

Non-linearly stable reduced-order models for incompressible flow with energy-conserving finite volume methods

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Abstract

A novel reduced order model (ROM) formulation for incompressible flows is presented with the key property that it possesses non-linear stability, independent of the mesh (of the full order model), the time step, the viscosity, and the number of modes. The two essential elements to non-linear stability are: (1) *first discretise* the full order model, and *then project* the discretised equations, and (2) use spatial and temporal discretisation schemes that are *globally energy-conserving* (in the limit of vanishing viscosity). For this purpose, as full order model a staggered-grid finite volume method in conjunction with an implicit Runge-Kutta method is employed. In addition, a new constrained singular value decomposition is proposed which enforces global momentum conservation. The resulting ROM is thus globally conserving mass, momentum and kinetic energy. For non-homogeneous boundary conditions, a (one-time) Poisson equation is solved that accounts for the boundary contribution. The stability of the proposed ROM is demonstrated in several test cases.

1. Introduction

The simulation of turbulent fluid flows is an ongoing challenge in the scientific community. The computational cost of Direct Numerical Simulation (DNS) or Large Eddy Simulation (LES) of turbulent flows quickly becomes imperative when one is interested in *control, design, optimization* and *uncertainty quantification* [7, 31]. For these purposes, a reduction in complexity of the full model is required to arrive at a computationally tractable model, a so-called reduced order model (ROM). Several techniques exist to construct a ROM, such as balanced truncation, Krylov subspace methods, and POD-Galerkin methods [4]. In this work we focus on the most popular technique, the POD-Galerkin method, in which the governing equations of the full model are projected onto a lower-dimensional space via a Galerkin step, with the projection basis determined from a proper orthogonal decomposition (POD) of snapshots of the full order model (FOM).

Projection-based models have been shown to work for a large class of problems, such as diffusion-dominated linear time-varying (LTI) systems, in which the input-output relation of the full model can be represented by a lower-dimensional model, due to rapid decay of the singular values of the Hankel matrix [7]. However, in turbulent flow, which is a nonlinear, convection-dominated problem, the construction of accurate and stable ROMs is still an open challenge. There are several (related) reasons why current ROMs have issues with accuracy and stability in case of turbulent flows: the Kolmogorov N -width decays too slowly; the non-linear dynamical system is very sensitive to perturbations; the modes with low energy (small scales, dissipation) which are neglected in the POD procedure are relevant for the dynamics of the large scales; the reduced model can have different stability characteristics [5, 9, 12, 21, 25, 32].

A number of approaches have been proposed to tackle these issues; we summarize the list in [12]: including dissipation via a closure model (see also [11]); modifying the POD basis by including functions that resolve a range of scales; using a minimum residual formulation [10]; using an inner product different from L^2 , e.g. based on H^1 . Another promising approach towards stable methods, which we follow in this work, is

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via structure-preserving model reduction, in which reduced order models are developed in such a way that invariants and/or symmetries of the full model are kept [2, 3, 10, 16, 23]. An example is a ROM that inherits the symplectic form of a Hamiltonian system, leading to a ROM that is applicable to long-time integration [23].

For the incompressible Navier-Stokes equations, which do not form a Hamiltonian system, the symmetries in the equations are tightly related to conservation of kinetic energy through the skew-symmetry of the convective operator, and the relation between the divergence and the gradient operator. Several other adaptations to the classic POD-Galerkin method were developed to take into account symmetry or invariance properties. For example, Balajewicz et al. [6] add a power balance equation for the resolved turbulent kinetic energy when solving for the POD basis functions and coefficients. Mohebujjaman et al. [19] employ a combined projection and data-driven approach in a finite-element context and obtain correction terms by solving a constrained minimization problem, using as Ansatz the negative definiteness of the diffusion operator and the energy-conserving property of the convection term. Mohebujjaman [20] investigate conservation of mass and energy of the ROM in the context of a finite element framework and discusses the treatment of non-homogeneous boundary conditions via a Stokes extension in order to mimic the continuous energy balance. Carlberg et al. [10] construct a momentum-conserving ROM in a finite-volume context. Rowley et al. [26] consider the choice of an appropriate inner product and corresponding energy norm for compressible flow. Kalashnikova et al. considered energy stability in terms of a continuous formulation [15].

