\} \;\textbf{ and }\; \delta \overset{\mathrm{sc}}{\models} \psi'''] \;\textbf{ for some }\; E$$

Then, by applying substitution, conclude:

$$E \in \left\{E' \,\middle|\, \begin{array}{l} E' = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \end{array}\right\} \;\textbf{ and }\; \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\mathsf{Port}(\psi)}$$

Then, by applying ZFC, conclude:

$$[\left[\begin{array}{l} E = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \end{array}\right] \;\textbf{ for some }\; P_+] \;\textbf{ and }\; \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\mathsf{Port}(\psi)}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{l} E = \{V \mid V = \{p\} \;\textbf{ and }\; p \in P_+\} \\ \textbf{and }\; P_+ \subseteq \mathsf{Port}(\psi) \;\textbf{ and }\; P_+ \in \mathcal{P} \\ \qquad \textbf{and }\; \delta \overset{\mathrm{sc}}{\models} [\![E]\!]_{\mathsf{Port}(\psi)} \end{array}\right] \;\textbf{ for some }\; P_+$$

Then, by applying substitution, conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ P_+ \in \mathcal{P} \ \textbf{and} \ \delta \overset{\text{sc}}{\models} [\![\{V \mid V = \{p\} \ \textbf{and} \ p \in P_+\}]\!]_{\mathsf{Port}(\psi)}$$

Then, by applying (X4), conclude:

$$P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ P_+ \in \mathcal{P} \ \textbf{and} \ \big[\big[\psi'' \equiv \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi''\big] \ \textbf{for some} \ \psi''\big]$$

Then, by applying standard inference rules, conclude:

$$\big[P_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ P_+ \in \mathcal{P} \ \textbf{and} \ \psi'' \equiv \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P_+) \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi''\big] \ \textbf{for some} \ \psi''$$

Then, by applying ZFC, conclude:

$$\psi'' \in \big\{\psi' \,\big|\, \psi' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \ \textbf{and} \ P'_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ P'_+ \in \mathcal{P}\big\} \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi''$$

Then, by introducing (X5), conclude:

$$\begin{gathered}\psi'' \in \big\{\psi' \,\big|\, \psi' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \ \textbf{and} \ P'_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ P'_+ \in \mathcal{P}\big\} \ \textbf{and} \ \delta \overset{\text{sc}}{\models} \psi'' \\ \textbf{and} \ \big\{\psi' \,\big|\, \psi' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \ \textbf{and} \ P'_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ P'_+ \in \mathcal{P}\big\} \in \wp(\mathbb{SC})\end{gathered}$$

Then, by applying Lemma 5:1, conclude:

$$\delta \overset{\text{sc}}{\models} \textstyle\sum(\big\{\psi' \,\big|\, \psi' \equiv_{\text{sc}} \mathsf{cp}(\mathsf{Port}(\psi)\,,\, P'_+) \ \textbf{and} \ P'_+ \subseteq \mathsf{Port}(\psi) \ \textbf{and} \ P'_+ \in \mathcal{P}\big\})$$

Then, by applying (A1), conclude $\delta \overset{\text{sc}}{\models} \psi$.

Now, prove the theorem by the following reduction. Recall from (Y8)(X6):

$$\begin{aligned} &\big[\big[\delta \overset{\text{sc}}{\models} \psi \ \textbf{implies} \ \delta \models [\![\mathsf{graph}(\psi)]\!]\big] \ \textbf{for all} \ \psi\big] \\ \textbf{and} \ &\big[\big[\delta \models [\![\mathsf{graph}(\psi)]\!] \ \textbf{implies} \ \delta \overset{\text{sc}}{\models} \psi\big] \ \textbf{for all} \ \psi\big]\end{aligned}$$

Then, by applying standard inference rules, conclude $\big[\big[\delta \overset{\text{sc}}{\models} \psi \ \textbf{iff} \ \delta \models [\![\mathsf{graph}(\psi)]\!]\big] \ \textbf{for all} \ \psi\big]$. Then, by applying Definition 10 of $\equiv_{\text{sc}}$, conclude $\psi \equiv_{\text{sc}} [\![\mathsf{graph}(\psi)]\!]$.

(| QED. |)

B.8 Lemma 9

Proof (of Lemma 9).

1. First, assume:

   (A1) $p \in V \in E \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$

   Next, observe:

   (Z1) Recall $V \in E \in \mathcal{E}$. Then, by applying ZFC, conclude $E\,,\,\{V\} \in \mathcal{E}$. Then, by applying ZFC, conclude $E \setminus \{V\} \in \mathcal{E}$. Then, by applying standard inference rules, conclude:

   $$T = E \setminus \{V\} \textbf{ for some } T$$

   Now, prove the lemma by the following reduction. Recall $p \in V \in E \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$ from (A1). Then, by applying (Z1), conclude $\big[[T = E \setminus \{V\} \textbf{ for some } T\big] \textbf{ and } p \in V \in E \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\big]$. Then, by applying standard inference rules, conclude:

   $$\big[T = E \setminus \{V\} \textbf{ and } p \in V \in E \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\big] \textbf{ for some } T$$

   Then, by applying ZFC, conclude:

   $$T = E \setminus \{V\} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T' = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}\}$$

   Then, by applying Definition 22 of Edge, conclude $\big[T = E \setminus \{V\} \textbf{ and } T \in \mathsf{Edge}(p\,,\,\mathcal{E})\big]$. Then, by applying substitution, conclude $E \setminus \{V\} \in \mathsf{Edge}(p\,,\,\mathcal{E})$.

   ⦇ QED. ⦈

2. First, assume:

   (B1) $p_1\,,\,p_2 \in P \in \bigstar(\mathcal{E})$

   Next, observe:

   (Y1) Recall $P \in \bigstar(\mathcal{E})$ from (B1). Then, by Definition 23 of $\bigstar$, conclude:

   $$P \in \{P' \mid P' \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[[p \in P' \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \textbf{ for all } p\big]\}$$

   Then, by applying ZFC, conclude:

   $$\big[[p \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \textbf{ for all } p\big] \textbf{ for some } \mathcal{T}$$

   Then, by applying standard inference rules, conclude:

   $$\big[p \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \textbf{ for all } p$$

   Now, prove the lemma by the following reduction. Recall $p_1\,,\,p_2 \in P$ from (B1). Then, by introducing (Y1), conclude:

   $$p_1\,,\,p_2 \in P' \textbf{ and } \big[\big[[p_2' \in P' \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p_2'\,,\,\mathcal{E})\big] \textbf{ for all } p_2'\big] \textbf{ for some } \mathcal{T}\big]$$

   Then, by applying standard inference rules, conclude:

   $$\big[p_1\,,\,p_2 \in P' \textbf{ and } \big[[p_2' \in P' \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p_2'\,,\,\mathcal{E})\big] \textbf{ for all } p_2'\big]\big] \textbf{ for some } \mathcal{T}$$

   Then, by applying standard inference rules, conclude $\big[\mathcal{T} = \mathsf{Edge}(p_1\,,\,\mathcal{E}) \textbf{ and } \mathcal{T} = \mathsf{Edge}(p_2\,,\,\mathcal{E})\big]$. Then, by applying substitution, conclude $\mathsf{Edge}(p_1\,,\,\mathcal{E}) = \mathsf{Edge}(p_2\,,\,\mathcal{E})$.

⦇ QED. ⦈

3. First, assume:

   (C1) $p_1 \in V_1 \in E_1 \in \mathcal{E}$

   (C2) $\mathsf{Edge}(p_1 , \mathcal{E}) = \mathsf{Edge}(p_2 , \mathcal{E})$

   Next, observe:

   (X1) Suppose $p_1 \in V_1 \in E_1 \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying Lemma 9:1, conclude:

   $$E_1 \setminus \{V_1\} \in \mathsf{Edge}(p_1 , \mathcal{E})$$

   Then, by introducing (C2), conclude $\big[E_1 \setminus \{V_1\} \in \mathsf{Edge}(p_1 , \mathcal{E})$ **and** $\mathsf{Edge}(p_1 , \mathcal{E}) = \mathsf{Edge}(p_2 , \mathcal{E})\big]$. Then, by applying substitution, conclude $E_1 \setminus \{V_1\} \in \mathsf{Edge}(p_2 , \mathcal{E})$. Then, by applying Definition 22 of Edge, conclude $\big[E_1 \setminus \{V_1\} \in \{T \mid T = E_2 \setminus \{V_2\}$ **and** $p_2 \in V_2 \in E_2 \in \mathcal{E}\}\big]$. Then, by applying ZFC, conclude:

   $$\big[E_1 \setminus \{V_1\} = E_2 \setminus \{V_2\} \textbf{ and } p_2 \in V_2 \in E_2 \in \mathcal{E}\big] \textbf{ for some } E_2 , V_2$$

   (X2) Suppose:

   $$\big[E_2' \setminus \{V_2'\} = E_1' \setminus \{V_1'\} \textbf{ and } V_1' \in E_1'\big] \textbf{ for some } E_1' , E_2' , V_1' , V_2'$$

   Then, by applying ZFC, conclude $\big[E_2' \setminus \{V_2'\} = E_1' \setminus \{V_1'\}$ **and** $E_1' = E_1' \cup \{V_1'\}\big]$. Then, by applying ZFC, conclude $\big[E_2' \setminus \{V_2'\} = E_1' \setminus \{V_1'\}$ **and** $E_1' = (E_1' \setminus \{V_1'\}) \cup \{V_1'\}\big]$. Then, by applying substitution, conclude $E_1' = (E_2' \setminus \{V_2'\}) \cup \{V_1'\}$.

   (X3) Suppose:

   $$\big[E_2' \setminus \{V_2'\} = E_1' \setminus \{V_1'\} \textbf{ and } V_2' \in E_2'\big] \textbf{ for some } E_1' , E_2' , V_1' , V_2'$$

   Then, by a reduction similar to (X2), conclude $E_2' = (E_1' \setminus \{V_1'\}) \cup \{V_2'\}$.

   (X4) Suppose:

   $$\big[V_1 \in E_1 \textbf{ and } V_2 \in E_2 \textbf{ and } E_1 \setminus \{V_1\} = E_2 \setminus \{V_2\}\big] \textbf{ for some } V_2 , E_2$$

   Then, by applying (X2)(X3), conclude $\big[E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\}$ **and** $E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\}\big]$. Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1 , V_1) \curlyvee (E_2 , V_2)$.

   Now, prove the lemma by the following reduction. Recall $\mathsf{Edge}(p_1 , \mathcal{E}) = \mathsf{Edge}(p_2 , \mathcal{E})$ from (C2). Then, by applying Definition 22 of Edge, conclude $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$. Then, by introducing (C1), conclude:

   $$p_1 \in V_1 \in E_1 \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$$

   Then, by applying (X1), conclude:

   $$V_1 \in E_1 \textbf{ and } \Big[\big[E_1 \setminus \{V_1\} = E_2 \setminus \{V_2\} \textbf{ and } p_2 \in V_2 \in E_2 \in \mathcal{E}\big] \textbf{ for some } E_2 , V_2\Big]$$

   Then, by applying standard inference rules, conclude:

   $$\big[V_1 \in E_1 \textbf{ and } E_1 \setminus \{V_1\} = E_2 \setminus \{V_2\} \textbf{ and } p_2 \in V_2 \in E_2 \in \mathcal{E}\big] \textbf{ for some } E_2 , V_2$$

   Then, by applying (X4), conclude $\big[p_2 \in V_2 \in E_2 \in \mathcal{E}$ **and** $(E_1 , V_1) \curlyvee (E_2 , V_2)\big]$.

   ⦇ QED. ⦈

4. Prove the lemma by a reduction similar to Lemma 9:3.

   ⦇ QED. ⦈

B.9 Lemma 10

Proof (of Lemma 10).

1. First, observe:

(Z1) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)\ \ \textbf{for some}\ \ T$$

Then, by applying Definition 22 of Edge, conclude:

$$p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T \in \big\{T' \,\big|\, T' = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big\}$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ \Big[\begin{bmatrix} T = E \setminus \{V\}\ \ \textbf{and} \\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \end{bmatrix}\ \ \textbf{for some}\ \ V\,,\,E\Big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and} \\ T = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \end{bmatrix}\ \ \textbf{for some}\ \ V\,,\,E$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T = E \setminus \{V\}\ \ \textbf{and}\ \ \begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \\ \textbf{or}\ \ p \in V \in E \in \mathcal{E}_2 \end{bmatrix}$$

Then, by applying standard inference rules, conclude:

$$\begin{matrix} & \big[p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1\big] \\ \textbf{or} & \big[p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_2\big] \end{matrix}$$

Then, by applying ZFC, conclude:

$$\begin{matrix} & \big[p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1\big] \\ \textbf{or} & \big[p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_2\big] \end{matrix}$$

Then, by applying ZFC, conclude:

$$\begin{matrix} & \big[p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T \in \{T' \mid T' = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E' \in \mathcal{E}_1\}\big] \\ \textbf{or} & \big[p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ T \in \{T' \mid T' = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E' \in \mathcal{E}_2\}\big] \end{matrix}$$

Then, by applying Definition 22 of Edge, conclude $\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1)\ \ \textbf{or}\ \ T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2)\big]$. Then, by applying ZFC, conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_2)$.

(Z2) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_2)\ \ \textbf{for some}\ \ T$$

Then, by applying ZFC, conclude $\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1)\ \ \textbf{or}\ \ T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2)\big]$. Then, by applying Definition 22 of Edge, conclude:

$$p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1\,,\,\mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ \begin{bmatrix} T \in \{T' \mid T' = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1\} \\ \textbf{or}\ \ T \in \{T' \mid T' = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_2\} \end{bmatrix}$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\mathrm{ORT}\ \ \textbf{and}\ \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ \begin{bmatrix} T \in \{T' \mid T' = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1\} \\ \textbf{or}\ \ T \in \{T' \mid T' = E \setminus \{V\}\ \ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_2\} \end{bmatrix}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c} p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \\ \left[\begin{array}{l} \left[\left[T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1\right] \ \textbf{for some} \ V, E\right] \\ \textbf{or} \ \left[\left[T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_2\right] \ \textbf{for some} \ V, E\right] \end{array}\right] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c} p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \\ \left[\left[\begin{array}{l} \quad \left[T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1\right] \\ \textbf{or} \ \left[T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_2\right] \end{array}\right] \ \textbf{for some} \ V, E\right] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c} p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \\ \left[\begin{array}{l} \quad \left[T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1\right] \\ \textbf{or} \ \left[T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_2\right] \end{array}\right] \end{array}\right] \ \textbf{for some} \ V, E$$

Then, by applying standard inference rules, conclude:

$$p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ \left[\begin{array}{l} \quad p \in V \in E \in \mathcal{E}_1 \\ \textbf{or} \ p \in V \in E \in \mathcal{E}_2 \end{array}\right]$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying ZFC, conclude:

$$\begin{array}{c} p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \\ T \in \{T' \mid T' = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\} \end{array}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2)$.

Now, prove the lemma by the following reduction. Recall from (Z1)(Z2):

$$\begin{array}{l} \quad\ \left[\left[T \in \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2) \ \textbf{implies} \ T \in \mathsf{Edge}(p, \mathcal{E}_1) \cup \mathsf{Edge}(p, \mathcal{E}_2)\right] \ \textbf{for all} \ T\right] \\ \textbf{and} \ \left[\left[T \in \mathsf{Edge}(p, \mathcal{E}_1) \cup \mathsf{Edge}(p, \mathcal{E}_2) \ \textbf{implies} \ T \in \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2)\right] \ \textbf{for all} \ T\right] \end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{l} \quad\ \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2) \subseteq \mathsf{Edge}(p, \mathcal{E}_1) \cup \mathsf{Edge}(p, \mathcal{E}_2) \\ \textbf{and} \ \mathsf{Edge}(p, \mathcal{E}_1) \cup \mathsf{Edge}(p, \mathcal{E}_2) \subseteq \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2) \end{array}$$

Then, by applying ZFC, conclude $\mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p, \mathcal{E}_1) \cup \mathsf{Edge}(p, \mathcal{E}_2)$.

(| QED. |)

2. First, assume:

(A1) $\checkmark(\mathcal{E})$

(A2) $\mathcal{E}_1, \mathcal{E}_2 \subseteq \mathcal{E}$

Next, observe:

(Y1) Suppose:

$$\begin{bmatrix} T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \ \textbf{and} \ E \notin \mathcal{E}_2 \\ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_2 \end{bmatrix} \ \textbf{for some} \ T\,,\, E\,,\, E'\,,\, V\,,\, V'$$

Then, by introducing (A2), conclude:

$$\mathcal{E}_1\,,\,\mathcal{E}_2 \subseteq \mathcal{E} \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \ \textbf{and} \ E \notin \mathcal{E}_2 \\ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_2$$

Then, by applying ZFC, conclude:

$$T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E} \ \textbf{and} \ E \notin \mathcal{E}_2 \\ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E} \ \textbf{and} \ E' \in \mathcal{E}_2$$

Then, by introducing (A1), conclude:

$$\checkmark(\mathcal{E}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E} \ \textbf{and} \ E \notin \mathcal{E}_2 \\ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E} \ \textbf{and} \ E' \in \mathcal{E}_2$$

Then, by applying Definition 24 of $\checkmark$, conclude:

$$[[\begin{bmatrix} p \in V_1 \in E_1 \in \mathcal{E} \\ \textbf{and} \ p \in V_2 \in E_2 \in \mathcal{E} \end{bmatrix} \ \textbf{implies} \ V_1 = V_2] \ \textbf{for all} \ p\,,\, V_1\,,\, V_2\,,\, E_1\,,\, E_2] \\ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E} \ \textbf{and} \ E \notin \mathcal{E}_2 \\ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E} \ \textbf{and} \ E' \in \mathcal{E}_2$$

Then, by applying standard inference rules, conclude:

$$V = V' \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ V \in E \ \textbf{and} \ E \notin \mathcal{E}_2 \\ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ V' \in E' \ \textbf{and} \ E' \in \mathcal{E}_2$$

Then, by applying substitution, conclude:

$$T = E \setminus \{V\} \ \textbf{and} \ V \in E \ \textbf{and} \ E \notin \mathcal{E}_2 \ \textbf{and} \ T = E' \setminus \{V\} \ \textbf{and} \ V \in E' \ \textbf{and} \ E' \in \mathcal{E}_2$$

Then, by applying substitution, conclude:

$$E' \setminus \{V\} = E \setminus \{V\} \ \textbf{and} \ V \in E \ \textbf{and} \ E \notin \mathcal{E}_2 \ \textbf{and} \ V \in E' \ \textbf{and} \ E' \in \mathcal{E}_2$$

Then, by applying ZFC, conclude $\big[E' = E \ \textbf{and} \ E \notin \mathcal{E}_2 \ \textbf{and} \ E' \in \mathcal{E}_2\big]$. Then, by applying substitution, conclude $\big[E \notin \mathcal{E}_2 \ \textbf{and} \ E \in \mathcal{E}_2\big]$. Then, by applying standard inference rules, conclude **false**.

(Y2) Recall $\checkmark(\mathcal{E})$ from (A1). Then, by applying Definition 24 of $\checkmark$, conclude $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$. Then, by introducing (A2), conclude $\big[\mathcal{E}_2 \subseteq \mathcal{E} \ \textbf{and} \ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\big]$. Then, by applying ZFC, conclude $\mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER})$.

(Y3) Suppose:

$$T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1 \setminus \mathcal{E}_2) \ \textbf{for some} \ T$$

Then, by applying Definition 22 of Edge, conclude:

$$p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ T \in \big\{T' \,\big|\, T' = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \setminus \mathcal{E}_2\big\}$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ [\begin{bmatrix} T = E \setminus \{V\} \ \textbf{and} \\ p \in V \in E \in \mathcal{E}_1 \setminus \mathcal{E}_2 \end{bmatrix} \ \textbf{for some} \ V\,,\, E]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \\ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \setminus \mathcal{E}_2 \end{bmatrix} \ \textbf{for some} \ V \,,\, E$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \ \textbf{and} \ E \notin \mathcal{E}_2$$

Then, by applying (Y1), conclude:

$$\begin{gathered} p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \\ \textbf{and} \ \big[\big[\textbf{not} \ \big[T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_2\big]\big] \ \textbf{for all} \ E' \,,\, V'\big] \end{gathered}$$

Then, by applying ZFC, conclude:

$$\begin{gathered} p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \\ \textbf{and} \ \big[\big[\textbf{not} \ \big[T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_2\big]\big] \ \textbf{for all} \ E' \,,\, V'\big] \end{gathered}$$

Then, by applying ZFC, conclude:

$$\begin{gathered} p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er}) \\ \textbf{and} \ T \in \{T' \mid T' = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1\} \\ \textbf{and} \ T \notin \{T' \mid T' = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_2\} \end{gathered}$$

Then, by introducing (Y2), conclude:

$$\begin{gathered} \mathcal{E}_2 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \ p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er}) \\ \textbf{and} \ T \in \{T' \mid T' = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1\} \\ \textbf{and} \ T \notin \{T' \mid T' = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_2\} \end{gathered}$$

Then, by applying Definition 22 of Edge, conclude $\big[T \in \mathsf{Edge}(p \,,\, \mathcal{E}_1) \ \textbf{and} \ T \notin \mathsf{Edge}(p \,,\, \mathcal{E}_2)\big]$. Then, by applying ZFC, conclude $T \in \mathsf{Edge}(p \,,\, \mathcal{E}_1) \setminus \mathsf{Edge}(p \,,\, \mathcal{E}_2)$.

(Y4) Suppose:

$$T \in \mathsf{Edge}(p \,,\, \mathcal{E}_1) \setminus \mathsf{Edge}(p \,,\, \mathcal{E}_2) \ \textbf{for some} \ T$$

Then, by applying ZFC, conclude $\big[T \in \mathsf{Edge}(p \,,\, \mathcal{E}_1) \ \textbf{and} \ T \notin \mathsf{Edge}(p \,,\, \mathcal{E}_2)\big]$. Then, by applying Definition 22 of Edge, conclude:

$$\begin{gathered} p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er}) \\ \textbf{and} \ T \in \{T' \mid T' = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1\} \\ \textbf{and} \ T \notin \{T' \mid T' = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_2\} \end{gathered}$$

Then, by applying ZFC, conclude:

$$\begin{gathered} p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \\ \textbf{and} \ \big[\big[\textbf{not} \ \big[T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_2\big]\big] \ \textbf{for all} \ E' \,,\, V'\big] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered} p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \\ \textbf{and} \ \big[\textbf{not} \ \big[T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_2\big]\big] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$p \in \mathbb{P}\textsc{ort} \ \textbf{and} \ \mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er}) \ \textbf{and} \ T = E \setminus \{V\} \ \textbf{and} \ p \in V \in E \in \mathcal{E}_1 \ \textbf{and} \ \begin{bmatrix} T \neq E \setminus \{V\} \\ \textbf{or} \ p \notin V \\ \textbf{or} \ V \notin E \\ \textbf{or} \ E \notin \mathcal{E}_2 \end{bmatrix}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c}\big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; T \neq E \setminus \{V\}\big] \\ \textbf{or}\; \big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; p \notin V\big] \\ \textbf{or}\; \big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; V \notin E\big] \\ \textbf{or}\; \big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; E \notin \mathcal{E}_2\big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c}\big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; \textsf{false}\big] \\ \textbf{or}\; \big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; V \in E \in \mathcal{E}_1 \;\textbf{and}\; \textsf{false}\big] \\ \textbf{or}\; \big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \;\textbf{and}\; E \in \mathcal{E}_1 \;\textbf{and}\; \textsf{false}\big] \\ \textbf{or}\; \big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; E \notin \mathcal{E}_2\big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c}\textsf{false} \;\textbf{or}\; \textsf{false} \;\textbf{or}\; \textsf{false} \;\textbf{or} \\ \big[p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; E \notin \mathcal{E}_2\big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; E \notin \mathcal{E}_2$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \;\textbf{and}\; E \notin \mathcal{E}_2$$

Then, by applying ZFC, conclude:

$$p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and}\; T = E \setminus \{V\} \;\textbf{and}\; p \in V \in E \in \mathcal{E}_1 \setminus \mathcal{E}_2$$

Then, by applying ZFC, conclude:

$$\begin{array}{c}p \in \mathbb{P}\text{ORT} \;\textbf{and}\; \mathcal{E}_1 \setminus \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER}) \;\textbf{and} \\ T \in \{T' \mid T' = E' \setminus \{V'\} \;\textbf{and}\; p \in V' \in E' \in \mathcal{E}_1 \setminus \mathcal{E}_2\}\end{array}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1 \setminus \mathcal{E}_2)$.

Now, prove the lemma by the following reduction. Recall from (Y3)(Y4):

$$\begin{array}{c}\big[\big[T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1 \setminus \mathcal{E}_2) \;\textbf{implies}\; T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1) \setminus \mathsf{Edge}(p\,,\, \mathcal{E}_2)\big] \;\textbf{for all}\; T\big] \\ \textbf{and}\; \big[\big[T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1) \setminus \mathsf{Edge}(p\,,\, \mathcal{E}_2) \;\textbf{implies}\; T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1 \setminus \mathcal{E}_2)\big] \;\textbf{for all}\; T\big]\end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c}\mathsf{Edge}(p\,,\, \mathcal{E}_1 \setminus \mathcal{E}_2) \subseteq \mathsf{Edge}(p\,,\, \mathcal{E}_1) \setminus \mathsf{Edge}(p\,,\, \mathcal{E}_2) \\ \textbf{and}\; \mathsf{Edge}(p\,,\, \mathcal{E}_1) \setminus \mathsf{Edge}(p\,,\, \mathcal{E}_2) \subseteq \mathsf{Edge}(p\,,\, \mathcal{E}_1 \setminus \mathcal{E}_2)\end{array}$$

Then, by applying ZFC, conclude $T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1 \setminus \mathcal{E}_2) = T \in \mathsf{Edge}(p\,,\, \mathcal{E}_1) \setminus \mathsf{Edge}(p\,,\, \mathcal{E}_2)$.

(| QED. |)

B.10 Lemma 11

Proof (of Lemma 11). First, assume:

(A1) $k \in \mathbb{N}^+$

(A2) $p \in \mathsf{Port}(\mathcal{E})$

Now, prove the lemma by induction on k.

- **Base:** $k = 1$
  First, observe:

  (Z1) Recall $p \in \mathsf{Port}(\mathcal{E})$ from (A2). Then, by applying ZFC, conclude $p \in \{p\} \in \wp^+(\mathsf{Port}(\mathcal{E}))$.

  (Z2) Recall $p \in \mathsf{Port}(\mathcal{E})$ from (A2). Then, by applying Definition 13 of Port, conclude:

  $$\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \text{ and } p \in \mathbb{P}\text{ORT}$$

  Then, by applying Definition 22 of Edge, conclude:

  $$\mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E}) \text{ for some } \mathcal{T}$$

  (Z3) Suppose:

  $$p' \in \{p\} \text{ for some } p'$$

  Then, by introducing (Z2), conclude $\big[p' \in \{p\} \text{ and } \big[\mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E}) \text{ for some } \mathcal{T}\big]\big]$. Then, by applying standard inference rules, conclude:

  $$\big[p' \in \{p\} \text{ and } \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \text{ for some } \mathcal{T}$$

  Then, by applying ZFC, conclude $\big[p' = p \text{ and } \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big]$. Then, by applying substitution, conclude $\mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})$.

  (Z4) Recall $|\{p\}| = 1$ from ZFC. Then, by applying **Base**, conclude $|\{p\}| = k$. Then, by applying standard inference rules, conclude $\big[|\{p\}| = k \text{ or false}\big]$. Then, by applying standard inference rules, conclude $\big[|\{p\}| = k \text{ or } \big[\big[\mathcal{T}' = \mathsf{Edge}(p'\,,\,\mathcal{E}) \text{ implies } p' \in \{p\}\big] \text{ for all } \mathcal{T}'\,,\,p'\big]\big]$.

  Now, prove the base case by the following reduction. Suppose:

  $$P = \{p\} \text{ for some } P$$

  Then, by introducing (Z1)(Z2)(Z4), conclude:

  $$\begin{gathered} P = \{p\} \text{ and } p \in \{p\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \text{and } \big[\big[\big[p' \in \{p\} \text{ implies } \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \text{ for all } p'\big] \text{ for some } \mathcal{T}\big] \text{ and} \\ \big[|\{p\}| = k \text{ or } \big[\big[\mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E}) \text{ implies } p' \in \{p\}\big] \text{ for all } \mathcal{T}'\,,\,p'\big]\big] \end{gathered}$$

  Then, by applying standard inference rules, conclude:

  $$\left[\begin{gathered} P = \{p\} \text{ and } p \in \{p\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \text{and } \big[\big[p' \in \{p\} \text{ implies } \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \text{ for all } p'\big] \text{ and} \\ \big[|\{p\}| = k \text{ or } \big[\big[\mathcal{T}' = \mathsf{Edge}(p'\,,\,\mathcal{E}')\text{ implies } p' \in \{p\}\big] \text{ for all } \mathcal{T}'\,,\,p'\big]\big] \end{gathered}\right] \text{ for some } \mathcal{T}$$

  Then, by applying standard inference rules, conclude:

  $$\begin{gathered} P = \{p\} \text{ and } p \in \{p\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \text{and } \big[\big[p' \in \{p\} \text{ implies } \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \text{ for all } p'\big] \text{ and} \\ \big[|\{p\}| = k \text{ or } \big[\big[\mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E}) \text{ implies } p' \in \{p\}\big] \text{ for all } p'\big]\big] \end{gathered}$$

  Then, by applying substitution, conclude:

  $$\begin{gathered} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \text{and } \big[\big[p' \in P \text{ implies } \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \text{ for all } p'\big] \text{ and} \\ \big[|P| = k \text{ or } \big[\big[\mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E}) \text{ implies } p' \in P\big] \text{ for all } p'\big]\big] \end{gathered}$$

– **IH:**

$$\left[\begin{array}{c}\hat{k} < k \ \textbf{implies} \\ \left[\left[\begin{array}{c}\hat{p} \in \hat{P} \in \wp^{+}(\mathsf{Port}(\hat{\mathcal{E}})) \\ \textbf{and} \ \left[\left[\hat{p}' \in \hat{P} \ \textbf{implies} \ \hat{\mathcal{T}} = \mathsf{Edge}(\hat{p}'\,,\,\hat{\mathcal{E}})\right] \ \textbf{for all} \ \hat{p}'\right] \ \textbf{and} \\ \left[|\hat{P}| = \hat{k} \ \textbf{or} \ \left[\left[\hat{\mathcal{T}} = \mathsf{Edge}(\hat{p}'\,,\,\hat{\mathcal{E}}) \ \textbf{implies} \ \hat{p}' \in \hat{P}\right] \ \textbf{for all} \ \hat{p}'\right]\right]\end{array}\right]\right. \\ \textbf{for some} \ \hat{P}\,,\,\hat{\mathcal{T}}\left.\right]\end{array}\right] \ \textbf{for all} \ \hat{k}\,,\,\hat{p}\,,\,\hat{\mathcal{E}}$$

– **Step:** $k > 1$
First, observe:

(Y1) Suppose:

$$p \in P^{\dagger} \in \wp^{+}(\mathsf{Port}(\mathcal{E})) \ \textbf{for some} \ P^{\dagger}$$

Then, by applying ZFC, conclude $p \in \mathsf{Port}(\mathcal{E})$. Then, by applying Definition 21 of Port, conclude $\big[\mathcal{E} \in \wp^{2}(\mathbb{V}\text{ER}) \ \textbf{and} \ p \in \{p' \mid p' \in V \in E \in \mathcal{E}\}\big]$. Then, by applying ZFC, conclude:

$$p \in V \in E \in \mathcal{E} \in \wp^{2}(\mathbb{V}\text{ER}) \ \textbf{for some} \ V\,,\,E$$

Then, by applying Lemma 9:1, conclude $E \setminus \{V\} \in \mathsf{Edge}(p\,,\,\mathcal{E})$. Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}) \neq \emptyset$.

(Y2) Suppose:

$$\big[p \in P^{\dagger} \in \wp^{+}(\mathsf{Port}(\mathcal{E})) \ \textbf{and} \ \big[\big[p' \in P^{\dagger} \ \textbf{implies} \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \ \textbf{for all} \ p'\big]\big] \ \textbf{for some} \ P^{\dagger}\,,\,\mathcal{T}$$

Then, by applying standard inference rules, conclude:

$$p \in P^{\dagger} \in \wp^{+}(\mathsf{Port}(\mathcal{E})) \ \textbf{and} \ \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by applying (Y1), conclude $\big[\mathsf{Edge}(p\,,\,\mathcal{E}) \neq \emptyset \ \textbf{and} \ \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big]$. Then, by applying substitution, conclude $\mathcal{T} \neq \emptyset$.

(Y3) Suppose:

$$\mathcal{T} = \mathsf{Edge}(p^{\dagger}\,,\,\mathcal{E}) \ \textbf{for some} \ p^{\dagger}\,,\,\mathcal{E}$$

Then, by applying standard inference rules, conclude:

$$\big[p' = p^{\dagger} \ \textbf{implies} \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \ \textbf{for all} \ p'$$

Then, by applying ZFC, conclude $\big[\big[p' \in \{p^{\dagger}\} \ \textbf{implies} \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \ \textbf{for all} \ p'\big]$.

(Y4) Suppose:

$$\big[\textbf{not} \ \big[\big[\mathcal{T} = \mathsf{Edge}(p^{\dagger}\,,\,\mathcal{E}) \ \textbf{implies} \ p^{\dagger} \in P^{\dagger}\big] \ \textbf{for all} \ p^{\dagger}\big]\big] \ \textbf{for some} \ P^{\dagger}\,,\,\mathcal{T}$$

Then, by applying standard inference rules, conclude:

$$\big[\textbf{not} \ \big[\big[\mathcal{T} = \mathsf{Edge}(p^{\dagger}\,,\,\mathcal{E})\big] \ \textbf{implies} \ p^{\dagger} \in P^{\dagger}\big]\big] \ \textbf{for some} \ p^{\dagger}$$

Then, by applying standard inference rules, conclude $\big[\mathcal{T} = \mathsf{Edge}(p^{\dagger}\,,\,\mathcal{E}) \ \textbf{and} \ p^{\dagger} \notin P^{\dagger}\big]$. Then, by applying (Y3), conclude:

$$\mathcal{T} = \mathsf{Edge}(p^{\dagger}\,,\,\mathcal{E}) \ \textbf{and} \ p^{\dagger} \notin P^{\dagger} \ \textbf{and} \ \big[\big[p' \in \{p^{\dagger}\} \ \textbf{implies} \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \ \textbf{for all} \ p'\big]$$

(Y5) Suppose:

$$\{\mathcal{W} \mid \mathcal{W} = E \setminus \{V\} \ \textbf{and} \ p^{\dagger} \in V \in E \in \mathcal{E}\} \neq \emptyset \ \textbf{for some} \ p^{\dagger}$$

Then, by applying ZFC, conclude:

$$p^{\dagger} \in V \in E \in \mathcal{E} \ \textbf{for some} \ V\,,\,E$$

Then, by applying ZFC, conclude $p^{\dagger} \in \{p^{\ddagger} \mid p^{\ddagger} \in V' \in E' \in \mathcal{E}\}$.

(Y6) Suppose:

$$\big[\mathcal{T} \neq \emptyset \ \textbf{and}\ \mathcal{T} = \mathsf{Edge}(p^\dagger, \mathcal{E})\big] \ \textbf{for some}\ \mathcal{T}, p^\dagger$$

Then, by applying substitution, conclude $\mathsf{Edge}(p^\dagger, \mathcal{E}) \neq \emptyset$. Then, by applying Definition 22 of Edge, conclude $\big[\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}\ \{\mathcal{W} \mid \mathcal{W} = E \setminus \{V\} \ \textbf{and}\ p^\dagger \in V \in E \in \mathcal{E}\} \neq \emptyset\big]$. Then, by applying (Y5), conclude $\big[\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}\ p^\dagger \in \{p^\ddagger \mid p^\ddagger \in V' \in E' \in \mathcal{E}\}\big]$. Then, by applying Definition 21 of Port, conclude $p^\dagger \in \mathsf{Port}(\mathcal{E})$. Then, by applying ZFC, conclude:

$$\{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E}))$$

(Y7) Suppose:

$$\big[p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{and}\ \mathcal{T} \neq \emptyset \ \textbf{and}\ \mathcal{T} = \mathsf{Edge}(p^\dagger, \mathcal{E})\big] \ \textbf{for some}\ P^\dagger, \mathcal{T}, p^\dagger$$

Then, by applying (Y6), conclude $\big[p \in P^\dagger \ \textbf{and}\ P^\dagger, \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E}))\big]$. Then, by applying ZFC, conclude $\big[p \in P^\dagger \cup \{p^\dagger\} \ \textbf{and}\ P^\dagger, \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E}))\big]$. Then, by applying ZFC, conclude $p \in P^\dagger \cup \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E}))$.

(Y8) Suppose:

$$\left[\begin{array}{l}\big[\big[p' \in P^\dagger \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'\big]\\ \big[\big[p' \in \{p^\dagger\} \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'\big]\end{array}\right] \ \textbf{for some}\ P^\dagger, p^\dagger$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{rl} & \big[p' \in P^\dagger \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big]\\ \textbf{and} & \big[p' \in \{p^\dagger\} \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big]\end{array}\right] \ \textbf{for all}\ p'$$

Then, by applying standard inference rules, conclude:

$$\big[\big[p' \in P^\dagger \ \textbf{or}\ p' \in \{p^\dagger\}\big] \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'$$

Then, by applying ZFC, conclude $\big[\big[\big[p' \in P^\dagger \cup \{p^\dagger\}\big] \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'\big]$.

(Y9) Suppose:

$$\big[|P^\dagger| = k - 1 \ \textbf{and}\ p^\dagger \notin P^\dagger\big] \ \textbf{for some}\ P^\dagger, p^\dagger$$

Then, by applying ZFC, conclude $\big[|P^\dagger| = k - 1 \ \textbf{and}\ |P^\dagger \cup \{p^\dagger\}| = |P^\dagger| + 1\big]$. Then, by applying substitution, conclude $|P^\dagger \cup \{p^\dagger\}| = k - 1 + 1$. Then, by applying PA, conclude $|P^\dagger \cup \{p^\dagger\}| = k$.

(Y0) Suppose:

$$\left[\begin{array}{l} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{and}\ \big[\big[p' \in P^\dagger \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'\big]\\ \textbf{and}\ |P^\dagger| = k - 1 \ \textbf{and}\ \big[\textbf{not}\ \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \ \textbf{implies}\ p' \in P^\dagger\big] \ \textbf{for all}\ p'\big]\big]\end{array}\right] \\ \textbf{for some}\ P^\dagger, \mathcal{T}$$

Then, by applying (Y2), conclude:

$$\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{and}\ \big[\big[p' \in P^\dagger \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'\big]\\ \textbf{and}\ |P^\dagger| = k - 1 \ \textbf{and}\ \big[\textbf{not}\ \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \ \textbf{implies}\ p' \in P^\dagger\big] \ \textbf{for all}\ p'\big]\big]\\ \textbf{and}\ \mathcal{T} \neq \emptyset\end{array}$$

Then, by applying (Y4), conclude:

$$\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{and}\ \big[\big[p' \in P^\dagger \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'\big]\\ \textbf{and}\ |P^\dagger| = k - 1 \ \textbf{and}\ \mathcal{T} \neq \emptyset \ \textbf{and}\\ \Big[\left[\begin{array}{c}\mathcal{T} = \mathsf{Edge}(p^\dagger, \mathcal{E}) \ \textbf{and}\ p^\dagger \notin P^\dagger \ \textbf{and}\\ \big[\big[p' \in \{p^\dagger\} \ \textbf{implies}\ \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \ \textbf{for all}\ p'\big]\end{array}\right] \ \textbf{for some}\ p^\dagger\Big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ |P^\dagger| = k - 1 \ \textbf{ and } \ \mathcal{T} \neq \emptyset \ \textbf{ and } \ \mathcal{T} = \mathsf{Edge}(p^\dagger , \mathcal{E}) \ \textbf{ and } \ p^\dagger \notin P^\dagger \ \textbf{ and } \\ \big[\big[p' \in \{p^\dagger\} \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \end{array}\right] \ \textbf{for some} \ p^\dagger$$

Then, by applying (Y7), conclude:

$$\begin{array}{c} p \in P^\dagger \cup \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ |P^\dagger| = k - 1 \ \textbf{ and } \ p^\dagger \notin P^\dagger \ \textbf{ and } \ \big[\big[p' \in \{p^\dagger\} \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \end{array}$$

Then, by applying (Y8), conclude:

$$\begin{array}{c} p \in P^\dagger \cup \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \cup \{p^\dagger\} \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ |P^\dagger| = k - 1 \ \textbf{ and } \ p^\dagger \notin P^\dagger \end{array}$$

Then, by applying (Y9), conclude:

$$\begin{array}{c} p \in P^\dagger \cup \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \cup \{p^\dagger\} \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ |P^\dagger \cup \{p^\dagger\}| = k \end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c} p \in P^\dagger \cup \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \cup \{p^\dagger\} \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ |P^\dagger \cup \{p^\dagger\}| = k \ \textbf{ and } \ \big[P = P^\dagger \cup \{p^\dagger\} \ \textbf{ for some } \ P\big] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c} p \in P^\dagger \cup \{p^\dagger\} \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \cup \{p^\dagger\} \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ |P^\dagger \cup \{p^\dagger\}| = k \ \textbf{ and } \ P = P^\dagger \cup \{p^\dagger\} \end{array}\right]$$

$$\textbf{for some} \ P$$

Then, by applying substitution, conclude:

$$\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ |P| = k \end{array}$$

(X1) Suppose:

$$\left[\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ \big[\big[\mathcal{T} = \mathsf{Edge}(p' , \mathcal{E}) \ \textbf{ implies } \ p' \in P^\dagger\big] \ \textbf{ for all } \ p'\big] \end{array}\right] \ \textbf{for some} \ P^\dagger$$

Then, by applying ZFC, conclude:

$$\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ \big[\big[\mathcal{T} = \mathsf{Edge}(p' , \mathcal{E}) \ \textbf{ implies } \ p' \in P^\dagger\big] \ \textbf{ for all } \ p'\big] \ \textbf{ and } \ \big[P = P^\dagger \ \textbf{ for some } \ P\big] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P^\dagger \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ \big[\big[\mathcal{T} = \mathsf{Edge}(p' , \mathcal{E}) \ \textbf{ implies } \ p' \in P^\dagger\big] \ \textbf{ for all } \ p'\big] \ \textbf{ and } \ P = P^\dagger \end{array}\right] \ \textbf{for some} \ P$$

Then, by applying substitution, conclude:

$$\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \ \textbf{ and } \ \big[\big[p' \in P \ \textbf{ implies } \ \mathcal{T} = \mathsf{Edge}(p' , \mathcal{E})\big] \ \textbf{ for all } \ p'\big] \\ \textbf{ and } \ \big[\big[\mathcal{T} = \mathsf{Edge}(p' , \mathcal{E}) \ \textbf{ implies } \ p' \in P\big] \ \textbf{ for all } \ p'\big] \end{array}$$

Now, prove the inductive step by the following reduction. Recall $k > 1$ from **Step**. Then, by applying PA, conclude $k-1 < k$. Then, by applying **IH**, conclude:

$$\left[\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \textbf{and } \big[\big[p' \in P^\dagger \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \textbf{ and} \\ \big[|P^\dagger| = k-1 \textbf{ or } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P^\dagger\big] \textbf{ for all } p'\big]\big] \end{array}\right] \textbf{ for some } P^\dagger, \mathcal{T}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P^\dagger \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \textbf{ and} \\ \left[\begin{array}{c} \big[|P^\dagger| = k-1 \textbf{ and } \big[\textbf{not } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P^\dagger\big] \textbf{ for all } p'\big]\big]\big] \\ \textbf{or } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P^\dagger\big] \textbf{ for all } p'\big] \end{array}\right] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{l} \left[\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P^\dagger \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } |P^\dagger| = k-1 \textbf{ and } \big[\textbf{not } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P^\dagger\big] \textbf{ for all } p'\big]\big] \end{array}\right] \\ \textbf{or } \left[\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P^\dagger \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P^\dagger\big] \textbf{ for all } p'\big] \end{array}\right] \end{array}$$

Then, by applying (Y0), conclude:

$$\begin{array}{l} \left[\left[\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } |P| = k \end{array}\right] \textbf{ for some } P\right] \\ \textbf{or } \left[\begin{array}{c} p \in P^\dagger \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P^\dagger \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P^\dagger\big] \textbf{ for all } p'\big] \end{array}\right] \end{array}$$

Then, by applying (X1), conclude:

$$\begin{array}{l} \left[\left[\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } |P| = k \end{array}\right] \textbf{ for some } P\right] \\ \textbf{or } \left[\left[\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P\big] \textbf{ for all } p'\big] \end{array}\right] \textbf{ for some } P\right] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{l} \left[\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } |P| = k \end{array}\right] \\ \textbf{or } \left[\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[\big[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \\ \textbf{and } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P\big] \textbf{ for all } p'\big] \end{array}\right] \end{array}\right] \textbf{ for some } P$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \textbf{and } \big[\big[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})\big] \textbf{ for all } p'\big] \textbf{ and} \\ \big[|P| = k \textbf{ or } \big[\big[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P\big] \textbf{ for all } p'\big]\big] \end{array}$$

⦇ QED. ⦈

B.11 Lemma 12

Proof (of Lemma 12).

1. First, observe:

   (Z1) Suppose:

   $$P \in \bigstar(\mathcal{E}) \ \ \textbf{for some}\ \ P$$

   Then, by applying Definition 23 of $\bigstar$, conclude:

   $$P \in \{P' \mid P' \in \wp^{+}(\mathsf{Port}(\mathcal{E})) \ \ \textbf{and}\ \ \big[[p \in P' \ \ \textbf{iff}\ \ \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \ \ \textbf{for all}\ \ p\big]\}$$

   Then, by applying ZFC, conclude $P \in \wp^{+}(\mathsf{Port}(\mathcal{E}))$. Then, by applying ZFC, conclude:

   $$P \subseteq \mathsf{Port}(\mathcal{E})$$

   (Z2) Suppose:

   $$p \in \bigcup \bigstar(\mathcal{E}) \ \ \textbf{for some}\ \ p$$

   Then, by applying ZFC, conclude:

   $$p \in P \in \bigstar(\mathcal{E}) \ \ \textbf{for some}\ \ P$$

   Then, by applying (Z1), $\big[p \in P \ \ \textbf{and}\ \ P \subseteq \mathsf{Port}(\mathcal{E})\big]$. Then, by applying ZFC, conclude:

   $$p \in \mathsf{Port}(\mathcal{E})$$

   (Z3) Suppose:

   $$p \in \mathsf{Port}(\mathcal{E}) \ \ \textbf{for some}\ \ p$$

   Then, by applying ZFC, conclude $\mathsf{Port}(\mathcal{E}) \neq \emptyset$. Then, by applying ZFC, conclude $|\mathsf{Port}(\mathcal{E})| > 0$. Then, by applying PA, conclude $|\mathsf{Port}(\mathcal{E})| \in \mathbb{N}^{+}$. Then, by applying PA, conclude:

   $$|\mathsf{Port}(\mathcal{E})| + 1 \in \mathbb{N}^{+}$$

   (Z4) Suppose:

   $$\big[P \in \wp^{+}(\mathsf{Port}(\mathcal{E})) \ \ \textbf{and}\ \ |P| = |\mathsf{Port}(\mathcal{E})| + 1\big] \ \ \textbf{for some}\ \ P$$

   Then, by applying ZFC, conclude $\big[|P| \leq |\mathsf{Port}(\mathcal{E})| \ \ \textbf{and}\ \ |P| = |\mathsf{Port}(\mathcal{E})| + 1\big]$. Then, by applying substitution, conclude $|\mathsf{Port}(\mathcal{E})| + 1 \leq |\mathsf{Port}(\mathcal{E})|$. Then, by applying ZFC, conclude:

   $$|\mathsf{Port}(\mathcal{E})| < |\mathsf{Port}(\mathcal{E})|$$

   Then, by applying ZFC, conclude **false**.

   (Z5) Suppose:

   $$\begin{bmatrix} & \big[[p' \in P \ \ \textbf{implies}\ \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \ \ \textbf{for all}\ \ p'\big] \\ \textbf{and} & \big[[\mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E}) \ \ \textbf{implies}\ \ p' \in P\big] \ \ \textbf{for all}\ \ p'\big] \end{bmatrix} \ \ \textbf{for some}\ \ P$$

   Then, by applying standard inference rules, conclude:

   $$\begin{bmatrix} & [p' \in P \ \ \textbf{implies}\ \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})] \\ \textbf{and} & [\mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E}) \ \ \textbf{implies}\ \ p' \in P] \end{bmatrix} \ \ \textbf{for all}\ \ p'$$

   Then, by applying standard inference rules, conclude $\big[[p' \in P \ \ \textbf{iff}\ \ \mathcal{T} = \mathsf{Edge}(p'\,,\,\mathcal{E})\big] \ \ \textbf{for all}\ \ p'\big]$.

(Z6) Suppose:

$$p \in \mathsf{Port}(\mathcal{E}) \textbf{ for some } p$$

Then, by applying (Z3), conclude $\big[p \in \mathsf{Port}(\mathcal{E}) \textbf{ and } |\mathsf{Port}(\mathcal{E})|+1 \in \mathbb{N}^+\big]$. Then, by applying Lemma 11, conclude:

$$\left[\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \textbf{and } \big[[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})] \textbf{ for all } p'\big] \textbf{ and} \\ \big[|P| = |\mathsf{Port}(\mathcal{E})| + 1 \textbf{ or } \big[[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P] \textbf{ for all } p'\big]\big] \end{array}\right] \textbf{ for some } P, \mathcal{T}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and} \\ |P| = |\mathsf{Port}(\mathcal{E})| + 1 \end{bmatrix} \textbf{ or } \left[\begin{array}{l} \qquad p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \textbf{and } \big[[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})] \textbf{ for all } p'\big] \\ \textbf{and } \big[[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P] \textbf{ for all } p'\big] \end{array}\right]$$

Then, by applying (Z4), conclude:

$$\textbf{false or } \left[\begin{array}{l} \qquad p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \textbf{and } \big[[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})] \textbf{ for all } p'\big] \\ \textbf{and } \big[[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P] \textbf{ for all } p'\big] \end{array}\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{l} \qquad p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \\ \textbf{and } \big[[p' \in P \textbf{ implies } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})] \textbf{ for all } p'\big] \\ \textbf{and } \big[[\mathcal{T} = \mathsf{Edge}(p', \mathcal{E}) \textbf{ implies } p' \in P] \textbf{ for all } p'\big] \end{array}$$

Then, by applying (Z5), conclude:

$$p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})] \textbf{ for all } p'\big]$$

Then, by applying Definition 21 of Port, conclude:

$$\begin{array}{c} p \in P \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})] \textbf{ for all } p'\big] \\ \textbf{and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c} p \in P \in \{P' \mid P' \in \wp^+(\mathsf{Port}(\mathcal{E})) \textbf{ and } \big[[p' \in P' \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p', \mathcal{E})] \textbf{ for all } p'\big]\} \\ \textbf{and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \end{array}$$

Then, by applying Definition 23 of $\bigstar$, conclude $p \in P \in \bigstar(\mathcal{E})$. Then, by applying ZFC, conclude $p \in \bigcup \bigstar(\mathcal{E})$.

Now, prove the lemma by the following reduction. Recall from (Z2)(Z6):

$$\begin{array}{l} \big[[p \in \bigcup \bigstar(\mathcal{E}) \textbf{ implies } p \in \mathsf{Port}(\mathcal{E})] \textbf{ for all } p\big] \\ \textbf{and } \big[[p \in \mathsf{Port}(\mathcal{E}) \textbf{ implies } p \in \bigcup \bigstar(\mathcal{E})] \textbf{ for all } p\big] \end{array}$$

Then, by applying ZFC, conclude $\big[\bigcup \bigstar(\mathcal{E}) \subseteq \mathsf{Port}(\mathcal{E}) \textbf{ and } \mathsf{Port}(\mathcal{E}) \subseteq \bigcup \bigstar(\mathcal{E})\big]$. Then, by applying ZFC, conclude $\bigcup \bigstar(\mathcal{E}) = \mathsf{Port}(\mathcal{E})$.

