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Technical Note TN 14

The generalized potential of an ellipsoid

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§ 1. Ellipsoid

The Newtonian attraction of an ellipsoid has been a famous problem of the past to which Newton, MacLaurin and Ivory have paid much attention *). The usual derivation of the potential of an ellipsoid at either an internal or an external point makes use of geometrical arguments. In the following lines a simple analytical derivation will be given by using the technique of Laplace transformation.

Let the ellipsoid be given by

$$(1.1)\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=r^2 ,$$

and let the generalized potential at P(x,y,z) due to a unit mass at (ξ,η,ζ) be given by

$$(1.2)\qquad \left\{(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2\right\}^{-\theta},$$

where $0 \leq \theta < \frac{3}{2}$. Then the generalized potential at P of the ellipsoid is

$$(1.3)\qquad V(r^2)=\iiint_{-\infty}^{\infty} U\left\{r^2-\frac{\xi^2}{a^2}-\frac{\eta^2}{b^2}-\frac{\zeta^2}{c^2}\right\}\left\{\Sigma(x-\xi)^2\right\}^{-\theta} d\xi\, d\eta\, d\zeta ,$$

where U(t) is the well-known unit function

$$(1.4)\qquad U(t)=\begin{cases}1 & \text{for} \quad t>0,\\ 0 & \text{for} \quad t<0.\end{cases}$$

If upon (1.3) Laplace transformation is applied with respect to r^2 we obtain

$$(1.5)\qquad \bar{V}(p)=p^{-1}\iiint_{-\infty}^{\infty} \exp-p\left(\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}+\frac{\zeta^2}{c^2}\right)\left\{\Sigma(x-\xi)^2\right\}^{-\theta} d\xi\, d\eta\, d\zeta .$$

By making the substitution

$$(1.6)\qquad \left\{\Sigma(x-\xi)^2\right\}^{-\theta}=\Gamma^{-1}(\theta)\int_0^{\infty} s^{-1-\theta}\exp\left\{\frac{-\Sigma(x-\xi)^2}{s}\right\}ds,$$

the last expression can be written in the form

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*) Cf. A.S. Ramsay. Newtonian attraction. Cambridge 1940.

(1.7) $$\Gamma(\theta)\,\bar{V}(p) = p^{-1}\int_0^\infty s^{-1-\theta}ds \iiint_{-\infty}^{\infty} \exp-\left\{p\sum\frac{\xi^2}{a^2}+\sum\frac{(x-\xi)^2}{s}\right\}d\xi\,d\eta\,d\zeta.$$

The triple integral on the right-hand side is a product of three integrals, the integration of which is elementary. The integration with respect to ξ gives e.g.

(1.8) $$\int_{-\infty}^{\infty}\exp-\left\{\frac{p}{a^2}\xi^2+\frac{1}{s}(\xi-x)^2\right\}d\xi = \frac{\pi^{\frac{1}{2}}a\ s^{\frac{1}{2}}}{(a^2+sp)^{\frac{1}{2}}}\exp-\frac{px^2}{sp+a^2},$$

so that (1.7) becomes

$$\Gamma(\theta)\,\bar{V}(p) = \pi^{3/2}abc\,p^{-1}\int_0^\infty s^{\frac{1}{2}-\theta}\frac{\exp-\sum\frac{px^2}{sp+a^2}}{\prod(a^2+sp)^{\frac{1}{2}}}ds =$$

(1.9)

$$= \pi^{3/2}\,abc\int_0^\infty \frac{s^{\frac{1}{2}-\theta}}{\prod(a^2+s)^{\frac{1}{2}}}\,p^{\theta-\frac{5}{2}}\exp-p\sum\frac{x^2}{a^2+s}\,ds.$$

Inverse transformation gives finally

(1.10) $$V(r^2) = \frac{\pi^{3/2}abc}{\Gamma(\theta)\Gamma(\frac{5}{2}-\theta)}\int_S^\infty \frac{\left(r^2-\frac{x^2}{a^2+s}-\frac{y^2}{b^2+s}-\frac{z^2}{c^2+s}\right)^{3/2-\theta}}{\sqrt{(a^2+s)(b^2+s)(c^2+s)}}\,s^{\frac{1}{2}-\theta}ds,$$

where the interval (S,∞) is determined by the requirement

(1.11) $$\frac{x^2}{a^2+s}+\frac{y^2}{b^2+s}+\frac{z^2}{c^2+s} \leqq r^2$$

with the equality sign for $s=S$.

The last expression holds for external points as well as for internal points. However, for an internal point it follows at once that $S=0$. The expression (1.10) holds for external points also if $\frac{3}{2}\leqq\theta<\frac{5}{2}$. For $\theta=5/2$ the result may be obtained at once by inversion of (1.9) viz.

(1.12) $$V(r^2) = \frac{\frac{4}{3}\pi abc}{S^2\sum\frac{x^2}{(a^2+S)^2}\prod\sqrt{a^2+S}}.$$

§ 2. Sphere

If a=b=c=1 and $R^2 \stackrel{def}{=} x^2+y^2+z^2$ the expression (1.10) becomes for external points

$$(2.1) \qquad V(r^2) = \frac{\pi^{3/2}}{\Gamma(\theta)\,\Gamma(\frac{5}{2}-\theta)} \int_S^{\infty} s^{\frac{1}{2}-\theta}\left(r^2-\frac{R^2}{1+s}\right)^{3/2-\theta}(1+s)^{-3/2}ds,$$

where $S = R^2/r^2-1$.