A method that combines global mass, momentum and kinetic energy conservation (in the inviscid limit) appears to be missing in literature. Furthermore, a detailed discussion of the non-linear stability of the *fully discrete* ROM (in a finite-volume context) is, to the author's knowledge, not available. In this paper we thus aim at developing a fully discrete reduced order model that possess non-linear stability, independent of the viscosity, mesh, time step or number of modes. Our approach to achieving non-linear stability in incompressible flow is through the use of a finite-volume based energy-conserving discretisation method in space [34] and time [28], which preserves the symmetries of the continuous equations. This symmetry-preserving discrete system is then projected via a new constrained SVD method with a weighted inner product, *in such a way that the reduced model is mass-, momentum- and energy-conserving (and thus stable)*. An important difference with existing work, e.g. [17, 32], is that we *first discretise the equations, and then perform the projection*, instead of the reverse order. This allows us to circumvent the inf-sup condition on the level of the ROM, and makes the boundary condition treatment straightforward. A graphical summary of the approach is shown in figure 1.

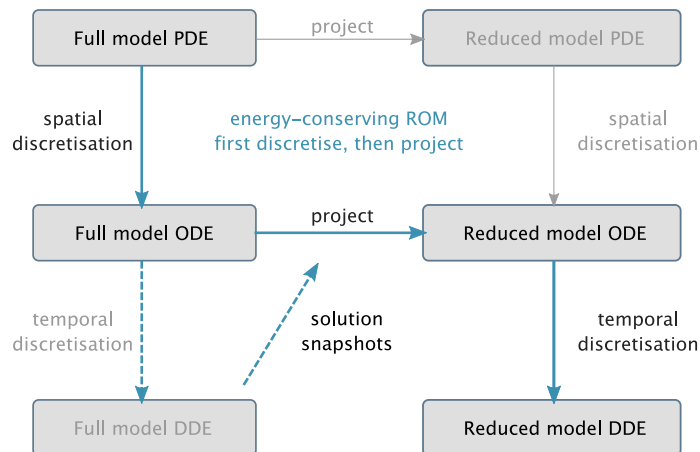


Figure 1: Our new approach to energy-stable reduced order models follows the blue arrows instead of the conventional route (in grey): first spatial discretisation, then projection (ODE = ordinary differential equation, DDE = discretised differential equation).

We limit ourselves in the analysis in several important aspects. First, we will consider the so-called *solution reproduction problem*, which is the first step before solving the full *parametric problem* [12]. Second,

we will focus mainly on the *non-linear stability* of the discrete ROM, and leave aside the discussion of consistency/accuracy. Given a non-linearly stable ROM, we have a framework in which we can in future work assess for example the accuracy of closure models, e.g. [11, 19, 35]. Our definition of stability should therefore be interpreted in the classical sense: a certain norm of the solution stays bounded in time. Note that this is different from the ‘stabilizing’ methods that have been proposed in the ROM community, which are required to counteract numerical oscillations [36]. This latter type of (in)stability is not the focus of this paper.

The novelty of this paper is threefold. First, we derive an energy-conserving reduced-order model, which possesses *nonlinear stability independent of the mesh and time step of the FOM, and independent of the time step and number of modes of the reduced order model*. Second, we enforce *global momentum conservation via a constrained singular value decomposition*. Third, we propose a new procedure to handle unsteady boundary conditions.

This paper is organized as follows. First, in section 2 we discuss symmetry and energy-conservation properties of the incompressible Navier-Stokes equations on continuous, semi-discrete and fully discrete level. In section 3 we construct the new POD-Galerkin method, which conserves mass, momentum and energy globally. In section 4 non-homogeneous boundary conditions are discussed. In section 5 the theoretical results are demonstrated for three cases: a shear layer roll-up, an actuator disk, and a lid-driven cavity flow.

2. Energy-conserving discretisation of the incompressible Navier-Stokes equations

In order to develop the ROM, the energy equalities of the FOM on the continuous and discrete level are needed. These are derived in this section.

2.1. Continuous energy estimate

The incompressible Navier-Stokes equations describe conservation of mass and momentum:

$$\nabla \cdot \mathbf{u} = 0, \quad (1)$$

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) = -\nabla p + \nu \nabla \cdot (\nabla \mathbf{u} + (\nabla \mathbf{u})^T), \quad (2)$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity field, $p(\mathbf{x}, t)$ is the reduced pressure, $\mathbf{x} \in \Omega \subset \mathbb{R}^d$ ($d = 2$ or 3), t denotes time, and ν the kinematic viscosity. The equations are supplemented with an initial condition

$$\mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad (3)$$

and boundary conditions, e.g. periodic boundary conditions or no-slip conditions

$$\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on} \quad \partial\Omega. \quad (4)$$

We introduce the convection and diffusion operators $C(\mathbf{u}, \mathbf{u}) := \nabla \cdot (\mathbf{u} \otimes \mathbf{u})$ and $D\mathbf{u} := \nabla \cdot (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$. Other forms of the convective operator are detailed in Appendix A.1.