(| QED. |)

2. First, assume:

(B1) $P_1 \neq P_2$

(B2) $P_1\,,\,P_2 \in \bigstar(\mathcal{E})$

Next, observe:

(Y1) Recall $P_1\,,\,P_2 \in \bigstar(\mathcal{E})$ from (B2) Then, by applying Definition 23 of $\bigstar$, conclude:

$$P_1\,,\,P_2 \in \{P \mid P \in \wp^+(\mathsf{Port}(\mathcal{E})) \;\textbf{and}\; \big[\big[p \in P \;\textbf{iff}\; \mathcal{T} = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big]\}$$

Then, by applying ZFC, conclude:

$$\begin{array}{l} \big[\big[\big[p \in P_1 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \;\textbf{for some}\; \mathcal{T}_1\big] \\ \textbf{and}\; \big[\big[\big[p \in P_2 \;\textbf{iff}\; \mathcal{T}_2 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \;\textbf{for some}\; \mathcal{T}_2\big] \end{array}$$

Now, prove the lemma by the following reduction. Suppose $P_1 \cap P_2 \neq \emptyset$. Then, by applying ZFC, conclude:

$$p \in P_1 \cap P_2 \;\textbf{for some}\; p$$

Then, by applying ZFC, conclude $\big[p \in P_1 \;\textbf{and}\; p \in P_2\big]$. Then, by introducing (Y1), conclude:

$$\begin{array}{c} p \in P_1 \;\textbf{and}\; p \in P_2 \textbf{and}\; \big[\big[\big[p \in P_1 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \;\textbf{for some}\; \mathcal{T}_1\big] \\ \textbf{and}\; \big[\big[\big[p \in P_2 \;\textbf{iff}\; \mathcal{T}_2 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \;\textbf{for some}\; \mathcal{T}_2\big] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{l} \qquad p \in P_1 \;\textbf{and}\; p \in P_2 \\ \textbf{and}\; \big[\big[p \in P_1 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \\ \textbf{and}\; \big[\big[p \in P_2 \;\textbf{iff}\; \mathcal{T}_2 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \end{array}\right] \;\textbf{for some}\; \mathcal{T}_1\,,\,\mathcal{T}_2$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{l} \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E}) \;\textbf{and}\; \mathcal{T}_2 = \mathsf{Edge}(p\,,\,\mathcal{E}) \\ \textbf{and}\; \big[\big[p \in P_1 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \\ \textbf{and}\; \big[\big[p \in P_2 \;\textbf{iff}\; \mathcal{T}_2 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \end{array}$$

Then, by applying substitution, conclude:

$$\begin{array}{l} \mathcal{T}_1 = \mathcal{T}_2 \\ \textbf{and}\; \big[\big[p \in P_1 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \\ \textbf{and}\; \big[\big[p \in P_2 \;\textbf{iff}\; \mathcal{T}_2 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \end{array}$$

Then, by applying substitution, conclude:

$$\begin{array}{l} \big[\big[p \in P_1 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \\ \textbf{and}\; \big[\big[p \in P_2 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{for all}\; p\big] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{l} \qquad \big[p \in P_1 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \\ \textbf{and}\; \big[p \in P_2 \;\textbf{iff}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \end{array}\right] \;\textbf{for all}\; p$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{l} \qquad \big[p \in P_1 \;\textbf{implies}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{and}\; \big[\mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E}) \;\textbf{implies}\; p \in P_1\big] \\ \textbf{and}\; \big[p \in P_2 \;\textbf{implies}\; \mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E})\big] \;\textbf{and}\; \big[\mathcal{T}_1 = \mathsf{Edge}(p\,,\,\mathcal{E}) \;\textbf{implies}\; p \in P_2\big] \end{array}\right] \;\textbf{for all}\; p$$

Then, by applying standard inference rules, conclude:

$$\big[\big[p \in P_1 \;\textbf{implies}\; p \in P_2\big] \;\textbf{and}\; \big[p \in P_2 \;\textbf{implies}\; p \in P_1\big]\big] \;\textbf{for all}\; p$$

Then, by applying standard inference rules, conclude:

$$\big[\big[p \in P_1 \;\textbf{implies}\; p \in P_2\big] \;\textbf{for all}\; p\big] \;\textbf{and}\; \big[\big[p \in P_2 \;\textbf{implies}\; p \in P_1\big] \;\textbf{for all}\; p\big]$$

Then, by applying ZFC, conclude $\big[P_1 \subseteq P_2 \;\textbf{and}\; P_2 \subseteq P_1\big]$. Then, by applying ZFC, conclude $P_1 = P_2$. Then, by introducing (B1), conclude $\big[P_1 \neq P_2 \;\textbf{and}\; P_1 = P_2\big]$. Then, by applying standard inference rules, conclude **false**.

⦇ QED. ⦈

B.12 Lemma 13

Proof (of Lemma 13).

1. First, assume:

   (A1) $\begin{bmatrix} (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix}$

   (A2) $\checkmark(\mathcal{E})$

   Next, observe:

   (Z1) Recall $\checkmark(\mathcal{E})$ from (A2). Then, by applying Definition 24 of $\checkmark$, conclude:

   $$\begin{gathered} \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and} \\ \left[\left[\begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{E} \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{E} \end{bmatrix} \textbf{ implies } V_1 = V_2\right] \textbf{ for all } p\,,\,V_1\,,\,V_2\,,\,E_1\,,\,E_2\right] \\ \textbf{and } \left[\left[V \in E \in \mathcal{E} \textbf{ implies } V \neq \emptyset\right] \textbf{ for all } V\,,\,E\right] \end{gathered}$$

   (Z2) Recall $(X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2)$ from (A1). Then, by applying Definition 19 of $\curlyvee$, conclude:

   $$Y \in \mathcal{E} \textbf{ and } (X\,,\,V_1) \curlyvee (Y\,,\,V_2)$$

   Then, by applying Lemma 6:1, conclude $V_2 \in Y \in \mathcal{E}$. Then, by applying (Z1), conclude:

   $$V_2 \in Y \in \mathcal{E} \textbf{ and } V_2 \neq \emptyset$$

   Then, by applying ZFC, conclude $\left[V_2 \in Y \in \mathcal{E} \textbf{ and } \left[p_2 \in V_2 \textbf{ for some } p_2\right]\right]$. Then, by applying standard inference rules, conclude:

   $$p_2 \in V_2 \in Y \in \mathcal{E} \textbf{ for some } p_2$$

   (Z3) Suppose

   $$p \in V \in E \in \mathcal{E} \textbf{ for some } p\,,\,V\,,\,E$$

   Then, by introducing (Z1), conclude $p \in V \in E \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $p \in V \in E \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $p \in V \in \mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $p \in V \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $p \in \mathbb{P}\text{ORT}$.

   (Z4) Suppose:

   $$p_1 \in V_1 \in E_1 \in \mathcal{E} \textbf{ for some } p_1\,,\,E_1$$

   Then, by applying ZFC, conclude $\left[\left[T = E \setminus \{V_1\} \textbf{ for some } T\right] \textbf{ and } p_1 \in V_1 \in E \in \mathcal{E}\right]$. Then, by applying standard inference rules, conclude:

   $$\left[T = E \setminus \{V_1\} \textbf{ and } p_1 \in V_1 \in E \in \mathcal{E}\right] \textbf{ for some } T$$

   Then, by introducing (Z1), conclude $\left[\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T = E \setminus \{V_1\} \textbf{ and } p_1 \in V_1 \in E \in \mathcal{E}\right]$. Then, by applying (Z3), conclude:

   $$p_1 \in \mathbb{P}\text{ORT} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T = E \setminus \{V_1\} \textbf{ and } p_1 \in V_1 \in E \in \mathcal{E}$$

   Then, by applying ZFC, conclude:

   $$\begin{gathered} p_1 \in \mathbb{P}\text{ORT} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T' = E' \setminus \{V_1\} \textbf{ and } p_1 \in V_1 \in E' \in \mathcal{E}\} \\ \textbf{and } T = E \setminus \{V_1\} \end{gathered}$$

   Then, by applying Definition 22 of Edge, conclude $\left[T \in \mathsf{Edge}(p_1\,,\,\mathcal{E}) \textbf{ and } T = E \setminus \{V_1\}\right]$.

(Z5) Suppose:

$$[p_1 \in V_1 \ \textbf{and} \ p_2 \in V_2] \ \textbf{for some} \ p_1 , p_2$$

Then, by applying ZFC, conclude $p_1 , p_2 \in V_1 \cup V_2$. Then, by introducing (A1), conclude:

$$V_1 \cup V_2 \subseteq P \ \textbf{and} \ P \in \bigstar(\mathcal{E}) \ \textbf{and} \ p_1 , p_2 \in V_1 \cup V_2$$

Then, by applying ZFC, conclude $p_1 , p_2 \in P \in \bigstar(\mathcal{E})$. Then, by applying Lemma 9:2, conclude $\mathsf{Edge}(p_1 , \mathcal{E}) = \mathsf{Edge}(p_2 , \mathcal{E})$.

(Z6) Suppose:

$$[T \in \mathsf{Edge}(p_2 , \mathcal{E}) \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E}] \ \textbf{for some} \ T , p_2$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E_2 \setminus \{V\} \ \textbf{and} \ p_2 \in V \in E_2 \in \mathcal{E}\} \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E}$$

Then, by applying ZFC, conclude:

$$[[T = E_2 \setminus \{V\} \ \textbf{and} \ p_2 \in V \in E_2 \in \mathcal{E}] \ \textbf{for some} \ E_2 , V] \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E}$$

Then, by applying standard inference rules, conclude:

$$[T = E_2 \setminus \{V\} \ \textbf{and} \ p_2 \in V \in E_2 \in \mathcal{E} \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E}] \ \textbf{for some} \ E_2 , V$$

Then, by applying (Z1), conclude $[V = V_2 \ \textbf{and} \ T = E_2 \setminus \{V\} \ \textbf{and} \ V \in E_2 \in \mathcal{E}]$. Then, by applying substitution, conclude $[T = E_2 \setminus \{V_2\} \ \textbf{and} \ V_2 \in E_2 \in \mathcal{E}]$.

(Z7) Recall $(X , V_1) \curlyvee_{\mathcal{E}} (Y , V_2)$ from (A1). Then, by applying Definition 19 of $\curlyvee$, conclude $V_1 , V_2 \in \mathbb{V}\text{ER}$.

(Z8) Suppose:

$$E_1 , E_2 \in \mathcal{E} \ \textbf{for some} \ E_1 , E_2$$

Then, by introducing (Z1), conclude $E_1 , E_2 \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $E_1 , E_2 \in \wp(\mathbb{V}\text{ER})$.

(Z9) Suppose:

$$[E_1 \setminus \{V_1\} = E_2 \setminus \{V_2\} \ \textbf{and} \ V_2 \in E_2 \in \mathcal{E} \ \textbf{and} \ V_1 \in E_1 \in \mathcal{E}] \ \textbf{for some} \ E_1 , E_2$$

Then, by applying ZFC, conclude:

$$E_1 \setminus \{V_1\} = E_2 \setminus \{V_2\} \ \textbf{and} \ E_2 = (E_2 \setminus \{V_2\}) \cup \{V_2\} \ \textbf{and} \ E_1 = (E_1 \setminus \{V_1\}) \cup \{V_1\}$$

Then, by applying substitution, conclude:

$$E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \ \textbf{and} \ E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\}$$

Then, by applying (Z7), conclude:

$$\begin{gathered} V_1 , V_2 \in \mathbb{V}\text{ER} \ \textbf{and} \\ E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \ \textbf{and} \ E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \end{gathered}$$

Then, by applying (Z8), conclude:

$$\begin{gathered} E_1 , E_2 \in \wp(\mathbb{V}\text{ER}) \ \textbf{and} \ V_1 , V_2 \in \mathbb{V}\text{ER} \ \textbf{and} \\ E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \ \textbf{and} \ E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \end{gathered}$$

Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1 , V_1) \curlyvee (E_2 , V_2)$.

(Z0) Recall $(X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2)$ from (A1). Then, by applying Lemma 7:1, conclude $V_1 \neq V_2$.

(Y1) Suppose:

$$(E_1, V_1) \curlyvee (E_2, V_2) \text{ for some } E_1, E_2, V_1, V_2$$

Then, by introducing (Z0), conclude $\big[(E_1, V_1) \curlyvee (E_2, V_2) \text{ and } V_1 \neq V_2\big]$. Then, by applying Lemma 6:2, conclude $E_1 \neq E_2$.

(Y2) Recall $(X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2)$ from (A1). Then, by applying Definition 19 of $\curlyvee$, conclude $V_1 \cap V_2 = \emptyset$.

(Y3) Suppose:

$$E_1 \in \{E_1' \mid V_1 \in E_1' \in \mathcal{E}\} \text{ for some } E_1$$

Then, by applying ZFC, conclude $V_1 \in E \in \mathcal{E}$. Then, by applying (Z1), conclude:

$$V_1 \neq \emptyset \text{ and } V_1 \in E_1 \in \mathcal{E}$$

Then, by applying ZFC, conclude $\big[\big[p_1 \in V_1 \text{ for some } p_1\big] \text{ and } V_1 \in E_1 \in \mathcal{E}\big]$. Then, by applying standard inference rules, conclude:

$$p_1 \in V_1 \in E \in \mathcal{E} \text{ for some } p_1$$

Then, by introducing (Z2), conclude $\big[\big[p_2 \in V_2 \in Y \in \mathcal{E} \text{ for some } p_2\big] \text{ and } p_1 \in V_1 \in E_1 \in \mathcal{E}\big]$. Then, by applying standard inference rules, conclude:

$$\big[p_2 \in V_2 \in Y \in \mathcal{E} \text{ and } p_1 \in V_1 \in E_1 \in \mathcal{E}\big] \text{ for some } p_2$$

Then, by applying (Z4), conclude:

$$\big[\big[T \in \mathsf{Edge}(p_1, \mathcal{E}) \text{ and } T = E_1 \setminus \{V_1\}\big] \text{ for some } T\big] \\ \text{and } p_2 \in V_2 \in Y \in \mathcal{E} \text{ and } V_1 \in E_1 \in \mathcal{E}$$

Then, by applying standard inference rules, conclude:

$$\big[T \in \mathsf{Edge}(p_1, \mathcal{E}) \text{ and } T = E_1 \setminus \{V_1\} \text{ and } p_2 \in V_2 \in Y \in \mathcal{E} \text{ and } V_1 \in E_1\big] \text{ for some } T$$

Then, by applying (Z5), conclude:

$$T \in \mathsf{Edge}(p_2, \mathcal{E}) \text{ and } T = E_1 \setminus \{V_1\} \text{ and } p_2 \in V_2 \in Y \in \mathcal{E} \text{ and } V_1 \in E_1 \in \mathcal{E}$$

Then, by applying (Z6), conclude:

$$\big[\big[T = E_2 \setminus \{V_2\} \text{ and } V_2 \in E_2 \in \mathcal{E}\big] \text{ for some } E_2\big] \text{ and } T = E_1 \setminus \{V_1\} \text{ and } V_1 \in E_1 \in \mathcal{E}$$

Then, by applying standard inference rules, conclude:

$$\big[T = E_2 \setminus \{V_2\} \text{ and } V_2 \in E_2 \in \mathcal{E} \text{ and } T = E_1 \setminus \{V_1\} \text{ and } V_1 \in E_1 \in \mathcal{E}\big] \text{ for some } E_2$$

Then, by applying substitution, conclude:

$$E_1 \setminus \{V_1\} = E_2 \setminus \{V_2\} \text{ and } V_2 \in E_2 \in \mathcal{E} \text{ and } V_1 \in E_1 \in \mathcal{E}$$

Then, by applying (Z9), conclude $\big[(E_1, V_1) \curlyvee (E_2, V_2) \text{ and } E_1, E_2 \in \mathcal{E}\big]$. Then, by applying (Y1), conclude $\big[E_1 \neq E_2 \text{ and } (E_1, V_1) \curlyvee (E_2, V_2) \text{ and } E_1, E_2 \in \mathcal{E}\big]$. Then, by introducing (Y2), conclude $\big[E_1 \neq E_2 \text{ and } (E_1, V_1) \curlyvee (E_2, V_2) \text{ and } E_1, E_2 \in \mathcal{E} \text{ and } V_1 \cap V_2 = \emptyset\big]$. Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$. Then, by applying ZFC, conclude:

$$E_1 \in \{E_1' \mid (E_1', V_1) \curlyvee_{\mathcal{E}} (E_2', V_2)\}$$

(Y4) Suppose:

$$E_1 \in \{E_1' \mid (E_1', V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\} \textbf{ for some } E_1$$

Then, by applying ZFC, conclude:

$$(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \textbf{ for some } E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $\big[E_1 \in \mathcal{E} \textbf{ and } (E_1, V_1) \curlyvee (E_2, V_2)\big]$. Then, by applying Lemma 6:1, conclude $V_1 \in E_1 \in \mathcal{E}$. Then, by applying ZFC, conclude:

$$E_1 \in \{E_1' \mid V_1 \in E_1' \in \mathcal{E}\}$$

Now, prove the lemma by the following reduction. Then, by introducing (Y3)(Y4), conclude:

$$\begin{array}{rl} & \big[\big[E \in \{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} \textbf{ implies } E \in \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\}\big] \textbf{ for all } E\big] \\ \textbf{and} & \big[\big[E \in \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\} \textbf{ implies } E \in \{E_1 \mid V_1 \in E_1 \in \mathcal{E}\}\big] \textbf{ for all } E\big] \end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{rl} & \{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} \subseteq \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\} \\ \textbf{and} & \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\} \subseteq \{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} \end{array}$$

Then, by applying ZFC, conclude $\{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\}$.

⦇ QED. ⦈

2. Prove the lemma by a reduction similar to Lemma 13:1.

⦇ QED. ⦈

3. First, assume:

(B1) $\begin{bmatrix} (X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix}$

(B2) $\checkmark(\mathcal{E})$

Next, observe:

(X1) Suppose:

$$V_1 \in E_1 \in \mathcal{E} \textbf{ for some } E_1$$

Then, by applying ZFC, conclude $E_1 \in \{E_1' \mid V_1 \in E_1' \in \mathcal{E}\}$. Then, by introducing (B2), conclude:

$$\checkmark(\mathcal{E}) \textbf{ and } E_1 \in \{E_1' \mid V_1 \in E_1' \in \mathcal{E}\}$$

Then, by introducing (B1), conclude:

$$\begin{bmatrix} (X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{ and } \checkmark(\mathcal{E}) \textbf{ and } E_1 \in \{E_1' \mid V_1 \in E_1' \in \mathcal{E}\}$$

Then, by applying 13:1, conclude $E_1 \in \{E_1' \mid (E_1', V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\}$. Then, by applying ZFC, conclude:

$$(E_1, V_1) \curlyvee (E_2, V_2) \textbf{ for some } E_2$$

(X2) Suppose:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\big] \textbf{ for some } E_\dagger, E_1, E_2$$

Then, by applying Lemma 7:2, conclude $\big[V_2 \notin E_1 \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})\big]$. Then, by applying ZFC, conclude $\big[V_2 \notin E_1 \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap (E_1 \setminus \{V_1\}))\big]$. Then, by applying ZFC, conclude $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}))$.

(X3) Suppose:

$$\begin{bmatrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\})) \\ \textbf{and } (E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \end{bmatrix} \textbf{ for some } E_\dagger\,,\, E_1\,,\, E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\})) \textbf{ and } (E_1\,,\, V_1) \curlyvee (E_2\,,\, V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}))$$

Then, by applying substitution, conclude $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$.

(X4) Suppose:

$$E_\dagger \in \{E'_\dagger \mid E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \textbf{ and } V_1 \in E_1 \in \mathcal{E}\} \textbf{ for some } E_\dagger$$

Then, by applying ZFC, conclude:

$$\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \textbf{ and } V_1 \in E_1 \in \mathcal{E}\right] \textbf{ for some } E_1$$

Then, by applying (X1), conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \textbf{ and } \left[(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \textbf{ for some } E_2\right]$$

Then, by applying standard inference rules, conclude:

$$\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2)\right] \textbf{ for some } E_2$$

Then, by applying (X2), conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\})) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2)$$

Then, by applying (X3), conclude $\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2)\right]$. Then, by applying ZFC, conclude:

$$E_\dagger \in \left\{E'_\dagger \,\middle|\, E'_\dagger = \{V_1 \cup V_2\} \cup (E'_1 \cap E'_2) \textbf{ and } (E'_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E'_2\,,\, V_2)\right\}$$

(X5) Suppose:

$$\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2)\right] \textbf{ for some } E_\dagger\,,\, E_1\,,\, E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\, V_1) \curlyvee (E_2\,,\, V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$$

Then, by applying substitution, conclude $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}))$.

(X6) Suppose:

$$\begin{bmatrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\})) \\ \textbf{and } (E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \end{bmatrix} \textbf{ for some } E_\dagger\,,\, E_1\,,\, E_2$$

Then, by applying Lemma 7:2, conclude:

$$V_2 \notin E_1 \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}))$$

Then, by applying ZFC, conclude $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap (E_1 \setminus \{V_1\}))$. Then, by applying ZFC, conclude $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})$.

(X7) Suppose:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $\left[E_1\in\mathcal{E}\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)\right]$. Then, by applying Lemma 6:1, conclude $V_1\in E_1\in\mathcal{E}$.

(X8) Suppose:

$$E_\dagger\in\{E'_\dagger\mid E'_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}\ \textbf{for some}\ E_\dagger$$

Then, by applying ZFC, conclude:

$$\left[E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\right]\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying (X5), conclude:

$$E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap((E_1\setminus\{V_1\})\cup\{V_2\}))\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)$$

Then, by applying (X6), conclude $\left[E_\dagger=\{V_1\cup V_2\}\cup(E_1\setminus\{V_1\})\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\right]$. Then, by applying (X7), conclude $\left[E_\dagger=\{V_1\cup V_2\}\cup(E_1\setminus\{V_1\})\ \textbf{and}\ V_1\in E_1\in\mathcal{E}\right]$. Then, by applying ZFC, conclude $E_\dagger\in\{E'_\dagger\mid E'_\dagger=\{V_1\cup V_2\}\cup(E'_1\setminus\{V_1\})\ \textbf{and}\ V_1\in E'_1\in\mathcal{E}\}$.

Now, prove the lemma by the following reduction. Recall from (X4)(X8):

$$\begin{aligned}&\left[\begin{bmatrix}E\in\left\{E_\dagger\,\middle|\,E_\dagger=\{V_1\cup V_2\}\cup(E_1\setminus\{V_1\})\ \textbf{and}\ V_1\in E_1\in\mathcal{E}\right\}\ \textbf{implies}\\ E\in\{E_\dagger\mid E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}\end{bmatrix}\ \textbf{for all}\ E\right]\\ \textbf{and}\ &\left[\begin{bmatrix}E\in\{E_\dagger\mid E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}\\ \textbf{implies}\ E\in\left\{E_\dagger\,\middle|\,E_\dagger=\{V_1\cup V_2\}\cup(E_1\setminus\{V_1\})\ \textbf{and}\ V_1\in E_1\in\mathcal{E}\right\}\end{bmatrix}\ \textbf{for all}\ E\right]\end{aligned}$$

Then, by applying ZFC, conclude:

$$\begin{aligned}&\left\{E_\dagger\,\middle|\,\begin{matrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\setminus\{V_1\})\\ \textbf{and}\ V_1\in E_1\in\mathcal{E}\end{matrix}\right\}\subseteq\left\{E_\dagger\,\middle|\,\begin{matrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\end{matrix}\right\}\\ \textbf{and}\ &\left\{E_\dagger\,\middle|\,\begin{matrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\end{matrix}\right\}\subseteq\left\{E_\dagger\,\middle|\,\begin{matrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\setminus\{V_1\})\\ \textbf{and}\ V_1\in E_1\in\mathcal{E}\end{matrix}\right\}\end{aligned}$$

Then, by applying ZFC, conclude:

$$\left\{E_\dagger\,\middle|\,\begin{matrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\setminus\{V_1\})\\ \textbf{and}\ V_1\in E_1\in\mathcal{E}\end{matrix}\right\}=\left\{E_\dagger\,\middle|\,\begin{matrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\end{matrix}\right\}$$

⦇ QED. ⦈

4. Prove the lemma by a reduction similar to Lemma 13:3.

⦇ QED. ⦈

5. First, assume:

(C1) $\begin{bmatrix}(X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)\ \textbf{and}\\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{E})\end{bmatrix}$

(C2) $\checkmark(\mathcal{E})$

(C3) $E\in\mathcal{E}$

Next, observe:

(W1) Recall $(X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)$ from (C1). Then, by applying Lemma 7:1, conclude $V_1\neq V_2$.

(W2) Recall $(X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)$ from (C1). Then, by applying Definition 19, conclude:

$$X\,,\,Y\in\mathcal{E}\ \ \textbf{and}\ \ (X\,,\,V_1)\curlyvee(Y\,,\,V_2)$$

Then, by applying Lemma 6:1, conclude $\big[V_1\in X\in\mathcal{E}\ \ \textbf{and}\ \ V_2\in Y\in\mathcal{E}\big]$.

(W3) Recall $P\in\bigstar(\mathcal{E})$ from (C1). Then, by applying Definition 23 of $\bigstar$, conclude $P\in\bigstar(\mathcal{E})\in\wp^2(\mathbb{P}\textsc{ort})$. Then, by applying ZFC, conclude $P\in\wp(\mathbb{P}\textsc{ort})$. Then, by introducing (C1), conclude:

$$V_1\cup V_2\subseteq P\ \ \textbf{and}\ \ P\in\wp(\mathbb{P}\textsc{ort})$$

Then, by applying ZFC, conclude $V_1\cup V_2\in\wp(\mathbb{P}\textsc{ort})$. Then, by applying ZFC, conclude:

$$V_1\,,\,V_2\in\wp(\mathbb{P}\textsc{ort})$$

Then, by applying ZFC, conclude:

$$\big[\big[p\in V_1\ \ \textbf{implies}\ \ p\in\mathbb{P}\textsc{ort}\big]\ \ \textbf{for all}\ \ p\big]\ \ \textbf{and}\ \ \big[\big[p\in V_2\ \ \textbf{implies}\ \ p\in\mathbb{P}\textsc{ort}\big]\ \ \textbf{for all}\ \ p\big]$$

(W4) Recall $\big[V_1\in X\in\mathcal{E}\ \ \textbf{and}\ \ V_2\in Y\in\mathcal{E}\big]$ from (W2). Then, by introducing (C2), conclude:

$$\checkmark(\mathcal{E})\ \ \textbf{and}\ \ V_1\in X\in\mathcal{E}\ \ \textbf{and}\ \ V_2\in Y\in\mathcal{E}$$

Then, by applying Definition 24 of $\checkmark$, conclude:

$$\big[\big[V\in E\in\mathcal{E}\ \ \textbf{implies}\ \ V\neq\emptyset\big]\ \ \textbf{for all}\ \ V\,,\,E\big]\ \ \textbf{and}\ \ V_1\in X\in\mathcal{E}\ \ \textbf{and}\ \ V_2\in Y\in\mathcal{E}$$

Then, by applying standard inference rules, conclude $V_1\,,\,V_2\neq\emptyset$. Then, by applying ZFC, conclude $\big[\big[p_1\in V_1\ \ \textbf{for some}\ \ p_1\big]\ \ \textbf{and}\ \ \big[p_2\in V_2\ \ \textbf{for some}\ \ p_2\big]\big]$. Then, by applying (W3), conclude:

$$\big[\big[p_1\in V_1\ \ \textbf{and}\ \ p_1\in\mathbb{P}\textsc{ort}\big]\ \ \textbf{for some}\ \ p_1\big]\ \ \textbf{and}\ \ \big[\big[p_2\in V_2\ \ \textbf{and}\ \ p_2\in\mathbb{P}\textsc{ort}\big]\ \ \textbf{for some}\ \ p_2\big]$$

(W5) Recall $P\in\bigstar(\mathcal{E})$ from (C1). Then, by applying Definition 23 of $\bigstar$, conclude $\mathcal{E}\in\wp^2(\mathbb{V}\textsc{er})$.

(W6) Suppose $V_1\,,\,V_2\in E$. Then, by applying ZFC, conclude:

$$\big[T_1=E\setminus\{V_1\}\ \ \textbf{for some}\ \ T_1\big]\ \ \textbf{and}\ \ V_1\,,\,V_2\in E$$

Then, by applying standard inference rules, conclude:

$$\big[T_1=E\setminus\{V_1\}\ \ \textbf{and}\ \ V_1\,,\,V_2\in E\big]\ \ \textbf{for some}\ \ T_1$$

Then, by introducing (W1), conclude $\big[V_1\neq V_2\ \ \textbf{and}\ \ T_1=E\setminus\{V_1\}\ \ \textbf{and}\ \ V_1\,,\,V_2\in E\big]$. Then, by applying ZFC, conclude $\big[V_2\in E\setminus\{V_1\}\ \ \textbf{and}\ \ T_1=E\setminus\{V_1\}\ \ \textbf{and}\ \ V_1\in E\big]$. Then, by applying substitution, conclude $\big[V_2\in T_1\ \ \textbf{and}\ \ T_1=E\setminus\{V_1\}\ \ \textbf{and}\ \ V_1\in E\big]$. Then, by introducing (W4), conclude:

$$\big[\big[p_1\in V_1\ \ \textbf{and}\ \ p_1\in\mathbb{P}\textsc{ort}\big]\ \ \textbf{for some}\ \ p_1\big]\ \ \textbf{and}\ \ V_2\in T_1\ \ \textbf{and}\ \ T_1=E\setminus\{V_1\}\ \ \textbf{and}\ \ V_1\in E$$

Then, by applying standard inference rules, conclude:

$$\big[p_1\in\mathbb{P}\textsc{ort}\ \ \textbf{and}\ \ V_2\in T_1\ \ \textbf{and}\ \ T_1=E\setminus\{V_1\}\ \ \textbf{and}\ \ p_1\in V_1\in E\big]\ \ \textbf{for some}\ \ p_1$$

Then, by introducing (C3), conclude:

$$p_1\in\mathbb{P}\textsc{ort}\ \ \textbf{and}\ \ V_2\in T_1\ \ \textbf{and}\ \ T_1=E\setminus\{V_1\}\ \ \textbf{and}\ \ p_1\in V_1\in E\in\mathcal{E}$$

Then, by applying ZFC, conclude:

$$p_1\in\mathbb{P}\textsc{ort}\ \ \textbf{and}\ \ V_2\in T_1\ \ \textbf{and}\ \ T_1\in\{T\mid T=E\setminus\{V\}\ \ \textbf{and}\ \ p_1\in V\in E\in\mathcal{E}\}\ \ \textbf{and}\ \ p_1\in V_1$$

Then, by introducing (W5), conclude:

$$\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}\ p_1 \in \mathbb{P}\text{ORT} \ \textbf{and}\ V_2 \in T_1 \ \textbf{and}$$
$$T_1 \in \{T \mid T = E \setminus \{V\} \ \textbf{and}\ p_1 \in V \in E \in \mathcal{E}\} \ \textbf{and}\ p_1 \in V_1$$

Then, by applying Definition 22 of Edge, conclude:

$$V_2 \in T_1 \ \textbf{and}\ T_1 \in \mathsf{Edge}(p_1\,,\,\mathcal{E}) \ \textbf{and}\ p_1 \in V_1$$

(W7) Recall from (W4):

$$\big[p_2 \in V_2 \ \textbf{and}\ p_2 \in \mathbb{P}\text{ORT}\big] \ \textbf{for some}\ p_2$$

Then, by introducing (W2), conclude $\big[p_2 \in V_2 \in Y \in \mathcal{E} \ \textbf{and}\ p_2 \in \mathbb{P}\text{ORT}\big]$. Then, by introducing (W5), conclude $\big[\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}\ p_2 \in V_2 \in Y \in \mathcal{E} \ \textbf{and}\ p_2 \in \mathbb{P}\text{ORT}\big]$. Then, by applying Definition 22 of Edge, conclude:

$$\mathsf{Edge}(p_2\,,\,\mathcal{E}) = \{T \mid T = E \setminus \{V\} \ \textbf{and}\ p_2 \in V \in E \in \mathcal{E}\} \ \textbf{and}\ p_2 \in V_2 \in Y \in \mathcal{E}$$

Then, by applying ZFC, conclude:

$$\Big[\big[T \in \mathsf{Edge}(p_2\,,\,\mathcal{E}) \ \textbf{implies}\ \big[\begin{bmatrix} T = E \setminus \{V\} \ \textbf{and} \\ p_2 \in V \in E \in \mathcal{E} \end{bmatrix} \ \textbf{for some}\ E\,,\,V\big]\big] \ \textbf{for all}\ T\Big]$$
$$\textbf{and}\ p_2 \in V_2 \in Y \in \mathcal{E}$$

Then, by introducing (C2), conclude

$$\checkmark(\mathcal{E}) \ \textbf{and}$$
$$\Big[\big[T \in \mathsf{Edge}(p_2\,,\,\mathcal{E}) \ \textbf{implies}\ \big[\begin{bmatrix} T = E \setminus \{V\} \ \textbf{and} \\ p_2 \in V \in E \in \mathcal{E} \end{bmatrix} \ \textbf{for some}\ E\,,\,V\big]\big] \ \textbf{for all}\ T\Big]$$
$$\textbf{and}\ p_2 \in V_2 \in Y \in \mathcal{E}$$

Then, by applying Definition 24 of ✓, conclude:

$$\Big[\big[\big[p \in W_1 \in E_1 \in \mathcal{E} \ \textbf{and}\ p \in W_2 \in E_2 \in \mathcal{E}\big] \ \textbf{implies}\ W_1 = W_2\big] \ \textbf{for all}\ p\,,\,W_1\,,\,W_2\,,\,E_1\,,\,E_2\Big]$$
$$\textbf{and}\ \Big[\big[T \in \mathsf{Edge}(p_2\,,\,\mathcal{E}) \ \textbf{implies}\ \big[\begin{bmatrix} T = E \setminus \{V\} \ \textbf{and} \\ p_2 \in V \in E \in \mathcal{E} \end{bmatrix} \ \textbf{for some}\ E\,,\,V\big]\big] \ \textbf{for all}\ T\Big]$$
$$\textbf{and}\ p_2 \in V_2 \in Y \in \mathcal{E}$$

Then, by applying standard inference rules, conclude:

$$\Big[\big[T \in \mathsf{Edge}(p_2\,,\,\mathcal{E}) \ \textbf{implies}\ \big[\big[V = V_2 \ \textbf{and}\ T = E \setminus \{V\}\big] \ \textbf{for some}\ E\,,\,V\big]\big] \ \textbf{for all}\ T\Big]$$
$$\textbf{and}\ p_2 \in V_2$$

Then, by applying substitution, conclude:

$$\Big[\big[T \in \mathsf{Edge}(p_2\,,\,\mathcal{E}) \ \textbf{implies}\ \big[T = E \setminus \{V_2\} \ \textbf{for some}\ E\big]\big] \ \textbf{for all}\ T\Big] \ \textbf{and}\ p_2 \in V_2$$

Then, by applying ZFC, conclude:

$$\big[\big[T \in \mathsf{Edge}(p_2\,,\,\mathcal{E}) \ \textbf{implies}\ V_2 \notin T\big] \ \textbf{for all}\ T\big] \ \textbf{and}\ p_2 \in V_2$$

Then, by applying standard inference rules, conclude:

$$\big[\big[V_2 \in T \ \textbf{implies}\ T \notin \mathsf{Edge}(p_2\,,\,\mathcal{E})\big] \ \textbf{for all}\ T\big] \ \textbf{and}\ p_2 \in V_2$$

(W8) Suppose:

$$\big[p_1 \in V_1 \;\; \textbf{and} \;\; p_2 \in V_2\big] \;\; \textbf{for some} \;\; p_1 \, , \, p_2$$

Then, by applying ZFC, conclude $p_1 \, , \, p_2 \in V_1 \cup V_2$. Then, by introducing (C1), conclude:

$$V_1 \cup V_2 \subseteq P \;\; \textbf{and} \;\; P \in \bigstar(\mathcal{E}) \;\; \textbf{and} \;\; p_1 \, , \, p_2 \in V_1 \cup V_2$$

Then, by applying ZFC, conclude $p_1 \, , \, p_2 \in P \in \bigstar(\mathcal{E})$. Then, by applying Lemma 9:2, conclude $\mathsf{Edge}(p_1 \, , \, \mathcal{E}) = \mathsf{Edge}(p_2 \, , \, \mathcal{E})$.

Now, prove the lemma by the following reduction. Suppose $V_1 \, , \, V_2 \in E$. Then, by applying (W6), conclude:

$$\big[V_2 \in T_1 \;\; \textbf{and} \;\; T_1 \in \mathsf{Edge}(p_1 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; p_1 \in V_1\big] \;\; \textbf{for some} \;\; T_1 \, , \, p_1$$

Then, by introducing (W7), conclude:

$$\begin{gathered}\Big[\Big[\Big[\big[V_2 \in T \;\; \textbf{implies} \;\; T \notin \mathsf{Edge}(p_2 \, , \, \mathcal{E})\big] \;\; \textbf{for all} \;\; T\Big] \;\; \textbf{and} \;\; p_2 \in V_2\Big] \;\; \textbf{for some} \;\; p_2\Big] \\ \textbf{and} \;\; V_2 \in T_1 \;\; \textbf{and} \;\; T_1 \in \mathsf{Edge}(p_1 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; p_1 \in V_1\end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{gathered}\Big[\big[V_2 \in T \;\; \textbf{implies} \;\; T \notin \mathsf{Edge}(p_2 \, , \, \mathcal{E})\big] \;\; \textbf{for all} \;\; T\Big] \;\; \textbf{and} \;\; p_2 \in V_2 \\ \textbf{and} \;\; V_2 \in T_1 \;\; \textbf{and} \;\; T_1 \in \mathsf{Edge}(p_1 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; p_1 \in V_1\end{gathered}\right] \;\; \textbf{for some} \;\; p_2$$

Then, by applying standard inference rules, conclude:

$$T_1 \notin \mathsf{Edge}(p_2 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; p_2 \in V_2 \;\; \textbf{and} \;\; T_1 \in \mathsf{Edge}(p_1 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; p_1 \in V_1$$

Then, by applying (W8), conclude:

$$\mathsf{Edge}(p_1 \, , \, \mathcal{E}) = \mathsf{Edge}(p_2 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; T_1 \notin \mathsf{Edge}(p_2 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; T_1 \in \mathsf{Edge}(p_1 \, , \, \mathcal{E})$$

Then, by applying substitution, conclude $\big[T_1 \notin \mathsf{Edge}(p_1 \, , \, \mathcal{E}) \;\; \textbf{and} \;\; T_1 \in \mathsf{Edge}(p_1 \, , \, \mathcal{E})\big]$. Then, by applying standard inference rules, conclude **false**.

(| QED. |)

6. First, assume:

(D1) $\begin{bmatrix}(X \, , \, V_1) \curlyvee_{\mathcal{E}} (Y \, , \, V_2) \;\; \textbf{and} \\ V_1 \cup V_2 \subseteq P \;\; \textbf{and} \;\; P \in \bigstar(\mathcal{E})\end{bmatrix}$

(D2) $\checkmark(\mathcal{E})$

(D3) $V_1 \in E_1 \in \mathcal{E}$

Next, observe:

(V1) Recall $\checkmark(\mathcal{E})$ from (D2). Then, by applying Definition 24 of $\checkmark$, conclude:

$$\begin{gathered}\mathcal{E} \in \wp^2(\mathbb{V}\mathrm{ER}) \;\; \textbf{and} \\ \Big[\Big[\begin{bmatrix} p \in V_1' \in E_1' \in \mathcal{E} \\ \textbf{and} \;\; p \in V_2' \in E_2' \in \mathcal{E}\end{bmatrix} \;\; \textbf{implies} \;\; V_1' = V_2'\Big] \;\; \textbf{for all} \;\; p \, , \, V_1' \, , \, V_2' \, , \, E_1' \, , \, E_2'\Big] \\ \textbf{and} \;\; \Big[\big[V \in E \in \mathcal{E} \;\; \textbf{implies} \;\; V \neq \emptyset\big] \;\; \textbf{for all} \;\; V \, , \, E\Big] \;\; \textbf{and}\end{gathered}$$

(V2) Recall $(X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2)$ from (D1). Then, by applying Definition 19 of $\curlyvee$, conclude:

$$X \in \mathcal{E} \ \textbf{and} \ (X, V_1) \curlyvee (Y, V_2)$$

Then, by applying Lemma 6:1, conclude $V_1 \in X \in \mathcal{E}$. Then, by applying (V1), conclude:

$$V_1 \in X \in \mathcal{E} \ \textbf{and} \ V_1 \neq \emptyset$$

Then, by applying ZFC, conclude $\big[V_1 \in X \in \mathcal{E} \ \textbf{and} \ \big[p_1 \in V_1 \ \textbf{for some} \ p_1\big]\big]$. Then, by applying standard inference rules, conclude:

$$p_1 \in V_1 \in X \in \mathcal{E} \ \textbf{for some} \ p_1$$

(V3) By a reduction similar to (V2), conclude:

$$p_2 \in V_2 \in Y \in \mathcal{E} \ \textbf{for some} \ p_2$$

(V4) Recall from (V2):

$$p_1 \in V_1 \ \textbf{for some} \ p_1$$

Then, by introducing (D3), conclude $p_1 \in V_1 \in E_1 \in \mathcal{E}$.

(V5) Suppose:

$$\big[p_1 \in V_1 \ \textbf{and} \ p_2 \in V_2\big] \ \textbf{for some} \ p_1, p_2$$

Then, by applying ZFC, conclude $p_1, p_2 \in V_1 \cup V_2$. Then, by introducing (D1), conclude:

$$p_1, p_2 \in V_1 \cup V_2 \ \textbf{and} \ V_1 \cup V_2 \subseteq P \ \textbf{and} \ P \in \bigstar(\mathcal{E})$$

Then, by applying ZFC, conclude $p_1, p_2 \in P \in \bigstar(\mathcal{E})$. Then, by applying Lemma 9:2, conclude $\mathsf{Edge}(p_1, \mathcal{E}) = \mathsf{Edge}(p_2, \mathcal{E})$.

(V6) Recall $(X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2)$ from (D1). Then, by applying Lemma 7:1, conclude $V_1 \neq V_2$.

(V7) Suppose:

$$\big[(E_1, V_1) \curlyvee (E_2, V_2') \ \textbf{and} \ V_2 = V_2'\big] \ \textbf{for some} \ E_2, V_2'$$

Then, by applying substitution, conclude $(E_1, V_1) \curlyvee (E_2, V_2)$. Then, by introducing (V6), conclude:

$$(E_1, V_1) \curlyvee (E_2, V_2) \ \textbf{and} \ V_1 \neq V_2$$

Then, by applying Lemma 6:2, conclude $\big[(E_1, V_1) \curlyvee (E_2, V_2) \ \textbf{and} \ E_1 \neq E_2\big]$.

(V8) Recall $(X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2)$ from (D1). Then, by applying Definition 19 of $\curlyvee$, conclude $V_1 \cap V_2 = \emptyset$.

Now, prove the lemma by the following reduction. Recall from (V4)(V3):

$$\big[p_1 \in V_1 \in E_1 \in \mathcal{E} \ \textbf{for some} \ p_1\big] \ \textbf{and} \ \big[p_2 \in V_2 \in Y \in \mathcal{E} \ \textbf{for some} \ p_2\big]$$

Then, by applying standard inference rules, conclude:

$$\big[p_1 \in V_1 \in E_1 \in \mathcal{E} \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E}\big] \ \textbf{for some} \ p_1, p_2$$

Then, by applying (V5), conclude:

$$p_1 \in V_1 \in E_1 \in \mathcal{E} \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E} \ \textbf{and} \ \mathsf{Edge}(p_1, \mathcal{E}) = \mathsf{Edge}(p_2, \mathcal{E})$$

Then, by applying Lemma 9:3, conclude:

$$E_1 \in \mathcal{E} \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E} \ \textbf{and} \ \Big[\begin{bmatrix} p_2 \in V_2' \in E_2 \in \mathcal{E} \ \textbf{and} \\ (E_1 , V_1) \curlyvee (E_2 , V_2') \end{bmatrix} \ \textbf{for some} \ V_2' , E_2\Big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} E_1 \in \mathcal{E} \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E} \ \textbf{and} \\ p_2 \in V_2' \in E_2 \in \mathcal{E} \ \textbf{and} \ (E_1 , V_1) \curlyvee (E_2 , V_2') \end{bmatrix} \ \textbf{for some} \ V_2' , E_2$$

Then, by applying (V1), conclude $\big[E_1 , E_2 \in \mathcal{E} \ \textbf{and} \ (E_1 , V_1) \curlyvee (E_2 , V_2') \ \textbf{and} \ V_2 = V_2'\big]$. Then, by applying (V7), conclude $\big[E_1 , E_2 \in \mathcal{E} \ \textbf{and} \ (E_1 , V_1) \curlyvee (E_2 , V_2) \ \textbf{and} \ E_1 \neq E_2\big]$. Then, by introducing (V8), conclude $\big[E_1 , E_2 \in \mathcal{E} \ \textbf{and} \ (E_1 , V_1) \curlyvee (E_2 , V_2) \ \textbf{and} \ E_1 \neq E_2 \ \textbf{and} \ V_1 \cap V_2 = \emptyset\big]$. Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)$.

(| QED. |)

7. Prove the lemma by a reduction similar to Lemma 13:6.

(| QED. |)

8. First, assume:

(E1) $\begin{bmatrix} (X , V_1) \curlyvee_{\mathcal{E}} (Y , V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and} \ P \in \bigstar(\mathcal{E}) \end{bmatrix}$

(E2) $\checkmark(\mathcal{E})$

(E3) $p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$

(E4) $V \notin \{V_1 , V_2\}$

(E5) $\mathsf{Edge}(p , \mathcal{E}) = \mathsf{Edge}(p' , \mathcal{E})$

(E6) $\mathcal{E}_1 = \{E_1 \mid (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)\}$

(E7) $\mathcal{E}_2 = \{E_2 \mid (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)\}$

Next, observe:

(U1) Recall $\mathcal{E}_1 = \big\{E_1 \,\big|\, (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)\big\}$ from (E6). Then, by applying Definition 19 of $\curlyvee$, conclude $\mathcal{E}_1 = \big\{E_1 \,\big|\, (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2) \ \textbf{and} \ E_1 \in \mathcal{E}\big\}$. Then, by applying ZFC, conclude $\mathcal{E}_1 \subseteq \mathcal{E}$.

(U2) By a reduction similar to (U1), conclude $\mathcal{E}_2 \subseteq \mathcal{E}$.

(U3) Recall $\mathcal{E}_1 , \mathcal{E}_2 \subseteq \mathcal{E}$ from (U1)(U2). Then, by applying ZFC, conclude $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E}$.

(U4) Recall $\checkmark(\mathcal{E})$ from (E2). Then, by applying Definition 24 of $\checkmark$, conclude $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$.

(U5) Recall $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$ from (U4). Then, by introducing (U3), conclude:

$$\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E} \ \textbf{and} \ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying ZFC, conclude $\mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER})$.

(U6) Recall $p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$ from (E3). Then, by introducing (U5), conclude:

$$p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying ZFC, conclude $p \in V \in E \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $p \in V \in \mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $p \in V \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $p \in \mathbb{P}\text{ORT}$.

(U7) Suppose:

$$E_1 \in \mathcal{E}_1 \;\; \textbf{for some} \;\; E_1$$

Then, by applying (E6), conclude $E_1 \in \{E_1' \mid (E_1' \,,\, V_1) \curlyvee_{\mathcal{E}} (E_2 \,,\, V_2)\}$. Then, by applying ZFC, conclude:

$$(E_1 \,,\, V_1) \curlyvee_{\mathcal{E}} (E_2 \,,\, V_2) \;\; \textbf{for some} \;\; E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1 \,,\, V_1) \curlyvee (E_2 \,,\, V_2)$. Then, by applying Lemma 6:1, conclude $V_1 \in E_1$. Then, by introducing (E4), conclude:

$$V \notin \{V_1 \,,\, V_2\} \;\; \textbf{and} \;\; V_1 \in E_1$$

Then, by applying ZFC, conclude $\big[V \neq V_1 \;\; \textbf{and} \;\; V_1 \in E_1\big]$. Then, by applying ZFC, conclude $V_1 \in E_1 \setminus \{V\}$.

(U8) Suppose:

$$E_2 \in \mathcal{E}_2 \;\; \textbf{for some} \;\; E_2$$

Then, by a reduction similar to (U7), conclude $V_2 \in E_2 \setminus \{V\}$.

(U9) Recall $E \in \mathcal{E}_1 \cup \mathcal{E}_2$ from (E3). Then, by applying ZFC, conclude $\big[E \in \mathcal{E}_1 \;\; \textbf{or} \;\; E \in \mathcal{E}_2\big]$. Then, by applying (U7)(U8), conclude $\big[V_1 \in E \setminus \{V\} \;\; \textbf{or} \;\; V_2 \in E \setminus \{V\}\big]$.

(U0) Recall $p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$ from (E3). Then, by applying ZFC, conclude:

$$\big[T = E \setminus \{V\} \;\; \textbf{for some} \;\; T\big] \;\; \textbf{and} \;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying standard inference rules, conclude:

$$\big[T = E \setminus \{V\} \;\; \textbf{and} \;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\; \textbf{for some} \;\; T$$

Then, by introducing (U3), conclude $\big[\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E} \;\; \textbf{and} \;\; T = E \setminus \{V\} \;\; \textbf{and} \;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big]$. Then, by applying ZFC, conclude $\big[T = E \setminus \{V\} \;\; \textbf{and} \;\; p \in V \in E \in \mathcal{E}\big]$. Then, by applying ZFC, conclude $\big[T \in \{T' \mid T' = E' \setminus \{V'\} \;\; \textbf{and} \;\; p \in V' \in E' \in \mathcal{E}\} \;\; \textbf{and} \;\; T = E \setminus \{V\}\big]$. Then, by applying substitution, conclude $E \setminus \{V\} \in \{T' \mid T' = E' \setminus \{V'\} \;\; \textbf{and} \;\; p \in V' \in E' \in \mathcal{E}\}$. Then, by introducing (U4), conclude:

$$\begin{gathered} \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \;\; \textbf{and} \\ E \setminus \{V\} \in \{T' \mid T' = E' \setminus \{V'\} \;\; \textbf{and} \;\; p \in V' \in E' \in \mathcal{E}\} \end{gathered}$$

Then, by introducing (U6), conclude:

$$\begin{gathered} p \in \mathbb{P}\text{ORT} \;\; \textbf{and} \;\; \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \;\; \textbf{and} \\ E \setminus \{V\} \in \{T' \mid T' = E' \setminus \{V'\} \;\; \textbf{and} \;\; p \in V' \in E' \in \mathcal{E}\} \end{gathered}$$

Then, by applying Definition 22 of Edge, conclude $E \setminus \{V\} \in \mathsf{Edge}(p \,,\, \mathcal{E})$. Then, by applying (E5), conclude $E \setminus \{V\} \in \mathsf{Edge}(p' \,,\, \mathcal{E})$. Then, by introducing (U9), conclude:

$$\big[V_1 \in E \setminus \{V\} \;\; \textbf{or} \;\; V_2 \in E \setminus \{V\}\big] \;\; \textbf{and} \;\; E \setminus \{V\} \in \mathsf{Edge}(p' \,,\, \mathcal{E})$$

Then, by applying standard inference rules, conclude:

$$V_1 \in E \setminus \{V\} \in \mathsf{Edge}(p' \,,\, \mathcal{E}) \;\; \textbf{or} \;\; V_2 \in E \setminus \{V\} \in \mathsf{Edge}(p' \,,\, \mathcal{E})$$

(T1) Recall $\checkmark(\mathcal{E})$ from (E2). Then, by introducing (E1), conclude:

$$\begin{bmatrix} X \,,\, Y \in \mathcal{E} \;\; \textbf{and} \;\; X \neq Y \\ \textbf{and} \;\; (X \,,\, V_1) \curlyvee (Y \,,\, V_2) \;\; \textbf{and} \\ V_1 \cup V_2 \subseteq P \;\; \textbf{and} \;\; P \in \bigstar(\mathcal{E}) \end{bmatrix} \;\; \textbf{and} \;\; \checkmark(\mathcal{E})$$

Then, by applying Lemma 13:1, conclude:

$$\{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} = \left\{ E_1 \,\middle|\, \begin{array}{l} E_1\,,\,E_2 \in \mathcal{E} \ \textbf{and}\ E_1 \neq E_2 \\ \textbf{and}\ (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2) \end{array} \right\}$$

Then, by applying (E6), conclude $\{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} = \mathcal{E}_1$.

(T2) By a reduction similar to (T1), conclude $\{E_2 \mid V_2 \in E_2 \in \mathcal{E}\} = \mathcal{E}_2$.

(T3) Suppose:

$$\big[V_1 \in E' \in \mathcal{E}\ \textbf{and}\ p' \in V' \in E'\big]\ \textbf{for some}\ E'\,,\,V'$$

Then, by applying ZFC, conclude $p' \in V' \in E' \in \{E_1 \mid V_1 \in E_1 \in \mathcal{E}\}$. Then, by applying (T1), conclude $p' \in V' \in E' \in \mathcal{E}_1$.

(T4) Suppose:

$$\big[V_2 \in E' \in \mathcal{E}\ \textbf{and}\ p' \in V' \in E'\big]\ \textbf{for some}\ E'\,,\,V'$$

Then, by a reduction similar to (T3), conclude $p' \in V' \in E' \in \mathcal{E}_2$.

(T5) Suppose:

$$V_1\,,\,V_2 \in E' \in \mathcal{E}\ \textbf{for some}\ E'$$

Then, by introducing (E2), conclude $\big[\checkmark(\mathcal{E})\ \textbf{and}\ V_1\,,\,V_2 \in E' \in \mathcal{E}\big]$. Then, by introducing (E1), conclude:

$$\begin{bmatrix} X\,,\,Y \in \mathcal{E}\ \textbf{and}\ X \neq Y \\ \textbf{and}\ (X\,,\,V_1) \curlyvee (Y\,,\,V_2)\ \textbf{and} \\ V_1 \cup V_2 \subseteq P\ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix}\ \textbf{and}\ \checkmark(\mathcal{E})\ \textbf{and}\ V_1\,,\,V_2 \in E' \in \mathcal{E}$$

Then, by applying Lemma 13:5, conclude $\big[\big[\textbf{not}\ V_1\,,\,V_2 \in E'\big]\ \textbf{and}\ V_1\,,\,V_2 \in E'\big]$. Then, by applying standard inference rules, conclude **false**.

(T6) Suppose:

$$\big[V_1 \in E' \setminus \{V'\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}\big]\ \textbf{for some}\ E'\,,\,V'$$

Then, by applying ZFC, conclude $\big[V_1 \in E' \in \mathcal{E}\ \textbf{and}\ p' \in V' \in E'\big]$. Then, by applying (T3), conclude $\big[V_1 \in E' \in \mathcal{E}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}_1\big]$. Then, by applying (T5), conclude:

$$V_2 \notin E'\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}_1$$

Then, by applying ZFC, conclude $\big[V_2 \neq V'\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}_1\big]$.

(T7) Suppose:

$$\big[V_2 \in E' \setminus \{V'\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}\big]\ \textbf{for some}\ E'\,,\,V'$$

Then, by a reduction similar to (T6), conclude $\big[V_1 \neq V'\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}_2\big]$.

(T8) Suppose $V_1 \in E \setminus \{V\} \in \mathsf{Edge}(p'\,,\,\mathcal{E})$. Then, by applying Definition 22 of Edge, conclude:

$$V_1 \in E \setminus \{V\} \in \{T' \mid T' = E' \setminus \{V'\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}\}$$

Then, by applying ZFC, conclude:

$$V_1 \in E \setminus \{V\}\ \textbf{and}\ \big[\big[T = E' \setminus \{V'\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}\big]\ \textbf{for some}\ E'\,,\,V'\big]$$

Then, by applying standard inference rules, conclude:

$$\big[V_1 \in E \setminus \{V\}\ \textbf{and}\ E \setminus \{V\} = E' \setminus \{V'\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}\big]\ \textbf{for some}\ E'\,,\,V'$$

Then, by applying substitution, conclude $\big[V_1 \in E' \setminus \{V'\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}\big]$. Then, by applying ZFC, conclude $\big[V_1 \neq V'\ \textbf{and}\ V_1 \in E' \setminus \{V'\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}\big]$. Then, by applying (T6), conclude $\big[V_1 \neq V'\ \textbf{and}\ V_2 \neq V'\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}_1\big]$. Then, by applying ZFC, conclude:

$$V' \notin \{V_1\,,\,V_2\}\ \textbf{and}\ p' \in V' \in E' \in \mathcal{E}_1$$

(T9) Suppose:

$$V_2 \in T \in \mathsf{Edge}(p' , \mathcal{E}) \ \textbf{for some} \ T$$

Then, by a reduction similar to (T8), conclude $\big[V' \notin \{V_1 , V_2\} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E}_2\big]$.