Substitution of $s=R^2/r^2u-1$ transforms this into

$$(2.2) \qquad V(r^2) = \frac{\pi^{3/2}\,r^3}{\Gamma(\theta)\Gamma(\frac{5}{2}-\theta)R^{2\theta}} \int_0^1 u^{\theta-1}(1-u)^{\frac{3}{2}-\theta}\left(1-\frac{r^2}{R^2}u\right)^{\frac{1}{2}-\theta}du,$$

so that

$$(2.3) \qquad V(r^2) = \frac{\frac{4}{3}\pi r^3}{R^{2\theta}}\,F(\theta,\theta-\tfrac{1}{2};\tfrac{5}{2};\tfrac{r^2}{R^2}), \qquad r \leq R.$$

For $\theta=\frac{1}{2}$ this reduces to the well-known Newtonian result

$$(2.4) \qquad V(r^2) = \frac{4/3\,\pi r^3}{R}$$

which equals the potential of a masspoint at the origin where the total mass of the sphere is concentrated.

For internal points the constant S in (2.1) equals zero. Substitution of $s=1/u-1$ gives

$$(2.5) \qquad V(r^2) = \frac{\pi^{3/2}r^{3-2\theta}}{\Gamma(\theta)\Gamma(\frac{5}{2}-\theta)} \int_0^1 u^{\theta-1}(1-u)^{\frac{1}{2}-\theta}\left(1-\frac{R^2}{r^2}u\right)^{\frac{3}{2}-\theta}du,$$

so that

$$(2.6) \qquad V(r^2) = \frac{4\pi r^{3-2\theta}}{3-2\theta}\,F(\theta,\theta-\tfrac{3}{2};\tfrac{3}{2};\tfrac{R^2}{r^2}), \qquad r \geq R.$$

At the surface (2.3) and (2.6) have the common value

$$(2.7) \qquad V(r^2) = \frac{\pi(2r)^{3-2\theta}}{(2-\theta)(3-2\theta)}, \qquad r = R.$$

For $\theta=\frac{1}{2}$ the expression (2.6) reduces to

$$(2.8) \qquad V(r^2) = 2\pi\left(r^2-\tfrac{1}{3}R^2\right), \qquad r \geq R.$$

§3. Rectangular block

Let the block be determined by

$$(3.1)\qquad |x| < a, \quad |y| < b, \quad |z| < c$$

then the generalized potential at (x,y,z) is given by

$$(3.2)\qquad V(a^2,b^2,c^2) = \iiint_{-\infty}^{\infty} U(a^2-\xi^2)U(b^2-\eta^2)U(c^2-\zeta^2)\left\{\sum(x-\xi)^2\right\}^{-\theta} d\xi\, d\eta\, d\zeta .$$

Repeated Laplace transformation with respect to a^2,b^2,c^2 gives

$$(3.3)\qquad \overline{\overline{\overline{V}}}(p,q,r) = \iiint_{-\infty}^{\infty} \frac{e^{-p\xi^2-q\eta^2-r\zeta^2}}{pqr}\left\{\sum(x-\xi)^2\right\}^{-\theta} d\xi\, d\eta\, d\zeta .$$

The substitution (1.6) changes this into

$$(3.4)\qquad \Gamma(\theta)\,\overline{\overline{\overline{V}}}(p,q,r) = p^{-1}q^{-1}r^{-1}\int_0^{\infty} s^{-1-\theta}ds \cdot \iiint_{-\infty}^{\infty}\exp -\left\{\sum p\xi^2+s^{-1}\sum(x-\xi)^2\right\} d\xi\, d\eta\, d\zeta .$$

Integration with respect to ξ gives in view of (1.8)

$$(3.5)\qquad \int_{-\infty}^{\infty} \exp-\left\{p\xi^2+s^{-1}(\xi-x)^2\right\} d\xi = \frac{\pi^{\frac{1}{2}}s^{\frac{1}{2}}}{(1+sp)^{\frac{1}{2}}}\exp - \frac{px^2}{1+sp} ,$$

so that with a similar result for η and ζ the expression (3.4) reduces to

$$(3.6)\qquad \Gamma(\theta)\,\overline{\overline{\overline{V}}}(p,q,r) = \pi^{3/2}\int_0^{\infty} s^{\frac{1}{2}-\theta}\,\frac{\exp-\sum\frac{px^2}{1+sp}}{\prod p(1+sp)^{\frac{1}{2}}}\,ds.$$

The inversion with respect to p involves the factor

$$(3.7)\qquad \frac{\exp-\frac{px^2}{1+sp}}{p(1+sp)^{\frac{1}{2}}} .$$

It can be verified without difficulty that the corresponding original is

$$(3.8)\qquad \tfrac{1}{2}\left\{\operatorname{erf}\frac{a+x}{s^{\frac{1}{2}}} + \operatorname{erf}\frac{a-x}{s^{\frac{1}{2}}}\right\} .$$

Hence the total inversion of (3.6) gives

$$(3.9)\qquad \Gamma(\theta)\,V(a^2,b^2,c^2) = \frac{\pi}{8}^{3/2}\int_0^{\infty} s^{\frac{1}{2}-\theta}\prod\left\{\operatorname{erf}\frac{a+x}{s^{\frac{1}{2}}} + \operatorname{erf}\frac{a-x}{s^{\frac{1}{2}}}\right\} ds.$$

Differentiation of V with respect to a,b and c gives

$$\sum \frac{1}{\Gamma(\theta)} \int_0^\infty s^{-1-\theta} \exp - s^{-1} \left\{ (x \pm a)^2 + (y \pm b)^2 + (z \pm c)^2 \right\} ds =$$

$$= \sum \left\{ (x \pm a)^2 + (y \pm b)^2 + (z \pm c)^2 \right\}^{-\theta}, \tag{3.10}$$

which represents the sum of the generalized potentials from the eight corners of the block. By inverting the argument the expression (3.9) may be obtained by integration of (3.10) with respect to a,b and c, which is another proof of the result.