To derive the kinetic energy equation, an inner product is needed. We choose the $L^2(\Omega)$ inner product and induced norm [8]:

$$(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, d\Omega, \quad \|\mathbf{u}\| := (\mathbf{u}, \mathbf{u})^{1/2}. \quad (5)$$

The kinetic energy is then defined as $K := \frac{1}{2}\|\mathbf{u}\|^2$. An equation for the evolution of K is derived by differentiating K in time and substituting the momentum equation:

$$2 \frac{dK}{dt} = \frac{d(\mathbf{u}, \mathbf{u})}{dt} = -(C(\mathbf{u}, \mathbf{u}), \mathbf{u}) - (\mathbf{u}, C(\mathbf{u}, \mathbf{u})) - (\nabla p, \mathbf{u}) - (\mathbf{u}, \nabla p) + (D\mathbf{u}, \mathbf{u}) + (\mathbf{u}, D\mathbf{u}). \quad (6)$$

The equation simplifies due to three symmetry considerations. First due to the skew-symmetry of $C(\mathbf{u}, \mathbf{u})$, we have $(C(\mathbf{u}, \mathbf{u}), \mathbf{u}) = 0$ for periodic or no-slip boundary conditions (see also Appendix A.1). Second, the

pressure gradient contribution disappears because $(\nabla p, \mathbf{u}) = (p, \nabla \cdot \mathbf{u}) = 0$. Third, due to the symmetry of the diffusive operator we can write $(D(\mathbf{u}, \mathbf{u}), \mathbf{u}) = -(\nabla \mathbf{u}, \nabla \mathbf{u})$. The kinetic energy balance then reduces to

$$\frac{dK}{dt} = -\nu \|\nabla \mathbf{u}\|^2. \quad (7)$$

Consequently, in viscous flow the kinetic energy of the flow can only decrease in time, and in inviscid flow it is conserved.

2.2. Spatial discretisation and semi-discrete energy equation

In order to construct a non-linearly stable ROM, we require that the spatial discretisation mimics the energy-conserving properties of the continuous equations just derived. To this end, we consider a finite volume discretisation on a staggered cartesian grid [14, 30, 34]. For simplicity, we restrict ourselves to a second-order method in two dimensions and partition the domain in $N_p = N_x \times N_y$ finite volumes. We introduce the (time-dependent) solution vectors $u_h(t) \in \mathbb{R}^{N_p}$, $v_h(t) \in \mathbb{R}^{N_p}$ and $p_h(t) \in \mathbb{R}^{N_p}$, which consist of the (time-dependent) unknowns $u_{i+1/2,j}(t)$, $v_{i,j+1/2}(t)$, and $p_{i,j}(t)$, respectively (for $i = 1 \dots N_x$, $j = 1 \dots N_y$). The explicit time-dependence is suppressed when no confusion can arise. The horizontal and vertical velocity components are gathered in the vector $V_h = \begin{pmatrix} u_h \\ v_h \end{pmatrix} \in \mathbb{R}^{N_V}$, with $N_V = 2N_p$.

We integrate the divergence-free constraint (1) over a finite volume centred around the unknown $p_{i,j}$, which yields

$$\bar{u}_{i+1/2,j} - \bar{u}_{i-1/2,j} + \bar{v}_{i,j+1/2} - \bar{v}_{i,j-1/2} = 0. \quad (8)$$

The notation $\bar{(\cdot)}$ indicates integration over a face of the finite volume, e.g. $\bar{u}_{i+1/2,j} = \Delta y u_{i+1/2,j}$. In matrix-vector notation, the above equation can be written for all pressure volumes as

$$M_h V_h = 0, \quad M_h \in \mathbb{R}^{N_p \times N_V}. \quad (9)$$

Next we integrate the horizontal component of the momentum equation over a finite volume centred around the unknown $u_{i+1/2,j}$. The convective term in divergence form is discretised by mesh-independent interpolation of the neighbouring fluxes (see Appendix A.2). The divergence form ensures that momentum is conserved. When ensuring that the velocity field is discretely divergence free, the divergence form can be rewritten into the skew-symmetric form (see Appendix A.2):

$$\begin{aligned} [C_h^u(V_h, u_h)]_{i+1/2,j} &:= \frac{1}{2} u_{i+3/2,j} \frac{1}{2} (\bar{u}_{i+1/2,j} + \bar{u}_{i+3/2,j}) - \frac{1}{2} u_{i-1/2,j} \frac{1}{2} (\bar{u}_{i-1/2,j} + \bar{u}_{i+1/2,j}) \\ &+ \frac{1}{2} u_{i+1/2,j+1} \frac{1}{2} (\bar{v}_{i,j+1/2} + \bar{v}_{i+1,j+1/2}) - \frac{1}{2} u_{i+1/2,j-1} \frac{1}{2} (\bar{v}_{i,j-1/2} + \bar{v}_{i+1,j-1/2}). \end{aligned} \quad (10)$$