Now, prove the lemma by the following reduction. Recall from (U0):

$$V_1 \in E \setminus \{V\} \in \mathsf{Edge}(p' , \mathcal{E}) \ \textbf{or} \ V_2 \in E \setminus \{V\} \in \mathsf{Edge}(p' , \mathcal{E})$$

Then, by applying (T8)(T9), conclude:

$$\begin{bmatrix} & \big[[V' \notin \{V_1 , V_2\} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E}_1\big] \ \textbf{for some} \ V' , E'\big] \\ \textbf{or} & \big[[V' \notin \{V_1 , V_2\} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E}_2\big] \ \textbf{for some} \ V' , E'\big] \end{bmatrix}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} & [V' \notin \{V_1 , V_2\} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E}_1] \\ \textbf{or} & [V' \notin \{V_1 , V_2\} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E}_2] \end{bmatrix} \ \textbf{for some} \ V' , E'$$

Then, by applying standard inference rules, conclude:

$$V' \notin \{V_1 , V_2\} \ \textbf{and} \ \big[p' \in V' \in E' \in \mathcal{E}_1 \ \textbf{or} \ p' \in V' \in E' \in \mathcal{E}_2\big]$$

Then, by applying ZFC, conclude $\big[V' \notin \{V_1 , V_2\} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\big]$.

⦇ QED. ⦈

9. First, assume:

(F1) $\begin{bmatrix} (X , V_1) \curlyvee_{\mathcal{E}} (Y , V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and} \ P \in \bigstar(\mathcal{E}) \end{bmatrix}$

(F2) $\checkmark(\mathcal{E})$

(F3) $\mathcal{E}' = (\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger$

(F4) $\mathcal{E}_1 = \{E_1 \mid (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)\}$

(F5) $\mathcal{E}_2 = \{E_2 \mid (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)\}$

(F6) $\mathcal{E}_\dagger = \left\{ E_\dagger \,\middle|\, \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and} \ (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2) \end{matrix} \right\}$

Next, observe:

(S1) Suppose:

$$\big[p \in V \in E \ \textbf{and} \ V = V_1\big] \ \textbf{for some} \ p , V , E$$

Then, by applying substitution, conclude $p \in V_1$. Then, by applying ZFC, conclude $p \in V_1 \cup V_2$. Then, by applying ZFC, conclude $p \in V_1 \cup V_2 \in \{V_1 \cup V_2\}$. Then, by applying ZFC, conclude $p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\})$.

(S2) Suppose:

$$\big[p \in V \in E \ \textbf{and} \ V = V_2\big] \ \textbf{for some} \ p , V , E$$

Then, by a reduction similar to (S1), conclude $p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_2\})$.

(S3) Suppose:

$$\big[p \in V \in E \ \textbf{and} \ V \neq V_1\big] \ \textbf{for some} \ p , V , E$$

Then, by applying ZFC, conclude $p \in V \in E \setminus \{V_1\}$. Then, by applying ZFC, conclude:

$$p \in V \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\})$$

(S4) Suppose:

$$\big[p \in V \in E \textbf{ and } V \neq V_2\big] \textbf{ for some } p\,,\, V\,,\, E$$

Then, by a reduction similar to (S3), conclude $p \in V \in \{V_1 \cup V_2\} \cup (E \setminus \{V_2\})$.

(S5) Suppose:

$$p \in V \in E \textbf{ for some } p\,,\, V\,,\, E$$

Then, by applying standard inference rules, conclude $\big[p \in V \in E \textbf{ and } \textbf{true}\big]$. Then, by applying standard inference rules, conclude $\big[p \in V \in E \textbf{ and } \big[V = V_1 \textbf{ or } V \neq V_1\big]\big]$. Then, by applying standard inference rules, conclude $\big[\big[p \in V \in E \textbf{ and } V = V_1\big] \textbf{ or } \big[p \in V \in E \textbf{ and } V \neq V_1\big]\big]$. Then, by applying (S1), conclude:

$$p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\}) \textbf{ or } \big[p \in V \in E \textbf{ and } V \neq V_1\big]$$

Then, by applying (S3), conclude:

$$p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\}) \textbf{ or } p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\})$$

Then, by applying standard inference rules, conclude $p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\})$.

(S6) Suppose:

$$p \in V \in E \textbf{ for some } p\,,\, V\,,\, E$$

Then, by a reduction similar to (S5), conclude $p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_2\})$.

(S7) Recall $(X\,,\, V_1) \curlyvee_{\mathcal{E}} (Y\,,\, V_2)$ from (F1) Then, by applying Definition 19 of $\curlyvee$, conclude $V_1\,,\, V_2 \in \mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1\,,\, V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1 \cup V_2 \in \mathbb{V}\text{ER}$. Then, by applying ZFC, conclude $\{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})$

(S8) Suppose:

$$(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \textbf{ for some } E_1\,,\, E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $E_1\,,\, E_2 \in \wp(\mathbb{V}\text{ER})$. Then, by introducing (S7), conclude $E_1\,,\, E_2\,,\, \{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude:

$$\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})$$

(S9) Suppose:

$$(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \textbf{ for some } E_1\,,\, E_2$$

Then, by applying (S8), conclude $\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ for some } E_\dagger$$

(S0) Suppose:

$$E_1 \in \mathcal{E}_1 \textbf{ for some } E_1$$

Then, by applying (F4), conclude $E_1 \in \{E_1' \mid (E_1'\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2)\}$. Then, by applying ZFC, conclude:

$$(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \textbf{ for some } E_2$$

Then, by applying (S8), conclude:

$$(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \textbf{ and } \big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ for some } E_\dagger\big]$$

Then, by applying standard inference rules, conclude:

$$\big[(E_1\,,\, V_1) \curlyvee_{\mathcal{E}} (E_2\,,\, V_2) \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big] \textbf{ for some } E_\dagger$$

Then, by applying ZFC, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \ \textbf{and} \ \ E_\dagger \in \left\{ E'_\dagger \,\middle|\, \begin{array}{l} E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and} \ \ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2) \end{array} \right\}$$

Then, by applying (F6), conclude $\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \ \textbf{and} \ \ E_\dagger \in \mathcal{E}_\dagger\right]$. Then, by applying substitution, conclude $\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \mathcal{E}_\dagger$. Then, by applying ZFC, conclude:

$$\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in (\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger$$

Then, by applying (F3), conclude $\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \mathcal{E}'$.

(R1) Suppose:

$$E_2 \in \mathcal{E}_2 \ \ \textbf{for some} \ \ E_2$$

Then, by a reduction similar to (S0), conclude $\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \mathcal{E}'$.

(R2) Suppose:

$$E_1 \in \mathcal{E}_1 \ \ \textbf{for some} \ \ E_1$$

Then, by applying (F4), conclude $E_1 \in \{E'_1 \mid (E'_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E'_2\,,\,V_2)\}$. Then, by applying ZFC, conclude $(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)$. Then, by applying Definition 19 of $\curlyvee$, conclude $E_1 \in \wp(\mathbb{V}\text{ER})$.

(R3) Suppose:

$$E_2 \in \mathcal{E}_2 \ \ \textbf{for some} \ \ E_2$$

Then, by a reduction similar to (R2), conclude $E_2 \in \wp(\mathbb{V}\text{ER})$.

(R4) Suppose:

$$E_\dagger \in \mathcal{E}_\dagger \ \ \textbf{for some} \ \ E_\dagger$$

Then, by applying (F4), conclude:

$$E_\dagger \in \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \ \textbf{and} \ \ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}$$

Then, by applying ZFC, conclude:

$$\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \ \textbf{and} \ \ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\right] \ \ \textbf{for some} \ \ E_1\,,\,E_2$$

Then, by applying (S8), conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \ \textbf{and} \ \ \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})$$

Then, by applying substitution, conclude $E_\dagger \in \wp(\mathbb{V}\text{ER})$.

(R5) Recall $\checkmark(\mathcal{E})$ from (F2). Then, by applying Definition 24 of $\checkmark$, conclude $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$.

(R6) Recall from (R2)(R3):

$$\begin{array}{rl} & \left[\left[E_1 \in \mathcal{E}_1 \ \ \textbf{implies} \ \ E_1 \in \wp(\mathbb{V}\text{ER})\right] \ \ \textbf{for all} \ \ E_1\right] \\ \textbf{and} & \left[\left[E_2 \in \mathcal{E}_2 \ \ \textbf{implies} \ \ E_2 \in \wp(\mathbb{V}\text{ER})\right] \ \ \textbf{for all} \ \ E_2\right] \end{array}$$

Then, by introducing (R4), conclude:

$$\begin{array}{rl} & \left[\left[E_1 \in \mathcal{E}_1 \ \ \textbf{implies} \ \ E_1 \in \wp(\mathbb{V}\text{ER})\right] \ \ \textbf{for all} \ \ E_1\right] \\ \textbf{and} & \left[\left[E_2 \in \mathcal{E}_2 \ \ \textbf{implies} \ \ E_2 \in \wp(\mathbb{V}\text{ER})\right] \ \ \textbf{for all} \ \ E_2\right] \\ \textbf{and} & \left[\left[E_\dagger \in \mathcal{E}_\dagger \ \ \textbf{implies} \ \ E_\dagger \in \wp(\mathbb{V}\text{ER})\right] \ \ \textbf{for all} \ \ E_\dagger\right] \end{array}$$

Then, by applying ZFC, conclude $\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger \subseteq \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude:

$$\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER})$$

Then, by introducing (R5), conclude $\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger\,,\,\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $(\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying (F3), conclude $\mathcal{E}' \in \wp^2(\mathbb{V}\text{ER})$.

(R7) Suppose:

$$p \in V \in E \in \mathcal{E}_1 \ \textbf{for some} \ p, V, E$$

Then, by applying (S5), conclude $\left[p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\}) \ \textbf{and} \ E \in \mathcal{E}_1\right]$. Then, by applying (S0), conclude $p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E \setminus \{V_1\}) \in \mathcal{E}'$. Then, by applying ZFC, conclude $p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}'\}$. Then, by introducing (R6), conclude:

$$p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}'\} \ \textbf{and} \ \mathcal{E}' \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying Definition 21 of Port, conclude $p \in \mathsf{Port}(\mathcal{E}')$.

(R8) Suppose:

$$p \in V \in E \in \mathcal{E}_2 \ \textbf{for some} \ p, V, E$$

Then, by a reduction similar to (R7), conclude $p \in \mathsf{Port}(\mathcal{E}')$

(R9) Suppose:

$$p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{for some} \ p, V, E$$

Then, by applying ZFC, conclude $\left[p \in V \in E \in \mathcal{E} \ \textbf{and} \ \left[E \in \mathcal{E}_1 \ \textbf{or} \ E \in \mathcal{E}_2\right]\right]$. Then, by applying standard inference rules, conclude $\left[p \in V \in E \in \mathcal{E}_1 \ \textbf{or} \ p \in V \in E \in \mathcal{E}_2\right]$. Then, by applying (R7)(R8), conclude $\left[p \in \mathsf{Port}(\mathcal{E}') \ \textbf{and} \ p \in \mathsf{Port}(\mathcal{E}')\right]$. Then, by applying standard inference rules, conclude $p \in \mathsf{Port}(\mathcal{E}')$.

(R0) Suppose:

$$\left[p \in V \in E \in \mathcal{E} \ \textbf{and} \ E \notin \mathcal{E}_1 \cup \mathcal{E}_2\right] \ \textbf{for some} \ p, V, E$$

Then, by applying ZFC, conclude $p \in V \in E \in \mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)$. Then, by applying ZFC, conclude $p \in V \in E \in (\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger$. Then, by applying (F6), conclude $p \in V \in E \in \mathcal{E}'$. Then, by applying ZFC, conclude $p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}'\}$. Then, by introducing (R6), conclude:

$$p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}'\} \ \textbf{and} \ \mathcal{E}' \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying Definition 21 of Port, conclude $p \in \mathsf{Port}(\mathcal{E}')$.

(Q1) Suppose:

$$p \in \mathsf{Port}(\mathcal{E}) \ \textbf{for some} \ p$$

Then, by applying Definition 21 of Port, conclude $p \in \{p' \mid p' \in V \in E \in \mathcal{E}\}$. Then, by applying ZFC, conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{for some} \ V, E$$

Then, by applying standard inference rules, conclude $\left[p \in V \in E \in \mathcal{E} \ \textbf{and} \ \textbf{true}\right]$. Then, by applying standard inference rules, conclude $\left[p \in V \in E \in \mathcal{E} \ \textbf{and} \ \left[E \in \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{or} \ E \notin \mathcal{E}_1 \cup \mathcal{E}_2\right]\right]$. Then, by applying standard inference rules, conclude:

$$p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{or} \ \left[p \in V \in E \in \mathcal{E} \ \textbf{and} \ E \notin \mathcal{E}_1 \cup \mathcal{E}_2\right]$$

Then, by applying (R9), conclude $\left[p \in \mathsf{Port}(\mathcal{E}') \ \textbf{or} \ \left[p \in V \in E \in \mathcal{E} \ \textbf{and} \ E \notin \mathcal{E}_1 \cup \mathcal{E}_2\right]\right]$. Then, by applying (R0), conclude $\left[p \in \mathsf{Port}(\mathcal{E}') \ \textbf{or} \ p \in \mathsf{Port}(\mathcal{E}')\right]$. Then, by applying standard inference rules, conclude $p \in \mathsf{Port}(\mathcal{E}')$.

(Q2) Suppose:

$$p \in V \in E \in \mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2) \ \textbf{for some} \ p, V, E$$

Then, by applying ZFC, conclude $p \in V \in E \in \mathcal{E}$. Then, by applying ZFC, conclude:

$$p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}\}$$

Then, by introducing (R5), conclude $\left[p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}\} \ \textbf{and} \ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\right]$. Then, by applying Definition 21 of Port, conclude $p \in \mathsf{Port}(\mathcal{E})$.

(Q3) Suppose:

$$\big[p \in V \in E \ \textbf{and}\ E = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big] \ \textbf{for some}\ E_1\,,\, E_2$$

Then, by applying substitution, conclude $p \in V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$. Then, by applying ZFC, conclude $\big[p \in V \in \{V_1 \cup V_2\}\ \textbf{or}\ p \in V \in E_1 \cap E_2\big]$. Then, by applying ZFC, conclude:

$$p \in V \in \{V_1\}\ \textbf{or}\ p \in V \in \{V_2\}\ \textbf{or}\ p \in V \in E_1 \cap E_2$$

(Q4) Suppose:

$$p \in V \in \{V'\}\ \textbf{for some}\ p\,,\, V\,,\, V'$$

Then, by applying ZFC, conclude $\big[p \in V\ \textbf{and}\ V = V'\big]$. Then, by applying substitution, conclude $p \in V'$.

(Q5) Suppose:

$$(E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\ \textbf{for some}\ E_1\,,\, E_2$$

Then, by applying Definition 19 of Υ, conclude $\big[E_1\,,\, E_2 \in \mathcal{E}\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon\ (E_2\,,\, V_2)\big]$. Then, by applying Lemma 6:1, conclude $\big[V_1 \in E_1 \in \mathcal{E}\ \textbf{and}\ V_2 \in E_2 \in \mathcal{E}\big]$.

(Q6) Suppose:

$$\big[p \in V \in \{V_1\}\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\big]\ \textbf{for some}\ p\,,\, V\,,\, E_1\,,\, E_2$$

Then, by applying (Q4), conclude $\big[p \in V_1\ \textbf{and}\ (E_1\,,\, V_1)\,\Upsilon_{\mathcal{E}}\,(E_2\,,\, V_2)\big]$. Then, by applying (Q5), conclude $p \in V_1 \in E_1 \in \mathcal{E}$. Then, by applying ZFC, conclude $p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}\}$. Then, by introducing (R5), conclude $\big[p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}\}\ \textbf{and}\ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\big]$. Then, by applying Definition 21 of Port, conclude $p \in \mathsf{Port}(\mathcal{E})$.

(Q7) Suppose:

$$\big[p \in V \in \{V_2\}\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\big]\ \textbf{for some}\ p\,,\, V\,,\, E_1\,,\, E_2$$

Then, by a reduction similar to (Q6), conclude $p \in \mathsf{Port}(\mathcal{E})$.

(Q8) Suppose:

$$\big[p \in V \in E_1 \cap E_2\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\big]\ \textbf{for some}\ p\,,\, V\,,\, E_1\,,\, E_2$$

Then, by applying ZFC, conclude $\big[p \in V \in E_1\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\big]$. Then, by introducing (Q5), conclude $p \in V \in E_1 \in \mathcal{E}$. Then, by applying ZFC, conclude $p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}\}$. Then, by introducing (R5), conclude $\big[p \in \{p' \mid p' \in V' \in E' \in \mathcal{E}\}\ \textbf{and}\ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\big]$. Then, by applying Definition 21 of Port, conclude $p \in \mathsf{Port}(\mathcal{E})$.

(Q9) Suppose:

$$p \in V \in E \in \mathcal{E}_\dagger\ \textbf{for some}\ p\,,\, V\,,\, E$$

Then, by applying (F6), conclude

$$p \in V \in E \in \{\mathcal{E}_\dagger \mid \mathcal{E}_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\}$$

Then, by applying ZFC, conclude:

$$p \in V \in E\ \textbf{and}\ \big[\big[E = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\big]\ \textbf{for some}\ E_1\,,\, E_2\big]$$

Then, by applying standard inference rules, conclude:

$$\big[p \in V \in E\ \textbf{and}\ E = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)\big]\ \textbf{for some}\ E_1\,,\, E_2$$

Then, by applying (Q3), conclude:

$$\big[p \in V \in \{V_1\}\ \textbf{or}\ p \in V \in \{V_2\}\ \textbf{or}\ p \in V \in E_1 \cap E_2\big]\ \textbf{and}\ (E_1\,,\, V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\, V_2)$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{rl} & [p \in V \in \{V_1\} \ \textbf{and} \ (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)] \\ \textbf{or} & [p \in V \in \{V_2\} \ \textbf{and} \ (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)] \\ \textbf{or} & [p \in V \in E_1 \cap E_2 \ \textbf{and} \ (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)] \end{array}$$

Then, by applying (Q6)(Q7), conclude:

$$p \in \mathsf{Port}(\mathcal{E}) \ \textbf{and} \ p \in \mathsf{Port}(\mathcal{E}) \ \textbf{and} \ [p \in V \in E_1 \cap E_2 \ \textbf{and} \ (E_1 , V_1) \curlyvee_{\mathcal{E}} (E_2 , V_2)]$$

Then, by applying (Q8), conclude $[p \in \mathsf{Port}(\mathcal{E}) \ \textbf{and} \ p \in \mathsf{Port}(\mathcal{E}) \ \textbf{and} \ p \in \mathsf{Port}(\mathcal{E})]$. Then, by applying standard inference rules, conclude $p \in \mathsf{Port}(\mathcal{E})$.

(Q0) Suppose:

$$p \in \mathsf{Port}(\mathcal{E}') \ \textbf{for some} \ p$$

Then, by applying Definition 21 of Port, conclude $p \in \{p' \mid p' \in V \in E \in \mathcal{E}'\}$. Then, by applying ZFC, conclude:

$$p \in V \in E \in \mathcal{E}' \ \textbf{for some} \ V , E$$

Then, by applying (F3), conclude $p \in V \in E \in (\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger$. Then, by applying ZFC, conclude $[p \in V \in E \in \mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2) \ \textbf{or} \ p \in V \in E \in \mathcal{E}_\dagger]$. Then, by applying (Q2), conclude:

$$p \in \mathsf{Port}(\mathcal{E}) \ \textbf{or} \ p \in V \in E \in \mathcal{E}_\dagger$$

Then, by applying (Q9), conclude $[p \in \mathsf{Port}(\mathcal{E}) \ \textbf{or} \ p \in \mathsf{Port}(\mathcal{E})]$. Then, by applying standard inference rules, conclude $p \in \mathsf{Port}(\mathcal{E})$.

Now, prove the lemma by the following reduction. Recall from (Q1)(Q0):

$$\begin{array}{rl} & [[p \in \mathsf{Port}(\mathcal{E}) \ \textbf{implies} \ p \in \mathsf{Port}(\mathcal{E}')] \ \textbf{for all} \ p] \\ \textbf{and} & [[p \in \mathsf{Port}(\mathcal{E}') \ \textbf{implies} \ p \in \mathsf{Port}(\mathcal{E})] \ \textbf{for all} \ p] \end{array}$$

Then, by applying ZFC, conclude $[\mathsf{Port}(\mathcal{E}) \subseteq \mathsf{Port}(\mathcal{E}') \ \textbf{and} \ \mathsf{Port}(\mathcal{E}') \subseteq \mathsf{Port}(\mathcal{E})]$. Then, by applying ZFC, conclude $\mathsf{Port}(\mathcal{E}) = \mathsf{Port}(\mathcal{E}')$.

⦇ QED. ⦈

B.13 Lemma 14

Proof (of Lemma 14).

1. First, assume:

(A1) $\begin{bmatrix} (X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)\ \textbf{and} \\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{E}) \end{bmatrix}$

(A2) $\checkmark(\mathcal{E})$

(A3) $p\in V\in E\in\mathcal{E}_1\cup\mathcal{E}_2$

(A4) $V\in\{V_1\,,\,V_2\}$

(A5) $\mathcal{E}_1=\{E_1\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}$

(A6) $\mathcal{E}_2=\{E_2\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}$

(A7) $\mathcal{E}_\dagger=\left\{E_\dagger\ \middle|\ \begin{matrix} E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2) \\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\end{matrix}\right\}$

Next, observe:

(Z1) Recall $\mathcal{E}_1=\{E_1\,|\,(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}$ from (A5). Then, by applying Definition 19 of $\curlyvee$, conclude $\mathcal{E}_1=\{E_1\,|\,(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ E_1\in\mathcal{E}\}$. Then, by applying ZFC, conclude $\mathcal{E}_1\subseteq\mathcal{E}$.

(Z2) By a reduction similar to (Z1), conclude $\mathcal{E}_2\subseteq\mathcal{E}$.

(Z3) Recall $\mathcal{E}_1\,,\,\mathcal{E}_2\subseteq\mathcal{E}$ from (Z1)(Z2). Then, by applying ZFC, conclude $\mathcal{E}_1\cup\mathcal{E}_2\subseteq\mathcal{E}$.

(Z4) Recall $p\in V\in E\in\mathcal{E}_1\cup\mathcal{E}_2$ from (A3). Then, by introducing (Z3), conclude:

$$p\in V\in E\in\mathcal{E}_1\cup\mathcal{E}_2\ \textbf{and}\ \mathcal{E}_1\cup\mathcal{E}_2\subseteq\mathcal{E}$$

Then, by applying ZFC, conclude $p\in V\in E\in\mathcal{E}$.

(Z5) Recall $\checkmark(\mathcal{E})$ from (A2). Then, by applying Definition 24 of $\checkmark$, conclude:

$$\begin{gathered}\mathcal{E}\in\wp^2(\mathbb{V}\text{ER})\ \textbf{and} \\ \left[\left[\begin{bmatrix} p'\in V_1'\in E_1\in\mathcal{E} \\ \textbf{and}\ p'\in V_2'\in E_2\in\mathcal{E}\end{bmatrix}\ \textbf{implies}\ V_1'=V_2'\right]\ \textbf{for all}\ p'\,,\,V_1'\,,\,V_2'\,,\,E_1\,,\,E_2\right] \\ \textbf{and}\ \left[\left[V\in E\in\mathcal{E}\ \textbf{implies}\ V\neq\emptyset\right]\ \textbf{for all}\ V\,,\,E\right]\end{gathered}$$

(Z6) Suppose:

$$\left[p\in V_1\ \textbf{and}\ p\in V_1\cup V_2\in E_1\cap E_2\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\right]\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying ZFC, conclude:

$$p\in V_1\ \textbf{and}\ p\in V_1\cup V_2\in E_1\in\mathcal{E}\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$p\in V_1\ \textbf{and}\ p\in V_1\cup V_2\in E_1\in\mathcal{E}\ \textbf{and}\ E_1\,,\,E_2\in\mathcal{E}\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$$

Then, by applying Lemma 6:1, conclude:

$$V_2\in E_2\in\mathcal{E}\ \textbf{and}\ p\in V_1\in E_1\in\mathcal{E}\ \textbf{and}\ p\in V_1\cup V_2\in E_1\in\mathcal{E}$$

Then, by applying (Z5), conclude $\left[V_1=V_1\cup V_2\ \textbf{and}\ V_2\in E_2\in\mathcal{E}\right]$. Then, by applying (Z5), conclude $\left[V_1=V_1\cup V_2\ \textbf{and}\ V_2\neq\emptyset\right]$. Then, by applying ZFC, conclude:

$$V_1=V_1\cup V_2\ \textbf{and}\ V_1\neq V_1\cup V_2$$

Then, by applying standard inference rules, conclude **false**.

(Z7) Suppose:

$$\big[p \in V_1 \ \textbf{and}\ p \in V_1 \cup V_2 \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\,V_2)\big]\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by a reduction similar to (Z6), conclude **false**.

(Z8) Suppose:

$$\big[p \in V_1 \cup V_2 \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\,V_2)\big]\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying ZFC, conclude:

$$\begin{bmatrix} & p \in V_1 \\ \textbf{or} & p \in V_2 \end{bmatrix} \ \textbf{and}\ p \in V_1 \cup V_2 \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\,V_2)$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{ll} & \big[p \in V_1 \ \textbf{and}\ p \in V_1 \cup V_2 \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\,V_2)\big] \\ \textbf{or} & \big[p \in V_2 \ \textbf{and}\ p \in V_1 \cup V_2 \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\,V_2)\big] \end{array}$$

Then, by applying (Z6)(Z7), conclude $\big[$**false or false**$\big]$. Then, by applying standard inference rules, conclude **false**.

(Z9) Recall $(X\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (X\,,\,V_2)$ from (A1). Then, by applying Definition 19 of Υ, conclude $V_1\,,\,V_2 \in \mathbb{V}\text{ER}$. Then, by Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1\,,\,V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1 \cup V_2 \in \mathbb{V}\text{ER}$. Then, by applying ZFC, conclude $\{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})$.

(Z0) Suppose $E_\dagger \in \mathcal{E}_\dagger$. Then, by applying (A7), conclude:

$$E_\dagger \in \left\{ E'_\dagger \;\middle|\; \begin{array}{l} E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\,V_2) \end{array} \right\}$$

Then, by applying ZFC, conclude:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\ \Upsilon_{\mathcal{E}}\ (E_2\,,\,V_2)\big]\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying Definition 19 of Υ, conclude:

$$E_1\,,\,E_2 \in \wp(\mathbb{V}\text{ER})\ \textbf{and}\ E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$$

Then, by introducing (Z9), conclude $\big[E_1\,,\,E_2\,,\,\{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})\ \textbf{and}\ E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big]$. Then, by applying ZFC, conclude $\big[\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})\ \textbf{and}\ E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big]$. Then, by applying substitution, conclude $E_\dagger \in \wp(\mathbb{V}\text{ER})$.

(Y1) Recall $\big[\big[E_\dagger \in \mathcal{E}_\dagger\ \textbf{implies}\ E_\dagger \in \wp(\mathbb{V}\text{ER})\big]\ \textbf{for all}\ E_\dagger\big]$ from (Z0). Then, by applying ZFC, conclude $\mathcal{E}_\dagger \subseteq \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $\mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER})$.

(Y2) Recall $p \in V \in E \in \mathcal{E}$ from (Z4). Then, by introducing (Z5), conclude:

$$p \in V \in E \in \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying ZFC, conclude $p \in V \in E \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $p \in V \in \mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $p \in V \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $p \in \mathbb{P}\text{ORT}$.

(Y3) Suppose:

$$\big[V = V_1\ \textbf{and}\ T = E_1 \setminus \{V\}\ \textbf{and}\ p \in V \in E_1 \in \mathcal{E}_1\big]\ \textbf{for some}\ E_1$$

Then, by applying substitution, conclude $\big[T = E_1 \setminus \{V_1\}\ \textbf{and}\ p \in V_1 \in E_1 \in \mathcal{E}_1\big]$. Then, by applying ZFC, conclude $\big[T = E_1 \setminus \{V_1\}\ \textbf{and}\ p \in V_1 \cup V_2\ \textbf{and}\ E_1 \in \mathcal{E}_1\big]$. Then, by applying (A5), conclude $\big[$

$T = E_1 \setminus \{V_1\}$ **and** $p \in V_1 \cup V_2$ **and** $E_1 \in \{E_1' \mid (E_1', V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\}\big]$. Then, by applying ZFC, conclude:

$$T = E_1 \setminus \{V_1\} \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } \big[\big[(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\big] \textbf{ for some } E_2\big]$$

Then, by applying standard inference rules, conclude:

$$\big[T = E_1 \setminus \{V_1\} \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\big] \textbf{ for some } E_2$$

Then, by applying ZFC, conclude:

$$T = E_1 \cap (E_1 \setminus \{V_1\}) \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying Lemma 7:2, conclude:

$$V_2 \notin E_1 \textbf{ and } T = E_1 \cap (E_1 \setminus \{V_1\}) \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying ZFC, conclude:

$$T = E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}) \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying 19 of $\curlyvee$, conclude:

$$(E_1, V_1) \curlyvee (E_2, V_2) \textbf{ and } T = E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}) \textbf{ and } p \in V_1 \cup V_2 \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying 19 of $\curlyvee$, conclude:

$$E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \textbf{ and } T = E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}) \textbf{ and } p \in V_1 \cup V_2 \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying substitution, conclude:

$$T = E_1 \cap E_2 \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying (Z8), conclude:

$$V_1 \cup V_2 \notin E_1 \cap E_2 \textbf{ and } T = E_1 \cap E_2 \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying ZFC, conclude:

$$T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V_1 \cup V_2\} \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying ZFC, conclude:

$$T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V_1 \cup V_2\} \textbf{ and } p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying ZFC, conclude:

$$\big[\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ for some } E_\dagger\big]\big] \textbf{ and} \\ T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V_1 \cup V_2\} \textbf{ and } p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and} \\ T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V_1 \cup V_2\} \textbf{ and} \\ p \in V_1 \cup V_2 \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and} \\ (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \end{array}\right] \textbf{ for some } E_\dagger$$

Then, by applying substitution, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and} \\ T = E_\dagger \setminus \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_1 \cup V_2 \in E_\dagger \ \textbf{and} \\ (E_1 \,, V_1) \curlyvee_{\mathcal{E}} (E_2 \,, V_2)$$

Then, by applying ZFC, conclude:

$$T = E_\dagger \setminus \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_1 \cup V_2 \in E_\dagger \in \left\{ E'_\dagger \,\middle|\, \begin{array}{l} E'_\dagger = \{V_1 \cup V_2\} \cup (E'_1 \cap E'_2) \\ \textbf{and} \ (E'_1 \,, V_1) \curlyvee_{\mathcal{E}} (E'_2 \,, V_2) \end{array} \right\}$$

Then, by applying (A7), conclude $\big[T = E_\dagger \setminus \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_1 \cup V_2 \in E_\dagger \in \mathcal{E}_\dagger\big]$. Then, by applying ZFC, conclude $\big[T \in \{T' \mid T' = E' \setminus V' \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_\dagger\}\big]$. Then, by introducing (Y1), conclude $\big[\mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ T \in \{T' \mid T' = E' \setminus V' \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_\dagger\}\big]$. Then, by introducing (Y2), conclude:

$$p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ T \in \{T' \mid T' = E' \setminus V' \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_\dagger\}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(p \,, \mathcal{E}_\dagger)$.

(Y4) Suppose:

$$\big[V = V_2 \ \textbf{and} \ T = E_1 \setminus \{V\} \ \textbf{and} \ p \in V \in E_1 \in \mathcal{E}_1\big] \ \textbf{for some} \ E_1$$

Then, by applying standard inference rules, conclude $\big[V = V_2 \ \textbf{and} \ V \in E_1 \in \mathcal{E}_1\big]$. Then, by applying substitution, conclude $V_2 \in E_1 \in \mathcal{E}_1$. Then, by applying (A5), conclude:

$$V_2 \in E_1 \in \{E'_1 \mid (E'_1 \,, V_1) \curlyvee_{\mathcal{E}} (E_2 \,, V_2)\}$$

Then, by applying ZFC, conclude $\big[V_2 \in E_1 \ \textbf{and} \ \big[(E_1 \,, V_1) \curlyvee_{\mathcal{E}} (E_2 \,, V_2) \ \textbf{for some} \ E_2\big]\big]$. Then, by applying standard inference rules, conclude:

$$\big[V_2 \in E_1 \ \textbf{and} \ (E_1 \,, V_1) \curlyvee_{\mathcal{E}} (E_2 \,, V_2)\big] \ \textbf{for some} \ E_2$$

Then, by applying Lemma 7:2, conclude $\big[V_2 \notin E_1 \ \textbf{and} \ V_2 \in E_1\big]$. Then, by applying standard inference rules, conclude **false**.

(Y5) Suppose:

$$T \in \mathsf{Edge}(p \,, \mathcal{E}_1) \ \textbf{for some} \ T$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1\}$$

Then, by applying ZFC, conclude:

$$\big[T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1\big] \ \textbf{for some} \ E' \,, V'$$

Then, by introducing (Z4), conclude $\big[p \in V \in E \in \mathcal{E} \ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1\big]$. Then, by introducing (Z1), conclude:

$$\mathcal{E}_1 \subseteq \mathcal{E} \ \textbf{and} \ p \in V \in E \in \mathcal{E}^{V_1 \cup V_2} \ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1$$

Then, by applying standard inference rules, conclude:

$$p \in V' \in E' \in \mathcal{E} \ \textbf{and} \ p \in V \in E \in \mathcal{E} \ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1$$

Then, by applying (Z5), conclude $\big[V = V' \ \textbf{and} \ T = E' \setminus \{V'\} \ \textbf{and} \ p \in V' \in E' \in \mathcal{E}_1\big]$. Then, by applying substitution, conclude $\big[T = E' \setminus \{V\} \ \textbf{and} \ p \in V \in E' \in \mathcal{E}_1\big]$. Then, by introducing (A4), conclude $\big[V \in \{V_1 \,, V_2\} \ \textbf{and} \ T = E' \setminus \{V\} \ \textbf{and} \ p \in V \in E' \in \mathcal{E}_1\big]$. Then, by applying ZFC, conclude $\big[[V = V_1 \textbf{ or } V = V_2] \textbf{ and } T = E' \setminus \{V\} \textbf{ and } p \in V \in E' \in \mathcal{E}_1\big]$. Then, by applying standard inference rules, conclude:

$$\begin{array}{l} \quad [V = V_1 \textbf{ and } T = E' \setminus \{V\} \textbf{ and } p \in V \in E' \in \mathcal{E}_1] \\ \textbf{or } [V = V_2 \textbf{ and } T = E' \setminus \{V\} \textbf{ and } p \in V \in E' \in \mathcal{E}_1] \end{array}$$

Then, by applying (Y3), conclude:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ or } [V = V_2 \textbf{ and } T = E' \setminus \{V\} \textbf{ and } p \in V \in E' \in \mathcal{E}_1]$$

Then, by applying (Y4), conclude $[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ or } \textbf{false}]$. Then, by applying standard inference rules, conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$.

(Y6) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2) \textbf{ for some } T$$

Then, by a reduction similar to (Y5), conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$.

(Y7) Recall $\mathcal{E}_1 \subseteq \mathcal{E}$ from (Z1). Then, by introducing (Z5), conclude $[\mathcal{E}_1 \subseteq \mathcal{E} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\textsc{er})]$. Then, by applying ZFC, conclude $\mathcal{E}_1 \in \wp^2(\mathbb{V}\textsc{er})$.

(Y8) By a reduction similar to (Y7), conclude $\mathcal{E}_2 \in \wp^2(\mathbb{V}\textsc{er})$.

(Y9) Suppose:

$$[T = E_1 \cap E_2 \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)] \textbf{ for some } E_1\,,\,E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$(E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2) \textbf{ and } T = E_1 \cap E_2 \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \textbf{ and } T = E_1 \cap E_2 \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying substitution, conclude:

$$T = E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}) \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying Lemma 7:2, conclude:

$$V_2 \notin E_1 \textbf{ and } T = E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}) \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying ZFC, conclude:

$$T = E_1 \cap (E_1 \setminus \{V_1\}) \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying ZFC, conclude:

$$T = E_1 \setminus \{V_1\} \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$(E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2) \textbf{ and } T = E_1 \setminus \{V_1\} \textbf{ and } p \in V_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying Lemma 6:1, conclude:

$$T = E_1 \setminus \{V_1\} \textbf{ and } p \in V_1 \in E_1 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying ZFC, conclude:

$$T = E_1 \setminus \{V_1\} \textbf{ and } p \in V_1 \in E_1 \in \{E_1' \mid (E_1'\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2'\,,\,V_2)\}$$

Then, by applying (A5), conclude $\left[T = E_1 \setminus \{V_1\} \textbf{ and } p \in V_1 \in E_1 \in \mathcal{E}_1\right]$. Then, by applying ZFC, conclude $T \in \{T' \mid T' = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1\}$. Then, by introducing (Y7), conclude:

$$\mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T' = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1\}$$

Then, by introducing (Y2), conclude:

$$p \in \mathbb{P}\text{ORT} \textbf{ and } \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T' = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1\}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(p, \mathcal{E}_1)$.

(Y0) Suppose:

$$\left[T = E_1 \cap E_2 \textbf{ and } p \in V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\right] \textbf{ for some } E_1, E_2$$

Then, by a reduction similar to (Y9), conclude $T \in \mathsf{Edge}(p, \mathcal{E}_2)$.

(X1) Suppose:

$$\begin{bmatrix} T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and } p \in V' \in \{V_1 \cup V_2\} \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \end{bmatrix} \textbf{ for some } E_1, E_2, V'$$

Then, by applying ZFC, conclude:

$$T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and } p \in V' \textbf{ and } V' = V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying substitution, conclude:

$$T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V_1 \cup V_2\} \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying (Z8), conclude:

$$\begin{gathered} V_1 \cup V_2 \notin E_1 \cap E_2 \textbf{ and } T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V_1 \cup V_2\} \textbf{ and } p \in V_1 \cup V_2 \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \end{gathered}$$

Then, by applying ZFC, conclude:

$$T = E_1 \cap E_2 \textbf{ and } p \in V_1 \cup V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying ZFC, conclude:

$$T = E_1 \cap E_2 \textbf{ and } \left[p \in V_1 \textbf{ or } p \in V_2\right] \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying standard inference rules, conclude:

$$\begin{aligned} &\left[T = E_1 \cap E_2 \textbf{ and } p \in V_1 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\right] \\ \textbf{or } &\left[T = E_1 \cap E_2 \textbf{ and } p \in V_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\right] \end{aligned}$$

Then, by applying (Y9)(Y0), conclude $\left[T \in \mathsf{Edge}(p, \mathcal{E}_1) \textbf{ or } T \in \mathsf{Edge}(p, \mathcal{E}_2)\right]$. Then, by applying ZFC, conclude $T \in \mathsf{Edge}(p, \mathcal{E}_1) \cup \mathsf{Edge}(p, \mathcal{E}_2)$. Then, by applying Lemma 10:1, conclude:

$$T \in \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2)$$

(X2) Suppose:

$$\begin{bmatrix} T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and} \\ p \in V' \in E_1 \cap E_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \end{bmatrix} \textbf{ for some } E_1, E_2, V'$$

Then, by applying standard inference rules, conclude:

$$p \in V' \in E_1 \cap E_2 \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$$

Then, by applying ZFC, conclude:

$$p \in V' \in E_1 \ \textbf{and}\ V' \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$p \in V' \in E_1 \in \mathcal{E} \ \textbf{and}\ V' \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)$$

Then, by introducing (Z4), conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{and}\ p \in V' \in E_1 \in \mathcal{E} \ \textbf{and}\ V' \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)$$

Then, by applying (Z5), conclude $\big[V = V' \ \textbf{and}\ V' \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big]$. Then, by applying substitution, conclude $\big[V \in E_1 \cap E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big]$. Then, by applying ZFC, conclude $\big[V \in E_1 \ \textbf{and}\ V \in E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big]$. Then, by introducing (A4), conclude $\big[V \in \{V_1\,,\,V_2\} \ \textbf{and}\ V \in E_1 \ \textbf{and}\ V \in E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big]$. Then, by applying ZFC, conclude:

$$\big[V = V_1 \ \textbf{or}\ V = V_2\big] \ \textbf{and}\ V \in E_1 \ \textbf{and}\ V \in E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{rl} & \big[V = V_1 \ \textbf{and}\ V \in E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big] \\ \textbf{or} & \big[V = V_2 \ \textbf{and}\ V \in E_1 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big] \end{array}$$

Then, by applying substitution, conclude:

$$\big[V_1 \in E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big] \ \textbf{or}\ \big[V_2 \in E_2 \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2)\big]$$

Then, by applying Lemma 7:2, conclude:

$$\big[V_1 \in E_2 \ \textbf{and}\ V_1 \notin E_2\big] \ \textbf{or}\ \big[V_2 \in E_1 \ \textbf{and}\ V_2 \notin E_1\big]$$

Then, by applying standard inference rules, conclude $\big[\textbf{false}\ \textbf{or}\ \textbf{false}\big]$. Then, by applying standard inference rules, conclude **false**.

(X3) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \ \textbf{for some}\ T$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\} \ \textbf{and}\ p \in V' \in E' \in \mathcal{E}_\dagger\}$$

Then, by applying ZFC, conclude:

$$\big[T = E' \setminus \{V'\} \ \textbf{and}\ p \in V' \in E' \in \mathcal{E}_\dagger\big] \ \textbf{for some}\ E'\,,\,V'$$

Then, by applying (A7), conclude:

$$T = E' \setminus \{V'\} \ \textbf{and}\ p \in V' \in E' \in \left\{ E_\dagger \,\middle|\, \begin{array}{l} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2) \end{array} \right\}$$

Then, by applying ZFC, conclude:

$$T = E' \setminus \{V'\} \ \textbf{and}\ p \in V' \in E' \ \textbf{and}\ \left[\begin{bmatrix} E' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2) \end{bmatrix} \ \textbf{for some}\ E_1\,,\,E_2\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} T = E' \setminus \{V'\} \ \textbf{and}\ p \in V' \in E' \ \textbf{and} \\ E' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and}\ (E_1\,,\,V_1)\ \curlyvee_{\mathcal{E}}\ (E_2\,,\,V_2) \end{bmatrix} \ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying substitution, conclude:

$$T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and } p \in V' \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying ZFC, conclude:

$$T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and } \begin{bmatrix} p \in V' \in \{V_1 \cup V_2\} \\ \textbf{or } p \in V' \in E_1 \cap E_2 \end{bmatrix} \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying standard inference rules, conclude:

$$\big[T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and } p \in V' \in \{V_1 \cup V_2\} \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big] \\ \textbf{or } \big[T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and } p \in V' \in E_1 \cap E_2 \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big]$$

Then, by applying (X1), conclude:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \textbf{ or } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2) \\ \textbf{or } \begin{bmatrix} T = (\{V_1 \cup V_2\} \cup (E_1 \cap E_2)) \setminus \{V'\} \textbf{ and } p \in V' \in E_1 \cap E_2 \\ \textbf{and } E_1\,,\,E_2 \in \mathcal{E} \textbf{ and } E_1 \neq E_2 \textbf{ and } (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2) \end{bmatrix}$$

Then, by applying (X2), conclude $\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) \textbf{ or } \textbf{false}\big]$. Then, by applying standard inference rules, conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)$.

(X4) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ for some } T$$

Then, by applying (X3), conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)$. Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\} \textbf{ for some } T$$

Then, by applying ZFC, conclude:

$$\big[T = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for some } E'\,,\,V'$$

Then, by introducing (Z3), conclude $\big[\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E} \textbf{ and } T = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\big]$. Then, by applying ZFC, conclude $\big[T = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}\big]$. Then, by applying ZFC, conclude $T \in \{T' \mid T = E'' \setminus \{V''\} \textbf{ and } p \in V'' \in E'' \in \mathcal{E}\}$. Then, by introducing (Z5), conclude $\big[\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T = E'' \setminus \{V''\} \textbf{ and } p \in V'' \in E'' \in \mathcal{E}\}\big]$. Then, by introducing (Y1), conclude:

$$p \in \mathbb{P}\text{ORT} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T = E'' \setminus \{V''\} \textbf{ and } p \in V'' \in E'' \in \mathcal{E}\}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E})$.

(X5) Recall $\big[\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E})\big] \textbf{ for all } T\big]$. Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E})$.

Now, prove the lemma by the following reduction. Recall from (Y5)(Y6):

$$\big[\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big] \textbf{ for all } T\big] \\ \textbf{and } \big[\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big] \textbf{ for all } T\big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} & \big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big] \\ \textbf{and} & \big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big] \end{bmatrix} \textbf{ for all } T$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{bmatrix} T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \\ \textbf{or}\;\; T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2) \end{bmatrix} \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\right] \textbf{ for all } T$$

Then, by applying ZFC, conclude

$$\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_2) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big] \textbf{ for all } T$$

Then, by applying Lemma 10:1, conclude:

$$\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big] \textbf{ for all } T$$

Then, by introducing (X3), conclude:

$$\begin{array}{r l} & \big[\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big] \textbf{ for all } T\big] \\ \textbf{and} & \big[\big[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)\big] \textbf{ for all } T\big] \end{array}$$

Then, by applying ZFC, conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) \textbf{ and } \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$$

Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$. Then, by introducing (X5), conclude $\big[\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E}) \textbf{ and } \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big]$.

(| QED. |)

2. First, assume:

(B1) $\begin{bmatrix} (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix}$

(B2) $\checkmark(\mathcal{E})$

(B3) $p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$

(B4) $V \notin \{V_1\,,\,V_2\}$

(B5) $q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2$

(B6) $W \notin \{V_1\,,\,V_2\}$

(B7) $\mathsf{Edge}(p\,,\,\mathcal{E}) = \mathsf{Edge}(q\,,\,\mathcal{E})$

(B8) $\mathcal{E}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$

(B9) $\mathcal{E}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$

(B0) $\mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{array}{l} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \end{array}\right\}$

Next, observe:

(W1) Recall $\mathcal{E}_1 = \big\{E_1 \,\big|\, (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big\}$ from (A5). Then, by applying Definition 19 of $\curlyvee$, conclude $\mathcal{E}_1 = \big\{E_1 \,\big|\, (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \textbf{ and } E_1 \in \mathcal{E}\big\}$. Then, by applying ZFC, conclude $\mathcal{E}_1 \subseteq \mathcal{E}$.

(W2) By a reduction similar to (W1), conclude $\mathcal{E}_2 \subseteq \mathcal{E}$.

(W3) Recall $\mathcal{E}_1\,,\,\mathcal{E}_2 \subseteq \mathcal{E}$ from (W1)(W2). Then, by applying ZFC, conclude $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E}$.

(W4) Recall $\checkmark(\mathcal{E})$ from (B2). Then, by applying Definition 24 of $\checkmark$, conclude:

$$\begin{array}{c} \mathcal{E} \in \wp^2(\mathbb{V}\mathrm{ER}) \textbf{ and} \\ \left[\left[\begin{bmatrix} p' \in V_1' \in E_1 \in \mathcal{E} \\ \textbf{and } p' \in V_2' \in E_2 \in \mathcal{E} \end{bmatrix} \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p'\,,\,V_1'\,,\,V_2'\,,\,E_1\,,\,E_2\right] \\ \textbf{and } \big[\big[V \in E \in \mathcal{E} \textbf{ implies } V \neq \emptyset\big] \textbf{ for all } V\,,\,E\big] \end{array}$$

(W5) Suppose:

$$p \in V' \in E' \in \mathcal{E}_1 \textbf{ for some } V', E'$$

Then, by introducing (W1), conclude $\left[\mathcal{E}_1 \subseteq \mathcal{E} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1\right]$. Then, by applying ZFC, conclude $p \in V' \in E' \in \mathcal{E}$. Then, by introducing (B3), conclude:

$$p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } p \in V' \in E' \in \mathcal{E}$$

Then, by introducing (W3), conclude $\left[\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E} \textbf{ and } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } p \in V' \in E' \in \mathcal{E}\right]$. Then, by applying ZFC, conclude $\left[p \in V \in E \in \mathcal{E} \textbf{ and } p \in V' \in E' \in \mathcal{E}\right]$. Then, by applying (W4), conclude $V = V'$.

(W6) Suppose:

$$p \in V' \in E' \in \mathcal{E}_2 \textbf{ for some } V', E'$$

Then, by a reduction similar to (W5), conclude $V = V'$.

(W7) Suppose:

$$q \in W' \in F' \in \mathcal{E}_1 \textbf{ for some } W', F'$$

Then, by a reduction similar to (W5), conclude $W = W'$.

(W8) Suppose:

$$q \in W' \in F' \in \mathcal{E}_2 \textbf{ for some } W', F'$$

Then, by a reduction similar to (W5), conclude $W = W'$.

(W9) Suppose:

$$E_1 \in \mathcal{E}_1 \textbf{ for some } E_1$$

Then, by applying (B8), conclude $E_1 \in \{E_1' \mid (E_1', V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\}$. Then, by applying ZFC, conclude:

$$(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \textbf{ for some } E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1, V_1) \curlyvee (E_2, V_2)$. Then, by applying Lemma 6:1, conclude $V_1 \in E_1$.

(W0) Suppose:

$$E_2 \in \mathcal{E}_2 \textbf{ for some } E_2$$

Then, by a reduction similar to (W9), conclude $V_2 \in E_2$.

(V1) Suppose:

$$T \in \mathsf{Edge}(p, \mathcal{E}_1) \textbf{ for some } T$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1\}$$

Then, by applying ZFC, conclude:

$$\left[T = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_1\right] \textbf{ for some } E', V'$$

Then, by applying (W5), conclude $\left[V = V' \textbf{ and } T = E' \setminus \{V'\}\right]$. Then, by applying substitution, conclude $\left[T = E' \setminus \{V\} \textbf{ and } E' \in \mathcal{E}_1\right]$. Then, by applying (W9), conclude:

$$V_1 \in E' \textbf{ and } T = E' \setminus \{V\}$$

Then, by introducing (B4), conclude $\left[V \notin \{V_1, V_2\} \textbf{ and } V_1 \in E' \textbf{ and } T = E' \setminus \{V\}\right]$. Then, by applying ZFC, conclude $\left[V \neq V_1 \textbf{ and } V_1 \in E' \textbf{ and } T = E' \setminus \{V\}\right]$. Then, by applying ZFC, conclude $\left[V_1 \in E' \setminus \{V\} \textbf{ and } T = E' \setminus \{V\}\right]$ Then, by applying substitution, conclude $V_1 \in T$.

(V2) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2) \ \ \textbf{for some} \ \ T$$

Then, by a reduction similar to (V1), conclude $V_2 \in T$.

(V3) Suppose:

$$T \in \mathsf{Edge}(q\,,\,\mathcal{E}_1) \ \ \textbf{for some} \ \ T$$

Then, by a reduction similar to (V1), conclude $V_1 \in T$.

(V4) Suppose:

$$T \in \mathsf{Edge}(q\,,\,\mathcal{E}_2) \ \ \textbf{for some} \ \ T$$

Then, by a reduction similar to (V1), conclude $V_2 \in T$.

(V5) Recall $\mathsf{Edge}(p\,,\,\mathcal{E}) = \mathsf{Edge}(q\,,\,\mathcal{E})$ from (B7). Then, by applying Definition 22 of Edge, conclude:

$$p\,,\,q \in \mathbb{P}\text{ORT}$$

(V6) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \ \ \textbf{for some} \ \ T$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\} \ \ \textbf{and} \ \ p \in V' \in E' \in \mathcal{E}_1\}$$

Then, by applying ZFC, conclude:

$$\big[T = E' \setminus \{V'\} \ \ \textbf{and} \ \ p \in V' \in E' \in \mathcal{E}_1\big] \ \ \textbf{for some} \ \ V'\,,\,E'$$

Then, by introducing (W1), conclude $\big[\mathcal{E}_1 \subseteq \mathcal{E} \ \ \textbf{and} \ \ T = E' \setminus \{V'\} \ \ \textbf{and} \ \ p \in V' \in E' \in \mathcal{E}_1\big]$. Then, by applying ZFC, conclude $\big[T = E' \setminus \{V'\} \ \ \textbf{and} \ \ p \in V' \in E' \in \mathcal{E}\big]$. Then, by applying ZFC, conclude $T \in \{T' \mid T' = E'' \setminus \{V''\} \ \ \textbf{and} \ \ p \in V'' \in E'' \in \mathcal{E}\}$. Then, by introducing (W4), conclude:

$$\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \ \textbf{and} \ \ T \in \{T' \mid T' = E'' \setminus \{V''\} \ \ \textbf{and} \ \ p \in V'' \in E'' \in \mathcal{E}\}$$

Then, by introducing (V5), conclude:

$$p \in \mathbb{P}\text{ORT} \ \ \textbf{and} \ \ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \ \textbf{and} \ \ T \in \{T' \mid T' = E'' \setminus \{V''\} \ \ \textbf{and} \ \ p \in V'' \in E'' \in \mathcal{E}\}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E})$. Then, by applying (B7), conclude $T \in \mathsf{Edge}(q\,,\,\mathcal{E})$. Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = F' \setminus \{W'\} \ \ \textbf{and} \ \ q \in W' \in F' \in \mathcal{E}\}$$

Then, by applying ZFC, conclude:

$$\big[V_1 \in T \ \ \textbf{and} \ \ T = F' \setminus \{W'\} \ \ \textbf{and} \ \ q \in W' \in F' \in \mathcal{E}\big] \ \ \textbf{for some} \ \ W'\,,\,F'$$

(V7) Recall $\checkmark(\mathcal{E})$ from (B2). Then, by introducing (B1), conclude:

$$\begin{bmatrix} (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \ \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \ \textbf{and} \ \ P \in \bigstar(\mathcal{E}) \end{bmatrix} \ \ \textbf{and} \ \ \checkmark(\mathcal{E})$$

Then, by applying Lemma 13:1, conclude $\{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$. Then, by applying (B8), conclude $\{E_1 \mid V_1 \in E_2 \in \mathcal{E}\} = \mathcal{E}_1$.