As a consequence, the discretised convection operator is conserving both momentum and energy. A similar scheme can be derived for the vertical component. The full convection operator reads

$$C_h(V_h, V_h) = \begin{pmatrix} C_h^u(V_h, u_h) \\ C_h^v(V_h, v_h) \end{pmatrix} =: \tilde{C}_h(V_h) V_h. \quad (11)$$

The notation $\tilde{C}_h(V_h) V_h$ is useful for making a distinction between the *convecting* quantity (the quantity between brackets) and the *convected* quantity. The definition of \tilde{C}_h is possible due to the fact that the nonlinearity of the convective term is only quadratic. The skew-symmetry property can be expressed in terms of \tilde{C}_h as

$$\tilde{C}_h(V_h) = -\tilde{C}_h(V_h)^T. \quad (12)$$

The pressure gradient term in (2) is given by

$$G_h p_h, \quad \text{where} \quad G_h = -M_h^T. \quad (13)$$

No boundary conditions are required for the pressure (except on outflow boundaries) – they are implied by the boundary conditions for the velocity [33]. The diffusive operator is discretised by second order central approximations, and can be represented by

$$D_h V_h, \quad (14)$$

where D_h is a symmetric negative definite matrix, which can be written as $D_h = -Q_h^T Q_h$.

The semi-discrete system then reads

$$M_h V_h = 0, \quad (15)$$

$$\Omega_h \frac{dV_h}{dt} = F_h(V_h, p_h, t), \quad (16)$$

where $F_h(V_h, p_h, t)$ contains the convective, diffusive and pressure gradient contributions:

$$F_h(V_h, p_h, t) = -\tilde{C}_h(V_h)V_h - G_h p_h + \nu D_h V_h. \quad (17)$$

Ω_h is a (time-independent) diagonal matrix with the finite volume sizes on its diagonal, which is symmetric positive definite.

To arrive at a semi-discrete energy equation requires the choice of a discrete inner product, i.e. a discrete version of (5). The natural choice in a finite volume context is (for $V_h, W_h \in \mathbb{R}^{N_V}$):

$$(V_h, W_h)_h := V_h^T \Omega_h W_h, \quad \|V_h\|_h := (V_h, V_h)_h^{1/2}, \quad (18)$$

and the discrete energy is defined as $K_h := \frac{1}{2} \|V_h\|_h^2$. In absence of boundary contributions, the time evolution of K_h is given by

$$\begin{aligned} 2 \frac{dK_h}{dt} &= \frac{d}{dt} (V_h^T \Omega_h V_h) \\ &= -V_h^T (\tilde{C}_h(V_h)^T + \tilde{C}_h(V_h)) V_h - 2p_h^T M_h V_h - 2\nu \|Q_h V_h\|^2. \end{aligned} \quad (19)$$

Due to the skew-symmetry property of \tilde{C}_h (equation (12)) and the divergence-freeness of V_h (15), one obtains

$$\frac{dK_h}{dt} = -\nu \|Q_h V_h\|^2, \quad (20)$$

which is the semi-discrete counterpart of equation (7).

2.3. Time discretisation and fully discrete energy equation

We continue with the temporal discretisation of equations (15)-(16) with an implicit s -stage Runge-Kutta method [28]. The stage values follow from

$$M_h V_h^{n,i} = 0, \quad (21)$$

$$\Omega_h \frac{V_h^{n,i} - V_h^n}{\Delta t} = \sum_{j=1}^s a_{ij} (F_h(V_h^{n,j}, t^{n,j}) - G_h p_h^{n,j}), \quad (22)$$

and the solution at the next time step is a combination of the stage values:

$$M_h V_h^{n+1} = 0, \quad (23)$$

$$\Omega_h \frac{V_h^{n+1} - V_h^n}{\Delta t} = \sum_{i=1}^s b_i (F_h(V_h^{n,i}, t^{n,i}) - G_h p_h^{n,i}). \quad (24)$$

Here, V_h^n and p_h^n are approximations to $V_h(t^n)$ and $p_h(t^n)$ respectively, which will be collected into a snapshot matrix to be used in the ROM construction (detailed in the next section).

The coefficients a and b of the Runge-Kutta method are chosen such that the temporal discretisation keeps the energy conservation property in the inviscid limit. An example of Runge-Kutta methods that satisfy this property are the Gauss methods [28]. The lowest order Gauss method, obtained for $s = 1$, is the second order implicit midpoint method with the following Butcher tableau: $a_{11} = \frac{1}{2}$, $b_1 = 1$. For the Gauss methods, the fully discrete energy equation can be written as

$$\frac{K_h^{n+1} - K_h^n}{\Delta t} = -\nu \sum_{i=1}^s \|Q_h V_h^{n,i}\|^2, \quad (25)$$

and K_h^n is an approximation to $K_h(t^n)$. The careful choice of spatial and temporal discretisation methods has yielded energy equations (20) and (25) that closely mimic the continuous energy estimate (7). The fully discrete energy equation shows that, in the absence of boundary contributions, the energy of the solution can only decrease due to viscous dissipation, independent of the mesh, the time step, or the viscosity.