(V8) By a reduction similar to (V7), conclude $\{E_2 \mid V_2 \in E_2 \in \mathcal{E}\} = \mathcal{E}_2$.

(V9) By a reduction similar to (V7), conclude:

$$\{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \textbf{ and } V_1 \in E_1 \in \mathcal{E}\} = \mathcal{E}_\dagger$$

(V0) Recall $\mathcal{E}_1 \subseteq \mathcal{E}$ from (W1). Then, by introducing (W4), conclude $[\mathcal{E}_1 \subseteq \mathcal{E} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})]$. Then, by applying ZFC, conclude $\mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER})$.

(U1) By a reduction similar to (V0), conclude $\mathcal{E}_2 \in \wp^2(\mathbb{V}\text{ER})$.

(U2) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \textbf{ for some } T$$

Then, by applying (V1), conclude $V_1 \in T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1)$. Then, by applying (V6), conclude:

$$V_1 \in T \textbf{ and } [[T = F' \setminus \{W'\} \textbf{ and } q \in W' \in F' \in \mathcal{E}] \textbf{ for some } W'\,,\,F']$$

Then, by applying standard inference rules, conclude:

$$[V_1 \in T \textbf{ and } T = F' \setminus \{W'\} \textbf{ and } q \in W' \in F' \in \mathcal{E}] \textbf{ for some } W'\,,\,F'$$

Then, by applying substitution, conclude:

$$V_1 \in F' \setminus \{W'\} \textbf{ and } T = F' \setminus \{W'\} \textbf{ and } q \in W' \in F' \in \mathcal{E}$$

Then, by applying ZFC, conclude $[V_1 \in F' \in \mathcal{E} \textbf{ and } T = F' \setminus \{W'\} \textbf{ and } q \in W' \in F']$. Then, by applying ZFC, conclude $[T = F' \setminus \{W'\} \textbf{ and } q \in W' \in F' \in \{F_1 \mid V_1 \in F_1 \in \mathcal{E}\}]$. Then, by applying (V7), conclude $[T = F' \setminus \{W'\} \textbf{ and } q \in W' \in F' \in \mathcal{E}_1]$. Then, by applying ZFC, conclude $T \in \{T' \mid T' = F'' \setminus \{W''\} \textbf{ and } q \in W'' \in F'' \in \mathcal{E}_1\}$. Then, by introducing (V0), conclude $[\mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T' = F'' \setminus \{W''\} \textbf{ and } q \in W'' \in F'' \in \mathcal{E}_1\}]$. Then, by introducing (V5), conclude:

$$q \in \mathbb{P}\text{ORT} \textbf{ and } \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and } T \in \{T' \mid T' = F'' \setminus \{W''\} \textbf{ and } q \in W'' \in F'' \in \mathcal{E}_1\}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(q\,,\,\mathcal{E}_1)$.

(U3) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2) \textbf{ for some } T$$

Then, by a reduction similar to (U2), conclude $T \in \mathsf{Edge}(q\,,\,\mathcal{E}_2)$.

(U4) Suppose:

$$T \in \mathsf{Edge}(q\,,\,\mathcal{E}_1) \textbf{ for some } T$$

Then, by a reduction similar to (U2), conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1)$.

(U5) Suppose:

$$T \in \mathsf{Edge}(q\,,\,\mathcal{E}_2) \textbf{ for some } T$$

Then, by a reduction similar to (U2), conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2)$.

(U6) Recall from (U2)(U4):

$$\begin{array}{rl} & [[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1) \textbf{ implies } T \in \mathsf{Edge}(q\,,\,\mathcal{E}_1)] \textbf{ for all } T] \\ \textbf{and} & [[T \in \mathsf{Edge}(q\,,\,\mathcal{E}_1) \textbf{ implies } T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1)] \textbf{ for all } T] \end{array}$$

Then, by applying ZFC, conclude $[\mathsf{Edge}(p\,,\,\mathcal{E}_1) \subseteq \mathsf{Edge}(q\,,\,\mathcal{E}_1) \textbf{ and } \mathsf{Edge}(q\,,\,\mathcal{E}_1) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E}_1)]$. Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1) = \mathsf{Edge}(q\,,\,\mathcal{E}_1)$.

(U7) By a reduction similar to (U6), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_2) = \mathsf{Edge}(q\,,\,\mathcal{E}_2)$.

(U8) Suppose

$$E_\dagger \in \mathcal{E}_\dagger \;\; \textbf{for some} \;\; E_\dagger$$

Then, by applying (B0), conclude:

$$E_\dagger \in \left\{ E'_\dagger \;\middle|\; \begin{array}{l} E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \\ \textbf{and} \;\; (E_1 \, , \, V_1) \curlyvee_{\mathcal{E}} (E_2 \, , \, V_2) \end{array} \right\}$$

Then, by applying ZFC, conclude:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \;\; (E_1 \, , \, V_1) \curlyvee_{\mathcal{E}} (E_2 \, , \, V_2)\big] \;\; \textbf{for some} \;\; E_1 \, , \, E_2$$

(U9) Suppose:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \;\; (E_1 \, , \, V_1) \curlyvee_{\mathcal{E}} (E_2 \, , \, V_2)\big] \;\; \textbf{for some} \;\; E_1 \, , \, E_2 \, , \, E_\dagger$$

Then, by applying Lemma 7:2, conclude:

$$V_2 \notin E_1 \;\; \textbf{and} \;\; E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \;\; (E_1 \, , \, V_1) \curlyvee_{\mathcal{E}} (E_2 \, , \, V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$V_2 \notin E_1 \;\; \textbf{and} \;\; E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\; \textbf{and} \;\; (E_1 \, , \, V_1) \curlyvee (E_2 \, , \, V_2)$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_2 = (E_1 \setminus \{V_1\}) \cup \{V_2\} \;\; \textbf{and} \;\; V_2 \notin E_1 \;\; \textbf{and} \;\; E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$$

Then, by applying substitution, conclude:

$$V_2 \notin E_1 \;\; \textbf{and} \;\; E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap ((E_1 \setminus \{V_1\}) \cup \{V_2\}))$$

Then, by applying ZFC, conclude $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap (E_1 \setminus \{V_1\}))$. Then, by applying ZFC, conclude $E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})$.

(U0) Recall $(X \, , \, V_1) \curlyvee_{\mathcal{E}} (Y \, , \, V_2)$ from (B1). Then, by applying Definition 19 of $\curlyvee$, conclude:

$$X \, , \, Y \in \mathcal{E} \;\; \textbf{and} \;\; (X \, , \, V_1) \curlyvee (Y \, , \, V_2)$$

Then, by applying Lemma 6:1, conclude $\big[V_1 \in X \in \mathcal{E} \;\; \textbf{and} \;\; V_2 \in Y \in \mathcal{E}\big]$.

(T1) Recall $p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$ from (B3). Then, by introducing (W3), conclude:

$$\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E} \;\; \textbf{and} \;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying ZFC, conclude $p \in V \in E \in \mathcal{E}$.

(T2) By a reduction similar to (T1), conclude $q \in W \in F \in \mathcal{E}$.

(T3) Suppose $p \in V_1$. Then, by introducing (U0), conclude $p \in V_1 \in X \in \mathcal{E}$. Then, by introducing (T1), conclude $\big[p \in V \in E \in \mathcal{E} \;\; \textbf{and} \;\; p \in V_1 \in X \in \mathcal{E}\big]$. Then, by applying (W4), conclude $V = V_1$. Then, by applying ZFC, conclude $V \in \{V_1 \, , \, V_2\}$. Then, by introducing (B4), conclude:

$$V \notin \{V_1 \, , \, V_2\} \;\; \textbf{and} \;\; V \in \{V_1 \, , \, V_2\}$$

Then, by applying standard inference rules, conclude **false**.

(T4) Suppose $p \in V_2$. Then, by a reduction similar to (T3), conclude **false**.

(T5) Suppose $q \in V_1$. Then, by a reduction similar to (T3), conclude **false**.

(T6) Suppose $q \in V_2$. Then, by a reduction similar to (T3), conclude **false**.

(T7) Suppose:

$$p \in V' \in \{V_1 \cup V_2\} \ \textbf{for some} \ V'$$

Then, by applying ZFC, conclude $\big[p \in V' \ \textbf{and} \ V' = V_1 \cup V_2\big]$. Then, by applying substitution, conclude $p \in V_1 \cup V_2$. Then, by applying ZFC, conclude $\big[p \in V_1 \ \textbf{or} \ p \in V_2\big]$. Then, by applying (T3), conclude $\big[\textbf{false} \ \textbf{or} \ p \in V_2\big]$. Then, by applying (T4), conclude $\big[\textbf{false} \ \textbf{or} \ \textbf{false}\big]$. Then, by applying standard inference rules, conclude **false**.

(T8) Suppose:

$$q \in W' \in \{V_1 \cup V_2\} \ \textbf{for some} \ W'$$

Then, by a reduction similar to (T7), conclude **false**.

(T9) Suppose:

$$p \in V' \in \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \ \textbf{for some} \ V', E_1$$

Then, by applying ZFC, conclude $\big[p \in V' \in \{V_1 \cup V_2\} \ \textbf{or} \ p \in V' \in E_1 \setminus \{V_1\}\big]$. Then, by applying (T7), conclude $\big[\textbf{false} \ \textbf{or} \ p \in V' \in E_1 \setminus \{V_1\}\big]$. Then, by applying standard inference rules, conclude $p \in V' \in E_1 \setminus \{V_1\}$.

(T0) Suppose:

$$q \in W' \in \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \ \textbf{for some} \ W', E_1$$

Then, by a reduction similar to (T9), conclude $q \in W' \in E_1 \setminus \{V_1\}$.

(S1) Suppose:

$$\big[V' \in E_1 \setminus \{V_1\} \ \textbf{and} \ (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\big] \ \textbf{for some} \ V', E_1, E_2$$

Then, by applying Definition 19, conclude $\big[V' \in E_1 \setminus \{V_1\} \ \textbf{and} \ (E_1, V_1) \curlyvee (E_2, V_2)\big]$. Then, by applying Lemma 6:1, conclude $\big[V' \in E_1 \setminus \{V_1\} \ \textbf{and} \ V_1 \in E_1\big]$. Then, by applying ZFC, conclude $\big[V' \neq V_1 \ \textbf{and} \ V_1 \in E_1\big]$. Then, by applying ZFC, conclude $V_1 \in E_1 \setminus \{V'\}$.

(S2) Suppose:

$$p \in V' \ \textbf{for some} \ V'$$

Then, by applying (T7), conclude $V' \notin \{V_1 \cup V_2\}$. Then, by applying ZFC, conclude $V' \neq V_1 \cup V_2$.

(S3) Suppose:

$$q \in W' \ \textbf{for some} \ W'$$

Then, by a reduction similar to (S2), conclude $W' \neq V_1 \cup V_2$.

(S4) Recall $V_2 \in Y \in \mathcal{E}$ from (U0). Then, by applying (W4), conclude $V_2 \neq \emptyset$. Then, by applying ZFC, conclude:

$$p_2 \in V_2 \ \textbf{for some} \ p_2$$

(S5) Recall $(X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2)$ from (S5). Then, by applying Lemma 7:1, conclude $V_1 \neq V_2$.

(S6) Suppose $V_1 = V_1 \cup V_2$. Then, by applying ZFC, conclude $V_2 \subseteq V_1$. Then, by introducing (S4), conclude $\big[\big[p_2 \in V_2 \ \textbf{for some} \ p_2\big] \ \textbf{and} \ V_2 \subseteq V_1\big]$. Then, by applying standard inference rules, conclude:

$$\big[p_2 \in V_2 \ \textbf{and} \ V_2 \subseteq V_1\big] \ \textbf{for some} \ p_2$$

Then, by applying ZFC, $\big[p_2 \in V_1 \ \textbf{and} \ p_2 \in V_2\big]$. Then, by introducing (U0), conclude:

$$p_2 \in V_1 \in X \in \mathcal{E} \ \textbf{and} \ p_2 \in V_2 \in Y \in \mathcal{E}$$

Then, by applying (W4), conclude $V_1 = V_2$. Then, by introducing (S5), conclude:

$$V_1 \neq V_2 \ \textbf{and} \ V_1 = V_2$$

Then, by applying standard inference rules, conclude **false**.

(S7) Suppose:

$$\big[V' \neq V_1 \cup V_2 \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})) \setminus \{V'\}\big] \;\textbf{ for some }\; V', E_1, T$$

Then, by applying ZFC, conclude:

$$V' \neq V_1 \cup V_2 \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \setminus \{V'\}) \cup ((E_1 \setminus \{V_1\}) \setminus \{V'\})$$

Then, by applying ZFC, conclude $T = \{V_1 \cup V_2\} \cup ((E_1 \setminus \{V_1\}) \setminus \{V'\})$. Then, by introducing (S6), conclude $\big[V_1 \neq V_1 \cup V_2 \;\textbf{ and }\; T = \{V_1 \cup V_2\} \cup ((E_1 \setminus \{V_1\}) \setminus \{V'\})\big]$. Then, by applying ZFC, conclude $T = (\{V_1 \cup V_2\} \setminus \{V_1\}) \cup ((E_1 \setminus \{V_1\}) \setminus \{V'\})$. Then, by applying ZFC, conclude:

$$T = (\{V_1 \cup V_2\} \setminus \{V_1\}) \cup ((E_1 \setminus \{V'\}) \setminus \{V_1\})$$

Then, by applying ZFC, conclude $T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V'\})) \setminus \{V_1\}$.

(S8) Suppose:

$$\big[T_1 = E_1 \setminus \{V'\} \;\textbf{ and }\; p \in V' \in E_1 \in \mathcal{E}\big] \;\textbf{ for some }\; E_1, T_1, V'$$

Then, by applying ZFC, conclude $T_1 \in \{T_1' \mid T_1' = E_1' \setminus \{V''\} \;\textbf{ and }\; p \in V'' \in E_1' \in \mathcal{E}\}$. Then, by introducing (W4), conclude:

$$\mathcal{E} \in \wp^2(\mathbb{V}\textsc{er}) \;\textbf{ and }\; T_1 \in \{T_1' \mid T_1' = E_1' \setminus \{V''\} \;\textbf{ and }\; p \in V'' \in E_1' \in \mathcal{E}\}$$

Then, by introducing (V5), conclude:

$$p \in \mathbb{P}\textsc{ort} \;\textbf{ and }\; \mathcal{E} \in \wp^2(\mathbb{V}\textsc{er}) \;\textbf{ and }\; T_1 \in \{T_1' \mid T_1' = E_1' \setminus \{V''\} \;\textbf{ and }\; p \in V'' \in E_1' \in \mathcal{E}\}$$

Then, by applying Definition 22 of Edge, conclude $T_1 \in \mathsf{Edge}(p, \mathcal{E})$. Then, by applying (B7), conclude $T_1 \in \mathsf{Edge}(q, \mathcal{E})$. Then, by applying Definition 22 of Edge, conclude:

$$T_1 \in \{T_1' \mid T_1' = F' \setminus \{W'\} \;\textbf{ and }\; q \in W' \in F' \in \mathcal{E}\}$$

Then, by applying ZFC, conclude:

$$\big[T_1 = F' \setminus \{W'\} \;\textbf{ and }\; q \in W' \in F' \in \mathcal{E}\big] \;\textbf{ for some }\; F', W'$$

(S9) Suppose:

$$\big[T_1 = E_1 \setminus \{W'\} \;\textbf{ and }\; q \in W' \in E_1 \in \mathcal{E}\big] \;\textbf{ for some }\; E_1, T_1, W'$$

Then, by a reduction similar to (S8), conclude:

$$\big[T_1 = E' \setminus \{V'\} \;\textbf{ and }\; p \in V' \in E' \in \mathcal{E}\big] \;\textbf{ for some }\; E', V'$$

(S0) Suppose:

$$\big[q \in W' \in F' \;\textbf{ and }\; V_1 \in F' \setminus \{W'\}\big] \;\textbf{ for some }\; W', F'$$

Then, by applying ZFC, conclude $\big[q \in W' \in F' \;\textbf{ and }\; V_1 \neq W'\big]$. Then, by applying ZFC, conclude $q \in W' \in F' \setminus \{V_1\}$. Then, by applying ZFC, conclude $q \in W' \in \{V_1 \cup V_2\} \cup (F' \setminus \{V_1\})$.

(R1) Suppose:

$$\big[p \in V' \in E' \;\textbf{ and }\; V_1 \in E' \setminus \{V'\}\big] \;\textbf{ for some }\; V', E'$$

Then, by a reduction similar to (S0), conclude $p \in V' \in \{V_1 \cup V_2\} \cup (E' \setminus \{V_1\})$.

(R2) Recall $(X, V_1) \curlyvee_{\mathcal{E}} (X, V_2)$ from (B1). Then, by applying Definition 19 of $\curlyvee$, conclude $V_1, V_2 \in \mathbb{V}\textsc{er}$. Then, by Definition 15 of $\mathbb{V}\textsc{er}$, conclude $V_1, V_2 \in \wp(\mathbb{P}\textsc{ort})$. Then, by applying ZFC, conclude $V_1 \cup V_2 \in \wp(\mathbb{P}\textsc{ort})$. Then, by Definition 15 of $\mathbb{V}\textsc{er}$, conclude $V_1 \cup V_2 \in \mathbb{V}\textsc{er}$. Then, by applying ZFC, conclude $\{V_1 \cup V_2\} \in \wp(\mathbb{V}\textsc{er})$.

(R3) Suppose $E_\dagger \in \mathcal{E}_\dagger$. Then, by applying (B0), conclude:

$$E_\dagger \in \left\{ E'_\dagger \,\middle|\, \begin{array}{c} E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \end{array} \right\}$$

Then, by applying ZFC, conclude:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big]\ \ \textbf{for some}\ \ E_1\,,\,E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_1\,,\,E_2 \in \wp(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$$

Then, by introducing (R2), conclude $\big[E_1\,,\,E_2\,,\,\{V_1 \cup V_2\} \in \wp(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big]$. Then, by applying ZFC, conclude $\big[\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\mathrm{ER})\ \ \textbf{and}\ \ E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\big]$. Then, by applying substitution, conclude $E_\dagger \in \wp(\mathbb{V}\mathrm{ER})$.

(R4) Recall $\big[\big[E_\dagger \in \mathcal{E}_\dagger\ \ \textbf{implies}\ \ E_\dagger \in \wp(\mathbb{V}\mathrm{ER})\big]\ \ \textbf{for all}\ \ E_\dagger\big]$ from (R3). Then, by applying ZFC, conclude $\mathcal{E}_\dagger \subseteq \wp(\mathbb{V}\mathrm{ER})$. Then, by applying ZFC, conclude $\mathcal{E}_\dagger \in \wp^2(\mathbb{V}\mathrm{ER})$.

(R5) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\ \ \textbf{for some}\ \ T$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E' \in \mathcal{E}_\dagger\}$$

Then, by applying ZFC, conclude:

$$\big[T = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E' \in \mathcal{E}_\dagger\big]\ \ \textbf{for some}\ \ E'\,,\,V'$$

Then, by applying (U8), conclude:

$$\begin{array}{c} T = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E'\ \ \textbf{and} \\ \big[\big[E' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big]\ \ \textbf{for some}\ \ E_1\,,\,E_2\big] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c} T = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E'\ \ \textbf{and} \\ E' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\ \ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \end{array}\right]\ \ \textbf{for some}\ \ E_1\,,\,E_2$$

Then, by applying (U9), conclude:

$$\begin{array}{c} T = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E'\ \ \textbf{and} \\ E' = \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})\ \ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \end{array}$$

Then, by applying substitution, conclude:

$$\begin{array}{c} T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})) \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in \{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\}) \\ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \end{array}$$

Then, by applying (T9), conclude:

$$T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})) \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E_1 \setminus \{V_1\}\ \ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)$$

Then, by applying (S1), conclude:

$$V_1 \in E_1 \setminus \{V'\}\ \ \textbf{and}\ \ T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})) \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E_1 \setminus \{V_1\}$$

Then, by applying (S2), conclude:

$$V' \neq V_1 \cup V_2 \;\textbf{ and }\; V_1 \in E_1 \setminus \{V'\} \;\textbf{ and }\\ T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V_1\})) \setminus \{V'\} \;\textbf{ and }\; p \in V' \in E_1 \setminus \{V_1\}$$

Then, by applying (S7), conclude:

$$V_1 \in E_1 \setminus \{V'\} \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V'\})) \setminus \{V_1\} \;\textbf{ and }\; p \in V' \in E_1 \setminus \{V_1\}$$

Then, by applying ZFC, conclude:

$$V_1 \in E_1 \setminus \{V'\} \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V'\})) \setminus \{V_1\} \;\textbf{ and }\; p \in V' \in E_1 \in \mathcal{E}$$

Then, by applying ZFC, conclude:

$$\big[T_1 = E_1 \setminus \{V'\} \;\textbf{ for some }\; T_1\big] \;\textbf{ and }\; V_1 \in E_1 \setminus \{V'\} \;\textbf{ and }\\ T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V'\})) \setminus \{V_1\} \;\textbf{ and }\; p \in V' \in E_1 \in \mathcal{E}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} T_1 = E_1 \setminus \{V'\} \;\textbf{ and }\; V_1 \in E_1 \setminus \{V'\} \;\textbf{ and } \\ T = (\{V_1 \cup V_2\} \cup (E_1 \setminus \{V'\})) \setminus \{V_1\} \;\textbf{ and }\; p \in V' \in E_1 \in \mathcal{E} \end{bmatrix} \;\textbf{for some }\; T_1$$

Then, by applying substitution, conclude:

$$T_1 = E_1 \setminus \{V'\} \;\textbf{ and }\; V_1 \in T_1 \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup T_1) \setminus \{V_1\} \;\textbf{ and }\; p \in V' \in E_1 \in \mathcal{E}$$

Then, by applying (S8), conclude:

$$\big[\big[T_1 = F' \setminus \{W'\} \;\textbf{ and }\; q \in W' \in F' \in \mathcal{E}\big] \;\textbf{ for some }\; F', W'\big]\\ \textbf{and }\; V_1 \in T_1 \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup T_1) \setminus \{V_1\}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} T_1 = F' \setminus \{W'\} \;\textbf{ and }\; q \in W' \in F' \in \mathcal{E} \;\textbf{ and} \\ V_1 \in T_1 \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup T_1) \setminus \{V_1\} \end{bmatrix} \;\textbf{for some }\; F', W'$$

Then, by applying substitution, conclude:

$$q \in W' \in F' \in \mathcal{E} \;\textbf{ and }\; V_1 \in F' \setminus \{W'\} \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup F' \setminus \{W'\}) \setminus \{V_1\}$$

Then, by applying (S3), conclude:

$$W' \neq V_1 \cup V_2 \;\textbf{ and }\; q \in W' \in F' \in \mathcal{E} \;\textbf{ and }\\ V_1 \in F' \setminus \{W'\} \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup F' \setminus \{W'\}) \setminus \{V_1\}$$

Then, by applying (S7), conclude:

$$q \in W' \in F' \in \mathcal{E} \;\textbf{ and }\; V_1 \in F' \setminus \{W'\} \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup F' \setminus \{V_1\}) \setminus \{W'\}$$

Then, by applying ZFC, conclude:

$$V_1 \in F' \in \mathcal{E} \;\textbf{ and }\; q \in W' \in F' \;\textbf{ and }\; V_1 \in F' \setminus \{W'\} \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup F' \setminus \{V_1\}) \setminus \{W'\}$$

Then, by applying (S0), conclude:

$$q \in W' \in \{V_1 \cup V_2\} \cup (F' \setminus \{V_1\}) \;\textbf{ and }\; V_1 \in F' \in \mathcal{E} \;\textbf{ and }\; T = (\{V_1 \cup V_2\} \cup F' \setminus \{V_1\}) \setminus \{W'\}$$

Then, by applying ZFC, conclude:

$$\big[F_\dagger = \{V_1 \cup V_2\} \cup F' \setminus \{V_1\} \ \textbf{for some} \ F_\dagger\big] \ \textbf{and}$$
$$q \in W' \in \{V_1 \cup V_2\} \cup (F' \setminus \{V_1\}) \ \textbf{and} \ V_1 \in F' \in \mathcal{E} \ \textbf{and} \ T = (\{V_1 \cup V_2\} \cup F' \setminus \{V_1\}) \setminus \{W'\}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} F_\dagger = \{V_1 \cup V_2\} \cup F' \setminus \{V_1\} \ \textbf{and} \\ q \in W' \in \{V_1 \cup V_2\} \cup (F' \setminus \{V_1\}) \ \textbf{and} \ V_1 \in F' \in \mathcal{E} \\ \textbf{and} \ T = (\{V_1 \cup V_2\} \cup F' \setminus \{V_1\}) \setminus \{W'\} \end{bmatrix} \ \textbf{for some} \ F_\dagger$$

Then, by applying substitution, conclude:

$$F_\dagger = \{V_1 \cup V_2\} \cup F' \setminus \{V_1\} \ \textbf{and} \ q \in W' \in F_\dagger \ \textbf{and} \ V_1 \in F' \in \mathcal{E} \ \textbf{and} \ T = F_\dagger \setminus \{W'\}$$

Then, by applying ZFC, conclude:

$$F_\dagger \in \{F'_\dagger \mid F'_\dagger = \{V_1 \cup V_2\} \cup F'' \setminus \{V_1\} \ \textbf{and} \ V_1 \in F'' \in \mathcal{E}\}$$
$$\textbf{and} \ q \in W' \in F_\dagger \ \textbf{and} \ T = F_\dagger \setminus \{W'\}$$

Then, by applying (V9), conclude $\big[q \in W' \in F_\dagger \in \mathcal{E}_\dagger \ \textbf{and} \ T = F_\dagger \setminus \{W'\}\big]$. Then, by applying ZFC, conclude $T \in \{T' \mid T' = F'' \setminus \{W''\} \ \textbf{and} \ q \in W'' \in F'' \in \mathcal{E}_\dagger\}$. Then, by applying (R4), conclude $\big[\mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ T \in \{T' \mid T' = F'' \setminus \{W''\} \ \textbf{and} \ q \in W'' \in F'' \in \mathcal{E}_\dagger\}\big]$. Then, by applying (V5), conclude:

$$p \in \mathbb{P}\text{ORT} \ \textbf{and} \ \mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and} \ T \in \{T' \mid T' = F'' \setminus \{W''\} \ \textbf{and} \ q \in W'' \in F'' \in \mathcal{E}_\dagger\}$$

Then, by applying Definition 22 of Edge, conclude $T \in \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger)$.

(R6) Suppose:

$$T \in \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger) \ \textbf{for some} \ T$$

Then, by a reduction similar to (R5), conclude $T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$.

(R7) Recall from (R5)(R6):

$$\begin{matrix} & \big[[T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \ \textbf{implies} \ T \in \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger)] \ \textbf{for all} \ T\big] \\ \textbf{and} & \big[[T \in \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger) \ \textbf{implies} \ T \in \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)] \ \textbf{for all} \ T\big] \end{matrix}$$

Then, by applying ZFC, conclude $\big[\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger) \ \textbf{and} \ \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\big]$. Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) = \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger)$.

Now, prove the lemma by the following reduction. Recall from Lemma 10:1:

$$\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_1) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_2)$$

Then, by applying (U6)(U7), $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(q\,,\,\mathcal{E}_1) \cup \mathsf{Edge}(q\,,\,\mathcal{E}_2)$. Then, by applying Lemma 10:1, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(q\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)$. Then, by introducing (R7), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) = \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger) \ \textbf{and} \ \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(q\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)$$

⦇ QED. ⦈

3. First, assume:

(C1) $\begin{bmatrix} (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \ \textbf{and} \\ V_1 \cup V_2 \subseteq P \ \textbf{and} \ P \in \bigstar(\mathcal{E}) \end{bmatrix}$

(C2) $\checkmark(\mathcal{E})$

(C3) $\left[\textbf{not}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\right]$ **for all** V , E

(C4) $p \in \mathbb{P}\text{ORT}$

(C5) $\mathcal{E}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$

(C6) $\mathcal{E}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$

(C7) $\mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{array}{l} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \end{array}\right\}$

Next, observe:

(Q1) Recall $\checkmark(\mathcal{E})$ from (C2). Then, by applying Definition 24 of $\checkmark$, conclude $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$.

(Q2) Recall $\mathcal{E}_1 = \left\{E_1 \,\middle|\, (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\right\}$ from (C5). Then, by applying Definition 19 of $\curlyvee$, conclude $\mathcal{E}_1 = \left\{E_1 \,\middle|\, (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\ \textbf{and}\ E_1 \in \mathcal{E}\right\}$. Then, by applying ZFC, conclude $\mathcal{E}_1 \subseteq \mathcal{E}$. Then, by introducing (Q1), conclude $\left[\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\ \textbf{and}\ \mathcal{E}_1 \subseteq \mathcal{E}\right]$. Then, by applying ZFC, conclude $\mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER})$. Then, by introducing (C4), conclude $\left[p \in \mathbb{P}\text{ORT}\ \textbf{and}\ \mathcal{E}_1 \in \wp^2(\mathbb{V}\text{ER})\right]$. Then, by applying Definition 22 of Edge, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1) \in \wp^2(\mathbb{V}\text{ER})$.

(Q3) By a reduction similar to (Q2), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_2) \in \wp^2(\mathbb{V}\text{ER})$.

(Q4) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_1)\ \ \textbf{for some}\ \ T$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\}\ \ \textbf{and}\ \ p \in V' \in E' \in \mathcal{E}_1\}$$

Then, by applying ZFC, conclude:

$$p \in V' \in E' \in \mathcal{E}_1\ \ \textbf{for some}\ \ V'\,,\,E'$$

Then, by applying ZFC, conclude $p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2$. Then, by applying (C3), conclude:

$$\left[\textbf{not}\ \ p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\right]\ \ \textbf{and}\ \ p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying standard inference rules, conclude **false**.

(Q5) Suppose:

$$T \in \mathsf{Edge}(p\,,\,\mathcal{E}_2)\ \ \textbf{for some}\ \ T$$

Then, by a reduction similar to (Q4), conclude **false**.

(Q6) Recall $\mathsf{Edge}(p\,,\,\mathcal{E}_1) \in \wp^2(\mathbb{V}\text{ER})$ from (Q2). Then, by introducing (Q4), conclude:

$$\left[T \notin \mathsf{Edge}(p\,,\,\mathcal{E}_1)\ \ \textbf{for all}\ \ T\right]\ \ \textbf{and}\ \ \mathsf{Edge}(p\,,\,\mathcal{E}_1) \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1) = \emptyset$.

(Q7) By a reduction similar to (Q6), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_2) = \emptyset$.

(Q8) Recall $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_1) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_2)$ from Lemma 10:1. Then, by applying (Q6)(Q7), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \emptyset \cup \emptyset$. Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \emptyset$.

(Q9) Recall $(X\,,\,V_1) \curlyvee (X\,,\,V_2)$ from (C1). Then, by applying Definition 19 of $\curlyvee$, conclude $V_1\,,\,V_2 \in \mathbb{V}\text{ER}$. Then, by Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1\,,\,V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1 \cup V_2 \in \mathbb{V}\text{ER}$. Then, by applying ZFC, conclude $\{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})$.

(Q0) Suppose $E_\dagger \in \mathcal{E}_\dagger$. Then, by applying (C7), conclude:

$$E_\dagger \in \left\{ E'_\dagger \,\middle|\, \begin{matrix} E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1\,, V_1) \curlyvee_{\mathcal{E}} (E_2\,, V_2) \end{matrix} \right\}$$

Then, by applying ZFC, conclude:

$$\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,, V_1) \curlyvee_{\mathcal{E}} (E_2\,, V_2)\right] \textbf{ for some } E_1\,, E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_1\,, E_2 \in \wp(\mathbb{V}\text{ER}) \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$$

Then, by introducing (Q9), conclude $\left[E_1\,, E_2\,, \{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER}) \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\right]$. Then, by applying ZFC, conclude $\left[\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER}) \textbf{ and } E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\right]$. Then, by applying substitution, conclude $E_\dagger \in \wp(\mathbb{V}\text{ER})$.

(P1) Recall $\left[\left[E_\dagger \in \mathcal{E}_\dagger \textbf{ implies } E_\dagger \in \wp(\mathbb{V}\text{ER})\right] \textbf{ for all } E_\dagger\right]$ from (Q0). Then, by applying ZFC, conclude $\mathcal{E}_\dagger \subseteq \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $\mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER})$. Then, by introducing (C4), conclude $\left[p \in \mathbb{P}\text{ORT} \textbf{ and } \mathcal{E}_\dagger \in \wp^2(\mathbb{V}\text{ER})\right]$. Then, by applying Definition 22 of Edge, conclude $\mathsf{Edge}(p\,, \mathcal{E}_\dagger) \in \wp^2(\mathbb{V}\text{ER})$.

(P2) Recall $\checkmark(\mathcal{E})$ from (C2). Then, by introducing (C1), conclude:

$$\begin{bmatrix} (X\,, V_1) \curlyvee_{\mathcal{E}} (Y\,, V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{ and } \checkmark(\mathcal{E})$$

Then, by applying Lemma 13:1, conclude $\{E_1 \mid V_1 \in E_1 \in \mathcal{E}\} = \{E_1 \mid (E_1\,, V_1) \curlyvee_{\mathcal{E}} (E_2\,, V_2)\}$. Then, by applying (C5), conclude $\{E_1 \mid V_1 \in E_2 \in \mathcal{E}\} = \mathcal{E}_1$.

(P3) By a reduction similar to (P2), conclude $\{E_2 \mid V_2 \in E_2 \in \mathcal{E}\} = \mathcal{E}_2$.

(P4) Suppose:

$$V_1 \in E_1 \in \mathcal{E} \textbf{ for some } E_1$$

Then, by applying ZFC, conclude $E_1 \in \{E'_1 \mid V_1 \in E'_1 \in \mathcal{E}\}$. Then, by applying (P2), conclude $E_1 \in \mathcal{E}_1$. Then, by applying ZFC, conclude $E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2$.

(P5) Suppose:

$$V_2 \in E_2 \in \mathcal{E} \textbf{ for some } E_1$$

Then, by a reduction similar to (P4), conclude $E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2$.

(P6) Suppose:

$$E_\dagger \in E'_\dagger \textbf{ for some } E_\dagger$$

Then, by applying (C7), conclude:

$$E_\dagger \in \left\{ E'_\dagger \,\middle|\, \begin{matrix} E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1\,, V_1) \curlyvee_{\mathcal{E}} (E_2\,, V_2) \end{matrix} \right\}$$

Then, by applying ZFC, conclude:

$$\left[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,, V_1) \curlyvee_{\mathcal{E}} (E_2\,, V_2)\right] \textbf{ for some } E_1\,, E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } E_1\,, E_2 \in \mathcal{E} \textbf{ and } (E_1\,, V_1) \curlyvee (E_2\,, V_2)$$

Then, by applying Lemma 6:1, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } V_1 \in E_1 \in \mathcal{E} \textbf{ and } V_2 \in E_2 \in \mathcal{E}$$

Then, by applying (P4)(P5), conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2$$

(P7) Suppose:

$$V' \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ for some } V', E_1, E_2$$

Then, by applying ZFC, conclude $\left[V' \in \{V_1 \cup V_2\} \textbf{ or } V' \in E_1 \cap E_2\right]$. Then, by applying ZFC, conclude $\left[V' = V_1 \cup V_2 \textbf{ or } V' \in E_1 \cap E_2\right]$. Then, by applying ZFC, conclude:

$$V' = V_1 \cup V_2 \textbf{ or } V' \in E_1$$

(P8) Suppose:

$$p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ for some } V', E'$$

Then, by introducing (C3), conclude $\left[\left[\textbf{not } p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\right] \textbf{ and } p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2\right]$. Then, by applying standard inference rules, conclude **false**.

(P9) Suppose:

$$\left[V' = V_1 \cup V_2 \textbf{ and } p \in V' \textbf{ and } V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2\right] \textbf{ for some } E_1, E_2$$

Then, by applying substitution, conclude:

$$p \in V_1 \cup V_2 \textbf{ and } V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying ZFC, conclude:

$$\left[p \in V_1 \textbf{ or } p \in V_2\right] \textbf{ and } V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying standard inference rules, conclude:

$$p \in V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ or } p \in V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying (P8), conclude $\left[\textbf{false or false}\right]$. Then, by applying standard inference rules, conclude **false**.

(P0) Suppose:

$$T \in \mathsf{Edge}(p, \mathcal{E}_\dagger) \textbf{ for some } T$$

Then, by applying Definition 22 of Edge, conclude:

$$T \in \{T' \mid T' = E' \setminus \{V'\} \textbf{ and } p \in V' \in E' \in \mathcal{E}_\dagger\}$$

Then, by applying ZFC, conclude:

$$p \in V' \in E' \in \mathcal{E}_\dagger \textbf{ for some } V', E'$$

Then, by applying (P6), conclude:

$$p \in V' \in E' \textbf{ and } \left[\begin{bmatrix} E' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and} \\ V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2 \end{bmatrix} \textbf{ for some } E_1, E_2\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in V' \in E' \textbf{ and } E' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2 \end{bmatrix} \textbf{ for some } E_1, E_2$$

Then, by applying substitution, conclude

$$p \in V' \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and}\ V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{and}\ V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying (P7), conclude:

$$\big[V' = V_1 \cup V_2 \ \textbf{or}\ V' \in E_1\big] \ \textbf{and}\ p \in V' \ \textbf{and}\ V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{and}\ V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying standard inference rules, conclude:

$$\big[V' = V_1 \cup V_2 \ \textbf{and}\ p \in V' \ \textbf{and}\ V_1 \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{and}\ V_2 \in E_2 \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \\ \textbf{or}\ p \in V' \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying (P9), conclude $\big[\textbf{false}\ \textbf{or}\ p \in V' \in E_1 \in \mathcal{E}_1 \cup \mathcal{E}_2\big]$. Then, by applying (P8), conclude $\big[\textbf{false}\ \textbf{or}\ \textbf{false}\big]$. Then, by applying standard inference rules, conclude **false**.

(O1) Recall $\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \in \wp^2(\mathbb{V}\text{ER})$ from (P1). Then, by introducing (P0), conclude:

$$\big[T \notin \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \ \textbf{for all}\ T\big] \ \textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) = \emptyset$.

(O2) Recall $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$ from (Q1). Then, by introducing (C4), conclude $\big[p \in \mathbb{P}\text{ORT}\ \textbf{and}\ \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})\big]$. Then, by applying Definition 22 of Edge, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}) \in \wp^2(\mathbb{V}\text{ER})$.

(O3) Recall $\mathsf{Edge}(p\,,\,\mathcal{E}) \in \wp^2(\mathbb{V}\text{ER})$ from (O2). Then, by applying ZFC, conclude $\emptyset \subseteq \mathsf{Edge}(p\,,\,\mathcal{E})$. Then, by applying (O1), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E})$.

Now, prove the lemma by the following reduction. Recall $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \emptyset$ from (Q8). Then, by applying (O1), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$. Then, by introducing (O3), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E}) \ \textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$$

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4. First, assume:

(D1) $\begin{bmatrix} (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2)\ \textbf{and} \\ V_1 \cup V_2 \subseteq P\ \textbf{and}\ P \in \bigstar(\mathcal{E}) \end{bmatrix}$

(D2) $\checkmark(\mathcal{E})$

(D3) $\mathsf{Edge}(p\,,\,\mathcal{E}) = \mathsf{Edge}(q\,,\,\mathcal{E})$

(D4) $\mathcal{E}' = (\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger$

(D5) $\mathcal{E}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$

(D6) $\mathcal{E}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$

(D7) $\mathcal{E}_\dagger = \left\{ E_\dagger \;\middle|\; \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and}\ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \end{matrix} \right\}$

Next, observe:

(N1) Suppose:

$$p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{for some}\ V\,,\,E$$

Then, by applying standard inference rules, conclude $\big[\textbf{true}\ \textbf{and}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big]$. Then, by applying standard inference rules, conclude:

$$\big[V \in \{V_1\,,\,V_2\}\ \textbf{or}\ V \notin \{V_1\,,\,V_2\}\big] \ \textbf{and}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying standard inference rules, conclude:

$$\big[V \in \{V_1\,,\,V_2\}\ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big]\ \ \textbf{or}\ \ \big[V \notin \{V_1\,,\,V_2\}\ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big]$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{r}\big[\big[V \in \{V_1\,,\,V_2\}\ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big]\ \ \textbf{for some}\ \ V\,,\,E\big]\\ \textbf{or}\ \ \big[\big[V \notin \{V_1\,,\,V_2\}\ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big]\ \ \textbf{for some}\ \ V\,,\,E\big]\end{array}$$

(N2) Suppose:

$$q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\ \ \textbf{for some}\ \ W\,,\,F$$

Then, by a reduction similar to (N1), conclude:

$$\begin{array}{r}\big[\big[W \in \{V_1\,,\,V_2\}\ \textbf{and}\ \ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big]\ \ \textbf{for some}\ \ W\,,\,F\big]\\ \textbf{or}\ \ \big[\big[W \notin \{V_1\,,\,V_2\}\ \textbf{and}\ \ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big]\ \ \textbf{for some}\ \ W\,,\,F\big]\end{array}$$

(N3) Recall $\mathcal{E}_1 = \big\{E_1 \,\big|\, (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\big\}$ from (A5). Then, by applying Definition 19 of $\curlyvee$, conclude $\mathcal{E}_1 = \big\{E_1 \,\big|\, (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\ \textbf{and}\ \ E_1 \in \mathcal{E}\big\}$. Then, by applying ZFC, conclude $\mathcal{E}_1 \subseteq \mathcal{E}$.

(N4) By a reduction similar to (N3), conclude $\mathcal{E}_2 \subseteq \mathcal{E}$.

(N5) Recall $\mathcal{E}_1\,,\,\mathcal{E}_2 \subseteq \mathcal{E}$ from (N1)(N2). Then, by applying ZFC, conclude $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E}$.

(N6) Recall $\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E}$ from (N5). Then, by applying ZFC, conclude $\mathcal{E}\,,\,\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E}$. Then, by introducing (D2), conclude $\big[\checkmark(\mathcal{E})\ \textbf{and}\ \ \mathcal{E}\,,\,\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E}\big]$. Then, by applying Lemma 10:2, conclude:

$$\begin{array}{r}\mathsf{Edge}(p\,,\,\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) = \mathsf{Edge}(p\,,\,\mathcal{E}) \setminus \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)\\ \textbf{and}\ \ \mathsf{Edge}(q\,,\,\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) = \mathsf{Edge}(q\,,\,\mathcal{E}) \setminus \mathsf{Edge}(q\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)\end{array}$$

(N7) Recall $\mathsf{Edge}(p\,,\,(\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger) = \mathsf{Edge}(p\,,\,\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$ from Lemma 10:1. Then, by applying (D4), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(p\,,\,\mathcal{E} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$. Then, by applying (N6), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}') = (\mathsf{Edge}(p\,,\,\mathcal{E}) \setminus \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$.

(N8) By a reduction similar to (N7), conclude:

$$\mathsf{Edge}(q\,,\,\mathcal{E}') = (\mathsf{Edge}(q\,,\,\mathcal{E}) \setminus \mathsf{Edge}(q\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(q\,,\,\mathcal{E}_\dagger)$$

(N9) Suppose:

$$\big[p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\ \textbf{and}\ \ V \in \{V_1\,,\,V_2\}\big]\ \ \textbf{for some}\ \ V\,,\,E$$

Then, by introducing (D1)(D2)(D5)(D6)(D7), conclude:

$$\begin{array}{c}\left[\begin{array}{c}(X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2)\\ \textbf{and}\ \ V_1 \cup V_2 \subseteq P\ \ \textbf{and}\ \ P \in \bigstar(\mathcal{E})\end{array}\right]\ \ \textbf{and}\ \ \checkmark(\mathcal{E})\\ \textbf{and}\ \ \mathcal{E}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}\\ \textbf{and}\ \ \mathcal{E}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}\\ \textbf{and}\ \ \mathcal{E}_\dagger = \left\{E_\dagger \,\middle|\, \begin{array}{l}E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)\\ \textbf{and}\ \ (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\end{array}\right\}\\ \textbf{and}\ \ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\ \ \textbf{and}\ \ V \in \{V_1\,,\,V_2\}\end{array}$$

Then, by applying Lemma 14:1, conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\ \ \textbf{and}\ \ \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by introducing (N7), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = (\mathsf{Edge}(p\,,\,\mathcal{E}) \setminus \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$$
$$\textbf{and } \mathsf{Edge}(p\,,\,\mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ and } \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by applying substitution, conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = (\mathsf{Edge}(p\,,\,\mathcal{E}) \setminus \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ and } \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by applying ZFC, conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(p\,,\,\mathcal{E}) \cup \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \textbf{ and } \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger) \subseteq \mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(p\,,\,\mathcal{E})$.

(N0) Suppose:

$$\left[q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } W \in \{V_1\,,\,V_2\}\right] \textbf{ for some } W\,,\,F$$

Then, by a reduction similar to (N9), conclude $\mathsf{Edge}(q\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E})$.

(M1) Suppose:

$$\left[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } V\,,\,E\right] \textbf{ and } \left[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } W\,,\,F\right]$$

Then, by applying (N9)(N0), conclude $\left[\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(p\,,\,\mathcal{E}) \textbf{ and } \mathsf{Edge}(q\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E})\right]$. Then, by introducing (D3), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}) = \mathsf{Edge}(q\,,\,\mathcal{E}) \textbf{ and } \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(p\,,\,\mathcal{E}) \textbf{ and } \mathsf{Edge}(q\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E})$$

Then, by applying substitution, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}')$.

(M2) Suppose:

$$\left[p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V \notin \{V_1\,,\,V_2\}\right] \textbf{ for some } V\,,\,E$$

Then, by introducing (D1)(D2)(D5)(D6)(D3), conclude:

$$\begin{bmatrix} (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{ and } \checkmark(\mathcal{E})$$
$$\textbf{and } \mathcal{E}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$$
$$\textbf{and } \mathcal{E}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2)\}$$
$$\textbf{and } \mathsf{Edge}(p\,,\,\mathcal{E}) = \mathsf{Edge}(q\,,\,\mathcal{E}) \textbf{ and}$$
$$p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V \notin \{V_1\,,\,V_2\}$$

Then, by applying Lemma 13:8, conclude:

$$\left[q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } W \notin \{V_1\,,\,V_2\}\right] \textbf{ for some } W\,,\,F$$

(M3) Suppose:

$$\left[q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } W \notin \{V_1\,,\,V_2\}\right] \textbf{ for some } W\,,\,F$$

Then, by a reduction similar to (M2), conclude:

$$\left[p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V \notin \{V_1\,,\,V_2\}\right] \textbf{ for some } V\,,\,E$$

(M4) Suppose:

$$[p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \;\textbf{ and }\; p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2] \;\textbf{ for some }\; V, V', E, E'$$

Then, by introducing (N5), conclude:

$$\mathcal{E}_1 \cup \mathcal{E}_2 \subseteq \mathcal{E} \;\textbf{ and }\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \;\textbf{ and }\; p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2$$

Then, by applying ZFC, conclude $[p \in V \in E \in \mathcal{E} \;\textbf{ and }\; p \in V' \in E' \in \mathcal{E}]$. Then, by introducing (D2), conclude $[\checkmark(\mathcal{E}) \;\textbf{ and }\; p \in V \in E \in \mathcal{E} \;\textbf{ and }\; p \in V' \in E' \in \mathcal{E}]$. Then, by applying Definition 24 of $\checkmark$, conclude:

$$[[\begin{bmatrix} p' \in V_1' \in E_1 \in \mathcal{E} \\ \textbf{and } p' \in V_2' \in E_2 \in \mathcal{E} \end{bmatrix} \;\textbf{ implies }\; V_1' = V_2'] \;\textbf{ for all }\; p', V_1', V_2', E_1, E_2]$$
$$p \in V \in E \in \mathcal{E} \;\textbf{ and }\; p \in V' \in E' \in \mathcal{E}$$

Then, by applying standard inference rules, conclude $V = V'$.

(M5) Suppose:

$$[q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \;\textbf{ and }\; q \in W' \in F' \in \mathcal{E}_1 \cup \mathcal{E}_2] \;\textbf{ for some }\; W, W', F, F'$$

Then, by a reduction similar to (M4), conclude $W = W'$.

(M6) Suppose:

$$[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1, V_2\} \end{bmatrix} \;\textbf{ for some }\; V, E] \;\textbf{ and }\; [\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1, V_2\} \end{bmatrix} \;\textbf{ for some }\; W, F]$$

Then, by applying (M3), conclude:

$$[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1, V_2\} \end{bmatrix} \;\textbf{ for some }\; V, E] \;\textbf{ and }\; [\begin{bmatrix} p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V' \notin \{V_1, V_2\} \end{bmatrix} \;\textbf{ for some }\; V', E']$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \;\textbf{ and }\; V \in \{V_1, V_2\} \\ \textbf{and } p \in V' \in E' \in \mathcal{E}_1 \cup \mathcal{E}_2 \;\textbf{ and }\; V' \notin \{V_1, V_2\} \end{bmatrix} \;\textbf{ for some }\; V, V', E, E'$$

Then, by introducing (M4), conclude $[V = V' \;\textbf{ and }\; V \in \{V_1, V_2\} \;\textbf{ and }\; V' \notin \{V_1, V_2\}]$. Then, by applying substitution, conclude $[V \in \{V_1, V_2\} \;\textbf{ and }\; V \notin \{V_1, V_2\}]$. Then, by applying standard inference rules, conclude **false**.

(M7) Suppose:

$$[[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1, V_2\} \end{bmatrix} \;\textbf{ for some }\; V, E] \;\textbf{ and }\; [\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1, V_2\} \end{bmatrix} \;\textbf{ for some }\; W, F]]$$

Then, by a reduction similar to (M6), conclude **false**.

(M8) Recall $\mathsf{Edge}(p, \mathcal{E}) = \mathsf{Edge}(q, \mathcal{E})$ from (D3). Then, by applying Definition 22 of Edge, conclude:

$$p, q \in \mathbb{P}\textsc{ort}$$

(M9) Suppose $[[\textbf{not }\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \;\textbf{ for all }\; V, E]$. Then, by introducing (M8), conclude:

$$p \in \mathbb{P}\textsc{ort} \;\textbf{ and }\; [[\textbf{not }\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \;\textbf{ for all }\; V, E]$$

Then, by introducing (D1)(D2)(D5)(D6)(D7), conclude:

$$\begin{bmatrix}(X\,,\,V_1)\curlyvee_{\mathcal{E}}(Y\,,\,V_2)\ \textbf{and}\\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{E})\end{bmatrix}\ \textbf{and}\ \checkmark(\mathcal{E})$$
$$\textbf{and}\ \mathcal{E}_1=\{E_1\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}$$
$$\textbf{and}\ \mathcal{E}_2=\{E_2\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\}$$
$$\textbf{and}\ \mathcal{E}_\dagger=\left\{E_\dagger\;\middle|\;\begin{matrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\end{matrix}\right\}\ \textbf{and}$$
$$p\in\mathbb{P}\text{ORT}\ \textbf{and}\ \big[\big[\textbf{not}\ p\in V\in E\in\mathcal{E}_1\cup\mathcal{E}_2\big]\ \textbf{for all}\ V\,,\,E\big]$$

Then, by applying Lemma 14:3, conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}_1\cup\mathcal{E}_2)=\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\ \textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\subseteq\mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by introducing (N7), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}')=(\mathsf{Edge}(p\,,\,\mathcal{E})\setminus\mathsf{Edge}(p\,,\,\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)$$
$$\textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}_1\cup\mathcal{E}_2)=\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\ \textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\subseteq\mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by applying substitution, conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}')=(\mathsf{Edge}(p\,,\,\mathcal{E})\setminus\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger))\cup\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\ \textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\subseteq\mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by applying ZFC, conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}')=\mathsf{Edge}(p\,,\,\mathcal{E})\cup\mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\ \textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}_\dagger)\subseteq\mathsf{Edge}(p\,,\,\mathcal{E})$$

Then, by applying ZFC, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}')=\mathsf{Edge}(p\,,\,\mathcal{E})$.

(M0) Suppose $\big[\big[\textbf{not}\ q\in W\in F\in\mathcal{E}_1\cup\mathcal{E}_2\big]\ \textbf{for all}\ W\,,\,F\big]$. Then, by a reduction similar to (M9), conclude $\mathsf{Edge}(q\,,\,\mathcal{E}')=\mathsf{Edge}(q\,,\,\mathcal{E})$.

(L1) Suppose:

$$\Big[\begin{bmatrix}p\in V\in E\in\mathcal{E}_1\cup\mathcal{E}_2\\ \textbf{and}\ V\in\{V_1\,,\,V_2\}\end{bmatrix}\ \textbf{for some}\ V\,,\,E\Big]\ \textbf{and}\ \big[\big[\textbf{not}\ q\in W\in F\in\mathcal{E}_1\cup\mathcal{E}_2\big]\ \textbf{for all}\ W\,,\,F\big]$$

Then, by applying (N9), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}')=\mathsf{Edge}(p\,,\,\mathcal{E})\ \textbf{and}\ \big[\big[\textbf{not}\ q\in W\in F\in\mathcal{E}_1\cup\mathcal{E}_2\big]\ \textbf{for all}\ W\,,\,F\big]$$

Then, by applying (M0), conclude $\big[\mathsf{Edge}(p\,,\,\mathcal{E}')=\mathsf{Edge}(p\,,\,\mathcal{E})\ \textbf{and}\ \mathsf{Edge}(q\,,\,\mathcal{E}')=\mathsf{Edge}(q\,,\,\mathcal{E})\big]$. Then, by introducing (D3), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E})=\mathsf{Edge}(q\,,\,\mathcal{E})\ \textbf{and}\ \mathsf{Edge}(p\,,\,\mathcal{E}')=\mathsf{Edge}(p\,,\,\mathcal{E})\ \textbf{and}\ \mathsf{Edge}(q\,,\,\mathcal{E}')=\mathsf{Edge}(q\,,\,\mathcal{E})$$

Then, by applying substitution, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}')=\mathsf{Edge}(q\,,\,\mathcal{E}')$.

(L2) Suppose:

$$\Big[\big[\big[\textbf{not}\ p\in V\in E\in\mathcal{E}_1\cup\mathcal{E}_2\big]\ \textbf{for all}\ V\,,\,E\big]\ \textbf{and}\ \Big[\begin{bmatrix}q\in W\in F\in\mathcal{E}_1\cup\mathcal{E}_2\\ \textbf{and}\ W\in\{V_1\,,\,V_2\}\end{bmatrix}\ \textbf{for some}\ W\,,\,F\Big]\Big]$$

Then, by a reduction similar to (L1), conclude $\mathsf{Edge}(p\,,\,\mathcal{E}')=\mathsf{Edge}(q\,,\,\mathcal{E}')$.

(L3) Suppose:

$$[[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1, V_2\} \end{bmatrix} \textbf{ for some } V, E] \textbf{ and } [\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1, V_2\} \end{bmatrix} \textbf{ for some } W, F]]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V \notin \{V_1, V_2\} \\ \textbf{and } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } W \notin \{V_1, V_2\} \end{bmatrix} \textbf{ for some } V, W, E, F$$

Then, by introducing (D1)(D2)(D5)(D6)(D7)(D3), conclude:

$$\begin{bmatrix} (X, V_1) \curlyvee_{\mathcal{E}} (Y, V_2) \textbf{ and} \\ V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E}) \end{bmatrix} \textbf{ and } \checkmark(\mathcal{E})$$
$$\textbf{and } \mathcal{E}_1 = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\}$$
$$\textbf{and } \mathcal{E}_2 = \{E_2 \mid (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\}$$
$$\textbf{and } \mathcal{E}_\dagger = \left\{ E_\dagger \middle| \begin{matrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \end{matrix} \right\}$$
$$\textbf{and } \mathsf{Edge}(p, \mathcal{E}) = \mathsf{Edge}(q, \mathcal{E})$$
$$\textbf{and } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } V \notin \{V_1, V_2\}$$
$$\textbf{and } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ and } W \notin \{V_1, V_2\}$$

Then, by applying Lemma 14:2, conclude:

$$\mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(q, \mathcal{E}_1 \cup \mathcal{E}_2) \textbf{ and } \mathsf{Edge}(p, \mathcal{E}_\dagger) = \mathsf{Edge}(q, \mathcal{E}_\dagger)$$

Then, by introducing (N7), conclude:

$$\mathsf{Edge}(p, \mathcal{E}') = (\mathsf{Edge}(p, \mathcal{E}) \setminus \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(p, \mathcal{E}_\dagger)$$
$$\textbf{and } \mathsf{Edge}(p, \mathcal{E}_1 \cup \mathcal{E}_2) = \mathsf{Edge}(q, \mathcal{E}_1 \cup \mathcal{E}_2) \textbf{ and } \mathsf{Edge}(p, \mathcal{E}_\dagger) = \mathsf{Edge}(q, \mathcal{E}_\dagger)$$

Then, by applying substitution, conclude:

$$\mathsf{Edge}(p, \mathcal{E}') = (\mathsf{Edge}(p, \mathcal{E}) \setminus \mathsf{Edge}(q, \mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(p, \mathcal{E}_\dagger) \textbf{ and } \mathsf{Edge}(p, \mathcal{E}_\dagger) = \mathsf{Edge}(q, \mathcal{E}_\dagger)$$

Then, by applying substitution, conclude:

$$\mathsf{Edge}(p, \mathcal{E}') = (\mathsf{Edge}(p, \mathcal{E}) \setminus \mathsf{Edge}(q, \mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(q, \mathcal{E}_\dagger)$$

Then, by applying (D3), conclude:

$$\mathsf{Edge}(p, \mathcal{E}') = (\mathsf{Edge}(q, \mathcal{E}) \setminus \mathsf{Edge}(q, \mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathsf{Edge}(q, \mathcal{E}_\dagger)$$

Then, by applying (N8), conclude $\mathsf{Edge}(p, \mathcal{E}') = \mathsf{Edge}(q, \mathcal{E}')$.

(L4) Suppose:

$$[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1, V_2\} \end{bmatrix} \textbf{ for some } V, E] \textbf{ and } [[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \textbf{ for all } W, F]$$

Then, by applying (M2), conclude:

$$[q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \textbf{ for some } W, F] \textbf{ and } [[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \textbf{ for all } W, F]$$

Then, by applying standard inference rules, conclude **false**.

(L5) Suppose:

$$\left[\left[\left[\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\right] \textbf{ for all } V, E\right] \textbf{ and } \left[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1, V_2\}\end{bmatrix} \textbf{ for some } W, F\right]\right]$$

Then, by a reduction similar to (L4), conclude **false**.

(L6) Suppose:

$$\left[\left[\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\right] \textbf{ for all } V, E\right] \textbf{ and } \left[\left[\textbf{not}\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\right] \textbf{ for all } W, F\right]$$

Then, by applying (M9)(M0), conclude $\left[\mathsf{Edge}(p, \mathcal{E}') = \mathsf{Edge}(p, \mathcal{E}) \textbf{ and } \mathsf{Edge}(q, \mathcal{E}') = \mathsf{Edge}(q, \mathcal{E})\right]$. Then, by introducing (D3), conclude:

$$\mathsf{Edge}(p, \mathcal{E}) = \mathsf{Edge}(q, \mathcal{E}) \textbf{ and } \mathsf{Edge}(p, \mathcal{E}') = \mathsf{Edge}(p, \mathcal{E}) \textbf{ and } \mathsf{Edge}(q, \mathcal{E}') = \mathsf{Edge}(q, \mathcal{E})$$

Then, by applying substitution, conclude $\mathsf{Edge}(p, \mathcal{E}') = \mathsf{Edge}(q, \mathcal{E}')$.

Now, prove the lemma by the following reduction. Suppose **true**. Then, by applying standard inference rules, conclude $[\textbf{true and true}]$. Then, by applying standard inference rules, conclude:

$$\left[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{for some } V, E\end{bmatrix} \textbf{ or } \begin{bmatrix} [\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \\ \textbf{for all } V, E\end{bmatrix}\right]$$
$$\textbf{and}$$
$$\left[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{for some } W, F\end{bmatrix} \textbf{ or } \begin{bmatrix} [\textbf{not}\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \\ \textbf{for all } W, F\end{bmatrix}\right]$$

Then, by applying (N1), conclude:

$$\left[\begin{bmatrix} \begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1, V_2\}\end{bmatrix} \\ \textbf{for some } V, E\end{bmatrix} \textbf{ or } \begin{bmatrix} \begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1, V_2\}\end{bmatrix} \\ \textbf{for some } V, E\end{bmatrix} \textbf{ or } \begin{bmatrix} [\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \\ \textbf{for all } V, E\end{bmatrix}\right]$$
$$\textbf{and}$$
$$\left[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{for some } W, F\end{bmatrix} \textbf{ or } \begin{bmatrix} [\textbf{not}\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \\ \textbf{for all } W, F\end{bmatrix}\right]$$

Then, by applying (N2), conclude:

$$\left[\begin{bmatrix} \begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1, V_2\}\end{bmatrix} \\ \textbf{for some } V, E\end{bmatrix} \textbf{ or } \begin{bmatrix} \begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1, V_2\}\end{bmatrix} \\ \textbf{for some } V, E\end{bmatrix} \textbf{ or } \begin{bmatrix} [\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \\ \textbf{for all } V, E\end{bmatrix}\right]$$
$$\textbf{and}$$
$$\left[\begin{bmatrix} \begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1, V_2\}\end{bmatrix} \\ \textbf{for some } W, F\end{bmatrix} \textbf{ or } \begin{bmatrix} \begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1, V_2\}\end{bmatrix} \\ \textbf{for some } W, F\end{bmatrix} \textbf{ or } \begin{bmatrix} [\textbf{not}\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \\ \textbf{for all } W, F\end{bmatrix}\right]$$

Then, by applying standard inference rules, conclude:

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } V , E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } W , F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } V , E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } W , F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } V , E\Big] \textbf{ and } \Big[\big[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } W , F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } V , E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } W , F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } V , E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } W , F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } V , E\Big] \textbf{ and } \Big[\big[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } W , F\Big]\Big]$$

or

$$\Big[\Big[\big[\textbf{not } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } V , E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } W , F\Big]\Big]$$

or

$$\Big[\Big[\big[\textbf{not } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } V , E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1 , V_2\} \end{bmatrix} \textbf{ for some } W , F\Big]\Big]$$

or

$$\Big[\Big[\big[\textbf{not } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } V , E\Big] \textbf{ and } \Big[\big[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } W , F\Big]\Big]$$

Then, by applying (M1), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \;\textbf{ or}$$

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } V\,,\,E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } W\,,\,F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \in \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } V\,,\,E\Big] \textbf{ and } \big[\big[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } W\,,\,F\big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } V\,,\,E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } W\,,\,F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } V\,,\,E\Big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } W\,,\,F\Big]\Big]$$

or

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } V \notin \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } V\,,\,E\Big] \textbf{ and } \big[\big[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } W\,,\,F\big]\Big]$$

or

$$\Big[\big[\big[\textbf{not } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } V\,,\,E\big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \in \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } W\,,\,F\Big]\Big]$$

or

$$\Big[\big[\big[\textbf{not } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } V\,,\,E\big] \textbf{ and } \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and } W \notin \{V_1\,,\,V_2\} \end{bmatrix} \textbf{ for some } W\,,\,F\Big]\Big]$$

or

$$\Big[\big[\big[\textbf{not } p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } V\,,\,E\big] \textbf{ and } \big[\big[\textbf{not } q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \textbf{ for all } W\,,\,F\big]\Big]$$

Then, by applying (M6)(M7), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \;\;\textbf{or}\;\; \textbf{false} \;\;\textbf{or}$$

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; V \in \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\big[\textbf{not}\;\; q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}\;\; \textbf{false} \;\;\textbf{or}$$

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; V \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; W \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}$$

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; V \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\big[\textbf{not}\;\; q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}$$

$$\Big[\Big[\big[\textbf{not}\;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; W \in \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}$$

$$\Big[\Big[\big[\textbf{not}\;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; W \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}$$

$$\Big[\Big[\big[\textbf{not}\;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\big[\textbf{not}\;\; q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; W\,,\,F\Big]\Big]$$

Then, by applying (L1)(L2), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \;\;\textbf{or}\;\; \textbf{false} \;\;\textbf{or}\;\; \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \;\;\textbf{or}\;\; \textbf{false} \;\;\textbf{or}$$

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; V \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; W \in \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}$$

$$\Big[\Big[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; V \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; W \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}\;\; \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \;\;\textbf{or}$$

$$\Big[\Big[\big[\textbf{not}\;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; W \in \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}$$

$$\Big[\Big[\big[\textbf{not}\;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\;\; W \notin \{V_1\,,\,V_2\} \end{bmatrix} \;\textbf{for some}\;\; W\,,\,F\Big]\Big]$$

$$\textbf{or}$$

$$\Big[\Big[\big[\textbf{not}\;\; p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; V\,,\,E\Big] \;\;\textbf{and}\;\; \Big[\big[\textbf{not}\;\; q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2\big] \;\textbf{for all}\;\; W\,,\,F\Big]\Big]$$

Then, by applying (L3), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}\ \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}$$
$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}$$
$$[[\begin{bmatrix} p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\ V \notin \{V_1\,,\,V_2\}\end{bmatrix} \ \textbf{for some}\ V\,,\,E] \ \textbf{and}\ [[\textbf{not}\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \ \textbf{for all}\ W\,,\,F]]$$
$$\textbf{or}\ \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}$$
$$[[[\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \ \textbf{for all}\ V\,,\,E] \ \textbf{and}\ [\begin{bmatrix} q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2 \\ \textbf{and}\ W \notin \{V_1\,,\,V_2\}\end{bmatrix} \ \textbf{for some}\ W\,,\,F]]$$
$$\textbf{or}$$
$$[[[\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \ \textbf{for all}\ V\,,\,E] \ \textbf{and}\ [[\textbf{not}\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \ \textbf{for all}\ W\,,\,F]]$$

Then, by applying (L4)(L5), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}\ \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}$$
$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}\ \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}$$
$$[[[\textbf{not}\ p \in V \in E \in \mathcal{E}_1 \cup \mathcal{E}_2] \ \textbf{for all}\ V\,,\,E] \ \textbf{and}\ [[\textbf{not}\ q \in W \in F \in \mathcal{E}_1 \cup \mathcal{E}_2] \ \textbf{for all}\ W\,,\,F]]$$

Then, by applying (L6), conclude:

$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}\ \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}$$
$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}\ \mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}\ \textbf{or}$$
$$\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}')$$

Then, by applying standard inference rules, conclude $[\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}') \ \textbf{or}\ \textbf{false}]$. Then, by applying standard inference rules, conclude $\mathsf{Edge}(p\,,\,\mathcal{E}') = \mathsf{Edge}(q\,,\,\mathcal{E}')$.

⦇ QED. ⦈

B.14 Theorem 4

Proof (of Theorem 4).

1. First, assume:

   (A1) Pre

   Next, observe:

   (Z1) Recall Pre from (A1). Then, by applying Figure 15, conclude:

   $$(\mathcal{V}\,,\,\mathcal{E}_{\text{in}}) \in \mathbb{G}\text{RAPH} \textbf{ and } \checkmark(\mathcal{E}_{\text{in}}) \textbf{ and } \mathcal{E} = \mathcal{E}_{\text{in}}$$

   (Z2) Recall $(\mathcal{V}\,,\,\mathcal{E}_{\text{in}}) \in \mathbb{G}\text{RAPH}$ from (Z1). Then, by applying Definition 18 of $[\![\cdot]\!]$, conclude:

   $$[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \in \mathbb{NSC}$$

   Then, by applying standard inference rules, conclude $[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]$.

   (Z3) Recall $(\mathcal{V}\,,\,\mathcal{E}_{\text{in}}) \in \mathbb{G}\text{RAPH}$ from (Z1). Then, by applying Definition 16 of $\mathbb{G}\text{RAPH}$, conclude:

   $$\mathcal{E}_{\text{in}} \in \wp^2(\mathbb{V}\text{ER})$$

   Then, by applying Definition 23 of $\bigstar$, conclude $\bigstar(\mathcal{E}_{\text{in}}) \in \wp^2(\mathbb{P}\text{ORT})$. Then, by applying standard inference rules, conclude $\bigstar(\mathcal{E}_{\text{in}}) = \bigstar(\mathcal{E}_{\text{in}})$.

   Now, prove the theorem by the following reduction. Recall from (Z2)(Z3)(Z1):

   $$[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{E}_{\text{in}}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{E}_{\text{in}})$$

   Then, by applying (Z1), conclude $\big[[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{E}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{E})\big]$. Then, by applying Figure 15, conclude Inv_1.

   ⦇ QED. ⦈

2. First, assume:

   (B1) Inv_1

   (B2) Cond_1

   (B3) $|\mathcal{E}| = z_1$

   Next, observe:

   (Y1) Recall Inv_1 from (B1). Then, by applying Figure 15, conclude:

   $$[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{E}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{E})$$

   (Y2) Recall Cond_1 from (B2). Then, by applying Figure 15, conclude:

   $$\big[(X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E})\big] \textbf{ for some } X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P$$

   (Y3) Suppose:

   $$\mathcal{E} = \mathcal{F} \textbf{ for some } \mathcal{F}$$

   Then, by introducing (Y1)(B3)(Y2), conclude:

   $$\mathcal{E} = \mathcal{F} \textbf{ and } [\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{E}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{E}) \textbf{ and } |\mathcal{E}| = z_1 \textbf{ and } \big[\big[(X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{E})\big] \textbf{ for some } X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\big]$$

   Then, by applying substitution, conclude:

   $$[\![(\mathcal{V}\,,\,\mathcal{F})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and } |\mathcal{F}| = z_1 \textbf{ and } \big[\big[(X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F})\big] \textbf{ for some } X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\big]$$

(Y4) Suppose:

$$\mathcal{E} = \mathcal{F} \textbf{ for some } \mathcal{F}$$

Then, by introducing (Y1), conclude $\left[\mathcal{E} = \mathcal{F} \textbf{ and } [\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]\right]$. Then, by applying (Y3), conclude $\left[[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } [\![(\mathcal{V}\,,\,\mathcal{F})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]\right]$. Then, by applying substitution, conclude $[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{F})]\!]$.

(Y5) Suppose:

$$\left[\mathcal{E} = \mathcal{F} \textbf{ and } (X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2)\right] \textbf{ for some } \mathcal{F}\,,\,X\,,\,Y\,,\,V_1\,,\,V_2$$

Then, by applying substitution, conclude $\left[\mathcal{E} = \mathcal{F} \textbf{ and } (X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2)\right]$. Then, by applying Figure 15, conclude $\left[\mathcal{E} = \mathcal{F} \textbf{ and } \mathsf{Cond}_2\right]$. Then, by applying standard inference rules, conclude:

$$\left[\mathcal{E} = \mathcal{F} \textbf{ and } \mathsf{Cond}_2\right] \textbf{ or } \textbf{false}$$

Then, by applying standard inference rules, conclude $\left[\left[\mathcal{E} = \mathcal{F} \textbf{ and } \mathsf{Cond}_2\right] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\right]$.

(Y6) Suppose:

$$\mathcal{E} = \mathcal{F} \textbf{ for some } \mathcal{F}$$

Then, by applying ZFC, conclude $\mathcal{E} = \mathcal{F} \setminus \emptyset$. Then, by applying ZFC, conclude $\mathcal{E} = \mathcal{F} \setminus (\emptyset \cup \emptyset)$. Then, by applying ZFC, conclude $\mathcal{E} = (\mathcal{F} \setminus (\emptyset \cup \emptyset)) \cup \emptyset$.

(Y7) Suppose:

$$\left[\mathcal{E} = \mathcal{F} \textbf{ and } \mathcal{E}_1 = \emptyset \textbf{ and } \mathcal{E}_2 = \emptyset \textbf{ and } \mathcal{E}_\dagger = \emptyset\right] \textbf{ for some } \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger\,,\,\mathcal{F}$$

Then, by introducing (Y6), conclude $\left[\mathcal{E} = (\mathcal{F} \setminus (\emptyset \cup \emptyset)) \cup \emptyset \textbf{ and } \mathcal{E}_1 = \emptyset \textbf{ and } \mathcal{E}_2 = \emptyset \textbf{ and } \mathcal{E}_\dagger = \emptyset\right]$. Then, by applying substitution, conclude $\mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger$.

(Y8) Suppose:

$$\left[\mathcal{E}_1 = \emptyset \textbf{ and } \mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}\right] \textbf{ for some } \mathcal{E}_1\,,\,\mathcal{F}_1$$

Then, by applying ZFC, conclude

$$\mathcal{E}_1 = \emptyset \textbf{ and } \mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \textbf{ and } \emptyset \in \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}$$

Then, by applying substitution, conclude $\mathcal{E}_1 \subseteq \mathcal{F}_1$.

(Y9) Suppose:

$$\left[\mathcal{E}_2 = \emptyset \textbf{ and } \mathcal{F}_2 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}\right] \textbf{ for some } \mathcal{E}_2\,,\,\mathcal{F}_2$$

Then, by a reduction similar to (Y8), conclude $\mathcal{E}_2 \subseteq \mathcal{F}_2$.

(Y0) Suppose:

$$\left[\mathcal{E}_\dagger = \emptyset \textbf{ and } \mathcal{F}_\dagger = \left\{E_\dagger \;\middle|\; \begin{array}{l} E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \\ \textbf{and } (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2) \end{array}\right\}\right] \textbf{ for some } \mathcal{E}_\dagger\,,\,\mathcal{F}_\dagger$$

Then, by a reduction similar to (Y8), conclude $\mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger$.

(X1) Suppose:

$$\left[\mathcal{E}_1 = \emptyset \textbf{ and } E'_1 \in \mathcal{E}_1 \textbf{ and } (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\right] \textbf{ for some } \mathcal{E}_1\,,\,E'_1\,,\,E'_2$$

Then, by applying substitution, conclude $\left[E'_1 \in \emptyset \textbf{ and } (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\right]$. Then, by applying ZFC, conclude $\left[\textbf{false} \textbf{ and } (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\right]$. Then, by applying standard inference rules, conclude **false**. Then, by applying standard inference rules, conclude:

$$E'_2 \in \mathcal{E}_2 \textbf{ and } \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger$$

(X2) Suppose:

$$\big[\mathcal{E}_2 = \emptyset \ \textbf{and}\ E'_2 \in \mathcal{E}_2 \ \textbf{and}\ (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big] \ \textbf{for some}\ \mathcal{E}_2\,,\,E'_1\,,\,E'_2$$

Then, by a reduction similar to (X1), conclude:

$$E'_1 \in \mathcal{E}_1 \ \textbf{and}\ \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger$$

(X3) Suppose:

$$\begin{bmatrix} \mathcal{E} = \mathcal{F} \ \textbf{and}\ \mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and}\ \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \\ \textbf{and}\ \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and}\ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \end{bmatrix}$$
$$\textbf{for some}\ \mathcal{F}\,,\,\mathcal{F}_1\,,\,\mathcal{F}_2\,,\,\mathcal{F}_\dagger$$

Then, by applying ZFC, conclude:

$$\begin{gathered} \mathcal{E} = \mathcal{F} \ \textbf{and}\ \mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and}\ \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \\ \textbf{and}\ \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and}\ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and} \\ \big[\mathcal{E}_1 = \emptyset \ \textbf{for some}\ \mathcal{E}_1\big] \ \textbf{and}\ \big[\mathcal{E}_2 = \emptyset \ \textbf{for some}\ \mathcal{E}_2\big] \ \textbf{and}\ \big[\mathcal{E}_\dagger = \emptyset \ \textbf{for some}\ \mathcal{E}_\dagger\big] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \mathcal{E} = \mathcal{F} \ \textbf{and}\ \mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and}\ \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \\ \textbf{and}\ \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and}\ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and} \\ \mathcal{E}_1 = \emptyset \ \textbf{and}\ \mathcal{E}_2 = \emptyset \ \textbf{and}\ \mathcal{E}_\dagger = \emptyset \end{bmatrix}$$
$$\textbf{for some}\ \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (Y7), conclude:

$$\begin{gathered} \mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and}\ \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \\ \textbf{and}\ \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and}\ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and} \\ \mathcal{E}_1 = \emptyset \ \textbf{and}\ \mathcal{E}_2 = \emptyset \ \textbf{and}\ \mathcal{E}_\dagger = \emptyset \ \textbf{and}\ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \end{gathered}$$

Then, by applying (Y8)(Y9)(Y0), conclude:

$$\begin{gathered} \mathcal{E}_1 = \emptyset \ \textbf{and}\ \mathcal{E}_2 = \emptyset \ \textbf{and} \\ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and}\ \mathcal{E}_1 \subseteq \mathcal{F}_1 \ \textbf{and}\ \mathcal{E}_2 \subseteq \mathcal{F}_2 \ \textbf{and}\ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \end{gathered}$$

Then, by applying (X1)(X2), conclude:

$$\begin{gathered} \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and}\ \mathcal{E}_1 \subseteq \mathcal{F}_1 \ \textbf{and}\ \mathcal{E}_2 \subseteq \mathcal{F}_2 \ \textbf{and}\ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and}\ \left[\begin{bmatrix} \big[E'_1 \in \mathcal{E}_1 \ \textbf{and}\ (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big] \ \textbf{implies} \\ \big[E'_2 \in \mathcal{E}_2 \ \textbf{and}\ \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big] \end{bmatrix} \textbf{for all}\ E'_1\,,\,E'_2\right] \\ \textbf{and}\ \left[\begin{bmatrix} \big[E'_2 \in \mathcal{E}_2 \ \textbf{and}\ (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big] \ \textbf{implies} \\ \big[E'_1 \in \mathcal{E}_1 \ \textbf{and}\ \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big] \end{bmatrix} \textbf{for all}\ E'_1\,,\,E'_2\right] \end{gathered}$$

Now, prove the theorem by the following reduction. Suppose **true**. Then, by applying standard inference rules, conclude:

$$\mathcal{E} = \mathcal{F} \ \textbf{for some}\ \mathcal{F}$$

Then, by applying (Y3), conclude:

$$\begin{gathered} \mathcal{E} = \mathcal{F} \ \textbf{and}\ [\![(\mathcal{V}\,,\,\mathcal{F})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \ \textbf{and}\ \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \ \textbf{and}\ \checkmark(\mathcal{F}) \ \textbf{and}\ |\mathcal{F}| = z_1 \ \textbf{and} \\ \big[\big[(X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2) \ \textbf{and}\ V_1 \cup V_2 \subseteq P \ \textbf{and}\ P \in \bigstar(\mathcal{F})\big] \ \textbf{for some}\ X\,,\,Y\,,\,V_1\,,\,V_2\,,\,P\big] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \mathcal{E} = \mathcal{F} \textbf{ and } [\![(\mathcal{V}, \mathcal{F})]\!] = [\![(\mathcal{V}, \mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\ |\mathcal{F}| = z_1 \textbf{ and } (X, V_1) \curlyvee_{\mathcal{F}} (Y, V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \end{bmatrix}$$

$$\textbf{for some } X, Y, V_1, V_2, P$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered} \mathcal{E} = \mathcal{F} \textbf{ and } [\![(\mathcal{V}, \mathcal{F})]\!] = [\![(\mathcal{V}, \mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\ |\mathcal{F}| = z_1 \textbf{ and } (X, V_1) \curlyvee_{\mathcal{F}} (Y, V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \\ \textbf{and } \begin{bmatrix} \mathcal{F}_1 = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{for some } \mathcal{F}_1 \end{bmatrix} \textbf{ and } \begin{bmatrix} \mathcal{F}_2 = \{E_2 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{for some } \mathcal{F}_2 \end{bmatrix} \textbf{ and} \\ [\mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \textbf{ for some } \mathcal{F}_\dagger] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \mathcal{E} = \mathcal{F} \textbf{ and } [\![(\mathcal{V}, \mathcal{F})]\!] = [\![(\mathcal{V}, \mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\ |\mathcal{F}| = z_1 \textbf{ and } (X, V_1) \curlyvee_{\mathcal{F}} (Y, V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and} \\ \mathcal{F}_1 = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \textbf{ and } \mathcal{F}_2 = \{E_2 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{and } \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \end{bmatrix}$$

$$\textbf{for some } \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_\dagger$$

Then, by applying Y4, conclude:

$$\begin{gathered} \mathcal{E} = \mathcal{F} \textbf{ and } [\![(\mathcal{V}, \mathcal{F})]\!] = [\![(\mathcal{V}, \mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\ |\mathcal{F}| = z_1 \textbf{ and } (X, V_1) \curlyvee_{\mathcal{F}} (Y, V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and} \\ \mathcal{F}_1 = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \textbf{ and } \mathcal{F}_2 = \{E_2 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{and } \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{and } [\![(\mathcal{V}, \mathcal{E})]\!] = [\![(\mathcal{V}, \mathcal{F})]\!] \end{gathered}$$

Then, by applying Y5, conclude:

$$\begin{gathered} \mathcal{E} = \mathcal{F} \textbf{ and } [\![(\mathcal{V}, \mathcal{F})]\!] = [\![(\mathcal{V}, \mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\ |\mathcal{F}| = z_1 \textbf{ and } (X, V_1) \curlyvee_{\mathcal{F}} (Y, V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and} \\ \mathcal{F}_1 = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \textbf{ and } \mathcal{F}_2 = \{E_2 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{and } \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{and } [\![(\mathcal{V}, \mathcal{E})]\!] = [\![(\mathcal{V}, \mathcal{F})]\!] \textbf{ and } \big[[\mathsf{Cond}_2 \textbf{ and } \mathcal{E} = \mathcal{F}] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\big] \end{gathered}$$

Then, by applying X3, conclude:

$$\begin{gathered} [\![(\mathcal{V}, \mathcal{F})]\!] = [\![(\mathcal{V}, \mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\ |\mathcal{F}| = z_1 \textbf{ and } (X, V_1) \curlyvee_{\mathcal{F}} (Y, V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and} \\ \mathcal{F}_1 = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \textbf{ and } \mathcal{F}_2 = \{E_2 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \textbf{and } \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \textbf{ and} \\ \textbf{and } [\![(\mathcal{V}, \mathcal{E})]\!] = [\![(\mathcal{V}, \mathcal{F})]\!] \textbf{ and } \big[[\mathsf{Cond}_2 \textbf{ and } \mathcal{E} = \mathcal{F}] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\big] \textbf{ and} \\ \begin{bmatrix} \begin{bmatrix} \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and } \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and } \left[\begin{bmatrix} [E_1' \in \mathcal{E}_1 \textbf{ and } (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \textbf{ implies} \\ [E_2' \in \mathcal{E}_2 \textbf{ and } \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \textbf{ for all } E_1', E_2'\right] \\ \textbf{and } \left[\begin{bmatrix} [E_2' \in \mathcal{E}_2 \textbf{ and } (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \textbf{ implies} \\ [E_1' \in \mathcal{E}_1 \textbf{ and } \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \textbf{ for all } E_1', E_2'\right] \end{bmatrix} \\ \textbf{for some } \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger \end{bmatrix} \end{gathered}$$

Then, by applying Figure 15, conclude Inv_2.

⦇ QED. ⦈

3. First, assume:

(C1) Inv_2

(C2) Cond_2

(C3) $|\mathcal{E}| = z_2$

Next, observe:

(W1) Recall Inv_2 from (C1). Then, by applying Figure 15, conclude:

$$\begin{gathered}
[\![(\mathcal{V}\,,\,\mathcal{F})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\
|\mathcal{F}| = z_1 \textbf{ and } (X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and} \\
\mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \textbf{ and } \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \\
\textbf{and } \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \textbf{ and} \\
\textbf{and } [\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{F})]\!] \textbf{ and } \big[[\mathsf{Cond}_2 \textbf{ and } \mathcal{E} = \mathcal{F}] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\big] \textbf{ and} \\
\left[\begin{array}{c}
\left[\begin{array}{l}
\quad \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and } \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\
\textbf{and } \left[\left[\begin{array}{c}[E_1' \in \mathcal{E}_1 \textbf{ and } (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2)] \textbf{ implies} \\ [E_2' \in \mathcal{E}_2 \textbf{ and } \{V_1\,,\,V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger]\end{array}\right] \textbf{ for all } E_1'\,,\,E_2'\right] \\
\textbf{and } \left[\left[\begin{array}{c}[E_2' \in \mathcal{E}_2 \textbf{ and } (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2)] \textbf{ implies} \\ [E_1' \in \mathcal{E}_1 \textbf{ and } \{V_1\,,\,V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger]\end{array}\right] \textbf{ for all } E_1'\,,\,E_2'\right]
\end{array}\right] \\
\textbf{for some } \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger
\end{array}\right]
\end{gathered}$$

(W2) Suppose:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \textbf{ for some } E_1\,,\,E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \textbf{ and } E_1\,,\,E_2 \in \wp(\mathbb{V}\text{ER}) \textbf{ and } V_1\,,\,V_2 \in \mathbb{V}\text{ER} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying Definition 20 of $\sqcup$, conclude:

$$\begin{gathered}(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) \in \wp^2(\mathbb{V}\text{ER}) \textbf{ and} \\ (E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = \mathcal{E} \setminus \{E_1\,,\,E_2\} \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\end{gathered}$$

(W3) Suppose:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \textbf{ for some } E_1\,,\,E_2$$

Then, by applying Lemma 8:3, $[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!]$. Then, by applying (W1), conclude $[\![(\mathcal{V}\,,\,(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2))]\!] = [\![(\mathcal{V}\,,\,\mathcal{F})]\!]$.

(W4) Suppose:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \textbf{ for some } E_1\,,\,E_2$$

Then, by applying Lemma 8:4, conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{E}|$.

(W5) Suppose:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \textbf{ for some } E_1\,,\,E_2$$

Then, by applying (W4), conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{E}|$. Then, by introducing (W1), conclude $\big[|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{E}| \textbf{ and } [\mathcal{E} = \mathcal{F} \textbf{ or } |\mathcal{E}| < |\mathcal{F}|]\big]$. Then, by applying standard inference rules, conclude:

$$\big[|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{E}| \textbf{ and } \mathcal{E} = \mathcal{F}\big] \textbf{ or } |(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{E}| < |\mathcal{F}|$$

Then, by applying substitution, conclude:

$$|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{F}| \ \textbf{ or } \ |(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{E}| < |\mathcal{F}|$$

Then, by applying PA, conclude $\big[|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{F}| \textbf{ or } |(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{F}|\big]$. Then, by applying standard inference rules, conclude $|(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{F}|$. Then, by applying standard inference rules, conclude $\big[\textbf{false} \textbf{ or } |(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{F}|\big]$. Then, by applying standard inference rules, conclude:

$$\big[\mathsf{Cond}_2 \ \textbf{ and } \ (E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = \mathcal{F}\big] \ \textbf{ or } \ |(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2)| < |\mathcal{F}|$$

(W6) Suppose:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \ \textbf{ for some } \ E_1\,,\,E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $\big[E_1\,,\,E_2 \in \mathcal{E} \textbf{ and } (E_1\,,\,V_1) \curlyvee (E_2\,,\,V_2)\big]$. Then, by applying Lemma 6:1, conclude $\big[V_1 \in E_1 \in \mathcal{E} \textbf{ and } V_2 \in E_2 \in \mathcal{E}\big]$.

(W7) Suppose:

$$E_\dagger \in \mathcal{F}_\dagger \ \textbf{ for some } \ E_\dagger$$

Then, by applying (W1), conclude:

$$E_\dagger \in \{E'_\dagger \mid E'_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{ and } \ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}$$

Then, by applying ZFC, conclude:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{ and } \ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\big] \ \textbf{ for some } \ E_1\,,\,E_2$$

(W8) Recall $(X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2)$ from (W1). Then, by applying Definition 19 of $\curlyvee$, conclude:

$$(X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2) \ \textbf{ and } \ X\,,\,Y \in \mathcal{F}$$

Then, by applying Lemma 6:1, conclude $\big[V_1 \in X \in \mathcal{F} \textbf{ and } V_2 \in Y \in \mathcal{F}\big]$.

(W9) Recall $\checkmark(\mathcal{F})$ from (W1). Then, by applying Definition 24 of $\checkmark$, conclude:

$$\Big[\Big[\begin{bmatrix} & p' \in V'_1 \in E'_1 \in \mathcal{F} \\ \textbf{and} & p' \in V'_2 \in E'_2 \in \mathcal{F}\end{bmatrix} \ \textbf{ implies } \ V'_1 = V'_2\Big] \ \textbf{ for all } \ p'\,,\,V'_1\,,\,V'_2\,,\,E'_1\,,\,E'_2\Big]$$
$$\textbf{and} \ \Big[\big[V \in E \in \mathcal{F} \ \textbf{ implies } \ V \neq \emptyset\big] \ \textbf{ for all } \ V\,,\,E\Big]$$

(W0) Recall $V_1 \in X \in \mathcal{F}$ from (W8). Then, by applying (W9), conclude $\big[V_1 \in X \in \mathcal{F} \textbf{ and } V_1 \neq \emptyset\big]$. Then, by applying ZFC, conclude $\big[V_1 \in X \in \mathcal{F} \textbf{ and } \big[p_1 \in V_1 \textbf{ for some } p_1\big]\big]$. Then, by applying standard inference rules, conclude:

$$p_1 \in V_1 \in X \in \mathcal{F} \ \textbf{ for some } \ p_1$$

(V1) By a reduction similar to (W0), conclude:

$$p_2 \in V_2 \in Y \in \mathcal{F} \ \textbf{ for some } \ p_2$$

(V2) Recall $(X\,,\,V_1) \curlyvee_{\mathcal{E}} (Y\,,\,V_2)$ from (W1). Then, by applying Lemma 7:1, conclude $V_1 \neq V_2$.

(V3) Suppose $V_1 = V_1 \cup V_2$. Then, by applying ZFC, conclude $V_2 \subseteq V_1$. Then, by introducing (V1), conclude $\big[V_2 \subseteq V_1 \textbf{ and } \big[p \in V_2 \textbf{ for some } p\big]\big]$. Then, by applying standard inference rules, conclude:

$$\big[V_2 \subseteq V_1 \ \textbf{ and } \ p \in V_2\big] \ \textbf{ for some } \ p$$

Then, by applying standard inference rules, conclude $\big[p \in V_1 \textbf{ and } p \in V_2\big]$. Then, by introducing (W8), conclude $\big[p \in V_1 \in X \in \mathcal{F} \textbf{ and } p \in V_2 \in Y \in \mathcal{F}\big]$. Then, by applying (W9), conclude $V_1 = V_2$. Then, by introducing (V2), conclude $\big[V_1 = V_2 \textbf{ and } V_1 \neq V_2\big]$. Then, by applying standard inference rules, conclude **false**.

(V4) Suppose $V_2 = V_1 \cup V_2$. Then, by a reduction similar to (V3), conclude **false**.

(V5) Suppose:

$$\big[(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\textbf{ and }\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \;\textbf{ and }\; E_1 \in \mathcal{E}_\dagger\big] \;\textbf{ for some }\; E_1\,,\,E_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (W6), $\big[V_1 \in E_1 \in \mathcal{E} \;\textbf{ and }\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \;\textbf{ and }\; E_1 \in \mathcal{E}_\dagger\big]$. Then, by applying ZFC, conclude $\big[V_1 \in E_1 \in \mathcal{E} \;\textbf{ and }\; E_1 \in \mathcal{F}_\dagger\big]$. Then, by applying (W7), conclude:

$$V_1 \in E_1 \in \mathcal{E} \;\textbf{ and }\; E_1 = \{V_1 \cup V_2\} \cup (E_1 \cap E_2)$$

Then, by applying ZFC, conclude $\big[V_1 \in E_1 \in \mathcal{E} \;\textbf{ and }\; V_1 \cup V_2 \in E_1 \in \mathcal{E}\big]$. Then, by introducing (W0), conclude $\big[V_1 \in E_1 \in \mathcal{E} \;\textbf{ and }\; V_1 \cup V_2 \in E_1 \in \mathcal{E} \;\textbf{ and }\; [p \in V_1 \;\textbf{ for some }\; p]\big]$. Then, by applying standard inference rules, conclude:

$$\big[p \in V_1 \in E_1 \in \mathcal{E} \;\textbf{ and }\; p \in V_1 \cup V_2 \in E_1 \in \mathcal{E}\big] \;\textbf{ for some }\; p$$

Then, by applying (W9), conclude $V_1 = V_1 \cup V_2$. Then, by introducing (V3), conclude:

$$V_1 = V_1 \cup V_2 \;\textbf{ and }\; V_1 \neq V_1 \cup V_2$$

Then, by applying standard inference rules, conclude **false**.

(V6) Suppose:

$$\big[(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\textbf{ and }\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \;\textbf{ and }\; E_2 \in \mathcal{E}_\dagger\big] \;\textbf{ for some }\; E_1\,,\,E_2\,,\,\mathcal{E}_\dagger$$

Then, by a reduction similar to (V5), conclude **false**.

(V7) Suppose:

$$\big[(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\textbf{ and }\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger\big] \;\textbf{ for some }\; E_1\,,\,E_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (V5)(V6), conclude $E_1\,,\,E_2 \notin \mathcal{E}_\dagger$. Then, by applying ZFC, conclude:

$$E_1 \notin \mathcal{E}_\dagger \;\textbf{ and }\; E_2 \notin \mathcal{E}_\dagger \setminus \{E_1\}$$

Then, by applying ZFC, conclude $\big[E_1 \notin \mathcal{E}_\dagger \;\textbf{ and }\; E_2 \notin \mathcal{E}_\dagger \setminus \{E_1\}\big]$. Then, by applying ZFC, conclude $\big[\mathcal{E}_\dagger = \mathcal{E}_\dagger \setminus \{E_1\} \;\textbf{ and }\; \mathcal{E}_\dagger \setminus \{E_1\} = (\mathcal{E}_\dagger \setminus \{E_1\}) \setminus \{E_2\}\big]$. Then, by applying substitution, conclude $\mathcal{E}_\dagger = (\mathcal{E}_\dagger \setminus \{E_1\}) \setminus \{E_2\}$. Then, by applying ZFC, conclude $\mathcal{E}_\dagger = \mathcal{E}_\dagger \setminus \{E_1\,,\,E_2\}$.

(V8) Suppose:

$$\big[(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\textbf{ and }\; \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger\big] \;\textbf{ for some }\; E_1\,,\,E_2$$

Then, by applying (W2), conclude:

$$(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = \mathcal{E} \setminus \{E_1\,,\,E_2\} \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \;\textbf{ and }\; \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger$$

Then, by applying substitution, conclude:

$$(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = ((\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger) \setminus \{E_1\,,\,E_2\} \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying ZFC, conclude:

$$\begin{aligned}(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = {} & ((\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \setminus \{E_1\,,\,E_2\}) \cup (\mathcal{E}_\dagger \setminus \{E_1\,,\,E_2\}) \\ & \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\end{aligned}$$

Then, by applying ZFC, conclude:

$$(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2 \cup \{E_1\,,\,E_2\})) \cup (\mathcal{E}_\dagger \setminus \{E_1\,,\,E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying ZFC, conclude:

$$\begin{aligned}(E_1\,,\,V_1) \sqcup_{\mathcal{E}} (E_2\,,\,V_2) = {} & (\mathcal{F} \setminus (\mathcal{E}_1 \cup \{E_1\} \cup \mathcal{E}_2 \cup \{E_2\})) \\ & \cup (\mathcal{E}_\dagger \setminus \{E_1\,,\,E_2\}) \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\end{aligned}$$

(V9) Suppose:

$$\left[\begin{array}{c}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\ \textbf{and}\ \mathcal{E}_2'=\mathcal{E}_2\cup\{E_2\}\end{array}\right]\ \textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1'\,,\,\mathcal{E}_2'$$

Then, by applying (V8), conclude:

$$\begin{array}{c}(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1\cup\{E_1\}\cup\mathcal{E}_2\cup\{E_2\}))\\ \cup(\mathcal{E}_\dagger\setminus\{E_1\,,\,E_2\})\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\ \textbf{and}\ \mathcal{E}_2'=\mathcal{E}_2\cup\{E_2\}\end{array}$$

Then, by applying substitution, conclude:

$$(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup(\mathcal{E}_\dagger\setminus\{E_1\,,\,E_2\})\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}$$

(V0) Suppose:

$$\left[\begin{array}{c}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\ \textbf{and}\\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\ \textbf{and}\ \mathcal{E}_2'=\mathcal{E}_2\cup\{E_2\}\ \textbf{and}\ \mathcal{E}_\dagger'=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\end{array}\right]$$

$$\textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1'\,,\,\mathcal{E}_2'$$

Then, by applying (V9), conclude:

$$\begin{array}{c}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger'=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\ \textbf{and}\\ (E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup(\mathcal{E}_\dagger\setminus\{E_1\,,\,E_2\})\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\end{array}$$

Then, by applying (V7), conclude:

$$\begin{array}{c}\mathcal{E}_\dagger'=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\ \textbf{and}\\ (E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\end{array}$$

Then, by applying substitution, conclude $(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup\mathcal{E}_\dagger'$.

(U1) Suppose:

$$\left[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\right]$$

$$\textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (V5), conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger$$

Then, by applying (W6), conclude $\left[E_1\in\mathcal{E}\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]$. Then, by applying substitution, conclude $\left[E_1\in(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]$. Then, by applying ZFC, conclude $\left[\left[E_1\in\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2)\ \textbf{or}\ E_1\in\mathcal{E}_\dagger\right]\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]$. Then, by applying standard inference rules, conclude $\left[\left[E_1\in\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2)\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]\ \textbf{or}\ \left[E_1\in\mathcal{E}_\dagger\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]\right]$. Then, by applying standard inference rules, conclude $\left[\left[E_1\in\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2)\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]\ \textbf{or}\ \textbf{false}\right]$. Then, by applying standard inference rules, conclude $\left[E_1\in\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2)\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]$. Then, by applying ZFC, conclude $\left[E_1\in\mathcal{F}\ \textbf{and}\ E_1\notin\mathcal{E}_\dagger\right]$.

(U2) Suppose:

$$\left[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\right]$$

$$\textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by a reduction similar to (U2), conclude $\left[E_2\in\mathcal{F}\ \textbf{and}\ E_2\notin\mathcal{E}_\dagger\right]$.

(U3) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\big]$$
$$\textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (U1)(U2), conclude $\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ E_1\,,\,E_2\in\mathcal{F}\big]$. Then, by applying Definition 19 of $\curlyvee$, conclude $\big[E_1\neq E_2\ \textbf{and}\ V_1\cap V_2=\emptyset\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)\ \textbf{and}\ E_1\,,\,E_2\in\mathcal{F}\big]$. Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)$.

(U4) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\big]$$
$$\textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (U3), conclude $(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)$. Then, by applying ZFC, conclude:

$$E_1\in\{E_1'\mid(E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)\}$$

Then, by applying (W1), conclude $E_1\in\mathcal{F}_1$. Then, by applying ZFC, conclude $\{E_1\}\subseteq\mathcal{F}_1$.

(U5) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\big]$$
$$\textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by a reduction similar to (U4), conclude $E_2\in\mathcal{F}_2$.

(U6) Suppose:

$$\begin{bmatrix}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\end{bmatrix}\ \textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_1'\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (U3), conclude $\big[\mathcal{E}_1\,,\,\{E_1\}\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\big]$. Then, by applying ZFC, conclude $\big[\mathcal{E}_1\cup\{E_1\}\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\big]$. Then, by applying substitution, conclude $\mathcal{E}_1'\subseteq\mathcal{F}_1$.

(U7) Suppose:

$$\begin{bmatrix}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\ \textbf{and}\ \mathcal{E}_2'=\mathcal{E}_2\cup\{E_2\}\end{bmatrix}\ \textbf{for some}\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_2'\,,\,\mathcal{E}_\dagger$$

Then, by a reduction similar to (U6), conclude $\mathcal{E}_2'\subseteq\mathcal{F}_2$.

(U8) Suppose $V_1\,,\,V_2\in\mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1\,,\,V_2\in\wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $V_1\cup V_2\in\wp(\mathbb{P}\text{ORT})$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1\cup V_2\in\mathbb{V}\text{ER}$. Then, by applying ZFC, conclude $\{V_1\cup V_2\}\in\wp(\mathbb{V}\text{ER})$.

(U9) Suppose:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $\big[E_1\,,\,E_2\in\wp(\mathbb{V}\text{ER})\ \textbf{and}\ V_1\,,\,V_2\in\mathbb{V}\text{ER}\big]$. Then, by applying (U8), conclude $E_1\,,\,E_2\,,\,\{V_1\cup V_2\}\in\wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude:

$$\{V_1\cup V_2\}\cup(E_1\cap E_2)\in\wp(\mathbb{V}\text{ER})$$

(U0) Suppose:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying (U9), conclude $\{V_1\cup V_2\}\cup(E_1\cap E_2)\in\wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude:

$$E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{for some}\ E_\dagger$$

(T1) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{ and }\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{ and }\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\big]$$
$$\textbf{for some }\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (U3), conclude $(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)$. Then, by applying (U0), conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{ and }\ \big[E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{ for some }\ E_\dagger\big]$$

Then, by applying standard inference rules, conclude:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{ and }\ E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\big]\ \textbf{ for some }\ E_\dagger$$

Then, by applying ZFC, conclude:

$$E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{ and }$$
$$E_\dagger\in\{E'_\dagger\mid E'_\dagger=\{V_1\cup V_2\}\cup(E'_1\cap E'_2)\ \textbf{ and }\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\}$$

Then, by applying (W1), conclude $\big[E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{ and }\ E_\dagger\in\mathcal{F}_\dagger\big]$. Then, by applying substitution, conclude $\{V_1\cup V_2\}\cup(E_1\cap E_2)\in\mathcal{F}_\dagger$. Then, by applying ZFC, conclude:

$$\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\subseteq\mathcal{F}_\dagger$$

(T2) Suppose:

$$\begin{bmatrix}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{ and }\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\\ \textbf{and }\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\ \textbf{ and }\ \mathcal{E}'_\dagger=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\end{bmatrix}\ \textbf{ for some }\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger\,,\,\mathcal{E}'_\dagger$$

Then, by applying (T1), conclude:

$$\mathcal{E}_\dagger\,,\,\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\subseteq\mathcal{F}_1\ \textbf{ and }\ \mathcal{E}'_\dagger=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}$$

Then, by applying ZFC, conclude:

$$\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\subseteq\mathcal{F}_1\ \textbf{ and }\ \mathcal{E}'_\dagger=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}$$

Then, by applying substitution, conclude $\mathcal{E}'_\dagger\subseteq\mathcal{F}_\dagger$.

(T3) Suppose:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{ for some }\ E_1\,,\,E_2$$

Then, by applying (U9), conclude $\{V_1\cup V_2\}\cup(E_1\cap E_2)\in\wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $\{V_1\cup V_2\}\cup(E_1\cap E_2)\in\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}$.

(T4) Suppose:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{ for some }\ E_1\,,\,E_2\,,\,E'_1\,,\,E'_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)$. Then, by applying Lemma 6:1, conclude $\big[V_1\in E_1\ \textbf{ and }\ V_2\in E_2\big]$.

(T5) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{ and }\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big]\ \textbf{ for some }\ E_1\,,\,E_2\,,\,E'_1\,,\,E'_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $\big[(E_1\,,\,V_1)\curlyvee(E_2\,,\,V_2)\ \textbf{ and }\ (E_1\,,\,V_1)\curlyvee(E'_2\,,\,V_2)\big]$. Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_1=(E_2\setminus\{V_2\})\cup\{V_1\}\ \textbf{ and }\ E'_2=(E_1\setminus\{V_1\})\cup\{V_2\}$$

Then, by applying substitution, conclude $E'_2=(((E_2\setminus\{V_2\})\cup\{V_1\})\setminus\{V_1\})\cup\{V_2\}$. Then, by applying ZFC, conclude $E'_2=((E_2\setminus\{V_2\})\setminus\{V_1\})\cup(\{V_1\}\setminus\{V_1\})\cup\{V_2\}$. Then, by applying ZFC, conclude $E'_2=((E_2\setminus\{V_1\})\setminus\{V_2\})\cup(\{V_1\}\setminus\{V_1\})\cup\{V_2\}$. Then, by applying ZFC, conclude $E'_2=((E_2\setminus\{V_1\})\setminus\{V_2\})\cup\emptyset\cup\{V_2\}$. Then, by applying ZFC, conclude:

$$E'_2=((E_2\setminus\{V_1\})\setminus\{V_2\})\cup\{V_2\}$$

(T6) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\ \textbf{for some}\ E_1\,,\,E_2\,,\,E'_1\,,\,E'_2$$

Then, by a reduction similar to (T5), conclude $E'_1 = ((E_1 \setminus \{V_2\}) \setminus \{V_1\}) \cup \{V_1\}$.

(T7) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ E_1 \neq E_2\big]\ \textbf{for some}\ E_1\,,\,E_2\,,\,E'_1\,,\,E'_2$$

Then, by Lemma 7:2, conclude $\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ V_1 \notin E_2\big]$. Then, by applying (T4), conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ V_1 \notin E_2\ \textbf{and}\ V_2 \in E_2$$

Then, by applying (T5), conclude:

$$E'_2 = ((E_2 \setminus \{V_1\}) \setminus \{V_2\}) \cup \{V_2\}\ \textbf{and}\ V_1 \notin E_2\ \textbf{and}\ V_2 \in E_2$$

Then, by applying ZFC, conclude $\big[E'_2 = (E_2 \setminus \{V_2\}) \cup \{V_2\}\ \textbf{and}\ V_2 \in E_2\big]$. Then, by applying ZFC, conclude $E'_2 = E_2$.

(T8) Suppose:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_1 \neq E_2\big]\ \textbf{for some}\ E_1\,,\,E_2\,,\,E'_1\,,\,E'_2$$

Then, by a reduction similar to (T7), conclude $E'_1 = E_1$.