3. Energy-conserving POD-Galerkin method

3.1. Introduction

We will follow the ODE-based projection approach [7] in which the POD-Galerkin method is applied to the semi-discrete energy-conserving formulation, i.e. we project the FOM given by equations (15) - (16). This is not the only possibility; one can instead project the continuous equations (1) and (2) and then discretize the resulting system, see e.g. [18, 32]. We comment on the difference with this approach in section 3.6.

We make the Ansatz that the velocity field $V_h(t) \in \mathbb{R}^{N_V}$ can be approximated by

$$V_h(t) \approx V_r(t) := \Phi a(t), \quad (26)$$

where $\Phi \in \mathbb{R}^{N_V \times M}$, $a(t) \in \mathbb{R}^M$, and $M \ll N_V$. The subscript r denotes quantities associated to the ROM. Equation (26) is substituted into the FOM and then the equations are projected by left-multiplying with Φ^T . In the POD approach Φ is obtained by performing a singular-value decomposition (SVD) of a snapshot matrix X (this will be detailed below). X contains K snapshots of the velocity field V_h , i.e.

$$X = [V_h^1 \dots V_h^n \dots V_h^K], \quad (27)$$

where each velocity field is divergence free, so that $M_h X_j = 0$ for each column X_j of X . Φ is subject to the orthonormality condition

$$\Phi^T \Phi = I. \quad (28)$$

The snapshots are obtained from the solution of the fully discretised FOM, equations (23)-(24).

3.2. Construction of basis via weighted orthonormality condition

In this work, instead of condition (28), we use a weighted orthonormality condition, namely

$$\Phi^T \Omega_h \Phi = I. \quad (29)$$

This is consistent with equation (18) and with the form of the ROM momentum equation and the ROM kinetic energy equation, as we will demonstrate next. After substituting (26) in (16) and projecting with Φ^T , one obtains

$$\Phi^T \Omega_h \frac{d\Phi a(t)}{dt} = \Phi^T F_h(\Phi a(t)). \quad (30)$$

In other words, it is natural to require $\Phi^T \Omega_h \Phi = I_M$ (the identity matrix of dimension $M_V \times M_V$), as the ROM then simplifies to

$$\frac{da(t)}{dt} = \Phi^T F_h(\Phi a(t)). \quad (31)$$

With choice (29), the energy of the FOM is approximated by

$$K_h(t) \approx K_r(t) = \frac{1}{2}(\Phi a(t))^T \Omega_h \Phi a(t) = \frac{1}{2}a(t)^T \Phi^T \Omega_h \Phi a(t) = \frac{1}{2}a(t)^T a(t). \quad (32)$$

The choice (29) thus simplifies the expression for both momentum and energy considerably.

We now specify the construction of Φ . Given that the energy norm is chosen to be based on a weighted inner product that contains the finite volume sizes Ω_h , the basis Φ should be computed from the following minimization problem [1, 24]:

$$\Phi = \arg \min_{\Phi} \|(I - \Phi \Phi^T \Omega_h) X\|_F^2 \quad \text{subject to} \quad \Phi^T \Omega_h \Phi = I_M, \quad (33)$$

instead of the classical ‘unweighted’ minimization problem:

$$\hat{\Phi} = \arg \min_{\hat{\Phi}} \|(I - \hat{\Phi} \hat{\Phi}^T) X\|_F^2 \quad \text{subject to} \quad \hat{\Phi}^T \hat{\Phi} = I_M. \quad (34)$$

The solution of the weighted problem can be expressed in terms of the solution of the unweighted problem as follows. Let

$$\Phi = \Omega_h^{-1/2} \hat{\Phi}, \quad (35)$$

where $\hat{\Phi}$ follows from the SVD of the scaled snapshot matrix $\hat{X} = \Omega_h^{1/2} X$:

$$\hat{X} = \hat{\Phi} \Sigma \Psi^*. \quad (36)$$

Since Ω_h is a diagonal matrix, its matrix square root is trivial to compute. The dimensions of the matrices in the SVD are

$$\hat{\Phi}, \Phi \in \mathbb{R}^{N_v \times N_v}, \quad \Sigma \in \mathbb{R}^{N_v \times K}, \quad \Psi \in \mathbb{R}^{K \times K}. \quad (37)$$

The columns of $\hat{\Phi}$, denoted by $\hat{\Phi}_j$, are the eigenvectors of the correlation matrix $\hat{X} \hat{X}^T$, i.e.

$$\hat{X} \hat{X}^T \hat{\Phi}_j = \lambda_j \hat{\Phi}_j, \quad (38)$$

where the eigenvalues λ are related to the singular values σ (diagonal entries of Σ) by $\sqrt{\lambda_i(\hat{X} \hat{X}^T)} = \sigma_i(\hat{X})$.