(T9) Suppose:

$$\begin{bmatrix} E'_1 \in \{E_1\}\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and} \\ (E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \end{bmatrix}$$
$$\textbf{for some}\ E_1\,,\,E'_1\,,\,E_2\,,\,E'_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (U3), conclude:

$$E'_1 \in \{E_1\}\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)$$

Then, by applying ZFC, conclude:

$$E'_1 \in \{E_1\}\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_2 \in \{E_2\}$$

Then, by applying (T3), conclude:

$$E'_1 \in \{E_1\}\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)$$
$$\textbf{and}\ E_2 \in \{E_2\}\ \textbf{and}\ \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying ZFC, conclude:

$$E'_1 = E_1\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)$$
$$\textbf{and}\ E_2 \in \{E_2\}\ \textbf{and}\ \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying substitution, conclude:

$$E'_1 = E_1\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)$$
$$\textbf{and}\ E_2 \in \{E_2\}\ \textbf{and}\ \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying (T7), conclude:

$$E'_1 = E_1\ \textbf{and}\ E'_2 = E_2\ \textbf{and}\ E_2 \in \{E_2\}\ \textbf{and}\ \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying substitution, conclude:

$$E'_2 \in \{E_2\}\ \textbf{and}\ \{V_1 \cup V_2\} \cup (E'_1 \cap E'_2) \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

(T0) Suppose:

$$\begin{bmatrix} E_2' \in \{E_2\} \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2) \ \textbf{and} \\ (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \ \textbf{and} \ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and} \ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \end{bmatrix}$$

$$\textbf{for some} \ E_1, E_2, E_1', E_2', \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by a reduction similar to (T9), conclude:

$$E_1' \in \{E_1\} \ \textbf{and} \ \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

(S1) Suppose:

$$\begin{bmatrix} (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \ \textbf{and} \ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and} \ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \ \textbf{and} \\ \left[\begin{bmatrix} [E_1' \in \mathcal{E}_1 \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \mathcal{E}_2 \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \ \textbf{for all} \ E_1', E_2' \right] \end{bmatrix}$$

$$\textbf{for some} \ E_1, E_2, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by applying (T9), conclude:

$$\left[\begin{bmatrix} [E_1' \in \mathcal{E}_1 \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \mathcal{E}_2 \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \ \textbf{for all} \ E_1', E_2' \right] \ \textbf{and}$$

$$\left[\begin{bmatrix} [E_1' \in \{E_1\} \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \{E_2\} \ \textbf{and} \ \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}] \end{bmatrix} \ \textbf{for all} \ E_1', E_2' \right]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \begin{bmatrix} [E_1' \in \mathcal{E}_1 \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \mathcal{E}_2 \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \ \textbf{and} \\ \begin{bmatrix} [E_1' \in \{E_1\} \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \{E_2\} \ \textbf{and} \ \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}] \end{bmatrix} \end{bmatrix} \ \textbf{for all} \ E_1', E_2'$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \begin{bmatrix} [E_1' \in \mathcal{E}_1 \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \\ \textbf{or} \ [E_1' \in \{E_1\} \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \end{bmatrix} \ \textbf{implies} \\ \begin{bmatrix} [E_2' \in \mathcal{E}_2 \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \ \textbf{or} \\ [E_2' \in \{E_2\} \ \textbf{and} \ \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}] \end{bmatrix} \end{bmatrix} \ \textbf{for all} \ E_1', E_2'$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} [[E_1' \in \mathcal{E}_1 \ \textbf{or} \ E_1' \in \{E_1\}] \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ \begin{bmatrix} [E_2' \in \mathcal{E}_2 \ \textbf{or} \ E_2' \in \{E_2\}] \ \textbf{and} \ [\{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger \ \textbf{or} \ E_2' \in \{E_2\}] \\ \textbf{and} \ [E_2' \in \mathcal{E}_2 \ \textbf{or} \ \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}] \ \textbf{and} \\ [\{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger \ \textbf{or} \ \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}] \end{bmatrix} \end{bmatrix}$$

$$\textbf{for all} \ E_1', E_2'$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} [[E_1' \in \mathcal{E}_1 \ \textbf{or} \ E_1' \in \{E_1\}] \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ \begin{bmatrix} [E_2' \in \mathcal{E}_2 \ \textbf{or} \ E_2' \in \{E_2\}] \ \textbf{and} \\ [\{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger \ \textbf{or} \ \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \in \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}] \end{bmatrix} \end{bmatrix}$$

$$\textbf{for all} \ E_1', E_2'$$

Then, by applying ZFC, conclude:

$$\left[\begin{matrix}[E_1' \in \mathcal{E}_1 \cup \{E_1\} \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \mathcal{E}_2 \cup \{E_2\} \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}]\end{matrix}\right] \ \textbf{for all} \ E_1', E_2'$$

(S2) Suppose:

$$\left[\begin{matrix}(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \ \textbf{and} \ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and} \ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \ \textbf{and} \\ \left[\left[\begin{matrix}[E_2' \in \mathcal{E}_2 \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_1' \in \mathcal{E}_1 \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger]\end{matrix}\right] \ \textbf{for all} \ E_1', E_2'\right]\end{matrix}\right]$$
$$\textbf{for some} \ E_1, E_2, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by a reduction similar to (S1), conclude

$$\left[\begin{matrix}[E_2' \in \mathcal{E}_2 \cup \{E_2\} \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_1' \in \mathcal{E}_1 \cup \{E_1\} \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}]\end{matrix}\right] \ \textbf{for all} \ E_1', E_2'$$

(S3) Suppose:

$$\left[\begin{matrix}(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \ \textbf{and} \ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and} \ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \ \textbf{and} \\ \left[\left[\begin{matrix}[E_1' \in \mathcal{E}_1 \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \mathcal{E}_2 \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger]\end{matrix}\right] \ \textbf{for all} \ E_1', E_2'\right] \ \textbf{and} \\ \mathcal{E}_1' = \mathcal{E}_1 \cup \{E_1\} \ \textbf{and} \ \mathcal{E}_2' = \mathcal{E}_2 \cup \{E_2\} \ \textbf{and} \ \mathcal{E}_\dagger' = \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\end{matrix}\right]$$
$$\textbf{for some} \ E_1, E_2, \mathcal{E}_1, \mathcal{E}_1', \mathcal{E}_2, \mathcal{E}_2', \mathcal{E}_\dagger, \mathcal{E}_\dagger'$$

Then, by applying (S1), conclude:

$$\left[\begin{matrix}\left[\begin{matrix}[E_1' \in \mathcal{E}_1 \cup \{E_1\} \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \mathcal{E}_2 \cup \{E_2\} \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}]\end{matrix}\right] \\ \textbf{for all} \ E_1', E_2'\end{matrix}\right]$$
$$\mathcal{E}_1' = \mathcal{E}_1 \cup \{E_1\} \ \textbf{and} \ \mathcal{E}_2' = \mathcal{E}_2 \cup \{E_2\} \ \textbf{and} \ \mathcal{E}_\dagger' = \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}$$

Then, by applying substitution, conclude:

$$\left[\begin{matrix}[E_1' \in \mathcal{E}_1' \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_2' \in \mathcal{E}_2' \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger']\end{matrix}\right] \ \textbf{for all} \ E_1', E_2'$$

(S4) Suppose:

$$\left[\begin{matrix}(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \ \textbf{and} \ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \ \textbf{and} \ \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \ \textbf{and} \\ \left[\left[\begin{matrix}[E_2' \in \mathcal{E}_2 \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_1' \in \mathcal{E}_1 \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger]\end{matrix}\right] \ \textbf{for all} \ E_1', E_2'\right] \ \textbf{and} \\ \mathcal{E}_1' = \mathcal{E}_1 \cup \{E_1\} \ \textbf{and} \ \mathcal{E}_2' = \mathcal{E}_2 \cup \{E_2\} \ \textbf{and} \ \mathcal{E}_\dagger' = \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\}\end{matrix}\right]$$
$$\textbf{for some} \ E_1, E_2, \mathcal{E}_1, \mathcal{E}_1', \mathcal{E}_2, \mathcal{E}_2', \mathcal{E}_\dagger, \mathcal{E}_\dagger'$$

Then, by a reduction similar to (S3), conclude:

$$\left[\begin{matrix}[E_2' \in \mathcal{E}_2' \ \textbf{and} \ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \ \textbf{implies} \\ [E_1' \in \mathcal{E}_1' \ \textbf{and} \ \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger']\end{matrix}\right] \ \textbf{for all} \ E_1', E_2'$$

(S5) Suppose:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\; \textbf{for some} \;\; E_1\,,\,E_2$$

Then, by introducing (W1):

$$\begin{gathered}(E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\; \textbf{and} \\ \left[\begin{array}{c}\left[\begin{array}{l} \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\; \textbf{and} \;\; \mathcal{E}_1 \subseteq \mathcal{F}_1 \;\; \textbf{and} \;\; \mathcal{E}_2 \subseteq \mathcal{F}_2 \;\; \textbf{and} \;\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \end{array}\right] \\ \textbf{for some} \;\; \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger \end{array}\right]\end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered}\left[\begin{array}{c} (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\; \textbf{and} \\ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\; \textbf{and} \;\; \mathcal{E}_1 \subseteq \mathcal{F}_1 \;\; \textbf{and} \;\; \mathcal{E}_2 \subseteq \mathcal{F}_2 \;\; \textbf{and} \;\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \end{array}\right] \\ \textbf{for some} \;\; \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger\end{gathered}$$

Then, by applying ZFC, conclude:

$$\begin{gathered} (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\; \textbf{and} \\ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\; \textbf{and} \;\; \mathcal{E}_1 \subseteq \mathcal{F}_1 \;\; \textbf{and} \;\; \mathcal{E}_2 \subseteq \mathcal{F}_2 \;\; \textbf{and} \;\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \\ \textbf{and} \;\; \begin{bmatrix} \mathcal{E}'_1 = \mathcal{E}_1 \cup \{E_1\} \\ \textbf{for some} \;\; \mathcal{E}'_1 \end{bmatrix} \;\; \textbf{and} \;\; \begin{bmatrix} \mathcal{E}'_2 = \mathcal{E}_2 \cup \{E_2\} \\ \textbf{for some} \;\; \mathcal{E}'_2 \end{bmatrix} \;\; \textbf{and} \;\; \begin{bmatrix} \mathcal{E}'_\dagger = \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \\ \textbf{for some} \;\; \mathcal{E}'_\dagger \end{bmatrix} \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered}\left[\begin{array}{c} (E_1\,,\,V_1) \curlyvee_{\mathcal{E}} (E_2\,,\,V_2) \;\; \textbf{and} \\ \mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\; \textbf{and} \;\; \mathcal{E}_1 \subseteq \mathcal{F}_1 \;\; \textbf{and} \;\; \mathcal{E}_2 \subseteq \mathcal{F}_2 \;\; \textbf{and} \;\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \\ \textbf{and} \;\; \left[\begin{bmatrix} [E'_2 \in \mathcal{E}_2 \;\; \textbf{and} \;\; (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)] \;\; \textbf{implies} \\ [E'_1 \in \mathcal{E}_1 \;\; \textbf{and} \;\; \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger] \end{bmatrix} \;\; \textbf{for all} \;\; E'_1\,,\,E'_2\right] \\ \textbf{and} \;\; \mathcal{E}'_1 = \mathcal{E}_1 \cup \{E_1\} \;\; \textbf{and} \;\; \mathcal{E}'_2 = \mathcal{E}_2 \cup \{E_2\} \;\; \textbf{and} \;\; \mathcal{E}'_\dagger = \mathcal{E}_\dagger \cup \{\{V_1 \cup V_2\} \cup (E_1 \cap E_2)\} \end{array}\right] \\ \textbf{for some} \;\; \mathcal{E}'_1\,,\,\mathcal{E}'_2\,,\,\mathcal{E}'_\dagger\end{gathered}$$

Then, by applying (V0), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2\subseteq\mathcal{F}_2\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\\ \textbf{and}\ \left[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_2'\in\mathcal{E}_2\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \textbf{and}\ \left[\begin{bmatrix}[E_2'\in\mathcal{E}_2\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_1'\in\mathcal{E}_1\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\ \textbf{and}\ \mathcal{E}_2'=\mathcal{E}_2\cup\{E_2\}\ \textbf{and}\ \mathcal{E}_\dagger'=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\ \textbf{and}\\ (E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup\mathcal{E}_\dagger'\end{gathered}$$

Then, by applying (U6)(U7), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\\ \textbf{and}\ \left[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_2'\in\mathcal{E}_2\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \textbf{and}\ \left[\begin{bmatrix}[E_2'\in\mathcal{E}_2\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_1'\in\mathcal{E}_1\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\ \textbf{and}\ \mathcal{E}_2'=\mathcal{E}_2\cup\{E_2\}\ \textbf{and}\ \mathcal{E}_\dagger'=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\ \textbf{and}\\ (E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup\mathcal{E}_\dagger'\ \textbf{and}\ \mathcal{E}_1'\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2'\subseteq\mathcal{F}_2\end{gathered}$$

Then, by applying (T2), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\\ \mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\\ \textbf{and}\ \left[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_2'\in\mathcal{E}_2\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \textbf{and}\ \left[\begin{bmatrix}[E_2'\in\mathcal{E}_2\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_1'\in\mathcal{E}_1\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \textbf{and}\ \mathcal{E}_1'=\mathcal{E}_1\cup\{E_1\}\ \textbf{and}\ \mathcal{E}_2'=\mathcal{E}_2\cup\{E_2\}\ \textbf{and}\ \mathcal{E}_\dagger'=\mathcal{E}_\dagger\cup\{\{V_1\cup V_2\}\cup(E_1\cap E_2)\}\ \textbf{and}\\ (E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup\mathcal{E}_\dagger'\ \textbf{and}\ \mathcal{E}_1'\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2'\subseteq\mathcal{F}_2\ \textbf{and}\ \mathcal{E}_\dagger'\subseteq\mathcal{F}_\dagger\end{gathered}$$

Then, by applying (S3)(S4), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup\mathcal{E}_\dagger'\ \textbf{and}\ \mathcal{E}_1'\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2'\subseteq\mathcal{F}_2\ \textbf{and}\ \mathcal{E}_\dagger'\subseteq\mathcal{F}_\dagger\\ \left[\begin{bmatrix}[E_1'\in\mathcal{E}_1'\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_2'\in\mathcal{E}_2'\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger']\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \left[\begin{bmatrix}[E_2'\in\mathcal{E}_2'\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_1'\in\mathcal{E}_1'\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger']\end{bmatrix}\ \textbf{for all}\ E_1'\,,\,E_2'\right]\end{gathered}$$

(S6) Suppose:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by introducing (W1), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\\ [\![(\mathcal{V}\,,\,\mathcal{F})]\!]=[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]\ \textbf{and}\ \bigstar(\mathcal{F})=\bigstar(\mathcal{E}_{\text{in}})\ \textbf{and}\ \checkmark(\mathcal{F})\ \textbf{and}\\ |\mathcal{F}|=z_1\ \textbf{and}\ (X\,,\,V_1)\curlyvee_{\mathcal{F}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{F})\ \textbf{and}\\ \mathcal{F}_1=\{E_1\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\ \mathcal{F}_2=\{E_2\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\\ \textbf{and}\ \mathcal{F}_\dagger=\{E_\dagger\mid E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\end{gathered}$$

Then, by applying (W3), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\\ [\![(\mathcal{V}\,,\,\mathcal{F})]\!]=[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]\ \textbf{and}\ \bigstar(\mathcal{F})=\bigstar(\mathcal{E}_{\text{in}})\ \textbf{and}\ \checkmark(\mathcal{F})\ \textbf{and}\\ |\mathcal{F}|=z_1\ \textbf{and}\ (X\,,\,V_1)\curlyvee_{\mathcal{F}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{F})\ \textbf{and}\\ \mathcal{F}_1=\{E_1\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\ \mathcal{F}_2=\{E_2\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\\ \textbf{and}\ \mathcal{F}_\dagger=\{E_\dagger\mid E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\\ [\![(\mathcal{V}\,,\,(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2))]\!]=[\![(\mathcal{V}\,,\,\mathcal{F})]\!]\end{gathered}$$

Then, by applying (W5), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\\ [\![(\mathcal{V}\,,\,\mathcal{F})]\!]=[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]\ \textbf{and}\ \bigstar(\mathcal{F})=\bigstar(\mathcal{E}_{\text{in}})\ \textbf{and}\ \checkmark(\mathcal{F})\ \textbf{and}\\ |\mathcal{F}|=z_1\ \textbf{and}\ (X\,,\,V_1)\curlyvee_{\mathcal{F}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{F})\ \textbf{and}\\ \mathcal{F}_1=\{E_1\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\ \mathcal{F}_2=\{E_2\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\\ \textbf{and}\ \mathcal{F}_\dagger=\{E_\dagger\mid E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\\ [\![(\mathcal{V}\,,\,(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2))]\!]=[\![(\mathcal{V}\,,\,\mathcal{F})]\!]\ \textbf{and}\ \left[\begin{array}{c}[\mathsf{Cond}_2\ \textbf{and}\ (E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=\mathcal{F}]\\ \textbf{or}\ |(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)|<|\mathcal{F}|\end{array}\right]\end{gathered}$$

Then, by applying (S5) conclude:

$$\begin{gathered}[\![(\mathcal{V}\,,\,\mathcal{F})]\!]=[\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]\ \textbf{and}\ \bigstar(\mathcal{F})=\bigstar(\mathcal{E}_{\text{in}})\ \textbf{and}\ \checkmark(\mathcal{F})\ \textbf{and}\\ |\mathcal{F}|=z_1\ \textbf{and}\ (X\,,\,V_1)\curlyvee_{\mathcal{F}}(Y\,,\,V_2)\ \textbf{and}\ V_1\cup V_2\subseteq P\ \textbf{and}\ P\in\bigstar(\mathcal{F})\ \textbf{and}\\ \mathcal{F}_1=\{E_1\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\ \mathcal{F}_2=\{E_2\mid(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\\ \textbf{and}\ \mathcal{F}_\dagger=\{E_\dagger\mid E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\}\ \textbf{and}\\ [\![(\mathcal{V}\,,\,(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2))]\!]=[\![(\mathcal{V}\,,\,\mathcal{F})]\!]\ \textbf{and}\ \left[\begin{array}{c}[\mathsf{Cond}_2\ \textbf{and}\ (E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=\mathcal{F}]\\ \textbf{or}\ |(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)|<|\mathcal{F}|\end{array}\right]\\ \textbf{and}\\ \left[\begin{array}{c}\left[\begin{array}{c}(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)=(\mathcal{F}\setminus(\mathcal{E}_1'\cup\mathcal{E}_2'))\cup\mathcal{E}_\dagger'\ \textbf{and}\ \mathcal{E}_1'\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2'\subseteq\mathcal{F}_2\ \textbf{and}\ \mathcal{E}_\dagger'\subseteq\mathcal{F}_\dagger\\ \left[\left[\begin{array}{c}[E_1'\in\mathcal{E}_1'\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_2'\in\mathcal{E}_2'\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger']\end{array}\right]\ \textbf{for all}\ E_1'\,,\,E_2'\right]\\ \left[\left[\begin{array}{c}[E_2'\in\mathcal{E}_2'\ \textbf{and}\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{implies}\\ [E_1'\in\mathcal{E}_1'\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger']\end{array}\right]\ \textbf{for all}\ E_1'\,,\,E_2'\right]\end{array}\right]\\ \textbf{for some}\ \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger\end{array}\right]\end{gathered}$$

Then, by applying Figure 15, conclude $\mathsf{Inv}_2[\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)]$.

(S7) Suppose:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying (W4), conclude $|(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)|<|\mathcal{E}|$. Then, by introducing (W1), conclude $\big[|(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)|<|\mathcal{E}|\ \textbf{and}\ |\mathcal{E}|=z_2\big]$. Then, by applying substitution, conclude:

$$|(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)|<z_2$$

Then, by applying standard inference rules, conclude $(|\mathcal{E}|<z_2)[\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)]$.

Now, prove the theorem by the following reduction. Recall Cond_2 from (C2). Then, by applying Figure 15, conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying (S6), conclude $\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{E}}(E_2\,,\,V_2)\ \textbf{and}\ \mathsf{Inv}_2[\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)]\big]$. Then, by applying (S7), conclude $\big[\mathsf{Inv}_2[\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)]\ \textbf{and}\ (|\mathcal{E}|<z_2)[\mathcal{E}:=(E_1\,,\,V_1)\sqcup_{\mathcal{E}}(E_2\,,\,V_2)]\big]$.

(| QED. |)

4. First, assume:

(D1) Inv_2

(D2) **not** Cond_2

Next, observe:

(R1) Recall Inv_2 from (D1). Then, by applying Figure 15, conclude:

$$\begin{gathered}
[\![(\mathcal{V}\,,\,\mathcal{F})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and} \\
|\mathcal{F}| = z_1 \textbf{ and } (X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and} \\
\mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \textbf{ and } \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \\
\textbf{and } \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \textbf{ and} \\
\textbf{and } [\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{F})]\!] \textbf{ and } \big[\big[\mathsf{Cond}_2 \textbf{ and } \mathcal{E} = \mathcal{F}\big] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\big] \textbf{ and} \\
\left[\begin{gathered}
\left[\begin{gathered}
\mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and } \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\
\textbf{and } \Big[\left[\begin{gathered}\big[E'_1 \in \mathcal{E}_1 \textbf{ and } (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big] \textbf{ implies} \\ \big[E'_2 \in \mathcal{E}_2 \textbf{ and } \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{gathered}\right] \textbf{ for all } E'_1\,,\,E'_2\Big] \\
\textbf{and } \Big[\left[\begin{gathered}\big[E'_2 \in \mathcal{E}_2 \textbf{ and } (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big] \textbf{ implies} \\ \big[E'_1 \in \mathcal{E}_1 \textbf{ and } \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{gathered}\right] \textbf{ for all } E'_1\,,\,E'_2\Big]
\end{gathered}\right] \\
\textbf{for some } \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger
\end{gathered}\right]
\end{gathered}$$

(R2) Suppose:

$$E_1 \in \mathcal{F}_1 \textbf{ for some } E_1$$

Then, by applying (R1), conclude $E_1 \in \{E'_1 \mid (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}$. Then, by applying ZFC, conclude:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2) \textbf{ for some } E_2$$

(R3) Suppose:

$$E_2 \in \mathcal{F}_2 \textbf{ for some } E_2$$

Then, by a reduction similar to (R2), conclude:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2) \textbf{ for some } E_1$$

(R4) Suppose:

$$\begin{gathered}\Big[E_1 \notin \mathcal{E}_1 \textbf{ and } \Big[\left[\begin{gathered}\big[E'_2 \in \mathcal{E}_2 \textbf{ and } (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big] \textbf{ implies} \\ \big[E'_1 \in \mathcal{E}_1 \textbf{ and } \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{gathered}\right] \textbf{ for all } E'_1\,,\,E'_2\Big]\Big] \\ \textbf{for some } E_1\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger\end{gathered}$$

Then, by applying standard inference rules, conclude:

$$E_1 \notin \mathcal{E}_1 \textbf{ and } \Big[\left[\begin{gathered}\big[E'_1 \notin \mathcal{E}_1 \textbf{ or } \{V_1\,,\,V_2\} \cup (E'_1 \cap E'_2) \notin \mathcal{E}_\dagger\big] \\ \textbf{implies } \big[E'_2 \notin \mathcal{E}_2 \textbf{ or } \big[\textbf{not } (E'_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big]\big]\end{gathered}\right] \textbf{ for all } E'_1\,,\,E'_2\Big]$$

Then, by applying standard inference rules, conclude:

$$\big[E'_2 \notin \mathcal{E}_2 \textbf{ or } \big[\textbf{not } (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E'_2\,,\,V_2)\big]\big] \textbf{ for all } E'_2$$

(R5) Suppose:

$$\big[E_2 \notin \mathcal{E}_2 \ \textbf{and} \ \big[\begin{bmatrix}\big[E'_1 \in \mathcal{E}_1 \ \textbf{and} \ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big] \ \textbf{implies} \\ \big[E'_2 \in \mathcal{E}_2 \ \textbf{and} \ \{V_1\,,\,V_2\}\cup(E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{bmatrix} \ \textbf{for all} \ E'_1\,,\,E'_2\big]\big]$$

$$\textbf{for some} \ E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by a reduction similar to (R4), conclude:

$$\big[E'_1 \notin \mathcal{E}_1 \ \textbf{or} \ \big[\textbf{not} \ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big] \ \textbf{for all} \ E'_1$$

(R6) Suppose:

$$\begin{bmatrix} E_1 \in \mathcal{F}_1 \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \\ \big[\begin{bmatrix}\big[E'_2 \in \mathcal{E}_2 \ \textbf{and} \ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big] \ \textbf{implies} \\ \big[E'_1 \in \mathcal{E}_1 \ \textbf{and} \ \{V_1\,,\,V_2\}\cup(E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{bmatrix} \ \textbf{for all} \ E'_1\,,\,E'_2\big]\end{bmatrix} \ \textbf{for some} \ E_1\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (R2), conclude:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{for some} \ E_2\big] \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and}$$

$$\big[\begin{bmatrix}\big[E'_2 \in \mathcal{E}_2 \ \textbf{and} \ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big] \ \textbf{implies} \\ \big[E'_1 \in \mathcal{E}_1 \ \textbf{and} \ \{V_1\,,\,V_2\}\cup(E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{bmatrix} \ \textbf{for all} \ E'_1\,,\,E'_2\big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \\ \big[\begin{bmatrix}\big[E'_2 \in \mathcal{E}_2 \ \textbf{and} \ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big] \ \textbf{implies} \\ \big[E'_1 \in \mathcal{E}_1 \ \textbf{and} \ \{V_1\,,\,V_2\}\cup(E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{bmatrix} \ \textbf{for all} \ E'_1\,,\,E'_2\big]\end{bmatrix} \ \textbf{for some} \ E_2$$

Then, by applying (R4), conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ \big[E_2 \notin \mathcal{E}_2 \ \textbf{or} \ \big[\textbf{not} \ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]$$

Then, by applying standard inference rules, conclude:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2\big] \ \textbf{or}$$
$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ \big[\textbf{not} \ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]$$

Then, by applying standard inference rules, conclude:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2\big] \ \textbf{or} \ \big[E_1 \notin \mathcal{E}_1 \ \textbf{and} \ \textbf{false}\big]$$

Then, by applying standard inference rules, conclude:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2\big] \ \textbf{or} \ \textbf{false}$$

Then, by applying standard inference rules, conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2$$

(R7) Suppose:

$$\begin{bmatrix} E_2 \in \mathcal{F}_2 \ \textbf{and} \ E_2 \notin \mathcal{E}_2 \ \textbf{and} \\ \big[\begin{bmatrix}\big[E'_1 \in \mathcal{E}_1 \ \textbf{and} \ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big] \ \textbf{implies} \\ \big[E'_2 \in \mathcal{E}_2 \ \textbf{and} \ \{V_1\,,\,V_2\}\cup(E'_1 \cap E'_2) \in \mathcal{E}_\dagger\big]\end{bmatrix} \ \textbf{for all} \ E'_1\,,\,E'_2\big]\end{bmatrix} \ \textbf{for some} \ E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by a reduction similar to (R6), conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2$$

(R8) Suppose:

$$E_1 \in \mathcal{F}_2 \ \textbf{for some} \ E_1$$

Then, by applying (R1), conclude $E_1 \in \{E_2 \mid (E'_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\}$. Then, by applying ZFC, conclude:

$$(E'_1, V_1) \curlyvee_{\mathcal{F}} (E_1, V_2) \ \textbf{for some} \ E'_1$$

Then, by applying Definition 19 of $\curlyvee$, conclude $(E'_1, V_1) \curlyvee (E_1, V_2)$. Then, by applying Lemma 6:1, conclude $V_2 \in E_1$.

(R9) Suppose:

$$E_2 \in \mathcal{F}_1 \ \textbf{for some} \ E_2$$

Then, by a reduction similar to (R8), conclude $V_2 \in E_1$.

(R0) Suppose:

$$\big[(E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \ \textbf{and} \ E_1 \in \mathcal{F}_2\big] \ \textbf{for some} \ E_1, E_2$$

Then, by applying (R8), conclude $\big[(E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \ \textbf{and} \ V_2 \in E_1\big]$. Then, by applying Lemma 7:2, conclude $\big[V_2 \notin E_1 \ \textbf{and} \ V_2 \in E_1\big]$. Then, by applying standard inference rules, conclude **false**.

(Q1) Suppose:

$$\big[(E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \ \textbf{and} \ E_2 \in \mathcal{F}_1\big] \ \textbf{for some} \ E_1, E_2$$

Then, by a reduction similar to (R0), conclude **false**

(Q2) Suppose:

$$\big[\mathcal{E}_1 \subseteq \mathcal{F}_1 \ \textbf{and} \ \mathcal{E}_2 \subseteq \mathcal{F}_2 \ \textbf{and} \ (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2\big] \\ \textbf{for some} \ E_1, E_2, \mathcal{E}_1, \mathcal{E}_2$$

Then, by applying (R0)(Q1), conclude

$$\mathcal{E}_1 \subseteq \mathcal{F}_1 \ \textbf{and} \ \mathcal{E}_2 \subseteq \mathcal{F}_2 \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2 \ \textbf{and} \ E_1 \notin \mathcal{F}_2 \ \textbf{and} \ E_2 \notin \mathcal{F}_1$$

Then, by applying ZFC, conclude $\big[E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2 \ \textbf{and} \ E_1 \notin \mathcal{E}_2 \ \textbf{and} \ E_2 \notin \mathcal{E}_1\big]$. Then, by applying ZFC, conclude $\big[E_1 \notin \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{and} \ E_2 \notin \mathcal{E}_1 \cup \mathcal{E}_2\big]$.

(Q3) Suppose:

$$\big[\mathcal{E}_1 \subseteq \mathcal{F}_1 \ \textbf{and} \ \mathcal{E}_2 \subseteq \mathcal{F}_2 \ \textbf{and} \ (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2\big] \\ \textbf{for some} \ E_1, E_2, \mathcal{E}_1, \mathcal{E}_2$$

Then, by applying (Q2), conclude $\big[(E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{and} \ E_2 \notin \mathcal{E}_1 \cup \mathcal{E}_2\big]$. Then, by applying Definition 19 of $\curlyvee$, conclude $\big[E_1, E_2 \in \mathcal{F} \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \cup \mathcal{E}_2 \ \textbf{and} \ E_2 \notin \mathcal{E}_1 \cup \mathcal{E}_2\big]$. Then, by applying ZFC, conclude $E_1, E_2 \in \mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}$.

(Q4) Suppose:

$$\begin{bmatrix} \mathcal{E} = (\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}) \cup \mathcal{E}_\dagger \ \textbf{and} \ \mathcal{E}_1 \subseteq \mathcal{F}_1 \ \textbf{and} \ \mathcal{E}_2 \subseteq \mathcal{F}_2 \ \textbf{and} \\ \textbf{and} \ (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \ \textbf{and} \ E_1 \notin \mathcal{E}_1 \ \textbf{and} \ E_2 \notin \mathcal{E}_2 \end{bmatrix} \ \textbf{for some} \ E_1, E_2, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by applying (Q3), conclude $\big[\mathcal{E} = (\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}) \cup \mathcal{E}_\dagger \ \textbf{and} \ E_1, E_2 \in \mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}\big]$. Then, by applying ZFC, conclude $\big[\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\} \subseteq \mathcal{E} \ \textbf{and} \ E_1, E_2 \in \mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}\big]$. Then, by applying ZFC, conclude $E_1, E_2 \in \mathcal{E}$.

(Q5) Recall $\big[\textbf{not} \ \mathsf{Cond}_2\big]$ from (D2). Then, by applying Figure 15, conclude:

$$\textbf{not} \ \big[(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \ \textbf{for some} \ E_1, E_2\big]$$

Then, by applying standard inference rules, conclude:

$$\big[\textbf{not} \ (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)\big] \ \textbf{for all} \ E_1, E_2$$

(Q6) Suppose:

$$\begin{bmatrix} \mathcal{E} = (\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and} \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \textbf{ and } E_1 \notin \mathcal{E}_1 \textbf{ and } E_2 \notin \mathcal{E}_2 \end{bmatrix} \textbf{ for some } E_1, E_2, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by applying (Q4), conclude $[(E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \textbf{ and } E_1, E_2 \in \mathcal{E}]$. Then, by applying Definition 19 of $\curlyvee$, conclude $[E_1, E_2 \in \mathcal{E} \textbf{ and } E_1 \neq E_2 \textbf{ and } V_1 \cap V_2 = \emptyset \textbf{ and } (E_1, V_1) \curlyvee (E_2, V_2)]$. Then, by applying Definition 19 of $\curlyvee$, conclude $(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)$. Then, by applying (Q5), conclude:

$$(E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2) \textbf{ and } [\textbf{not } (E_1, V_1) \curlyvee_{\mathcal{E}} (E_2, V_2)]$$

Then, by applying standard inference rules, conclude **false**.

(Q7) Suppose:

$$\begin{bmatrix} E_1 \in \mathcal{F}_1 \textbf{ and } E_1 \notin \mathcal{E}_1 \\ \textbf{and } \mathcal{E} = (\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and} \\ \left[\begin{bmatrix} [E_2' \in \mathcal{E}_2 \textbf{ and } (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \textbf{ implies} \\ [E_1' \in \mathcal{E}_1 \textbf{ and } \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \textbf{ for all } E_1', E_2'\right] \end{bmatrix} \textbf{ for some } E_1, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by applying (R6), conclude:

$$\begin{gathered} \mathcal{E} = (\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and} \\ [[(E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \textbf{ and } E_1 \notin \mathcal{E}_1 \textbf{ and } E_2 \notin \mathcal{E}_2] \textbf{ for some } E_2] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \mathcal{E} = (\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \\ \textbf{and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2) \textbf{ and } E_1 \notin \mathcal{E}_1 \textbf{ and } E_2 \notin \mathcal{E}_2 \end{bmatrix} \textbf{ for some } E_2$$

Then, by applying (Q6), conclude **false**.

(Q8) Suppose:

$$\begin{bmatrix} E_2 \in \mathcal{F}_2 \textbf{ and } E_2 \notin \mathcal{E}_2 \\ \textbf{and } \mathcal{E} = (\mathcal{F} \setminus \{\mathcal{E}_1 \cup \mathcal{E}_2\}) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and} \\ \left[\begin{bmatrix} [E_1' \in \mathcal{E}_1 \textbf{ and } (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \textbf{ implies} \\ [E_2' \in \mathcal{E}_2 \textbf{ and } \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \textbf{ for all } E_1', E_2'\right] \end{bmatrix} \textbf{ for some } E_1, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by applying a reduction similar to (Q7), conclude **false**.

(Q9) Suppose:

$$E_\dagger \in \mathcal{F}_\dagger \textbf{ for some } E_\dagger$$

Then, by applying (R1), conclude:

$$E_\dagger \in \{E_\dagger' \mid E_\dagger' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\}$$

Then, by applying ZFC, conclude:

$$[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)] \textbf{ for some } E_1, E_2$$

(Q0) Suppose:

$$\begin{bmatrix} \{V_1, V_2\} \cup (E_1 \cap E_2) \notin \mathcal{E}_\dagger \textbf{ and} \\ \left[\begin{bmatrix} [E_2' \in \mathcal{E}_2 \textbf{ and } (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2)] \textbf{ implies} \\ [E_1' \in \mathcal{E}_1 \textbf{ and } \{V_1, V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger] \end{bmatrix} \textbf{ for all } E_1', E_2'\right] \end{bmatrix}$$
$$\textbf{for some } E_1, E_2, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_\dagger$$

Then, by applying standard inference rules, conclude:

$$\{V_1\,,\,V_2\}\cup(E_1\cap E_2)\notin\mathcal{E}_\dagger\ \textbf{ and }$$
$$\Big[\begin{bmatrix}[E_1'\notin\mathcal{E}_1\ \textbf{ or }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\notin\mathcal{E}_\dagger] \\ \textbf{implies }\ [E_2'\notin\mathcal{E}_2\ \textbf{ or }\ [\textbf{not }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]$$

Then, by applying standard inference rules, conclude $\big[E_2\notin\mathcal{E}_2\ \textbf{ or }\ \big[\textbf{not }\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]$.

(P1) Suppose:

$$\begin{bmatrix}\{V_1\,,\,V_2\}\cup(E_1\cap E_2)\notin\mathcal{E}_\dagger\ \textbf{ and } \\ \Big[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_2'\in\mathcal{E}_2\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]\end{bmatrix}$$
$$\textbf{for some }\ E_1\,,\,E_2\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying a reduction similar to (Q0), conclude $\big[E_1\notin\mathcal{E}_1\ \textbf{ or }\ \big[\textbf{not }\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]$.

(P2) Suppose:

$$\begin{bmatrix}E_\dagger\in\mathcal{F}_\dagger\ \textbf{ and }\ E_\dagger\notin\mathcal{E}_\dagger \\ \textbf{and }\ \Big[\begin{bmatrix}[E_2'\in\mathcal{E}_2\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_1'\in\mathcal{E}_1\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big] \\ \textbf{and }\ \Big[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_2'\in\mathcal{E}_2\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]\end{bmatrix}$$
$$\textbf{for some }\ E_\dagger\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (Q9), conclude:

$$\Big[\begin{bmatrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2) \\ \textbf{and }\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\end{bmatrix}\ \textbf{ for some }\ E_1\,,\,E_2\Big]\ \textbf{ and }\ E_\dagger\notin\mathcal{E}_\dagger\ \textbf{ and}$$
$$\textbf{and }\ \Big[\begin{bmatrix}[E_2'\in\mathcal{E}_2\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_1'\in\mathcal{E}_1\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]$$
$$\textbf{and }\ \Big[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_2'\in\mathcal{E}_2\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix}E_\dagger=\{V_1\cup V_2\}\cup(E_1\cap E_2)\ \textbf{ and }\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{ and }\ E_\dagger\notin\mathcal{E}_\dagger \\ \textbf{and }\ \Big[\begin{bmatrix}[E_2'\in\mathcal{E}_2\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_1'\in\mathcal{E}_1\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big] \\ \textbf{and }\ \Big[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_2'\in\mathcal{E}_2\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]\end{bmatrix}\ \textbf{ for some }\ E_1\,,\,E_2$$

Then, by applying substitution, conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{ and }\ \{V_1\cup V_2\}\cup(E_1\cap E_2)\notin\mathcal{E}_\dagger$$
$$\textbf{and }\ \Big[\begin{bmatrix}[E_2'\in\mathcal{E}_2\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_1'\in\mathcal{E}_1\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]$$
$$\textbf{and }\ \Big[\begin{bmatrix}[E_1'\in\mathcal{E}_1\ \textbf{ and }\ (E_1'\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2'\,,\,V_2)]\ \textbf{ implies} \\ [E_2'\in\mathcal{E}_2\ \textbf{ and }\ \{V_1\,,\,V_2\}\cup(E_1'\cap E_2')\in\mathcal{E}_\dagger]\end{bmatrix}\ \textbf{ for all }\ E_1'\,,\,E_2'\Big]$$

Then, by applying (Q0)(P1), conclude:

$$\begin{gathered}(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\\ \big[E_2\notin\mathcal{E}_2\ \textbf{or}\ \big[\textbf{not}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]\ \textbf{and}\ \big[E_1\notin\mathcal{E}_1\ \textbf{or}\ \big[\textbf{not}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]\end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered}\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_2\notin\mathcal{E}_2\ \textbf{and}\ E_1\notin\mathcal{E}_1\big]\\ \textbf{or}\ \big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_2\notin\mathcal{E}_2\ \textbf{and}\ \big[\textbf{not}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]\\ \textbf{or}\ \big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ \big[\textbf{not}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\ \textbf{and}\ E_1\notin\mathcal{E}_1\big]\\ \textbf{or}\ \big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ \big[\textbf{not}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\ \textbf{and}\ \big[\textbf{not}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]\end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered}\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_2\notin\mathcal{E}_2\ \textbf{and}\ E_1\notin\mathcal{E}_1\big]\\ \textbf{or}\ \big[E_2\notin\mathcal{E}_2\ \textbf{and}\ \textbf{false}\big]\ \textbf{or}\ \big[\textbf{false}\ \textbf{and}\ E_1\notin\mathcal{E}_1\big]\\ \textbf{or}\ \big[\textbf{false}\ \textbf{and}\ \big[\textbf{not}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\big]\big]\end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_2\notin\mathcal{E}_2\ \textbf{and}\ E_1\notin\mathcal{E}_1\big]\ \textbf{or}\ \textbf{false}\ \textbf{or}\ \textbf{false}\ \textbf{or}\ \textbf{false}$$

Then, by applying standard inference rules, conclude:

$$(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_1\notin\mathcal{E}_1\ \textbf{and}\ E_2\notin\mathcal{E}_2$$

(P3) Suppose:

$$\left[\begin{gathered}E_\dagger\in\mathcal{F}_\dagger\ \textbf{and}\ E_\dagger\notin\mathcal{E}_\dagger\\ \textbf{and}\ \mathcal{E}=(\mathcal{F}\setminus\{\mathcal{E}_1\cup\mathcal{E}_2\})\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2\subseteq\mathcal{F}_2\ \textbf{and}\\ \textbf{and}\ \left[\left[\begin{gathered}\big[E'_2\in\mathcal{E}_2\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big]\ \textbf{implies}\\ \big[E'_1\in\mathcal{E}_1\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E'_1\cap E'_2)\in\mathcal{E}_\dagger\big]\end{gathered}\right]\ \textbf{for all}\ E'_1\,,\,E'_2\right]\\ \textbf{and}\ \left[\left[\begin{gathered}\big[E'_1\in\mathcal{E}_1\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big]\ \textbf{implies}\\ \big[E'_2\in\mathcal{E}_2\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E'_1\cap E'_2)\in\mathcal{E}_\dagger\big]\end{gathered}\right]\ \textbf{for all}\ E'_1\,,\,E'_2\right]\end{gathered}\right]$$
$$\textbf{for some}\ E_\dagger\,,\,\mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (P2), conclude:

$$\begin{gathered}\mathcal{E}=(\mathcal{F}\setminus\{\mathcal{E}_1\cup\mathcal{E}_2\})\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2\subseteq\mathcal{F}_2\ \textbf{and}\\ \big[\big[(E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_1\notin\mathcal{E}_1\ \textbf{and}\ E_2\notin\mathcal{E}_2\big]\ \textbf{for some}\ E_1\,,\,E_2\big]\end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{gathered}\mathcal{E}=(\mathcal{F}\setminus\{\mathcal{E}_1\cup\mathcal{E}_2\})\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2\subseteq\mathcal{F}_2\\ \textbf{and}\ (E_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E_2\,,\,V_2)\ \textbf{and}\ E_1\notin\mathcal{E}_1\ \textbf{and}\ E_2\notin\mathcal{E}_2\end{gathered}\right]\ \textbf{for some}\ E_1\,,\,E_2$$

Then, by applying (Q6), conclude **false**.

(P4) Recall from (R1):

$$\left[\begin{gathered}\mathcal{E}=(\mathcal{F}\setminus(\mathcal{E}_1\cup\mathcal{E}_2))\cup\mathcal{E}_\dagger\ \textbf{and}\ \mathcal{E}_1\subseteq\mathcal{F}_1\ \textbf{and}\ \mathcal{E}_2\subseteq\mathcal{F}_2\ \textbf{and}\ \mathcal{E}_\dagger\subseteq\mathcal{F}_\dagger\\ \textbf{and}\ \left[\left[\begin{gathered}\big[E'_1\in\mathcal{E}_1\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big]\ \textbf{implies}\\ \big[E'_2\in\mathcal{E}_2\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E'_1\cap E'_2)\in\mathcal{E}_\dagger\big]\end{gathered}\right]\ \textbf{for all}\ E'_1\,,\,E'_2\right]\\ \textbf{and}\ \left[\left[\begin{gathered}\big[E'_2\in\mathcal{E}_2\ \textbf{and}\ (E'_1\,,\,V_1)\curlyvee_{\mathcal{F}}(E'_2\,,\,V_2)\big]\ \textbf{implies}\\ \big[E'_1\in\mathcal{E}_1\ \textbf{and}\ \{V_1\,,\,V_2\}\cup(E'_1\cap E'_2)\in\mathcal{E}_\dagger\big]\end{gathered}\right]\ \textbf{for all}\ E'_1\,,\,E'_2\right]\end{gathered}\right]$$
$$\textbf{for some}\ \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger$$

Then, by applying (Q7)(Q8), conclude:

$$\begin{array}{c}\mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\textbf{ and }\; \mathcal{E}_1 \subseteq \mathcal{F}_1 \;\textbf{ and }\; \mathcal{E}_2 \subseteq \mathcal{F}_2 \;\textbf{ and }\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and } \Big[\Big[\begin{array}{c}[E'_1 \in \mathcal{E}_1 \textbf{ and } (E'_1, V_1) \curlyvee_{\mathcal{F}} (E'_2, V_2)] \textbf{ implies} \\ [E'_2 \in \mathcal{E}_2 \textbf{ and } \{V_1, V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger]\end{array}\Big] \textbf{ for all } E'_1, E'_2\Big] \\ \textbf{and } \Big[\Big[\begin{array}{c}[E'_2 \in \mathcal{E}_2 \textbf{ and } (E'_1, V_1) \curlyvee_{\mathcal{F}} (E'_2, V_2)] \textbf{ implies} \\ [E'_1 \in \mathcal{E}_1 \textbf{ and } \{V_1, V_2\} \cup (E'_1 \cap E'_2) \in \mathcal{E}_\dagger]\end{array}\Big] \textbf{ for all } E'_1, E'_2\Big] \\ \textbf{and } \big[[E_1 \in \mathcal{F}_1 \textbf{ implies } E_1 \in \mathcal{E}_1] \textbf{ for all } E_1\big] \\ \textbf{and } \big[[E_2 \in \mathcal{F}_2 \textbf{ implies } E_2 \in \mathcal{E}_2] \textbf{ for all } E_2\big]\end{array}$$

Then, by applying (P3), conclude:

$$\begin{array}{c}\mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\textbf{ and }\; \mathcal{E}_1 \subseteq \mathcal{F}_1 \;\textbf{ and }\; \mathcal{E}_2 \subseteq \mathcal{F}_2 \;\textbf{ and }\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and } \big[[E_1 \in \mathcal{F}_1 \textbf{ implies } E_1 \in \mathcal{E}_1] \textbf{ for all } E_1\big] \\ \textbf{and } \big[[E_2 \in \mathcal{F}_2 \textbf{ implies } E_2 \in \mathcal{E}_2] \textbf{ for all } E_2\big] \\ \textbf{and } \big[[E_\dagger \in \mathcal{F}_\dagger \textbf{ implies } E_\dagger \in \mathcal{E}_\dagger] \textbf{ for all } E_\dagger\big]\end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c}\mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\textbf{ and }\; \mathcal{E}_1 \subseteq \mathcal{F}_1 \;\textbf{ and }\; \mathcal{E}_2 \subseteq \mathcal{F}_2 \;\textbf{ and }\; \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\ \textbf{and } \mathcal{F}_1 \subseteq \mathcal{E}_1 \textbf{ and } \mathcal{F}_2 \subseteq \mathcal{E}_2 \textbf{ and } \mathcal{F}_\dagger \subseteq \mathcal{E}_\dagger\end{array}$$

Then, by applying ZFC, conclude:

$$\mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \;\textbf{ and }\; \mathcal{E}_1 = \mathcal{F}_1 \;\textbf{ and }\; \mathcal{E}_2 = \mathcal{F}_2 \;\textbf{ and }\; \mathcal{E}_\dagger = \mathcal{F}_\dagger$$

Then, by applying substitution, conclude $\mathcal{E} = (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger$. Then, by applying Figure 15, conclude Interm.

(P5) Recall $\big[[\mathsf{Cond}_2 \textbf{ and } \mathcal{E} = \mathcal{F}] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\big]$ from (R1). Then, by introducing (D2), conclude:

$$\big[\mathsf{Cond}_2 \textbf{ and } \mathcal{E} = \mathcal{F} \textbf{ and } [\textbf{not } \mathsf{Cond}_2]\big] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|$$

Then, by applying standard inference rules, conclude $\big[[\mathcal{E} = \mathcal{F} \textbf{ and false}] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\big]$. Then, by applying standard inference rules, conclude $[\textbf{false or } |\mathcal{E}| < |\mathcal{F}|]$. Then, by applying standard inference rules, conclude $|\mathcal{E}| < |\mathcal{F}|$. Then, by introducing (R1), conclude $\big[|\mathcal{E}| < |\mathcal{F}| \textbf{ and } |\mathcal{F}| = z_1\big]$. Then, by applying substitution, conclude $|\mathcal{E}| < z_1$.

Now, prove the theorem by the following reduction. Recall Inv_2 from (D1). Then, by introducing (P4), conclude $[\mathsf{Inv}_2 \textbf{ and } \mathsf{Interm}]$. Then, by introducing (P5), conclude $[\mathsf{Inv}_2 \textbf{ and } \mathsf{Interm} \textbf{ and } |\mathcal{E}| < z_1]$.

⦇ QED. ⦈

5. First, assume:

(E1) Inv_2

(E2) Interm

(E3) $|\mathcal{E}| < z_1$

Next, observe:

(01) Recall Inv_2 from (E1). Then, by applying Figure 15, conclude:

$$\begin{gathered}
[\![(\mathcal{V}\,,\,\mathcal{F})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } \bigstar(\mathcal{F}) = \bigstar(\mathcal{E}_{\text{in}}) \textbf{ and } \checkmark(\mathcal{F}) \textbf{ and } \\
|\mathcal{F}| = z_1 \textbf{ and } (X\,,\,V_1) \curlyvee_{\mathcal{F}} (Y\,,\,V_2) \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and } \\
\mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \textbf{ and } \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \\
\textbf{and } \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \textbf{ and } \\
\textbf{and } [\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{F})]\!] \textbf{ and } \big[[\mathsf{Cond}_2 \textbf{ and } \mathcal{E} = \mathcal{F}] \textbf{ or } |\mathcal{E}| < |\mathcal{F}|\big] \textbf{ and } \\
\left[\begin{array}{c}
\left[\begin{array}{l}
\mathcal{E} = (\mathcal{F} \setminus (\mathcal{E}_1 \cup \mathcal{E}_2)) \cup \mathcal{E}_\dagger \textbf{ and } \mathcal{E}_1 \subseteq \mathcal{F}_1 \textbf{ and } \mathcal{E}_2 \subseteq \mathcal{F}_2 \textbf{ and } \mathcal{E}_\dagger \subseteq \mathcal{F}_\dagger \\
\textbf{and } \Big[\left[\begin{array}{c}[E_1' \in \mathcal{E}_1 \textbf{ and } (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2)] \textbf{ implies} \\ [E_2' \in \mathcal{E}_2 \textbf{ and } \{V_1\,,\,V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger]\end{array}\right] \textbf{ for all } E_1'\,,\,E_2'\Big] \\
\textbf{and } \Big[\left[\begin{array}{c}[E_2' \in \mathcal{E}_2 \textbf{ and } (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2)] \textbf{ implies} \\ [E_1' \in \mathcal{E}_1 \textbf{ and } \{V_1\,,\,V_2\} \cup (E_1' \cap E_2') \in \mathcal{E}_\dagger]\end{array}\right] \textbf{ for all } E_1'\,,\,E_2'\Big]
\end{array}\right] \\
\textbf{for some } \mathcal{E}_1\,,\,\mathcal{E}_2\,,\,\mathcal{E}_\dagger
\end{array}\right]
\end{gathered}$$

(02) Recall $\big[[\![(\mathcal{V}\,,\,\mathcal{F})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \textbf{ and } [\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{F})]\!]\big]$ from (01). Then, by applying substitution, conclude $[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]$.

(03) Suppose:

$$P \in \wp^+(\mathsf{Port}(\mathcal{F})) \textbf{ for some } P$$

Then, by applying ZFC, conclude $P \neq \emptyset$. Then, by applying ZFC, conclude:

$$p \in P \textbf{ for some } p$$

(04) Suppose:

$$E_1 \in \mathcal{F}_1 \textbf{ for some } E_1$$

Then, by applying (01), conclude $E_1 \in \{E_1' \mid (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2)\}$. Then, by applying ZFC, conclude $(E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)$. Then, by applying Definition 19 of $\curlyvee$, conclude $E_1 \in \wp(\mathbb{V}\text{ER})$.

(05) Suppose:

$$E_2 \in \mathcal{F}_2 \textbf{ for some } E_2$$

Then, by a reduction similar to (04), conclude $E_2 \in \wp(\mathbb{V}\text{ER})$.

(06) Suppose $V_1\,,\,V_2 \in \mathbb{V}\text{ER}$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1\,,\,V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by applying ZFC, conclude $V_1 \cup V_2 \in \wp(\mathbb{P}\text{ORT})$. Then, by applying Definition 15 of $\mathbb{V}\text{ER}$, conclude $V_1 \cup V_2 \in \mathbb{V}\text{ER}$. Then, by applying ZFC, conclude $\{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})$.

(07) Suppose:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2) \textbf{ for some } E_1\,,\,E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude $\big[E_1\,,\,E_2 \in \wp(\mathbb{V}\text{ER}) \textbf{ and } V_1\,,\,V_2 \in \mathbb{V}\text{ER}\big]$. Then, by applying (06), conclude $E_1\,,\,E_2\,,\,\{V_1 \cup V_2\} \in \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude:

$$\{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})$$

(08) Suppose:

$$E_\dagger \in \mathcal{F}_\dagger \textbf{ for some } E_\dagger$$

Then, by applying (01), conclude $E_\dagger \in \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}$. Then, by applying ZFC, conclude:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\big] \textbf{ for some } E_1\,,\,E_2$$

Then, by applying (07), conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \in \wp(\mathbb{V}\text{ER})$$

Then, by applying substitution, conclude $E_\dagger \in \wp(\mathbb{V}\text{ER})$.

(09) Recall $\checkmark(\mathcal{F})$ from (01). Then, by applying Definition 24 of $\checkmark$, conclude $\mathcal{F} \in \wp^2(\mathbb{V}\text{ER})$.

(00) Recall Interm from (E2). Then, by applying Figure 15, conclude $\mathcal{E} = (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger$.

(N1) Recall from (04)(05):

$$\big[\big[E_1 \in \mathcal{E}_1 \textbf{ implies } E_1 \in \wp(\mathbb{V}\text{ER})\big] \textbf{ for all } E_1\big] \\ \textbf{and } \big[\big[E_2 \in \mathcal{E}_2 \textbf{ implies } E_2 \in \wp(\mathbb{V}\text{ER})\big] \textbf{ for all } E_2\big]$$

Then, by introducing (08), conclude:

$$\big[\big[E_1 \in \mathcal{F}_1 \textbf{ implies } E_1 \in \wp(\mathbb{V}\text{ER})\big] \textbf{ for all } E_1\big] \\ \textbf{and } \big[\big[E_2 \in \mathcal{F}_2 \textbf{ implies } E_2 \in \wp(\mathbb{V}\text{ER})\big] \textbf{ for all } E_2\big] \\ \textbf{and } \big[\big[E_\dagger \in \mathcal{F}_\dagger \textbf{ implies } E_\dagger \in \wp(\mathbb{V}\text{ER})\big] \textbf{ for all } E_\dagger\big]$$

Then, by applying ZFC, conclude $\mathcal{F}_1 , \mathcal{F}_2 , \mathcal{F}_\dagger \subseteq \wp(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude:

$$\mathcal{F}_1 , \mathcal{F}_2 , \mathcal{F}_\dagger \in \wp^2(\mathbb{V}\text{ER})$$

Then, by introducing (09), conclude $\mathcal{F}_1 , \mathcal{F}_2 , \mathcal{F}_\dagger , \mathcal{F} \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying ZFC, conclude $(\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying (00), conclude $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$.