The basis for the ROM is obtained by taking the first M_V columns of Φ . M_V is typically prescribed by analysing the decay of the singular values σ .

In summary, the sequence to obtain Φ is: gather snapshots of the velocity field in X ; compute \hat{X} ; compute the SVD of \hat{X} to get $\hat{\Phi}$; compute Φ ; truncate Φ .

3.3. Mass conservation of the ROM

It is well-known that the mass conservation equation is identically satisfied by the ROM approximation, if the boundary conditions are no-slip or periodic [21]. In a finite volume context, this is shown as follows. The divergence-free condition (15) becomes

$$M_h \Phi a(t) = 0. \quad (39)$$

Rewriting equation (38) yields

$$X X^T \Omega_h \Phi_j = \lambda_j \Phi_j. \quad (40)$$

Left-multiplying this equation with M_h and using that the snapshots collected in X are divergence-free ($M_h X_j = 0$) yields

$$\lambda_j M_h \Phi_j = 0. \quad (41)$$

In other words, the ROM velocity field V_r satisfies the divergence-free condition (39), independent of the value of the coefficients $a(t)$. For non-homogeneous boundary conditions this is however not the case and we will present a boundary condition treatment in section 4.

3.4. Momentum conservation of the ROM

The ROM momentum equation was given by equation (31):

$$\begin{aligned}\frac{da(t)}{dt} &= \Phi^T F_h(\Phi a(t)) \\ &= \Phi^T (-\tilde{C}_h(\Phi a(t))\Phi a(t) + \nu D_h \Phi_h a(t)).\end{aligned}\tag{42}$$

Note that the pressure gradient term disappears because the ROM velocity field is already divergence-free and the pressure gradient is linked to the divergence operator (equation (13)): $\Phi^T G_h = (G_h^T \Phi)^T = -(M_h \Phi)^T = 0$. Momentum is, unlike mass, not a locally conserved quantity. However, momentum *is* globally conserved in case of periodic boundary conditions (see Appendix B), and we want the ROM to satisfy this property. Define the FOM global momentum of each velocity component as

$$P_h^u(t) = e_u^T \Omega_h V_h(t),\tag{43}$$

$$P_h^v(t) = e_v^T \Omega_h V_h(t),\tag{44}$$

where $e_u, e_v \in \mathbb{R}^{N_v}$. e_u contains a 1 for indices associated with the u -velocity component, and e_v contains a 1 for indices associated with the v -velocity component, such that $e = e_u + e_v = [1, 1, \dots, 1]^T$. Evolution of the u -component of the FOM global momentum is given by

$$\frac{dP_h^u}{dt} = e_u^T \Omega_h \frac{dV_h(t)}{dt} = e_u^T F_h(V_h, t) = 0,\tag{45}$$

with a similar expression for P_h^v . This expression evaluates to zero because of the telescoping property of finite volume methods in combination with a periodic domain.

Evolution of global momentum predicted by the ROM reads

$$\frac{dP_h^u}{dt} \approx \frac{dP_h^u}{dt} = e_u^T \Omega_h \Phi \frac{da(t)}{dt} = e_u^T \Omega_h \Phi \Phi^T F_h(\Phi a(t)).\tag{46}$$

In order to obtain global conservation of momentum of the ROM, we want to enforce the basis vectors Φ to satisfy

$$e_u^T \Omega_h \Phi \Phi^T = e_u^T,\tag{47}$$

with a similar expression for e_v . In other words, the projection of the vectors e_u and e_v by $\Omega_h \Phi \Phi^T$ should be exact. When performing the SVD without truncation this property can be easily achieved by adding these vectors to the snapshot matrix, since a property of the SVD is that the projection of vectors in the snapshot matrix is exact. However, upon truncating the decomposition to arrive at a reduced dimension, this property is generally lost. We therefore propose to enforce property (47) via a novel constrained SVD approach. First, we collect the vectors that should be exactly projected by the truncated SVD in the matrix E :

$$E = [e_u \ e_v],\tag{48}$$

scaled to have norm equal to 1. Subsequently, we perform an update of the snapshot matrix X :

$$\tilde{X} = X - EE^T X,\tag{49}$$

and we determine its SVD

$$\tilde{X} = \tilde{\Phi} \tilde{\Sigma} \tilde{\Psi}.\tag{50}$$

Consequently, we add E to $\tilde{\Phi}$, and then we truncate:

$$\bar{\Phi} = [E \ \tilde{\Phi}]_{M_V},\tag{51}$$

where the subscript M_V indicates that the first M_V columns are used. The resulting $\bar{\Phi}$ satisfies equation (47) (and a similar equation for e_v). Note that, when M_V is given, enforcing global momentum comes at the price of losing two or three of the modes present in $\tilde{\Phi}$ (depending on the spatial dimension of the problem).