(N2) Suppose:

$$\mathcal{T} = \mathsf{Edge}(p , \mathcal{F}) \textbf{ for some } \mathcal{T} , p$$

Then, by applying Definition 22 of Edge, conclude $p \in \mathbb{P}\text{ORT}$. Then, by introducing (N1), conclude:

$$p \in \mathbb{P}\text{ORT} \textbf{ and } \mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$$

Then, by applying Definition 22 of Edge, conclude $\mathsf{Edge}(p , \mathcal{E}) \in \wp^2(\mathbb{V}\text{ER})$. Then, by applying standard inference rules, conclude:

$$\mathcal{T}' = \mathsf{Edge}(p , \mathcal{E}) \textbf{ for some } \mathcal{T}'$$

(N3) Suppose:

$$\big[P \in \wp^+(\mathsf{Port}(\mathcal{F})) \textbf{ and } \big[\big[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p' , \mathcal{F})\big] \textbf{ for all } p'\big]\big] \textbf{ for some } P , \mathcal{T}$$

Then, by applying (03), conclude:

$$\big[p \in P \textbf{ for some } p\big] \textbf{ and } \big[\big[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p' , \mathcal{F})\big] \textbf{ for all } p'\big]$$

Then, by applying standard inference rules, conclude:

$$\big[p \in P \textbf{ and } \big[\big[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p' , \mathcal{F})\big] \textbf{ for all } p'\big]\big] \textbf{ for some } p$$

Then, by applying standard inference rules, conclude:

$$\big[\big[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p' , \mathcal{F})\big] \textbf{ for all } p'\big] \textbf{ and } \mathcal{T} = \mathsf{Edge}(p , \mathcal{F})$$

Then, by applying (N2), conclude:

$$\big[\big[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p' , \mathcal{F})\big] \textbf{ for all } p'\big] \textbf{ and} \\ \mathcal{T} = \mathsf{Edge}(p , \mathcal{F}) \textbf{ and } \big[\mathcal{T}' = \mathsf{Edge}(p , \mathcal{E}) \textbf{ for some } \mathcal{T}'\big]$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{l}\big[\big[p' \in P \textbf{ iff } \mathcal{T} = \mathsf{Edge}(p' , \mathcal{F})\big] \textbf{ for all } p'\big] \\ \textbf{and } \mathcal{T} = \mathsf{Edge}(p , \mathcal{F}) \textbf{ and } \mathcal{T}' = \mathsf{Edge}(p , \mathcal{E})\end{array}\right] \textbf{ for some } \mathcal{T}'$$

Then, by applying substitution, conclude:

$$\big[\big[p' \in P \;\textbf{iff}\; \mathsf{Edge}(p\,,\mathcal{F}) = \mathsf{Edge}(p'\,,\mathcal{F})\big] \;\textbf{for all}\; p'\big] \;\textbf{and}\; \mathcal{T}' = \mathsf{Edge}(p\,,\mathcal{E})$$

Then, by introducing (O1), conclude:

$$\begin{gathered}\big[\big[p' \in P \;\textbf{iff}\; \mathsf{Edge}(p\,,\mathcal{F}) = \mathsf{Edge}(p'\,,\mathcal{F})\big] \;\textbf{for all}\; p'\big] \;\textbf{and}\; \mathcal{T}' = \mathsf{Edge}(p\,,\mathcal{E}) \\ \textbf{and}\; \checkmark(\mathcal{F}) \;\textbf{and}\; (X\,,V_1) \curlyvee_{\mathcal{F}} (Y\,,V_2) \;\textbf{and}\; V_1 \cup V_2 \subseteq P \;\textbf{and}\; P \in \bigstar(\mathcal{F}) \;\textbf{and} \\ \mathcal{F}_1 = \{E_1 \mid (E_1\,,V_1) \curlyvee_{\mathcal{F}} (E_2\,,V_2)\} \;\textbf{and}\; \mathcal{F}_2 = \{E_2 \mid (E_1\,,V_1) \curlyvee_{\mathcal{F}} (E_2\,,V_2)\} \\ \textbf{and}\; \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\textbf{and}\; (E_1\,,V_1) \curlyvee_{\mathcal{F}} (E_2\,,V_2)\}\end{gathered}$$

Then, by introducing (O0), conclude:

$$\begin{gathered}\big[\big[p' \in P \;\textbf{iff}\; \mathsf{Edge}(p\,,\mathcal{F}) = \mathsf{Edge}(p'\,,\mathcal{F})\big] \;\textbf{for all}\; p'\big] \;\textbf{and}\; \mathcal{T}' = \mathsf{Edge}(p\,,\mathcal{E}) \\ \textbf{and}\; \checkmark(\mathcal{F}) \;\textbf{and}\; (X\,,V_1) \curlyvee_{\mathcal{F}} (Y\,,V_2) \;\textbf{and}\; V_1 \cup V_2 \subseteq P \;\textbf{and}\; P \in \bigstar(\mathcal{F}) \;\textbf{and} \\ \mathcal{F}_1 = \{E_1 \mid (E_1\,,V_1) \curlyvee_{\mathcal{F}} (E_2\,,V_2)\} \;\textbf{and}\; \mathcal{F}_2 = \{E_2 \mid (E_1\,,V_1) \curlyvee_{\mathcal{F}} (E_2\,,V_2)\} \\ \textbf{and}\; \mathcal{F}_\dagger = \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\textbf{and}\; (E_1\,,V_1) \curlyvee_{\mathcal{F}} (E_2\,,V_2)\} \\ \textbf{and}\; \mathcal{E} = (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger\end{gathered}$$

Then, by applying Lemma 14:4, conclude:

$$\big[\big[p' \in P \;\textbf{iff}\; \mathsf{Edge}(p\,,\mathcal{E}) = \mathsf{Edge}(p'\,,\mathcal{E})\big] \;\textbf{for all}\; p'\big] \;\textbf{and}\; \mathcal{T}' = \mathsf{Edge}(p\,,\mathcal{E})$$

Then, by applying substitution, conclude $\big[\big[p' \in P \;\textbf{iff}\; \mathcal{T}' = \mathsf{Edge}(p'\,,\mathcal{E})\big] \;\textbf{for all}\; p'\big]$.

(N4) Suppose:

$$P \in \bigstar(\mathcal{F}) \;\textbf{for some}\; P$$

Then, by applying Definition 23 of $\bigstar$, conclude:

$$P \in \{P' \mid P' \in \wp^+(\mathsf{Port}(\mathcal{F})) \;\textbf{and}\; \big[\big[p' \in P' \;\textbf{iff}\; \mathcal{T} = \mathsf{Edge}(p'\,,\mathcal{F})\big] \;\textbf{for all}\; p'\big]\}$$

Then, by applying ZFC, conclude:

$$\big[P \in \wp^+(\mathsf{Port}(\mathcal{F})) \;\textbf{and}\; \big[\big[p' \in P \;\textbf{iff}\; \mathcal{T} = \mathsf{Edge}(p'\,,\mathcal{F})\big] \;\textbf{for all}\; p'\big]\big] \;\textbf{for some}\; \mathcal{T}$$

Then, by applying (N3), conclude:

$$P \in \wp^+(\mathsf{Port}(\mathcal{F})) \;\textbf{and}\; \big[\big[\big[p' \in P \;\textbf{iff}\; \mathcal{T}' = \mathsf{Edge}(p'\,,\mathcal{E})\big] \;\textbf{for all}\; p'\big] \;\textbf{for some}\; \mathcal{T}'\big]$$

Then, by applying standard inference rules, conclude:

$$\big[P \in \wp^+(\mathsf{Port}(\mathcal{F})) \;\textbf{and}\; \big[\big[p' \in P \;\textbf{iff}\; \mathcal{T}' = \mathsf{Edge}(p'\,,\mathcal{E})\big] \;\textbf{for all}\; p'\big]\big] \;\textbf{for some}\; \mathcal{T}'$$

Then, by applying ZFC, conclude:

$$P \in \{P' \mid P' \in \wp^+(\mathsf{Port}(\mathcal{F})) \;\textbf{and}\; \big[\big[p' \in P' \;\textbf{iff}\; \mathcal{T}'' = \mathsf{Edge}(p'\,,\mathcal{E})\big] \;\textbf{for all}\; p'\big]\}$$

Then, by introducing (N1), conclude:

$$\begin{gathered}P \in \{P' \mid P' \in \wp^+(\mathsf{Port}(\mathcal{F})) \;\textbf{and}\; \big[\big[p' \in P' \;\textbf{iff}\; \mathcal{T}'' = \mathsf{Edge}(p'\,,\mathcal{E})\big] \;\textbf{for all}\; p'\big]\} \\ \textbf{and}\; \mathcal{E} \in \wp^2(\mathbb{V}\textsc{er})\end{gathered}$$

Then, by applying Definition 23 of $\bigstar$, conclude $P \in \bigstar(\mathcal{E})$.

(N5) Recall $\big[\big[P \in \bigstar(\mathcal{F}) \;\textbf{implies}\; P \in \bigstar(\mathcal{E})\big] \;\textbf{for all}\; P\big]$ from (N4). Then, by applying ZFC, conclude:

$$\bigstar(\mathcal{F}) \subseteq \bigstar(\mathcal{E})$$

(N6) Suppose:

$$[P' \in \bigstar(\mathcal{F}) \text{ and } P \notin \bigstar(\mathcal{F}) \text{ and } P = P'] \text{ for some } P, P'$$

Then, by applying substitution, conclude $[P \in \bigstar(\mathcal{F}) \text{ and } P \notin \bigstar(\mathcal{F})]$. Then, by applying standard inference rules, conclude **false**.

(N7) Suppose:

$$[P \in \bigstar(\mathcal{E}) \text{ and } P \notin \bigstar(\mathcal{F}) \text{ and } P' \in \bigstar(\mathcal{F})] \text{ for some } P, P'$$

Then, by introducing (N5), conclude:

$$P \in \bigstar(\mathcal{E}) \text{ and } P \notin \bigstar(\mathcal{F}) \text{ and } P' \in \bigstar(\mathcal{F}) \text{ and } \bigstar(\mathcal{F}) \subseteq \bigstar(\mathcal{E})$$

Then, by applying ZFC, conclude $[P, P' \in \bigstar(\mathcal{E}) \text{ and } P \notin \bigstar(\mathcal{F})]$. Then, by applying (N6), conclude $[P, P' \in \bigstar(\mathcal{E}) \text{ and } P \neq P']$. Then, by applying Lemma 12:2, conclude $P \cap P' = \emptyset$.

(N8) Suppose:

$$P \in \bigstar(\mathcal{E}) \text{ for some } P$$

Then, by applying Definition 23 of $\bigstar$, conclude:

$$P \in \{P' \mid P' \in \wp^{+}(\mathsf{Port}(\mathcal{E})) \text{ and } [[p' \in P' \text{ iff } \mathcal{T} = \mathsf{Edge}(p', \mathcal{F})] \text{ for all } p']\}$$

Then, by applying ZFC, conclude $P \in \wp^{+}(\mathsf{Port}(\mathcal{E}))$. Then, by applying ZFC, conclude $P \neq \emptyset$. Then, by applying ZFC, conclude:

$$p \in P \text{ for some } p$$

(N9) Recall from (00) $\mathcal{E} = (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_{\dagger}$. Then, by introducing (01), conclude:

$$\begin{gathered}\mathcal{E} = (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_{\dagger} \text{ and} \\ \checkmark(\mathcal{F}) \text{ and } (X, V_1) \curlyvee_{\mathcal{F}} (Y, V_2) \text{ and } V_1 \cup V_2 \subseteq P \text{ and } P \in \bigstar(\mathcal{F}) \text{ and} \\ \mathcal{F}_1 = \{E_1 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \text{ and } \mathcal{F}_2 = \{E_2 \mid (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\} \\ \text{and } \mathcal{F}_{\dagger} = \{E_{\dagger} \mid E_{\dagger} = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \text{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\}\end{gathered}$$

Then, by applying Lemma 13:9, conclude $\mathsf{Port}(\mathcal{E}) = \mathsf{Port}(\mathcal{F})$.

(N0) Suppose:

$$[P \in \bigstar(\mathcal{E}) \text{ and } P \notin \bigstar(\mathcal{F})] \text{ for some } P$$

Then, by applying (N7), conclude:

$$P \in \bigstar(\mathcal{E}) \text{ and } [[P' \in \bigstar(\mathcal{F}) \text{ implies } P \cap P' = \emptyset] \text{ for all } P']$$

Then, by applying (N8), conclude:

$$P \in \bigstar(\mathcal{E}) \text{ and } [[P' \in \bigstar(\mathcal{F}) \text{ implies } P \cap P' = \emptyset] \text{ for all } P'] \text{ and } [p \in P \text{ for some } p]$$

Then, by applying standard inference rules, conclude:

$$[p \in P \in \bigstar(\mathcal{E}) \text{ and } [[P' \in \bigstar(\mathcal{F}) \text{ implies } P \cap P' = \emptyset] \text{ for all } P']] \text{ for some } p$$

Then, by applying ZFC, conclude:

$$p \in P \in \bigstar(\mathcal{E}) \text{ and } [[P' \in \bigstar(\mathcal{F}) \text{ implies } p \notin P'] \text{ for all } P']$$

Then, by applying ZFC, conclude $[p \in P \in \bigstar(\mathcal{E}) \text{ and } p \notin \bigcup \bigstar(\mathcal{F})]$. Then, by applying ZFC, conclude $[p \in \bigcup \bigstar(\mathcal{E}) \text{ and } p \notin \bigcup \bigstar(\mathcal{F})]$. Then, by applying Lemma 12:1, conclude:

$$p \in \mathsf{Port}(\mathcal{E}) \text{ and } p \notin \mathsf{Port}(\mathcal{F})$$

Then, by introducing (N9), conclude $[p \in \mathsf{Port}(\mathcal{E}) \text{ and } p \notin \mathsf{Port}(\mathcal{F}) \text{ and } \mathsf{Port}(\mathcal{E}) = \mathsf{Port}(\mathcal{F})]$. Then, by applying substitution, conclude $[p \in \mathsf{Port}(\mathcal{E}) \text{ and } p \notin \mathsf{Port}(\mathcal{E})]$. Then, by applying standard inference rules, conclude **false**.

(M1) Recall $\big[\big[P \in \bigstar(\mathcal{E})$ **implies** $P \in \bigstar(\mathcal{F})\big]$ **for all** $P\big]$ from (N0). Then, by applying ZFC, conclude:

$$\bigstar(\mathcal{E}) \subseteq \bigstar(\mathcal{F})$$

Then, by introducing (N5), conclude $\big[\bigstar(\mathcal{E}) \subseteq \bigstar(\mathcal{F})$ **and** $\bigstar(\mathcal{F}) \subseteq \bigstar(\mathcal{E})\big]$. Then, by applying ZFC, conclude $\bigstar(\mathcal{E}) = \bigstar(\mathcal{F})$.

(M2) Recall $\bigstar(\mathcal{E}) = \bigstar(\mathcal{F})$ from (M1). Then, by applying (O1), conclude $\bigstar(\mathcal{E}) = \bigstar(\mathcal{E}_{\text{in}})$.

(M3) Recall $\checkmark(\mathcal{F})$ from (O1). Then, by applying Definition 24 of $\checkmark$, conclude:

$$\Big[\Big[\begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F} \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F} \end{bmatrix} \textbf{ implies } V_1' = V_2'\Big] \textbf{ for all } p\,,\, V_1'\,,\, V_2'\,,\, E_1\,,\, E_2\Big]$$
$$\textbf{and } \big[\big[V \in E \in \mathcal{F} \textbf{ implies } V \neq \emptyset\big] \textbf{ for all } V\,,\, E\big] \textbf{ and}$$
$$\Big[\Big[V \in E \in \mathcal{F} \textbf{ implies } \begin{bmatrix} \big[V \subseteq P' \textbf{ and } P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some } P' \end{bmatrix}\Big] \textbf{ for all } V\,,\, E\Big]$$

(M4) Suppose:

$$\begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{bmatrix} \textbf{ for some } p\,,\, V_1'\,,\, V_2'\,,\, E_1\,,\, E_2$$

Then, by applying ZFC, conclude $\big[p \in V_1' \in E_1 \in \mathcal{F}$ **and** $p \in V_2' \in E_2 \in \mathcal{F}\big]$. Then, by applying (M3), conclude $V_1' = V_2'$.

(M5) Suppose:

$$E_\dagger \in \{E_\dagger' \mid E_\dagger' = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2)\} \textbf{ for some } E_\dagger$$

Then, by applying ZFC, conclude:

$$\big[E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2)\big] \textbf{ for some } E_1\,,\, E_2$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2) \textbf{ and } E_1\,,\, E_2 \in \mathcal{F}$$

Then, by applying Lemma 6:1, conclude:

$$E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2) \textbf{ and } V_1 \in E_1 \in \mathcal{F} \textbf{ and } V_2 \in E_2 \in \mathcal{F}$$

(M6) Suppose:

$$p \in V \in \{V_1 \cup V_2\} \textbf{ for some } p\,,\, V$$

Then, by applying ZFC, conclude $\big[p \in V_1 \cup V_2\big]$. Then, by applying ZFC, conclude:

$$p \in V_1 \textbf{ or } p \in V_2$$

(M7) Suppose:

$$p \in V \in \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \textbf{ for some } p\,,\, V\,,\, E_1'\,,\, E_2'$$

Then, by applying ZFC, conclude $\big[p \in V \in \{V_1 \cup V_2\}$ **or** $p \in V \in E_1' \cap E_2'\big]$. Then, by applying (M6), conclude $\big[p \in V_1$ **or** $p \in V_2$ **or** $p \in V \in E_1' \cap E_2'\big]$. Then, by applying ZFC, conclude:

$$p \in V_1 \textbf{ or } p \in V_2 \textbf{ or } p \in V \in E_1'$$

(M8) Suppose:

$$p \in V \in E_\dagger \in \mathcal{F}_\dagger \ \textbf{for some} \ p\,,\, V\,,\, E_\dagger$$

Then, by applying (O1), conclude:

$$p \in V \in E_\dagger \in \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \ \textbf{and} \ (E_1'\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\, V_2)\}$$

Then, by applying (M5), conclude:

$$p \in V \in E_\dagger \ \textbf{and} \ \left[\begin{bmatrix} E_\dagger = \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \\ \textbf{and} \ (E_1'\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\, V_2) \ \textbf{and} \\ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F} \end{bmatrix} \textbf{for some} \ E_1'\,,\, E_2' \right]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p' \in V \in E_\dagger \ \textbf{and} \ E_\dagger = \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \\ \textbf{and} \ (E_1'\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\, V_2) \ \textbf{and} \\ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F} \ \textbf{and} \end{bmatrix} \textbf{for some} \ E_1'\,,\, E_2'$$

Then, by applying substitution, conclude:

$$p \in V \in \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \ \textbf{and} \ (E_1'\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\, V_2) \ \textbf{and} \ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F}$$

(M9) Suppose:

$$p \in V \in E_\dagger \in \mathcal{F}_\dagger \ \textbf{for some} \ p\,,\, V\,,\, E_\dagger$$

Then, by applying (M8), conclude:

$$p \in V \in \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \ \textbf{and} \ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F}$$

Then, by applying (M7), conclude:

$$\left[p \in V_1 \ \textbf{or} \ p \in V_2 \ \textbf{or} \ p \in V \in E_1'\right] \ \textbf{and} \ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered} \left[p \in V_1 \ \textbf{and} \ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F}\right] \\ \textbf{or} \ \left[p \in V_2 \ \textbf{and} \ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F}\right] \\ \textbf{or} \ \left[p \in V \in E_1' \ \textbf{and} \ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F}\right] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$p \in V_1 \in E_1' \in \mathcal{F} \ \textbf{or} \ p \in V_2 \in E_2' \in \mathcal{F} \ \textbf{or} \ p \in V \in E_1' \in \mathcal{F}$$

(M0) Suppose:

$$\left[p \in V' \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and} \ p \in V \in E' \in \mathcal{F}\right] \ \textbf{for some} \ p\,,\, V\,,\, V'\,,\, E\,,\, E'$$

Then, by applying ZFC, conclude $\left[p \in V' \in E \in \mathcal{F} \ \textbf{and} \ p \in V \in E' \in \mathcal{F}\right]$. Then, by applying (M3), conclude $V' = V$.

(L1) Suppose:

$$p_1 \in V_1 \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{for some} \ p_1\,,\, E_1$$

Then, by applying ZFC, conclude $p_1 \in V_1 \in E_1 \in \mathcal{F}$. Then, by introducing (O1), conclude:

$$\begin{gathered} p_1 \in V_1 \in E_1 \in \mathcal{F} \ \textbf{and} \\ \checkmark(\mathcal{F}) \ \textbf{and} \ (X\,,\, V_1) \curlyvee_{\mathcal{F}} (Y\,,\, V_2) \ \textbf{and} \ V_1 \cup V_2 \subseteq P \ \textbf{and} \ P \in \bigstar(\mathcal{F}) \end{gathered}$$

Then, by applying Lemma 13:6, conclude:

$$(E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2) \ \textbf{for some}\ E_2$$

Then, by applying ZFC, conclude $E_1 \in \{E_1' \mid (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2)\}$. Then, by introducing (O1), conclude:

$$E_1 \in \{E_1' \mid (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2)\} \ \textbf{and}$$
$$\mathcal{F}_1 = \{E_1 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\} \ \textbf{and}\ \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}$$

Then, by applying substitution, conclude:

$$E_1 \in \mathcal{F}_1 \ \textbf{and}\ \mathcal{F}_2 = \{E_2 \mid (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\}$$

Then, by applying ZFC, conclude $E_1 \in \mathcal{F}_1 \cup \mathcal{F}_2$.

(L2) Suppose:

$$p_2 \in V_2 \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{for some}\ p_2\,,\,E_2$$

Then, by a reduction similar to (L1), conclude $E_2 \in \mathcal{F}_1 \cup \mathcal{F}_2$.

(L3) Suppose:

$$\big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ p \in V_1 \in E_1' \in \mathcal{F}\big] \ \textbf{for some}\ p\,,\,V_1'\,,\,E_1\,,\,E_1'$$

Then, by applying (M0), conclude $\big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ V_1' = V_1\big]$. Then, by applying substitution, conclude $p \in V_1 \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$. Then, by applying (L1), conclude:

$$E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ E_1 \in \mathcal{F}_1 \cup \mathcal{F}_2$$

Then, by applying ZFC, conclude $\big[E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ E_1 \notin \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)\big]$. Then, by applying standard inference rules, conclude **false**.

(L4) Suppose:

$$\big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ p \in V_2 \in E_2' \in \mathcal{F}\big] \ \textbf{for some}\ p\,,\,V_1'\,,\,E_1\,,\,E_2'$$

Then, by a reduction similar to (L3), conclude **false**

(L5) Suppose:

$$\big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ p \in V_2' \in E_2 \in \mathcal{F}_\dagger\big] \ \textbf{for some}\ p\,,\,V_1'\,,\,V_2'\,,\,E_1\,,\,E_2$$

Then, by applying (M9), conclude:

$$p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}$$
$$\big[\big[p \in V_1 \in E_1' \in \mathcal{F} \ \textbf{or}\ p \in V_2 \in E_2' \in \mathcal{F} \ \textbf{or}\ p \in V_2' \in E_1' \in \mathcal{F}\big] \ \textbf{for some}\ E_1'\,,\,E_2'\big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and} \\ p \in V_1 \in E_1' \in \mathcal{F} \ \textbf{or}\ p \in V_2 \in E_2' \in \mathcal{F} \ \textbf{or}\ p \in V_2' \in E_1' \in \mathcal{F} \end{bmatrix} \ \textbf{for some}\ E_1'\,,\,E_2'$$

Then, by applying standard inference rules, conclude:

$$\big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ p \in V_1 \in E_1' \in \mathcal{F}\big]$$
$$\textbf{or}\ \big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ p \in V_2 \in E_2' \in \mathcal{F}\big]$$
$$\textbf{or}\ \big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ p \in V_2' \in E_1' \in \mathcal{F}\big]$$

Then, by applying (L3)(L4), conclude:

$$\textbf{false}\ \textbf{or}\ \textbf{false}\ \textbf{or}\ \big[p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and}\ p \in V_2' \in E_1' \in \mathcal{F}\big]$$

Then, by applying standard inference rules, conclude:

$$p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{and} \ p \in V_2' \in E_1' \in \mathcal{F}$$

Then, by applying ZFC, conclude $\big[p \in V_1' \in E_1 \in \mathcal{F} \ \textbf{and} \ p \in V_2' \in E_1' \in \mathcal{F}\big]$. Then, by applying (M3), conclude $V_1' = V_2'$.

(L6) Suppose:

$$\big[p \in V_1' \in E_1 \in \mathcal{F}_\dagger \ \textbf{and} \ p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)\big] \ \textbf{for some} \ p\,,\, V_1'\,,\, V_2'\,,\, E_1\,,\, E_2$$

Then, by a reduction similar to (L5), conclude $V_1' = V_2'$.

(L7) Suppose:

$$\big[p \in V_1' \in \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_2' \in \{V_1 \cup V_2\}\big] \ \textbf{for some} \ p\,,\, V_1'\,,\, V_2'$$

Then, by applying ZFC, conclude $\big[V_1' = V_1 \cup V_2 \ \textbf{and} \ V_2' = V_1 \cup V_2\big]$. Then, by applying substitution, conclude $V_1' = V_2'$.

(L8) Suppose:

$$\big[p \in V \in \{V_1 \cup V_2\} \ \textbf{and} \ p \in V' \in E_1 \cap E_2 \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2 \in \mathcal{F}\big] \\ \textbf{for some} \ V\,,\, V'\,,\, E_1\,,\, E_2$$

Then, by applying ZFC, conclude:

$$p \in V_1 \cup V_2 \ \textbf{and} \ p \in V' \in E_2 \in \mathcal{F} \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2 \in \mathcal{F}$$

Then, by applying ZFC, conclude:

$$\big[p \in V_1 \ \textbf{or} \ p \in V_2\big] \ \textbf{and} \ p \in V' \in E_2 \in \mathcal{F} \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2 \in \mathcal{F}$$

Then, by applying standard inference rules, conclude:

$$\big[p \in V_1 \in E_1 \in \mathcal{F} \ \textbf{and} \ p \in V' \in E_2 \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2 \in \mathcal{F}\big] \\ \textbf{or} \ \big[p \in V_2 \in E_2 \in \mathcal{F} \ \textbf{and} \ p \in V' \in E_2 \in \mathcal{F} \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big]$$

Then, by applying standard inference rules, conclude:

$$\big[p \in V_1 \in E_1 \in \mathcal{F} \ \textbf{and} \ p \in V' \in E_2 \in \mathcal{F}\big] \ \textbf{or} \ \big[p \in V_2 \in E_2 \in \mathcal{F} \ \textbf{and} \ p \in V' \in E_2 \in \mathcal{F}\big]$$

Then, by applying (M3), conclude $\big[V_1 = V' \ \textbf{or} \ V_2 = V'\big]$.

(L9) Suppose:

$$\begin{bmatrix} p \in V \in \{V_1 \cup V_2\} \ \textbf{and} \ p \in V' \in E_1 \cap E_2 \\ \textbf{and} \ (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2) \ \textbf{and} \\ V_1 \in E_1 \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2 \in \mathcal{F} \end{bmatrix} \ \textbf{for some} \ V\,,\, V'\,,\, E_1\,,\, E_2$$

Then, by applying (L8), conclude

$$V' \in E_1 \cap E_2 \ \textbf{and} \ (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2) \ \textbf{and} \ \big[V_1 = V' \ \textbf{or} \ V_2 = V'\big]$$

Then, by applying standard inference rules, conclude:

$$\big[V' \in E_1 \cap E_2 \ \textbf{and} \ (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2) \ \textbf{and} \ V_1 = V'\big] \\ \textbf{or} \ \big[V' \in E_1 \cap E_2 \ \textbf{and} \ (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2) \ \textbf{and} \ V_2 = V'\big]$$

Then, by applying substitution, conclude:

$$\big[V_1 \in E_1 \cap E_2 \ \textbf{and} \ (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2)\big] \\ \textbf{or} \ \big[V_2 \in E_1 \cap E_2 \ \textbf{and} \ (E_1\,,\, V_1) \curlyvee_{\mathcal{F}} (E_2\,,\, V_2)\big]$$

Then, by applying ZFC, conclude:

$$\begin{array}{c}\big[V_1 \in E_2 \ \textbf{and} \ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\big] \\ \textbf{or} \ \big[V_2 \in E_1 \ \textbf{and} \ (E_1\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2\,,\,V_2)\big]\end{array}$$

Then, by applying Lemma 7:2, conclude:

$$\big[V_1 \in E_2 \ \textbf{and} \ V_1 \notin E_2\big] \ \textbf{or} \ \big[V_2 \in E_1 \ \textbf{and} \ V_2 \notin E_1\big]$$

Then, by applying standard inference rules, conclude $\big[\textbf{false} \ \textbf{or} \ \textbf{false}\big]$. Then, by applying standard inference rules, conclude **false**.

(L0) Suppose:

$$\begin{array}{c}\big[p \in V_1' \in E_1' \cap E_2' \ \textbf{and} \ p \in V_2' \in E_1'' \cap E_2'' \ \textbf{and} \ E_1' \in \mathcal{F} \ \textbf{and} \ E_2'' \in \mathcal{F}\big] \\ \textbf{for some} \ V_1'\,,\,V_2'\,,\,E_1'\,,\,E_1''\,,\,E_2'\,,\,E_2''\end{array}$$

Then, by applying ZFC, conclude $\big[p \in V_1' \in E_1' \in \mathcal{F} \ \textbf{and} \ p \in V_2' \in E_2'' \in \mathcal{F}\big]$. Then, by applying (M3), conclude $V_1' = V_2'$.

(K1) Suppose:

$$\big[p \in V_1' \in E_1 \in \mathcal{F}_\dagger \ \textbf{and} \ p \in V_2' \in E_2 \in \mathcal{F}_\dagger\big] \ \textbf{for some} \ p\,,\,V_1'\,,\,V_2'\,,\,E_1\,,\,E_2$$

Then, by applying (M8), conclude:

$$\left[\begin{array}{c}\left[\begin{array}{c}p \in V_1' \in \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \\ \textbf{and} \ (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2) \ \textbf{and} \\ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F}\end{array}\right] \\ \textbf{for some} \ E_1'\,,\,E_2'\end{array}\right] \ \textbf{and} \ \left[\begin{array}{c}\left[\begin{array}{c}p \in V_2' \in \{V_1 \cup V_2\} \cup (E_1'' \cap E_2'') \\ \textbf{and} \ (E_1''\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2''\,,\,V_2) \ \textbf{and} \\ V_1 \in E_1'' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2'' \in \mathcal{F}\end{array}\right] \\ \textbf{for some} \ E_1''\,,\,E_2''\end{array}\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c}\left[\begin{array}{c}p \in V_1' \in \{V_1 \cup V_2\} \cup (E_1' \cap E_2') \ \textbf{and} \ p \in V_2' \in \{V_1 \cup V_2\} \cup (E_1'' \cap E_2'') \\ \textbf{and} \ (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2) \ \textbf{and} \ (E_1''\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2''\,,\,V_2) \ \textbf{and} \\ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F} \ \textbf{and} \ V_1 \in E_1'' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2'' \in \mathcal{F}\end{array}\right] \\ \textbf{for some} \ E_1'\,,\,E_1''\,,\,E_2'\,,\,E_2''\end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c}\big[p \in V_1' \in \{V_1 \cup V_2\} \ \textbf{or} \ p \in V_1' \in E_1' \cap E_2'\big] \ \textbf{and} \ \big[p \in V_2' \in \{V_1 \cup V_2\} \ \textbf{or} \ p \in V_2' \in E_1'' \cap E_2''\big] \\ \textbf{and} \ (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2) \ \textbf{and} \ (E_1''\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2''\,,\,V_2) \ \textbf{and} \\ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F} \ \textbf{and} \ V_1 \in E_1'' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2'' \in \mathcal{F}\end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c}\left[\begin{array}{c}\big[p \in V_1' \in \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_2' \in \{V_1 \cup V_2\}\big] \\ \textbf{or} \ \big[p \in V_1' \in \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_2' \in E_1'' \cap E_2''\big] \\ \textbf{or} \ \big[p \in V_1' \in E_1' \cap E_2' \ \textbf{and} \ p \in V_2' \in \{V_1 \cup V_2\}\big] \\ \textbf{or} \ \big[p \in V_1' \in E_1' \cap E_2' \ \textbf{and} \ p \in V_2' \in E_1'' \cap E_2''\big]\end{array}\right] \\ \textbf{and} \ (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2) \ \textbf{and} \ (E_1''\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2''\,,\,V_2) \ \textbf{and} \\ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2' \in \mathcal{F} \ \textbf{and} \ V_1 \in E_1'' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2'' \in \mathcal{F}\end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c}\big[p \in V_1' \in \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_2' \in \{V_1 \cup V_2\}\big] \\ \textbf{or} \ \left[\begin{array}{c}p \in V_1' \in \{V_1 \cup V_2\} \ \textbf{and} \ p \in V_2' \in E_1'' \cap E_2'' \ \textbf{and} \\ (E_1''\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2''\,,\,V_2) \ \textbf{and} \ V_1 \in E_1'' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2'' \in \mathcal{F}\end{array}\right] \\ \textbf{or} \ \left[\begin{array}{c}p \in V_1' \in E_1' \cap E_2' \ \textbf{and} \ p \in V_2' \in \{V_1 \cup V_2\} \ \textbf{and} \\ (E_1'\,,\,V_1) \curlyvee_{\mathcal{F}} (E_2'\,,\,V_2) \ \textbf{and} \ V_1 \in E_1' \in \mathcal{F} \ \textbf{and} \ V_2 \in E_2'\end{array}\right] \\ \textbf{or} \ \big[p \in V_1' \in E_1' \cap E_2' \ \textbf{and} \ p \in V_2' \in E_1'' \cap E_2'' \ \textbf{and} \ E_1' \in \mathcal{F} \ \textbf{and} \ E_2'' \in \mathcal{F}\big]\end{array}$$

Then, by applying (L7), conclude:

$$\begin{array}{c} V_1' = V_2' \\ \textbf{or} \left[\begin{array}{c} p \in V_1' \in \{V_1 \cup V_2\} \textbf{ and } p \in V_2' \in E_1'' \cap E_2'' \textbf{ and} \\ (E_1'', V_1) \curlyvee_{\mathcal{F}} (E_2'', V_2) \textbf{ and } V_1 \in E_1'' \in \mathcal{F} \textbf{ and } V_2 \in E_2'' \in \mathcal{F} \end{array}\right] \\ \textbf{or} \left[\begin{array}{c} p \in V_1' \in E_1' \cap E_2' \textbf{ and } p \in V_2' \in \{V_1 \cup V_2\} \textbf{ and} \\ (E_1', V_1) \curlyvee_{\mathcal{F}} (E_2', V_2) \textbf{ and } V_1 \in E_1' \in \mathcal{F} \textbf{ and } V_2 \in E_2' \end{array}\right] \\ \textbf{or } \left[p \in V_1' \in E_1' \cap E_2' \textbf{ and } p \in V_2' \in E_1'' \cap E_2'' \textbf{ and } E_1' \in \mathcal{F} \textbf{ and } E_2'' \in \mathcal{F}\right] \end{array}$$

Then, by applying (L9), conclude:

$$\begin{array}{c} V_1' = V_2' \textbf{ or false or false or} \\ \left[p \in V_1' \in E_1' \cap E_2' \textbf{ and } p \in V_2' \in E_1'' \cap E_2'' \textbf{ and } E_1' \in \mathcal{F} \textbf{ and } E_2'' \in \mathcal{F}\right] \end{array}$$

Then, by applying (L0), conclude $\left[V_1' = V_2' \textbf{ or false or false or } V_1' = V_2'\right]$. Then, by applying standard inference rules, conclude $V_1' = V_2'$.

(K2) Recall from (M4):

$$\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2$$

Then, by introducing (L5)(L6), conclude:

$$\begin{array}{l} \left[\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2\right] \\ \textbf{and } \left[\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F}_\dagger \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2\right] \\ \textbf{and } \left[\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F}_\dagger \textbf{ and} \\ p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2\right] \end{array}$$

Then, by introducing (K1), conclude:

$$\begin{array}{l} \left[\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2\right] \\ \textbf{and } \left[\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F}_\dagger \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2\right] \\ \textbf{and } \left[\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F}_\dagger \textbf{ and} \\ p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2\right] \\ \textbf{and } \left[\left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F}_\dagger \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F}_\dagger \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \textbf{ for all } p, V_1', V_2', E_1, E_2\right] \end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{l} \left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \\ \textbf{and } \left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F}_\dagger \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \\ \textbf{and } \left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F}_\dagger \textbf{ and} \\ p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \\ \textbf{and } \left[\left[\begin{array}{c} p \in V_1' \in E_1 \in \mathcal{F}_\dagger \\ \textbf{and } p \in V_2' \in E_2 \in \mathcal{F}_\dagger \end{array}\right] \textbf{ implies } V_1' = V_2'\right] \end{array}\right] \textbf{ for all } p, V_1', V_2', E_1, E_2$$

Then, by applying standard inference rules, conclude:

$$\Big[\begin{bmatrix}\begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } \ p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)\end{bmatrix} \ \textbf{or} \ \begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{and } \ p \in V_2' \in E_2 \in \mathcal{F}_\dagger\end{bmatrix} \\ \textbf{or} \ \begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F}_\dagger \ \textbf{and} \\ p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)\end{bmatrix} \ \textbf{or} \ \begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F}_\dagger \\ \textbf{and } \ p \in V_2' \in E_2 \in \mathcal{F}_\dagger\end{bmatrix}\end{bmatrix} \ \textbf{implies} \ V_1' = V_2'\Big]$$
$$\textbf{for all } \ p \, , \, V_1' \, , \, V_2' \, , \, E_1 \, , \, E_2$$

Then, by applying standard inference rules, conclude:

$$\Big[\Big[\begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{or } \ p \in V_1' \in E_1 \in \mathcal{F}_\dagger\end{bmatrix} \ \textbf{and} \ \begin{bmatrix} p \in V_2' \in E_2 \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{or } \ p \in V_2' \in E_2 \in \mathcal{F}_\dagger\end{bmatrix}\Big] \ \textbf{implies} \ V_1' = V_2'\Big]$$
$$\textbf{for all } \ p \, , \, V_1' \, , \, V_2' \, , \, E_1 \, , \, E_2$$

Then, by applying ZFC, conclude:

$$\Big[\begin{bmatrix} p \in V_1' \in E_1 \in (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger \\ \textbf{and } \ p \in V_2' \in E_2 \in (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger\end{bmatrix} \ \textbf{implies} \ V_1' = V_2'\Big] \ \textbf{for all } \ p \, , \, V_1' \, , \, V_2' \, , \, E_1 \, , \, E_2$$

Then, by applying (O0), conclude:

$$\Big[\begin{bmatrix} p \in V_1' \in E_1 \in \mathcal{E} \\ \textbf{and } \ p \in V_2' \in E_2 \in \mathcal{E}\end{bmatrix} \ \textbf{implies} \ V_1' = V_2'\Big] \ \textbf{for all } \ p \, , \, V_1' \, , \, V_2' \, , \, E_1 \, , \, E_2$$

(K3) Suppose:

$$V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \ \textbf{for some } \ V \, , \, E$$

Then, by applying ZFC, conclude $V \in E \in \mathcal{F}$. Then, by applying (M3), conclude $V \neq \emptyset$.

(K4) Suppose:

$$\big[V \in \{V_1 \cup V_2\} \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big] \ \textbf{for some } \ V \, , \, E_1$$

Then, by applying ZFC, conclude $\big[V = V_1 \cup V_2 \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big]$. Then, by applying (M3), conclude $\big[V = V_1 \cup V_2 \ \textbf{and} \ V_1 \neq \emptyset\big]$. Then, by applying ZFC, conclude $V \neq \emptyset$.

(K5) Suppose:

$$\big[V \in E_1 \cap E_2 \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big] \ \textbf{for some } \ V \, , \, E_1 \, , \, E_2$$

Then, by applying ZFC, conclude $V \in E_1 \in \mathcal{F}$. Then, by applying (M3), conclude $V \neq \emptyset$.

(K6) Suppose:

$$V \in E \in \mathcal{F}_\dagger \ \textbf{for some } \ V \, , \, E$$

Then, by applying (O1), conclude:

$$V \in E \in \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and} \ (E_1 \, , \, V_1) \curlyvee_{\mathcal{F}} (E_2 \, , \, V_2)\}$$

Then, by applying (M5), conclude:

$$V \in E \ \textbf{and} \ \big[\big[E = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big] \ \textbf{for some } \ E_1 \, , \, E_2\big]$$

Then, by applying standard inference rules, conclude:

$$\big[V \in E \ \textbf{and} \ E = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big] \ \textbf{for some } \ E_1 \, , \, E_2$$

Then, by applying substitution, conclude $\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big]$. Then, by applying ZFC, conclude $\big[\big[V \in \{V_1 \cup V_2\} \ \textbf{or} \ V \in E_1 \cap E_2\big] \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big]$. Then, by applying standard inference rules, conclude:

$$\big[V \in \{V_1 \cup V_2\} \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big] \ \textbf{or} \ \big[V \in E_1 \cap E_2 \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big]$$

Then, by applying (K4), conclude $\big[V \neq \emptyset \ \textbf{and} \ \big[V \in E_1 \cap E_2 \ \textbf{and} \ V_1 \in E_1 \in \mathcal{F}\big]\big]$. Then, by applying (K5), conclude $\big[V \neq \emptyset \ \textbf{and} \ V \neq \emptyset\big]$. Then, by applying standard inference rules, conclude $V \neq \emptyset$.

(K7) Recall $\big[\big[V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)$ **implies** $V \neq \emptyset\big]$ **for all** $V, E\big]$ from (K3). Then, by introducing (K6), conclude:

$$\big[\big[V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \textbf{ implies } V \neq \emptyset\big] \textbf{ for all } V, E\big] \\ \textbf{and } \big[\big[V \in E \in \mathcal{F}_\dagger \textbf{ implies } V \neq \emptyset\big] \textbf{ for all } V, E\big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \big[V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \textbf{ implies } V \neq \emptyset\big] \\ \textbf{and } \big[V \in E \in \mathcal{F}_\dagger \textbf{ implies } V \neq \emptyset\big] \end{bmatrix} \textbf{ for all } V, E$$

Then, by applying standard inference rules, conclude:

$$\Bigg[\begin{bmatrix} V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{or } V \in E \in \mathcal{F}_\dagger \end{bmatrix} \textbf{ implies } V \neq \emptyset\Bigg] \textbf{ for all } V, E$$

Then, by applying ZFC, conclude:

$$\big[V \in E \in (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger \textbf{ implies } V \neq \emptyset\big] \textbf{ for all } V, E$$

Then, by applying (O0), conclude $\big[\big[V \in E \in \mathcal{E}$ **implies** $V \neq \emptyset\big]$ **for all** $V, E\big]$.

(K8) Suppose:

$$V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \textbf{ for some } V, E$$

Then, by applying ZFC, conclude $V \in E \in \mathcal{F}$. Then, by applying (M3), conclude:

$$\big[V \subseteq P' \textbf{ and } P' \in \bigstar(\mathcal{F})\big] \textbf{ for some } P'$$

(K9) Suppose:

$$V \in \{V_1 \cup V_2\} \textbf{ for some } V$$

Then, by applying ZFC, conclude $V = V_1$. Then, by applying (O1), conclude:

$$V = V_1 \cup V_2 \textbf{ and } V_1 \cup V_2 \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F})$$

Then, by applying substitution, conclude $\big[V \subseteq P$ **and** $P \in \bigstar(\mathcal{F})\big]$. Then, by applying ZFC, conclude $\big[V \subseteq P$ **and** $P \in \bigstar(\mathcal{F})$ **and** $\big[P' = P$ **for some** $P'\big]\big]$. Then, by applying standard inference rules, conclude

$$\big[V \subseteq P \textbf{ and } P \in \bigstar(\mathcal{F}) \textbf{ and } P' = P\big] \textbf{ for some } P'$$

Then by applying substitution, conclude $\big[V \subseteq P'$ **and** $P' \in \bigstar(\mathcal{F})\big]$.

(K0) Suppose:

$$\big[V \in E_1 \cap E_2 \textbf{ and } V_1 \in E_1 \in \mathcal{F}\big] \textbf{ for some } V, E_1, E_2$$

Then, by applying ZFC, conclude $V \in E_1 \in \mathcal{F}$. Then, by applying (M3), conclude:

$$\big[V \subseteq P' \textbf{ and } P' \in \bigstar(\mathcal{F})\big] \textbf{ for some } P'$$

(J1) Suppose:

$$V \in E \in \mathcal{F}_\dagger \textbf{ for some } V, E$$

Then, by applying (O1), conclude:

$$V \in E \in \{E_\dagger \mid E_\dagger = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } (E_1, V_1) \curlyvee_{\mathcal{F}} (E_2, V_2)\}$$

Then, by applying (M5), conclude:

$$V \in E \textbf{ and } \big[\big[E = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \textbf{ and } V_1 \in E_1 \in \mathcal{F}\big] \textbf{ for some } E_1, E_2\big]$$

Then, by applying standard inference rules, conclude:

$$\big[V \in E \;\textbf{ and }\; E = \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\textbf{ and }\; V_1 \in E_1 \in \mathcal{F}\big] \;\textbf{ for some }\; E_1\,,\, E_2$$

Then, by applying substitution, conclude $\big[V \in \{V_1 \cup V_2\} \cup (E_1 \cap E_2) \;\textbf{ and }\; V_1 \in E_1 \in \mathcal{F}\big]$. Then, by applying ZFC, conclude $\big[\big[V \in \{V_1 \cup V_2\} \;\textbf{ or }\; V \in E_1 \cap E_2\big] \;\textbf{ and }\; V_1 \in E_1 \in \mathcal{F}\big]$. Then, by applying standard inference rules, conclude:

$$V \in \{V_1 \cup V_2\} \;\textbf{ or }\; \big[V \in E_1 \cap E_2 \;\textbf{ and }\; V_1 \in E_1 \in \mathcal{F}\big]$$

Then, by applying (K4), conclude:

$$\big[\big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \;\textbf{ for some }\; P'\big] \;\textbf{ and }\; \big[V \in E_1 \cap E_2 \;\textbf{ and }\; V_1 \in E_1 \in \mathcal{F}\big]$$

Then, by applying (K5), conclude:

$$\big[\big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \;\textbf{ for some }\; P'\big] \;\textbf{ and }\; \big[\big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \;\textbf{ for some }\; P'\big]$$

Then, by applying standard inference rules, conclude:

$$\big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \;\textbf{ for some }\; P'$$

(J2) Recall from (K8).

$$\Big[\big[V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\big] \;\textbf{ for all }\; V\,,\, E\Big]$$

Then, by introducing (J1), conclude:

$$\Big[\big[V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\big] \;\textbf{ for all }\; V\,,\, E\Big]$$
$$\textbf{and }\; \Big[\big[V \in E \in \mathcal{F}_\dagger \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\big] \;\textbf{ for all }\; V\,,\, E\Big]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} \big[V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\big] \\ \textbf{and }\; \big[V \in E \in \mathcal{F}_\dagger \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\big] \end{bmatrix} \;\textbf{ for all }\; V\,,\, E$$

Then, by applying standard inference rules, conclude:

$$\Big[\begin{bmatrix} V \in E \in \mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2) \\ \textbf{or }\; V \in E \in \mathcal{F}_\dagger \end{bmatrix} \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\Big] \;\textbf{ for all }\; V\,,\, E$$

Then, by applying ZFC, conclude:

$$\Big[V \in E \in (\mathcal{F} \setminus (\mathcal{F}_1 \cup \mathcal{F}_2)) \cup \mathcal{F}_\dagger \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\Big] \;\textbf{ for all }\; V\,,\, E$$

Then, by applying (O0), conclude:

$$\Big[\big[V \in E \in \mathcal{E} \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{F})\big] \\ \textbf{for some }\; P' \end{bmatrix}\big] \;\textbf{ for all }\; V\,,\, E\Big]$$

Then, by applying (M1), conclude:

$$\Big[\big[V \in E \in \mathcal{E} \;\textbf{ implies }\; \begin{bmatrix} \big[V \subseteq P' \;\textbf{ and }\; P' \in \bigstar(\mathcal{E})\big] \\ \textbf{for some }\; P' \end{bmatrix}\big] \;\textbf{ for all }\; V\,,\, E\Big]$$

(J3) Recall $\mathcal{E} \in \wp^2(\mathbb{V}\text{ER})$ from (N1). Then, by introducing (K2), conclude:

$$\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}$$
$$\left[\left[\begin{bmatrix} & p \in V_1' \in E_1 \in \mathcal{E} \\ \textbf{and} & p \in V_2' \in E_2 \in \mathcal{E}\end{bmatrix} \ \textbf{implies} \ V_1' = V_2'\right] \ \textbf{for all} \ p\,,\, V_1'\,,\, V_2'\,,\, E_1\,,\, E_2\right]$$

Then, by introducing (K7), conclude:

$$\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}$$
$$\left[\left[\begin{bmatrix} & p \in V_1' \in E_1 \in \mathcal{E} \\ \textbf{and} & p \in V_2' \in E_2 \in \mathcal{E}\end{bmatrix} \ \textbf{implies} \ V_1' = V_2'\right] \ \textbf{for all} \ p\,,\, V_1'\,,\, V_2'\,,\, E_1\,,\, E_2\right]$$
$$\textbf{and} \ \left[\left[V \in E \in \mathcal{E} \ \textbf{implies} \ V \neq \emptyset\right] \ \textbf{for all} \ V\,,\, E\right]$$

Then, by introducing (J2), conclude:

$$\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}$$
$$\left[\left[\begin{bmatrix} & p \in V_1' \in E_1 \in \mathcal{E} \\ \textbf{and} & p \in V_2' \in E_2 \in \mathcal{E}\end{bmatrix} \ \textbf{implies} \ V_1' = V_2'\right] \ \textbf{for all} \ p\,,\, V_1'\,,\, V_2'\,,\, E_1\,,\, E_2\right]$$
$$\textbf{and} \ \left[\left[V \in E \in \mathcal{E} \ \textbf{implies} \ V \neq \emptyset\right] \ \textbf{for all} \ V\,,\, E\right]$$
$$\left[\left[V \in E \in \mathcal{E} \ \textbf{implies} \ \begin{bmatrix}\left[V \subseteq P' \ \textbf{and} \ P' \in \bigstar(\mathcal{E})\right] \\ \textbf{for some} \ P'\end{bmatrix}\right] \ \textbf{for all} \ V\,,\, E\right]$$

Then, by applying Definition 24 of $\checkmark$, conclude $\checkmark(\mathcal{E})$.

Now, prove the theorem by the following reduction. Recall $[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]$ from (O2). Then, by introducing (M2), conclude $\left[[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \ \textbf{and} \ \bigstar(\mathcal{E}) = \bigstar(\mathcal{E}_{\text{in}})\right]$. Then, by introducing (J3), conclude $\left[[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \ \textbf{and} \ \bigstar(\mathcal{E}) = \bigstar(\mathcal{E}_{\text{in}}) \ \textbf{and} \ \checkmark(\mathcal{E})\right]$. Then, by applying Figure 15, conclude Inv_1. Then, by introducing (E3), conclude $\left[\mathsf{Inv}_1 \ \textbf{and} \ |\mathcal{E}| < z_1\right]$.

(| QED. |)

6. First, assume:

(F1) Inv_1

(F2) **not** Cond_1

Next, observe:

(I1) Recall Inv_1 from (F1). Then, by applying Figure 15, conclude:

$$[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \ \textbf{and} \ \bigstar(\mathcal{E}) = \bigstar(\mathcal{E}_{\text{in}}) \ \textbf{and} \ \checkmark(\mathcal{E})$$

(I2) Recall $\checkmark(\mathcal{E})$ from (I1). Then, by applying Definition 24 of $\checkmark$, conclude:

$$\mathcal{E} \in \wp^2(\mathbb{V}\text{ER}) \ \textbf{and}$$
$$\left[\left[\begin{bmatrix} & p \in V_1 \in E_1 \in \mathcal{E} \\ \textbf{and} & p \in V_2 \in E_2 \in \mathcal{E}\end{bmatrix} \ \textbf{implies} \ V_1 = V_2\right] \ \textbf{for all} \ p\,,\, V_1\,,\, V_2\,,\, E_1\,,\, E_2\right]$$
$$\textbf{and} \ \left[\left[V \in E \in \mathcal{E} \ \textbf{implies} \ V \neq \emptyset\right] \ \textbf{for all} \ V\,,\, E\right] \ \textbf{and}$$
$$\left[\left[V \in E \in \mathcal{E} \ \textbf{implies} \ \begin{bmatrix}\left[V \subseteq P' \ \textbf{and} \ P' \in \bigstar(\mathcal{E})\right] \\ \textbf{for some} \ P'\end{bmatrix}\right] \ \textbf{for all} \ V\,,\, E\right]$$

(I3) Suppose:

$$\left[V \notin \bigstar(\mathcal{E}) \text{ and } V \subseteq P' \text{ and } P' \in \bigstar(\mathcal{E})\right] \textbf{ for some } V\,,\,P'$$

Then, by applying ZFC, conclude $\left[V \subseteq P' \text{ and } P' \in \bigstar(\mathcal{E}) \text{ and } V \neq P'\right]$. Then, by applying ZFC, conclude $\left[V \subset P' \text{ and } P' \in \bigstar(\mathcal{E})\right]$. Then, by applying ZFC, conclude:

$$\left[\left[p' \notin V \text{ and } p' \in P'\right] \textbf{ for some } p'\right] \text{ and } P' \in \bigstar(\mathcal{E})$$

Then, by applying standard inference rules, conclude:

$$\left[p' \notin V \text{ and } p' \in P' \in \bigstar(\mathcal{E})\right] \textbf{ for some } p'$$

(I4) Suppose:

$$\begin{bmatrix} p \in V \in E \in \mathcal{E} \text{ and } p' \notin V \text{ and} \\ p \in V_1 \in E_1 \in \mathcal{E} \text{ and } p' \in V_2 \text{ and } V_1 = V_2 \end{bmatrix} \textbf{ for some } p\,,\,p'\,,\,V\,,\,V_1\,,\,V_2\,,\,E\,,\,E_1$$

Then, by applying (I2), conclude $\left[p' \notin V \text{ and } p' \in V_2 \text{ and } V_1 = V_2 \text{ and } V = V_1\right]$. Then, by applying substitution, conclude $\left[p' \notin V_1 \text{ and } p' \in V_1\right]$. Then, by applying standard inference rules, conclude **false**.

(I5) Suppose:

$$V \cap V' \neq \emptyset \textbf{ for some } V\,,\,V'$$

Then, by applying ZFC, conclude:

$$p \in V \cap V' \textbf{ for some } p$$

Then, by applying ZFC, conclude $\left[p \in V \text{ and } p \in V'\right]$.

(I6) Suppose:

$$\left[V \in E \in \mathcal{E} \text{ and } V' \in E' \in \mathcal{E} \text{ and } V \neq V' \text{ and } V \cap V' \neq \emptyset\right] \textbf{ for some } V\,,\,V'\,,\,E\,,\,E'$$

Then, by applying I5, conclude:

$$V \in E \in \mathcal{E} \text{ and } V' \in E' \in \mathcal{E} \text{ and } V \neq V' \text{ and } \left[\left[p \in V \text{ and } p \in V'\right] \textbf{ for some } p\right]$$

Then, by applying standard inference rules, conclude:

$$\left[p \in V \in E \in \mathcal{E} \text{ and } p \in V' \in E' \in \mathcal{E} \text{ and } V \neq V'\right] \textbf{ for some } p$$

Then, by applying (I2), conclude $\left[V = V' \text{ and } V \neq V'\right]$. Then, by applying standard inference rules, conclude **false**.