The initial condition for a is given by

$$a(0) = \Phi^T \Omega_h V_h(0). \quad (52)$$

Consequently, the initial momentum (u -component) is given by

$$P_r^u(0) = e_u^T \Omega_h \Phi a(0) = e_u^T \Omega_h \Phi \Phi^T \Omega_h V_h(0), \quad (53)$$

whereas the initial momentum of the FOM is

$$P_r^u(0) = e_u^T \Omega_h V_h(0). \quad (54)$$

The error between the two is

$$\epsilon_P(0) = e^T (I - \Omega_h \Phi \Phi^T) \Omega_h V_h(0) = 0, \quad (55)$$

when the constrained SVD is employed: the total momentum of the ROM is constant in time and equals the total momentum of the FOM, for the case of periodic boundary conditions.

3.5. Energy conservation of the ROM

One of the key questions in this paper is whether the kinetic energy of the ROM can be bounded in a similar way as the energy of the FOM (equations (7), (20), (25)). To this end, we differentiate the expression for the energy of the ROM, as given by equation (32), and simplify by using equations (39) and (42):

$$2 \frac{dK_r(t)}{dt} = \frac{da^T}{dt} a + a^T \frac{da}{dt} \quad (56)$$

$$= -(\Phi^T \tilde{C}_h(\Phi a) \Phi a)^T a - a^T (\Phi^T \tilde{C}_h(\Phi a) \Phi a) + \nu (\Phi^T D \Phi a)^T a + a^T \nu \Phi^T D_h \Phi a \quad (57)$$

$$= -a^T \Phi^T (\tilde{C}_h(\Phi a))^T + \tilde{C}_h(\Phi a) \Phi a - 2\nu \|Q_h \Phi a\|^2 \quad (58)$$

$$= -2\nu \|Q \Phi a\|^2. \quad (59)$$

The crucial steps in the derivation are the properties of the spatial discretisation operators: $G_h^T = -M_h$; $\tilde{C}_h(\Phi a)$ is skew-symmetric; D_h is symmetric negative definite. In summary, the energy evolution of the ROM in absence of boundary contributions is given by

$$\boxed{\frac{dK_r(t)}{dt} = -\nu \|Q_h \Phi a\|^2.} \quad (60)$$

Consequently, the ROM is stable, independent of the number of POD modes used.

In the inviscid limit, we have

$$K_r(t) = K_r(0), \quad (61)$$

where

$$K_r(0) = \frac{1}{2} a(0)^T a(0) = \frac{1}{2} (\Phi^T \Omega_h V_h(0))^T \Phi^T \Omega_h V_h(0) = \frac{1}{2} V_h(0)^T \Omega_h \Phi \Phi^T \Omega_h V_h(0). \quad (62)$$

The exact energy is given by

$$K_h(0) = \frac{1}{2} V_h(0)^T \Omega V_h(0), \quad (63)$$

and the error in the ROM (due to truncation) is therefore given by

$$\epsilon_K = \frac{1}{2} V(0)_h^T (I - \Omega_h \Phi \Phi^T) \Omega_h V_h(0). \quad (64)$$

Similar to the addition of the global momentum constraint, it is possible to add $V_h(0)$ to the truncated SVD so that it is projected exactly by $\Omega_h \Phi \Phi^T$. This guarantees that in the inviscid case the kinetic energy of the ROM remains equal to the FOM. However, for viscous simulations, the ROM will not reproduce the kinetic energy evolution exactly (compare equation (60) to (20)).

The last step in obtaining the ROM is to specify a time discretisation for equation (31) such that a fully discrete equivalent of (32) is obtained. The key is, not surprisingly, to use the energy-conserving Runge-Kutta time discretisation methods introduced in section 2.2. For example, the implicit midpoint method applied to (31) reads

$$\frac{a^{n+1} - a^n}{\Delta t} = \Phi^T F_h(\Phi a^{n+1/2}), \quad (65)$$

where $a^{n+1/2} = \frac{1}{2}(a^n + a^{n+1})$. The corresponding energy evolution is

$$\boxed{\frac{K_r^{n+1} - K_r^n}{\Delta t} = -\nu \|Q_h \Phi a^{n+1/2}\|^2}, \quad (66)$$

which is strictly decreasing in time when the viscosity ν is nonzero, and hence the fully discrete solution is stable.