(I7) Suppose:

$$\left[V_1 \in E_1 \text{ and } V_2 \in E_2 \text{ and } E_1 = (E_2 \setminus \{V_2\}) \cup \{V_1\} \text{ and } V_1 \neq V_2 \text{ and } E_1 = E_2\right] \\ \textbf{for some } V_1\,,\,V_2\,,\,E_1\,,\,E_2$$

Then, by applying ZFC, conclude:

$$V_1 \in E_1 \text{ and } V_2 \in E_2 \text{ and } E_1 = (E_2 \cup \{V_1\}) \setminus \{V_2\} \text{ and } E_1 = E_2$$

Then, by applying substitution, conclude $\left[V_1 \in E_1 \text{ and } V_2 \in E_1 \text{ and } E_1 = (E_1 \cup \{V_1\}) \setminus \{V_2\}\right]$. Then, by applying ZFC, conclude $\left[V_2 \in E_1 \text{ and } E_1 = E_1 \setminus \{V_2\}\right]$. Then, by applying ZFC, conclude $\left[E_1 \neq E_1 \setminus \{V_2\} \text{ and } E_1 = E_1 \setminus \{V_2\}\right]$. Then, by applying standard inference rules, conclude **false**.

(I8) Suppose:

$$\big[p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \notin V \ \textbf{and} \ p\,,\, p' \in P' \in \bigstar(\mathcal{E})\big] \ \textbf{for some} \ p\,,\, p'\,,\, V\,,\, E\,,\, P'$$

Then, by applying Lemma 9:2, conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \notin V \ \textbf{and} \ \mathsf{Edge}(p\,,\, \mathcal{E}) = \mathsf{Edge}(p'\,,\, \mathcal{E})$$

Then, by applying Lemma 9:3, conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \notin V \ \textbf{and} \ \left[\begin{bmatrix} p' \in V' \in E' \in \mathcal{E} \ \textbf{and} \\ (E\,,\, V) \curlyvee (E'\,,\, V') \end{bmatrix} \ \textbf{for some} \ V'\,,\, E'\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \notin V \ \textbf{and} \\ p' \in V' \in E' \in \mathcal{E} \ \textbf{and} \ (E\,,\, V) \curlyvee (E'\,,\, V') \end{bmatrix} \ \textbf{for some} \ V'\,,\, E'$$

Then, by applying ZFC, conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E} \ \textbf{and} \ (E\,,\, V) \curlyvee (E'\,,\, V') \ \textbf{and} \ V \neq V'$$

Then, by applying (I6), conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E} \ \textbf{and} \ (E\,,\, V) \curlyvee (E'\,,\, V') \ \textbf{and} \ V \neq V' \ \textbf{and} \ V \cap V' = \emptyset$$

Then, by applying (I7), conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E} \ \textbf{and} \ (E\,,\, V) \curlyvee (E'\,,\, V') \ \textbf{and} \ E \neq E' \ \textbf{and} \ V \cap V' = \emptyset$$

Then, by applying Definition 19 of $\curlyvee$, conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{and} \ p' \in V' \in E' \in \mathcal{E} \ \textbf{and} \ (E\,,\, V) \curlyvee_{\mathcal{E}} (E'\,,\, V')$$

(I9) Suppose:

$$\big[p\,,\, p' \in P' \ \textbf{and} \ p \in P_1 \ \textbf{and} \ p' \in P_2\big] \ \textbf{for some} \ p\,,\, p'\,,\, P_1\,,\, P_2$$

Then, by applying ZFC, conclude $\big[P' \cap P_1 \neq \emptyset \ \textbf{and} \ P' \cap P_2 \neq \emptyset\big]$. Then, by applying Lemma 12:2, conclude $\big[\big[P' = P_1 \ \textbf{or} \ \big[\textbf{not} \ P'\,,\, P_1 \in \bigstar(\gamma)\big]\big] \ \textbf{and} \ \big[P' = P_2 \ \textbf{or} \ \big[\textbf{not} \ P'\,,\, P_2 \in \bigstar(\gamma)\big]\big]\big]$.

(I0) Suppose:

$$\big[p\,,\, p' \in P' \ \textbf{and} \ P'\,,\, P_1\,,\, P_2 \in \bigstar(\mathcal{E}) \ \textbf{and} \ p \in P_1 \ \textbf{and} \ p' \in P_2\big] \ \textbf{for some} \ p\,,\, p'\,,\, P_1\,,\, P_2\,,\, P'$$

Then, by applying (I9), conclude:

$$\begin{gathered} P'\,,\, P_1\,,\, P_2 \in \bigstar(\mathcal{E}) \ \textbf{and} \\ \big[P' = P_1 \ \textbf{or} \ \big[\textbf{not} \ P'\,,\, P_1 \in \bigstar(\gamma)\big]\big] \ \textbf{and} \ \big[P' = P_2 \ \textbf{or} \ \big[\textbf{not} \ P'\,,\, P_2 \in \bigstar(\gamma)\big]\big] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\begin{gathered} \big[P' = P_1 \ \textbf{or} \ \big[\big[\textbf{not} \ P'\,,\, P_1 \in \bigstar(\gamma)\big] \ \textbf{and} \ P'\,,\, P_1 \in \bigstar(\mathcal{E})\big]\big] \\ \textbf{and} \ \big[P' = P_2 \ \textbf{or} \ \big[\big[\textbf{not} \ P'\,,\, P_2 \in \bigstar(\gamma)\big] \ \textbf{and} \ P'\,,\, P_2 \in \bigstar(\mathcal{E})\big]\big] \end{gathered}$$

Then, by applying standard inference rules, conclude:

$$\big[P' = P_1 \ \textbf{or} \ \textbf{false}\big] \ \textbf{and} \ \big[P' = P_2 \ \textbf{or} \ \textbf{false}\big]$$

Then, by applying standard inference rules, conclude $\big[P' = P_1 \ \textbf{and} \ P' = P_2\big]$.

(H1) Suppose:

$$\big[p\,,\,p' \in P' \in \bigstar(\mathcal{E}) \ \textbf{ and } \ p \in V_1 \in E_1 \in \mathcal{E} \ \textbf{ and } \ p' \in V_2 \in E_2 \in \mathcal{E}\big] \ \textbf{for some} \ V_1\,,\,V_2\,,\,E_1\,,\,E_2$$

Then, by applying (I2), conclude:

$$\begin{aligned} &p\,,\,p' \in P' \in \bigstar(\mathcal{E}) \ \textbf{ and } \ p \in V_1 \ \textbf{ and } \ p' \in V_2 \\ \textbf{and} \ &\big[\big[V_1 \subseteq P_1 \ \textbf{ and } \ P_1 \in \bigstar(\mathcal{E})\big] \ \textbf{ for some } \ P_1\big] \\ \textbf{and} \ &\big[\big[V_2 \subseteq P_2 \ \textbf{ and } \ P_2 \in \bigstar(\mathcal{E})\big] \ \textbf{ for some } \ P_2\big] \end{aligned}$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p\,,\,p' \in P' \in \bigstar(\mathcal{E}) \ \textbf{ and } \ p \in V_1 \ \textbf{ and } \ p' \in V_2 \ \textbf{ and } \\ V_1 \subseteq P_1 \ \textbf{ and } \ P_1 \in \bigstar(\mathcal{E}) \ \textbf{ and } \ V_2 \subseteq P_2 \ \textbf{ and } \ P_2 \in \bigstar(\mathcal{E}) \end{bmatrix} \ \textbf{for some} \ P_1\,,\,P_2$$

Then, by applying ZFC, conclude:

$$p\,,\,p' \in P' \ \textbf{ and } \ P'\,,\,P_1\,,\,P_2 \in \bigstar(\mathcal{E}) \ \textbf{ and } \ p \in P_1 \ \textbf{ and } \ p' \in P_2 \ \textbf{ and } \ V_1 \subseteq P_1 \ \textbf{ and } \ V_2 \subseteq P_2$$

Then, by applying (I0), conclude $\big[V_1 \subseteq P_1 \ \textbf{ and } \ V_2 \subseteq P_2 \ \textbf{ and } \ P' = P_1 \ \textbf{ and } \ P' = P_2\big]$. Then, by applying substitution, conclude $V_1\,,\,V_2 \subseteq P'$. Then, by applying ZFC, conclude $V_1 \cup V_2 \subseteq P'$.

(H2) Suppose:

$$\big[E \in \mathcal{E} \ \textbf{ and } \ E \not\subseteq \bigstar(\mathcal{E})\big] \ \textbf{for some} \ E$$

Then, by applying ZFC, conclude:

$$E \in \mathcal{E} \ \textbf{ and } \ \big[\big[V \in E \ \textbf{ and } \ V \notin \bigstar(\mathcal{E})\big] \ \textbf{ for some } \ V\big]$$

Then, by applying standard inference rules, conclude:

$$\big[V \in E \in \mathcal{E} \ \textbf{ and } \ V \notin \bigstar(\mathcal{E})\big] \ \textbf{for some} \ V$$

Then, by applying (I2), conclude:

$$V \in E \in \mathcal{E} \ \textbf{ and } \ V \notin \bigstar(\mathcal{E}) \ \textbf{ and } \ V \neq \emptyset \ \textbf{ and } \ \big[\big[V \subseteq P' \ \textbf{ and } \ P' \in \bigstar(\mathcal{E})\big] \ \textbf{ for some } \ P'\big]$$

Then, by applying standard inference rules, conclude:

$$\big[V \in E \in \mathcal{E} \ \textbf{ and } \ V \notin \bigstar(\mathcal{E}) \ \textbf{ and } \ V \neq \emptyset \ \textbf{ and } \ V \subseteq P' \ \textbf{ and } \ P' \in \bigstar(\mathcal{E})\big] \ \textbf{for some} \ P'$$

Then, by applying ZFC, conclude:

$$V \in E \in \mathcal{E} \ \textbf{ and } \ V \notin \bigstar(\mathcal{E}) \ \textbf{ and } \ \big[p \in V \ \textbf{ for some } \ p\big] \ \textbf{ and } \ V \subseteq P' \ \textbf{ and } \ P' \in \bigstar(\mathcal{E})$$

Then, by applying standard inference rules, conclude:

$$\big[p \in V \in E \in \mathcal{E} \ \textbf{ and } \ V \notin \bigstar(\mathcal{E}) \ \textbf{ and } \ V \subseteq P' \ \textbf{ and } \ P' \in \bigstar(\mathcal{E})\big] \ \textbf{for some} \ p$$

Then, by applying ZFC, conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{ and } \ V \notin \bigstar(\mathcal{E}) \ \textbf{ and } \ V \subseteq P' \ \textbf{ and } \ P' \in \bigstar(\mathcal{E}) \ \textbf{ and } \ p \in P'$$

Then, by applying (I3), conclude:

$$p \in V \in E \in \mathcal{E} \ \textbf{ and } \ p \in P' \ \textbf{ and } \ \big[\big[p' \notin V \ \textbf{ and } \ p' \in P' \in \bigstar(\mathcal{E})\big] \ \textbf{ for some } \ p'\big]$$

Then, by applying standard inference rules, conclude:

$$\big[p \in V \in E \in \mathcal{E} \ \textbf{ and } \ p' \notin V \ \textbf{ and } \ p\,,\,p' \in P' \in \bigstar(\mathcal{E})\big] \ \textbf{for some} \ p'$$

Then, by applying (I8), conclude:

$$p\,,\,p' \in P' \in \bigstar(\mathcal{E}) \;\textbf{ and }\; \left[\begin{bmatrix} p \in V \in E \in \mathcal{E} \;\textbf{ and }\; p' \in V' \in E' \in \mathcal{E} \\ \textbf{and }\; (E\,,\,V) \curlyvee_{\mathcal{E}} (E'\,,\,V') \end{bmatrix} \;\textbf{ for some }\; V'\,,\,E'\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{bmatrix} p\,,\,p' \in P' \in \bigstar(\mathcal{E}) \;\textbf{ and} \\ p \in V \in E \in \mathcal{E} \;\textbf{ and }\; p' \in V' \in E' \in \mathcal{E} \\ \textbf{and }\; (E\,,\,V) \curlyvee_{\mathcal{E}} (E'\,,\,V') \end{bmatrix} \;\textbf{ for some }\; V'\,,\,E'$$

Then, by applying (H1), conclude:

$$P' \in \bigstar(\mathcal{E}) \;\textbf{ and }\; (E\,,\,V) \curlyvee_{\mathcal{E}} (E'\,,\,V') \;\textbf{ and }\; V \cup V' \subseteq P'$$

Then, by applying Figure 15, conclude Cond. Then, by introducing (F2), conclude:

$$\left[\textbf{not }\; \mathsf{Cond}\right] \;\textbf{ and }\; \mathsf{Cond}$$

Then, by applying standard inference rules, conclude **false**.

(H3) Recall $\left[\left[E \in \mathcal{E} \;\textbf{ implies }\; E \subseteq \bigstar(\mathcal{E})\right] \;\textbf{ for all }\; E\right]$ from (H2). Then, by applying (I1), conclude:

$$\left[E \in \mathcal{E} \;\textbf{ implies }\; E \subseteq \bigstar(\mathcal{E}_{\text{in}})\right] \;\textbf{ for all }\; E$$

Now, prove the theorem by the following reduction. Recall $[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!]$ from (I1). Then, by introducing (H3), conclude $\left[[\![(\mathcal{V}\,,\,\mathcal{E})]\!] = [\![(\mathcal{V}\,,\,\mathcal{E}_{\text{in}})]\!] \;\textbf{ and }\; \left[\left[E \in \mathcal{E} \;\textbf{ implies }\; E \subseteq \bigstar(\mathcal{E}_{\text{in}})\right] \;\textbf{ for all }\; E\right]\right]$. Then, by applying Figure 15, conclude Post.

⦇ QED. ⦈

B.15 Lemma 15

Proof (of Lemma 15). First, assume:

(A1) $\psi \equiv_{\text{sc}} \psi'$

(A2) $\phi \equiv_{\text{dc}} \phi'$

(A3) $(\psi' , \phi') \in X \in \wp(\mathbb{SC} \times \mathbb{DC})$

Next, observe:

(Z1) Recall $\psi \equiv_{\text{sc}} \psi'$ from (A1). Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\psi , \psi' \in \mathbb{SC}$.

(Z2) Recall $\phi \equiv_{\text{dc}} \phi'$ from (A2). Then, by applying Definition 25 of $\mathbb{DC}$, conclude $\phi , \phi' \in \mathbb{DC}$.

(Z3) Recall $\big[\psi , \psi' \in \mathbb{SC}$ **and** $\phi , \phi' \in \mathbb{DC}\big]$ from (Z1)(Z2). Then, by applying ZFC, conclude:

$$(\psi , \phi) , (\psi' , \phi') \in \mathbb{SC} \times \mathbb{DC}$$

(Z4) Suppose:

$$\delta \models^{\text{sc}} \psi \ \textbf{for some} \ \delta$$

Then, by introducing (A1), conclude $\big[\delta \models^{\text{sc}} \psi$ **and** $\psi \equiv_{\text{sc}} \psi'\big]$. Then, by applying Definition 10 of $\mathbb{SC}$, conclude $\delta \models^{\text{sc}} \psi'$.

(Z5) Suppose:

$$\delta \models^{\text{dc}} \phi \ \textbf{for some} \ \delta$$

Then, by introducing (A2), conclude $\big[\delta \models^{\text{dc}} \phi$ **and** $\phi \equiv_{\text{sc}} \phi'\big]$. Then, by applying Definition 25 of $\mathbb{DC}$, conclude $\delta \models^{\text{dc}} \phi'$.

(Z6) Suppose:

$$\big[\delta \models^{\text{sc}} \psi \ \textbf{and} \ \delta \models^{\text{sc}} \phi\big] \ \textbf{for some} \ \delta$$

Then, by applying (Z4), conclude $\big[\delta \models^{\text{sc}} \psi'$ **and** $\delta \models^{\text{sc}} \phi\big]$. Then, by applying (Z5), conclude:

$$\delta \models^{\text{sc}} \psi' \ \textbf{and} \ \delta \models^{\text{sc}} \phi'$$

Then, by introducing (A3), conclude $\big[\delta \models^{\text{sc}} \psi'$ **and** $\delta \models^{\text{sc}} \phi'$ **and** $(\psi' , \phi') \in X\big]$.

(Z7) Recall from (Z6):

$$\Big[\big[\delta \models^{\text{sc}} \psi \ \textbf{and} \ \delta \models^{\text{sc}} \phi\big] \ \textbf{implies} \ \begin{bmatrix}\delta \models^{\text{sc}} \psi' \ \textbf{and} \ \delta \models^{\text{sc}} \phi' \\ \textbf{and} \ (\psi' , \phi') \in X\end{bmatrix}\Big] \ \textbf{for all} \ \delta$$

Then, by applying standard inference rules, conclude:

$$\Big[\big[\delta \models^{\text{sc}} \psi \ \textbf{and} \ \delta \models^{\text{sc}} \phi\big] \ \textbf{implies} \ \begin{bmatrix}\begin{bmatrix}\delta \models^{\text{sc}} \psi'' \ \textbf{and} \ \delta \models^{\text{sc}} \phi'' \\ \textbf{and} \ (\psi'' , \phi'') \in X\end{bmatrix} \\ \textbf{for some} \ \psi'' , \phi''\end{bmatrix}\Big] \ \textbf{for all} \ \delta$$

Now, prove the lemma by the following reduction. Recall $(\psi , \phi) , (\psi' , \phi') \in \mathbb{SC} \times \mathbb{DC}$ from (Z3). Then, by introducing (A3), conclude $\big[(\psi , \phi) , (\psi' , \phi') \in \mathbb{SC} \times \mathbb{DC}$ **and** $X \in \wp(\mathbb{SC} \times \mathbb{DC})\big]$. Then, by introducing (Z7), conclude:

$$(\psi , \phi) , (\psi' , \phi') \in \mathbb{SC} \times \mathbb{DC} \ \textbf{and} \ X \in \wp(\mathbb{SC} \times \mathbb{DC}) \ \textbf{and}$$

$$\Big[\Big[\big[\delta \models^{\text{sc}} \psi \ \textbf{and} \ \delta \models^{\text{sc}} \phi\big] \ \textbf{implies} \ \begin{bmatrix}\begin{bmatrix}\delta \models^{\text{sc}} \psi'' \ \textbf{and} \ \delta \models^{\text{sc}} \phi'' \\ \textbf{and} \ (\psi'' , \phi'') \in X\end{bmatrix} \\ \textbf{for some} \ \psi'' , \phi''\end{bmatrix}\Big] \ \textbf{for all} \ \delta\Big]$$

Then, by applying Definition 30 of $\trianglelefteq$, conclude $(\psi , \phi) \trianglelefteq X$.

⦇ QED. ⦈

B.16 Lemma 16

Proof (of Lemma 16). First, assume:

(A1) $\psi_1 \equiv_{\text{sc}} \psi_2$

(A2) $\phi_1 \equiv_{\text{dc}} \phi_2$

(A3) $\alpha \in \mathbb{CA}$

(A4) $\mathsf{Port}(\psi_2) \subseteq P$

(A5) $\mathsf{Port}(\phi_2) \subseteq P$

Next, observe:

(Z1) Recall $\alpha \in \mathbb{CA}$. Then, by applying Definition 29 of $\mathbb{CA}$, conclude:

$$\big[\alpha \in \mathbb{CA} \text{ and } \alpha = (Q\,,\,P\,,\,\longrightarrow\,,\,\imath) \text{ and } Q \subseteq \mathbb{S}\text{TATE} \text{ and } \longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \text{ and } \imath \in Q\big] \text{ for some } Q\,,\,P\,,\,\longrightarrow\,,\,\imath$$

Then, by applying substitution, conclude:

$$\begin{gathered}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath) \in \mathbb{CA} \text{ and } \alpha = (Q\,,\,P\,,\,\longrightarrow\,,\,\imath) \text{ and} \\ Q \subseteq \mathbb{S}\text{TATE} \text{ and } \longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \text{ and } \imath \in Q\end{gathered}$$

(Z2) Suppose:

$$\begin{bmatrix}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil \in \mathbb{CA} \text{ and} \\ (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil = (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\end{bmatrix} \text{ for some } Q\,,\,P\,,\,\longrightarrow\,,\,\longrightarrow'\,,\,\imath$$

Then, by applying substitution, conclude $(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA}$. Then, by applying Definition 29 of $\mathbb{CA}$, conclude $\big[(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA} \text{ and } \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q\big]$.

(Z3) Suppose:

$$(Q\,,\,P\,,\,\longrightarrow\,,\,\imath) \in \mathbb{CA} \text{ for some } Q\,,\,P\,,\,\longrightarrow\,,\,\imath$$

Then, by introducing (A4)(A5), conclude $\big[(Q\,,\,P\,,\,\longrightarrow\,,\,\imath) \in \mathbb{CA} \text{ and } \mathsf{Port}(\psi_2) \subseteq P \text{ and } \mathsf{Port}(\phi_2) \subseteq P\big]$. Then, by applying Definition 33 of $\longrightarrow\lceil\cdot/\cdot\rceil\ \cdot$, conclude:

$$\left[\begin{gathered}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil \in \mathbb{CA} \text{ and} \\ (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil = (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \text{ and} \\ \left[\begin{aligned}\longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \text{ and } q \xrightarrow{\psi,\phi} q'\}\end{aligned}\right]\end{gathered}\right] \text{ for some } \longrightarrow'$$

Then, by applying (Z2), conclude:

$$\begin{gathered}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil = (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \text{ and} \\ \left[\begin{aligned}\longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \text{ and } q \xrightarrow{\psi,\phi} q'\}\end{aligned}\right] \\ \text{and } (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA} \text{ and } \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q\end{gathered}$$

(Z4) Suppose:

$$Q \subseteq \mathbb{S}\text{TATE} \text{ for some } Q$$

Then, by applying standard inference rules, conclude $\big[Q \subseteq \mathbb{S}\text{TATE} \text{ and } \{(q\,,\,q) \mid q \in Q\} \in \Omega\big]$. Then, by applying ZFC, conclude $\big[Q \subseteq \mathbb{S}\text{TATE} \text{ and } \{(q\,,\,q) \mid q \in Q \text{ and } q \in \mathbb{S}\text{TATE}\} \in \Omega\big]$. Then, by applying

ZFC, conclude $\big[Q \subseteq \mathbb{S}\text{TATE} \textbf{ and } \{(q\,,\,q) \mid q \in Q \textbf{ and } q \in \mathbb{S}\text{TATE} \textbf{ and } (q\,,\,q) \in \mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}\} \in \Omega\big]$. Then, by applying ZFC, conclude:

$$Q \subseteq \mathbb{S}\text{TATE} \textbf{ and } \{(q\,,\,q) \mid q \in Q \textbf{ and } q \in \mathbb{S}\text{TATE} \textbf{ and } (q\,,\,q) \in \mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}\} \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE})$$

Then, by applying ZFC, conclude:

$$Q \subseteq \mathbb{S}\text{TATE} \textbf{ and } \{(q\,,\,q) \mid q \in Q \textbf{ and } q \in \mathbb{S}\text{TATE}\} \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE})$$

Then, by applying ZFC, conclude $\{(q\,,\,q) \mid q \in Q\} \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE})$.

Z5 Suppose:

$$Q \subseteq \mathbb{S}\text{TATE} \textbf{ for some } Q$$

Then, by applying ZFC, conclude $\big[Q \subseteq \mathbb{S}\text{TATE} \textbf{ and } \big[R = \{(q\,,\,q) \mid q \in Q\} \textbf{ for some } R\big]\big]$. Then, by applying standard inference rules, conclude:

$$\big[Q \subseteq \mathbb{S}\text{TATE} \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\}\big] \textbf{ for some } R$$

Then, by applying Z4, conclude $\big[\{(q\,,\,q) \mid q \in Q\} \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}) \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\}\big]$. Then, by applying substitution, conclude $\big[R \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}) \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\}\big]$.

Z6 Suppose **true**. Then, by applying standard inference rules, conclude $\{(q\,,\,q) \mid q \in Q\} \in \Omega$. Then, by applying ZFC, conclude $\{(q\,,\,q) \mid q \in Q \textbf{ and } (q\,,\,q) \in Q \times Q\} \in \Omega$. Then, by applying ZFC, conclude:

$$\{(q\,,\,q) \mid q \in Q \textbf{ and } r \in Q \times Q\} \in \wp(Q \times Q)$$

Then, by applying ZFC, conclude $\{(q\,,\,q) \mid q \in Q\} \in \wp(Q \times Q)$.

Z7 Suppose:

$$R = \{(q\,,\,q) \mid q \in Q\} \textbf{ for some } R\,,\,Q$$

Then, by introducing Z6, conclude $\big[R = \{(q\,,\,q) \mid q \in Q\} \textbf{ and } \{(q\,,\,q) \mid q \in Q\} \in \wp(Q \times Q)\big]$. Then, by applying substitution, conclude $R \in \wp(Q \times Q)$. Then, by applying ZFC, conclude $R \subseteq Q \times Q$.

Z8 Suppose:

$$\big[R = \{(q'\,,\,q') \mid q' \in Q\} \textbf{ and } q \in Q\big] \textbf{ for some } R\,,\,q\,,\,Q$$

Then, by applying ZFC, conclude $\big[R = \{(q'\,,\,q') \mid q' \in Q\} \textbf{ and } (q\,,\,q) \in \{(q'\,,\,q') \mid q' \in Q\}\big]$. Then, by applying substitution, conclude $q\ R\ q$.

Z9 Suppose:

$$\big[(\psi\,,\,\phi) = (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\big] \textbf{ for some } \psi\,,\,\phi\,,\,q\,,\,q'\,,\,\longrightarrow$$

Then, by applying substitution, conclude $q \xrightarrow{\psi_1,\phi_1} q'$. Then, by applying ZFC, conclude:

$$(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \in \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\}$$

Then, by applying ZFC, conclude:

$$(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \in \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\}$$

Z0 Suppose:

$$\left[\begin{array}{c} (\psi\,,\,\phi) = (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q' \textbf{ and } \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \quad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\}\end{array}\right]\end{array}\right] \textbf{ for some } \psi\,,\,\phi\,,\,q\,,\,q'\,,\,\longrightarrow\,,\,\longrightarrow'$$

Then, by applying (Z9), conclude:

$$\left[(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \in \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\}\right]$$
$$\textbf{and } \left[\longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\}\right]$$

Then, by applying substitution, conclude $q \xrightarrow{\psi_2,\phi_2}{}' q'$.

(Y1) Suppose:

$$\left[\begin{array}{c} q \xrightarrow{\psi,\phi} q' \textbf{ and} \\ \longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \\ \textbf{and } R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right] \textbf{ for some } q\,,\,q'\,,\,\longrightarrow,\,\psi\,,\,\phi\,,\,Q\,,\,P\,,\,R$$

Then, by applying ZFC, conclude $[q\,,\,q' \in Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\}]$. Then, by applying ZFC, conclude $[q\;R\;q \textbf{ and } q'\;R\;q']$.

(Y2) Suppose:

$$[q \xrightarrow{\psi,\phi} q' \textbf{ and } \longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q] \textbf{ for some } q\,,\,q'\,,\,\longrightarrow,\,\psi\,,\,\phi\,,\,Q\,,\,P$$

Then, by applying ZFC, conclude $[(q\,,\,\psi\,,\,\phi\,,\,q') \in \longrightarrow \in \wp(Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q)]$. Then, by applying ZFC, conclude $[\psi \in \mathsf{SC}(P) \textbf{ and } \phi \in \mathsf{DC}(P)]$. Then, by applying Definition 14 of SC, conclude:

$$\psi \in \mathbb{SC} \textbf{ and } \phi \in \mathsf{DC}(P)$$

Then, by applying Definition 27 of DC, conclude $[\psi \in \mathbb{SC} \textbf{ and } \phi \in \mathbb{DC}]$. Then, by applying ZFC, conclude $(\psi\,,\,\phi) \in \mathbb{SC} \times \mathbb{DC}$.

(Y3) Suppose:

$$\longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ for some } q\,,\,q'\,,\,\longrightarrow,\,Q\,,\,P$$

Then, by applying standard inference rules, conclude:

$$\longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q'\;R\;q'\} \in \Omega$$

Then, by applying (Y2), conclude:

$$\begin{array}{c} \longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and} \\ \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q'\;R\;q' \textbf{ and } (\psi''\,,\,\phi'') \in \mathbb{SC} \times \mathbb{DC}\} \in \Omega \end{array}$$

Then, by applying ZFC, conclude:

$$\begin{array}{c} \longrightarrow \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and} \\ \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q'\;R\;q' \textbf{ and } (\psi''\,,\,\phi'') \in \mathbb{SC} \times \mathbb{DC}\} \in \wp(\mathbb{SC} \times \mathbb{DC}) \end{array}$$

Then, by applying (Y2), conclude $\{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q'\;R\;q'\} \in \wp(\mathbb{SC} \times \mathbb{DC})$.

(Y4) Suppose:

$$\left[\begin{array}{c} q \xrightarrow{\psi,\phi} q' \textbf{ and } q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and} \\ \longrightarrow,\,\longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \\ \textbf{and } R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right] \textbf{ for some } q\,,\,q'\,,\,\longrightarrow,\,\longrightarrow',\,\psi\,,\,\psi'\,,\,\phi\,,\,\phi'\,,\,Q\,,\,P\,,\,R$$

Then, by applying (Y1), conclude $\left[q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } q' \; R \; q'\right]$. Then, by applying ZFC, conclude:

$$\longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } (\psi' , \phi') \in \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \textbf{ and } q' \; R \; q'\}$$

Then, by applying (Y3), conclude $(\psi' , \phi') \in \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \textbf{ and } q' \; R \; q'\} \in \wp(\mathbb{SC} \times \mathbb{DC})$.

(Y5) Suppose:

$$\left[\begin{array}{c} q \xrightarrow{\psi,\phi} q' \textbf{ and } q \xrightarrow{\psi_2,\phi_2}{}' q' \textbf{ and } \\ \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \\ \textbf{and } R = \{(q , q) \mid q \in Q\} \end{array}\right] \textbf{for some } q , q' , \longrightarrow , \longrightarrow' , \psi , \phi , Q , P , R$$

Then, by applying (Y4), conclude $(\psi_2 , \phi_2) \in \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \textbf{ and } q' \; R \; q'\} \in \wp(\mathbb{SC} \times \mathbb{DC})$. Then, by introducing (A1)(A2), conclude:

$$(\psi_2 , \phi_2) \in \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \textbf{ and } q' \; R \; q'\} \in \wp(\mathbb{SC} \times \mathbb{DC}) \textbf{ and } \psi_1 \equiv_{\mathrm{sc}} \psi_2 \textbf{ and } \phi_1 \equiv_{\mathrm{dc}} \phi_2$$

Then, by applying Lemma 15, conclude $(\psi_1 , \phi_1) \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \textbf{ and } q' \; R \; q'\}$.

(Y6) Suppose:

$$\left[\begin{array}{c} q \xrightarrow{\psi_1,\phi_1} q' \textbf{ and } q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } \\ \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \\ \textbf{and } R = \{(q , q) \mid q \in Q\} \end{array}\right] \textbf{for some } q , q' , \longrightarrow , \longrightarrow' , \psi , \phi , Q , P , R$$

Then, by a reduction similar to (Y5), conclude $(\psi_2 , \phi_2) \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q' \; R \; q'\}$.

(Y7) Suppose:

$$\left[\begin{array}{c} (\psi , \phi) = (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q' \textbf{ and } \\ \left[\begin{array}{l} \longrightarrow' = \{(q , \psi_2 , \phi_2 , q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q , \psi , \phi , q') \mid (\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \textbf{and} \\ \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q , q) \mid q \in Q\} \end{array}\right]$$
$$\textbf{for some } \psi , \phi , q , q' , \longrightarrow , \longrightarrow' , Q , P , R$$

Then, by applying (Z0), conclude:

$$(\psi , \phi) = (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q' \textbf{ and } \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q$$
$$\textbf{and } R = \{(q , q) \mid q \in Q\} \textbf{ and } q \xrightarrow{\psi_2,\phi_2}{}' q'$$

Then, by applying (Y5), conclude:

$$(\psi , \phi) = (\psi_1 , \phi_1) \textbf{ and } (\psi_1 , \phi_1) \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \textbf{ and } q' \; R \; q'\}$$

Then, by applying substitution, conclude $(\psi , \phi) \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \textbf{ and } q' \; R \; q'\}$.

(Y8) Suppose:

$$\left[(\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\right] \textbf{for some } \psi , \phi , q , q' , \longrightarrow$$

Then, by applying ZFC, conclude $(q , \psi , \phi , q') \in \{(q , \psi , \phi , q') \mid (\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi_1,\phi_1} q'\}$. Then, by applying ZFC, conclude:

$$(q , \psi , \phi , q') \in \{(q , \psi_2 , \phi_2 , q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup$$
$$\{(q , \psi , \phi , q') \mid (\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\}$$

(Y9) Suppose:

$$\left[\begin{array}{c}(\psi\,,\,\phi)=(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\end{array}\right]\ \textbf{for some}\ \psi\,,\,\phi\,,\,q\,,\,q'\,,\,\longrightarrow\,,\,\longrightarrow'$$

Then, by applying (Y8), conclude:

$$\left[\begin{array}{l}(q\,,\,\psi\,,\,\phi\,,\,q')\in\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]$$
$$\textbf{and}\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]$$

Then, by applying substitution, conclude $q\xrightarrow{\psi,\phi}{}'\,q'$.

(Y0) Suppose:

$$\left[q\xrightarrow{\psi,\phi}q'\ \textbf{and}\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\ \textbf{and}\ q\xrightarrow{\psi,\phi}{}'\,q'\right]$$
$$\textbf{for some}\ q\,,\,q'\,,\,\longrightarrow\,,\,\longrightarrow'\,,\,\psi\,,\,\phi\,,\,Q\,,\,P\,,\,R$$

Then, by applying (Y4), conclude $(\psi\,,\,\phi)\in\{(\psi''\,,\,\phi'')\mid q\xrightarrow{\psi'',\phi''}{}'\,q'\ \textbf{and}\ q'\ R\ q'\}\in\wp(\mathbb{SC}\times\mathbb{DC})$. Then, by applying Definition 10 of $\equiv_{\text{sc}}$, conclude:

$$(\psi\,,\,\phi)\in\{(\psi''\,,\,\phi'')\mid q\xrightarrow{\psi'',\phi''}{}'\,q'\ \textbf{and}\ q'\ R\ q'\}\in\wp(\mathbb{SC}\times\mathbb{DC})\ \textbf{and}\ \psi\equiv_{\text{sc}}\psi$$

Then, by applying Definition 25 of $\equiv_{\text{dc}}$, conclude:

$$(\psi\,,\,\phi)\in\{(\psi''\,,\,\phi'')\mid q\xrightarrow{\psi'',\phi''}{}'\,q'\ \textbf{and}\ q'\ R\ q'\}\in\wp(\mathbb{SC}\times\mathbb{DC})\ \textbf{and}\ \psi\equiv_{\text{sc}}\psi\ \textbf{and}\ \phi\equiv_{\text{dc}}\phi$$

Then, by applying Lemma 15, conclude $(\psi\,,\,\phi)\trianglelefteq\{(\psi''\,,\,\phi'')\mid q\xrightarrow{\psi'',\phi''}{}'\,q'\ \textbf{and}\ q'\ R\ q'\}$.

(X1) Suppose:

$$\left[\begin{array}{c}(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\ \textbf{and}\\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\end{array}\right]$$
$$\textbf{for some}\ \psi\,,\,\phi\,,\,q\,,\,q'\,,\,\longrightarrow\,,\,\longrightarrow'\,,\,Q\,,\,P\,,\,R$$

Then, by applying (Y9), conclude:

$$q\xrightarrow{\psi,\phi}q'\ \textbf{and}\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\ \textbf{and}\ q\xrightarrow{\psi,\phi}{}'\,q'$$

Then, by applying (Y0), conclude $(\psi\,,\,\phi)\trianglelefteq\{(\psi''\,,\,\phi'')\mid q\xrightarrow{\psi'',\phi''}{}'\,q'\ \textbf{and}\ q'\ R\ q'\}$.

(X2) Suppose:

$$\left[\begin{array}{c}q\xrightarrow{\psi,\phi}q'\ \textbf{and}\ q\ R\ q\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\ \textbf{and}\\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\end{array}\right]$$
$$\textbf{for some}\ \psi\,,\,\phi\,,\,q\,,\,q'\,,\,R\,,\,\longrightarrow\,,\,\longrightarrow'\,,\,Q\,,\,P$$

Then, by applying standard inference rules:

$$\begin{array}{c} q \xrightarrow{\psi,\phi} q' \ \textbf{and} \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \ \textbf{and} \\ \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \ \textbf{and} \ R = \{(q\,,\,q) \mid q \in Q\} \end{array}$$

Then, by applying standard inference rules:

$$\textbf{true and} \left[\begin{array}{c} q \xrightarrow{\psi,\phi} q' \ \textbf{and} \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \ \textbf{and} \\ \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \ \textbf{and} \ R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right]$$

Then, by applying standard inference rules:

$$\left[\begin{array}{l} \quad (\psi\,,\,\phi) = (\psi_1\,,\,\phi_1) \\ \textbf{or} \ (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \end{array}\right] \textbf{and} \left[\begin{array}{c} q \xrightarrow{\psi,\phi} q' \ \textbf{and} \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \ \textbf{and} \\ \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \ \textbf{and} \ R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right]$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{l} \left[\begin{array}{c} (\psi\,,\,\phi) = (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q' \ \textbf{and} \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \ \textbf{and} \\ \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \ \textbf{and} \ R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right] \\ \textbf{or} \left[\begin{array}{c} (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q' \ \textbf{and} \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \ \textbf{and} \\ \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \ \textbf{and} \ R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right] \end{array}$$

Then, by applying Y7, conclude:

$$\begin{array}{c} \left[(\psi\,,\,\phi) \trianglelefteq \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \ \textbf{and} \ q' \ R \ q'\}\right] \ \textbf{or} \\ \left[\begin{array}{c} (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q' \ \textbf{and} \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \ \textbf{and} \ q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \ \textbf{and} \\ \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \ \textbf{and} \ R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right] \end{array}$$

Then, by applying X1, conclude:

$$\begin{array}{l} \quad \left[(\psi\,,\,\phi) \trianglelefteq \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \ \textbf{and} \ q' \ R \ q'\}\right] \\ \textbf{or} \ \left[(\psi\,,\,\phi) \trianglelefteq \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \ \textbf{and} \ q' \ R \ q'\}\right] \end{array}$$

Then, by applying standard inference rules, conclude $(\psi\,,\,\phi) \trianglelefteq \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''}{}' q' \ \textbf{and} \ q' \ R \ q'\}$.

(X3) Suppose:

$$\left[\begin{array}{c} (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA} \;\; \textbf{and} \;\; R \in \wp(\mathbb{S}\textsc{tate} \times \mathbb{S}\textsc{tate}) \;\; \textbf{and} \\ R = \{(q\,,\,q) \mid q \in Q\} \;\; \textbf{and} \;\; \longrightarrow\,,\,\longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \;\; \textbf{and} \;\; \imath \in Q \\ \textbf{and} \;\; \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \;\; \textbf{and} \;\; q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \end{array}\right]$$

$$\textbf{for some} \;\; Q\,,\,P\,,\,\longrightarrow\,,\,\longrightarrow'\,,\,\imath\,,\,R$$

Then, by applying (Z7), conclude:

$$\begin{array}{c} (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA} \;\; \textbf{and} \;\; R \in \wp(\mathbb{S}\textsc{tate} \times \mathbb{S}\textsc{tate}) \;\; \textbf{and} \\ R = \{(q\,,\,q) \mid q \in Q\} \;\; \textbf{and} \;\; \longrightarrow\,,\,\longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \;\; \textbf{and} \;\; \imath \in Q \\ \textbf{and} \;\; \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \;\; \textbf{and} \;\; q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \\ \textbf{and} \;\; R \subseteq Q \times Q \end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c} (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA} \;\; \textbf{and} \;\; R \in \wp(\mathbb{S}\textsc{tate} \times \mathbb{S}\textsc{tate}) \;\; \textbf{and} \\ R = \{(q\,,\,q) \mid q \in Q\} \;\; \textbf{and} \;\; \longrightarrow\,,\,\longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \;\; \textbf{and} \;\; \imath \in Q \\ \textbf{and} \;\; \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \;\; \textbf{and} \;\; q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \\ \textbf{and} \;\; R \subseteq Q \times Q \;\; \textbf{and} \;\; P = P \end{array}$$

Then, by applying (Z8), conclude:

$$\begin{array}{c} (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA} \;\; \textbf{and} \;\; R \in \wp(\mathbb{S}\textsc{tate} \times \mathbb{S}\textsc{tate}) \\ \textbf{and} \;\; R = \{(q\,,\,q) \mid q \in Q\} \;\; \textbf{and} \;\; \longrightarrow\,,\,\longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \\ \textbf{and} \;\; \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \;\; \textbf{and} \;\; q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \\ \textbf{and} \;\; R \subseteq Q \times Q \;\; \textbf{and} \;\; P = P \;\; \textbf{and} \;\; \imath \; R \; \imath \end{array}$$

Then, by applying (X2), conclude:

$$\begin{array}{c} (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \in \mathbb{CA} \;\; \textbf{and} \;\; R \in \wp(\mathbb{S}\textsc{tate} \times \mathbb{S}\textsc{tate}) \\ \textbf{and} \;\; R \subseteq Q \times Q \;\; \textbf{and} \;\; P = P \;\; \textbf{and} \;\; \imath \; R \; \imath \;\; \textbf{and} \\ \left[\begin{array}{l} q \xrightarrow{\psi,\phi} q' \\ \textbf{and} \;\; q \; R \; q \end{array}\right] \;\; \textbf{implies} \;\; (\psi\,,\,\phi) \trianglelefteq \left\{ (\psi''\,,\,\phi'') \,\middle|\, \begin{array}{l} q \xrightarrow{\psi'',\phi''}{}' \; q' \\ \textbf{and} \;\; q' \; R \; q' \end{array} \right\} \end{array}$$

Then, by applying Definition 31 of $\preceq$, conclude $(Q\,,\,P\,,\,\longrightarrow\,,\,\imath) \preceq^R (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)$

(X4) Suppose:

$$\left[q \xrightarrow{\psi',\phi'}{}' \; q' \;\; \textbf{and} \;\; \left[\begin{array}{l} (q\,,\,\psi'\,,\,\phi'\,,\,q') \in \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \;\; \textbf{and} \;\; q \xrightarrow{\psi,\phi} q'\} \end{array}\right]\right]$$

$$\textbf{for some} \;\; q\,,\,q'\,,\,\longrightarrow'\,,\,\psi'\,,\,\phi'$$

Then, by applying substitution, conclude:

$$\begin{array}{l} (q\,,\,\psi'\,,\,\phi'\,,\,q') \in \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \;\; \textbf{and} \;\; q \xrightarrow{\psi,\phi} q'\} \end{array}$$

Then, by applying ZFC, conclude:

$$(q\,,\,\psi'\,,\,\phi'\,,\,q') \in \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \textbf{ or }$$
$$(q\,,\,\psi'\,,\,\phi'\,,\,q') \in \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\}$$

Then, by applying ZFC, conclude $\big[[(\psi'\,,\,\phi') = (\psi_2\,,\,\phi_2) \textbf{ and } q \xrightarrow{\psi_1,\phi_1} q'] \textbf{ or } q \xrightarrow{\psi',\phi'} q'\big]$.

(X5) Suppose:

$$\left[\begin{array}{c} q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\} \\ \textbf{and } (\psi'\,,\,\phi') = (\psi_2\,,\,\phi_2) \textbf{ and } q \xrightarrow{\psi_1,\phi_1} q' \end{array}\right]$$
$$\textbf{for some } q\,,\,q'\,,\,\longrightarrow,\,\longrightarrow'\,,\,\psi'\,,\,\phi'\,,\,Q\,,\,P\,,\,R$$

Then, by applying substitution, conclude:

$$q \xrightarrow{\psi_2,\phi_2}{}' q' \textbf{ and } \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\}$$
$$\textbf{and } (\psi'\,,\,\phi') = (\psi_2\,,\,\phi_2) \textbf{ and } q \xrightarrow{\psi_1,\phi_1} q'$$

Then, by applying (Y6), conclude:

$$(\psi'\,,\,\phi') = (\psi_2\,,\,\phi_2) \textbf{ and } (\psi_2\,,\,\phi_2) \trianglelefteq \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q' \mathrel{R} q'\}$$

Then, by applying substitution, conclude $(\psi'\,,\,\phi') \trianglelefteq \{(\psi''\,,\,\phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q' \mathrel{R} q'\}$.

(X6) Suppose:

$$\left[\begin{array}{c} q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } q \mathrel{R} q \textbf{ and} \\ \left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \textbf{and} \\ \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\} \end{array}\right]$$
$$\textbf{for some } \psi'\,,\,\phi'\,,\,q\,,\,q'\,,\,R\,,\,\longrightarrow,\,\longrightarrow'\,,\,Q\,,\,P$$

Then, by applying standard inference rules:

$$q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and}$$
$$\left[\begin{array}{l} \longrightarrow' = \{(q\,,\,\psi_2\,,\,\phi_2\,,\,q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \qquad \{(q\,,\,\psi\,,\,\phi\,,\,q') \mid (\psi\,,\,\phi) \neq (\psi_1\,,\,\phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \textbf{and}$$
$$\longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\}$$

Then, by applying (X4), conclude:

$$q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\}$$
$$\textbf{and } \big[[(\psi'\,,\,\phi') = (\psi_2\,,\,\phi_2) \textbf{ and } q \xrightarrow{\psi_1,\phi_1} q'] \textbf{ or } q \xrightarrow{\psi',\phi'} q'\big]$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c} q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\} \\ \textbf{and } (\psi'\,,\,\phi') = (\psi_2\,,\,\phi_2) \textbf{ and } q \xrightarrow{\psi_1,\phi_1} q' \end{array}\right]$$
$$\textbf{or } \left[\begin{array}{c} q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } \longrightarrow, \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q\,,\,q) \mid q \in Q\} \\ \textbf{and } q \xrightarrow{\psi',\phi'} q' \end{array}\right]$$

Then, by applying (X5), conclude:

$$\left[(\psi' , \phi') \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q' \mathrel{R} q'\}\right] \textbf{ or}$$
$$\left[\begin{array}{c} q \xrightarrow{\psi',\phi'}{}' q' \textbf{ and } \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } R = \{(q , q) \mid q \in Q\} \\ \textbf{and } q \xrightarrow{\psi',\phi'} q' \end{array}\right]$$

Then, by applying (Y0), conclude:

$$\begin{array}{c} (\psi' , \phi') \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q' \mathrel{R} q'\} \\ \textbf{or } (\psi' , \phi') \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q' \mathrel{R} q'\} \end{array}$$

Then, by applying standard inference rules, conclude $(\psi' , \phi') \trianglelefteq \{(\psi'' , \phi'') \mid q \xrightarrow{\psi'',\phi''} q' \textbf{ and } q' \mathrel{R} q'\}$.

(X7) Suppose:

$$\left[\begin{array}{c} (Q , P , \longrightarrow , \imath) , (Q , P , \longrightarrow' , \imath) \in \mathbb{CA} \textbf{ and } R \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}) \textbf{ and} \\ R = \{(q , q) \mid q \in Q\} \textbf{ and } \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } \imath \in Q \\ \textbf{and } \left[\begin{array}{r} \longrightarrow' = \{(q , \psi_2 , \phi_2 , q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \{(q , \psi , \phi , q') \mid (\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \end{array}\right]$$
$$\textbf{for some } Q , P , \longrightarrow , \longrightarrow' , \imath , R$$

Then, by applying (Z7), conclude:

$$\begin{array}{c} (Q , P , \longrightarrow , \imath) , (Q , P , \longrightarrow' , \imath) \in \mathbb{CA} \textbf{ and } R \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}) \textbf{ and} \\ R = \{(q , q) \mid q \in Q\} \textbf{ and } \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } \imath \in Q \\ \textbf{and } \left[\begin{array}{r} \longrightarrow' = \{(q , \psi_2 , \phi_2 , q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \{(q , \psi , \phi , q') \mid (\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \\ \textbf{and } R \subseteq Q \times Q \end{array}$$

Then, by applying standard inference rules, conclude:

$$\begin{array}{c} (Q , P , \longrightarrow , \imath) , (Q , P , \longrightarrow' , \imath) \in \mathbb{CA} \textbf{ and } R \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}) \textbf{ and} \\ R = \{(q , q) \mid q \in Q\} \textbf{ and } \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \textbf{ and } \imath \in Q \\ \textbf{and } \left[\begin{array}{r} \longrightarrow' = \{(q , \psi_2 , \phi_2 , q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \{(q , \psi , \phi , q') \mid (\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \\ \textbf{and } R \subseteq Q \times Q \textbf{ and } P = P \end{array}$$

Then, by applying (Z8), conclude:

$$\begin{array}{c} (Q , P , \longrightarrow , \imath) , (Q , P , \longrightarrow' , \imath) \in \mathbb{CA} \textbf{ and } R \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}) \\ \textbf{and } R = \{(q , q) \mid q \in Q\} \textbf{ and } \longrightarrow , \longrightarrow' \subseteq Q \times \mathsf{SC}(P) \times \mathsf{DC}(P) \times Q \\ \textbf{and } \left[\begin{array}{r} \longrightarrow' = \{(q , \psi_2 , \phi_2 , q') \mid q \xrightarrow{\psi_1,\phi_1} q'\} \cup \\ \{(q , \psi , \phi , q') \mid (\psi , \phi) \neq (\psi_1 , \phi_1) \textbf{ and } q \xrightarrow{\psi,\phi} q'\} \end{array}\right] \\ \textbf{and } R \subseteq Q \times Q \textbf{ and } P = P \textbf{ and } \imath \mathrel{R} \imath \end{array}$$

Then, by applying (X6), conclude:

$$\begin{array}{c} (Q , P , \longrightarrow , \imath) , (Q , P , \longrightarrow' , \imath) \in \mathbb{CA} \textbf{ and } R \in \wp(\mathbb{S}\text{TATE} \times \mathbb{S}\text{TATE}) \\ \textbf{and } R \subseteq Q \times Q \textbf{ and } P = P \textbf{ and } \imath \mathrel{R} \imath \textbf{ and} \\ \left[\begin{array}{l} q \xrightarrow{\psi',\phi'}{}' q' \\ \textbf{and } q \mathrel{R} q \end{array}\right] \textbf{ implies } (\psi' , \phi') \trianglelefteq \left\{ (\psi'' , \phi'') \,\middle|\, \begin{array}{l} q \xrightarrow{\psi'',\phi''} q' \\ \textbf{and } q' \mathrel{R} q' \end{array} \right\} \end{array}$$

Then, by applying Definition 31 of $\preceq$, conclude $(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath) \preceq^{R} (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)$

Now, prove the lemma by the following reduction. Recall from (Z1):

$$\left[\begin{array}{c}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\\ Q\subseteq\mathbb{S}\text{TATE}\ \textbf{and}\ \longrightarrow\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ \imath\in Q\end{array}\right]\ \textbf{for some}\ Q\,,\,P\,,\,\longrightarrow\,,\,\imath$$

Then, by applying (Z3), conclude:

$$\begin{array}{c}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\\ Q\subseteq\mathbb{S}\text{TATE}\ \textbf{and}\ \longrightarrow\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ \imath\in Q\ \textbf{and}\\ \Big[\left[\begin{array}{c}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil=(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\\ \textbf{and}\ (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\ \longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\end{array}\right]\ \textbf{for some}\ \longrightarrow'\Big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\ (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil=(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\\ \textbf{and}\ Q\subseteq\mathbb{S}\text{TATE}\ \textbf{and}\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ \imath\in Q\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\end{array}\right]$$
$$\textbf{for some}\ \longrightarrow'$$

Then, by applying substitution, conclude:

$$\begin{array}{c}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\ \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil=(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\ \textbf{and}\\ Q\subseteq\mathbb{S}\text{TATE}\ \textbf{and}\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ \imath\in Q\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\end{array}$$

Then, by applying (Z5), conclude:

$$\begin{array}{c}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\ \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil=(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\ \textbf{and}\\ Q\subseteq\mathbb{S}\text{TATE}\ \textbf{and}\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ \imath\in Q\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\\ \big[\big[R\in\wp(\mathbb{S}\text{TATE}\times\mathbb{S}\text{TATE})\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\ \textbf{for some}\ R\big]\big]\end{array}$$

Then, by applying standard inference rules, conclude:

$$\left[\begin{array}{c}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\ \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil=(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\\ \textbf{and}\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ \imath\in Q\ \textbf{and}\\ \left[\begin{array}{l}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{array}\right]\\ R\in\wp(\mathbb{S}\text{TATE}\times\mathbb{S}\text{TATE})\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\end{array}\right]\ \textbf{for some}\ R$$

Then, by applying (X3), conclude:

$$\begin{gathered}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\ \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil=(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\\ \textbf{and}\ \longrightarrow\,,\,\longrightarrow'\subseteq Q\times\mathsf{SC}(P)\times\mathsf{DC}(P)\times Q\ \textbf{and}\ \imath\in Q\ \textbf{and}\\ \begin{bmatrix}\longrightarrow'=\{(q\,,\,\psi_2\,,\,\phi_2\,,\,q')\mid q\xrightarrow{\psi_1,\phi_1}q'\}\cup\\ \qquad\{(q\,,\,\psi\,,\,\phi\,,\,q')\mid(\psi\,,\,\phi)\neq(\psi_1\,,\,\phi_1)\ \textbf{and}\ q\xrightarrow{\psi,\phi}q'\}\end{bmatrix}\\ R\in\wp(\mathbb{S}\text{TATE}\times\mathbb{S}\text{TATE})\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\\ \textbf{and}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\preceq^R(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\end{gathered}$$

Then, by applying (X7), conclude:

$$\begin{gathered}(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\,,\,(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\in\mathbb{CA}\ \textbf{and}\\ \alpha=(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\ \textbf{and}\ \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil=(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\\ \textbf{and}\ (Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\preceq^R(Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\ \textbf{and}\ (Q\,,\,P\,,\,\longrightarrow'\,,\,\imath)\preceq^R(Q\,,\,P\,,\,\longrightarrow\,,\,\imath)\end{gathered}$$

Then, by applying substitution, conclude:

$$\begin{gathered}\alpha\,,\,\alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil\in\mathbb{CA}\ \textbf{and}\ R=\{(q\,,\,q)\mid q\in Q\}\ \textbf{and}\\ \alpha\preceq^R\alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil\ \textbf{and}\ \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil\preceq^R\alpha\end{gathered}$$

Then, by applying ZFC, conclude:

$$\alpha\,,\,\alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil\in\mathbb{CA}\ \textbf{and}\ \alpha\preceq^R\alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil\ \textbf{and}\ \alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil\preceq^{R^{-1}}\alpha$$

Then, by applying Definition 32 of $\sim$, conclude $\alpha\sim\alpha\lceil(\psi_2\,,\,\phi_2)/(\psi_1\,,\,\phi_1)\rceil$.

(| QED. |)