3.6. Remarks

Some remarks are in place considering the unconditionally non-linearly stable ROMs proposed in the previous sections:

- The derivation is valid for homogeneous (no-slip, periodic) boundary conditions. For more generic boundary conditions, such as inflow conditions, the energy of the flow is not strictly decreasing in time. In this case, the ROM estimate should be such that it mimics the energy estimate of the FOM (see e.g. [30] for the boundary contributions to the energy equation). Further details on general boundary conditions are given in section 4.
- First discretising in space, and then performing the projection of the semi-discrete equations (instead of first projecting and then discretising the resulting equations, as performed for example in [32]) has several advantages: (i) the treatment of boundary conditions is straightforward, and only has to be done once (when spatially discretising the FOM); and (ii) the inf-sup condition does not need to be satisfied by the ROM.
- The energy-conserving property expressed by (60) or (66) is *independent* of whether the snapshot matrix has been generated using an energy-conserving discretisation method. The only condition on the snapshot matrix is that the snapshots are divergence-free.
- From an implementation point of view, the implicit time integration methods discussed here (on both the FOM and ROM level) might seem daunting at first sight. In a practical situation, however, we can advise the following strategy. First, solve the FOM with a time integration method of choice (e.g. explicit, IMEX, etc.) – this does not affect energy conservation of the ROM as long as the spatial discretization is energy-conserving (see previous remark). Second, use linear stability theory to estimate the eigenvalues of the ROM operator and use this to determine an efficient time integrator and time step for the ROM. Although in this second step the non-linear stability property will be lost, linear stability of the ROM in combination with adaptive time stepping could be an efficient solution in a practical situation. Of course, this depends on the application under consideration.
- Energy-conservation is obtained because the skew-symmetry property of the convective term is unchanged upon projection. However, this means that the POD-Galerkin method in its basic form is not suitable as a reduced model for turbulent flows, as there is no additional energy dissipation coming from projecting the convective terms.
- Although the pressure is not part of the solution, it can be obtained as a post-processing step once the velocity field is known by solving a Poisson equation.

4. Non-homogeneous boundary conditions

In this section we extend the results of the previous section to the more generic case of (stationary) non-homogeneous boundary conditions and forcing terms, which is known to be non-trivial (see for example [22, 17, 36]). In this case, the semi-discrete equations (15)-(16) read:

$$M_h V_h(t) = b_h, \quad (67)$$

$$\Omega_h \frac{dV_h(t)}{dt} = -\tilde{C}_h(V_h(t))V_h(t) - G_h p_h(t) + \nu D_h V_h(t) + r_h. \quad (68)$$

where the terms $b_h \in \mathbb{R}^{N_p}$ and $r_h \in \mathbb{R}^{N_v}$ are described for example in [30]. This formulation holds for both inflow, outflow, symmetry, periodic, and no-slip conditions, and also encompasses the case of body forces.

We approximate the FOM velocity field with a ROM with homogeneous boundary conditions and a term that incorporates the boundary conditions [13]:

$$V_h(t) \approx V_r(t) + V_{bc} = \Phi a(t) + V_{bc}. \quad (69)$$

V_{bc} is chosen such that

$$M_h(V_r(t) + V_{bc}) = b_h, \quad (70)$$

which reduces to $M_h V_{bc} = b_h$, since $V_r(t)$ satisfies homogeneous boundary conditions. The solution for V_{bc} follows as

$$V_{bc} = G_h \zeta_h, \quad \text{where} \quad M_h G_h \zeta_h = b_h. \quad (71)$$

Thus, the solution of one Poisson equation is needed to find the boundary terms V_{bc} . Given V_{bc} , the snapshot matrix (27) is changed into

$$X = [V_h^1 - V_{bc} \dots V_h^n - V_{bc} \dots V_h^K - V_{bc}], \quad (72)$$

which satisfies $M_h X_j = 0$ for each column j of X , and the projection matrix Φ thus satisfies $M_h \Phi_j = 0$. Consequently, with this construction the pressure gradient term still disappears from the momentum equation (like in the homogeneous case) and reads

$$\frac{da(t)}{dt} = \Phi^T \left(-\tilde{C}_h(\Phi a(t) + V_{bc})(\Phi a(t) + V_{bc}) + \nu D_h(\Phi a(t) + V_{bc}) \right). \quad (73)$$

5. Results

In this section we show the results of three test cases. In the first test case, we demonstrate the stability and energy conservation properties of the ROM through an inviscid simulation of a shear-layer roll-up. In the second test case, we demonstrate the treatment of non-homogeneous boundary conditions by simulating the flow through an actuator disk. In the third test case we consider the simulation of a lid-driven cavity, where we show that even though our POD-Galerkin method is *stable*, the quest for *accuracy* is still ongoing, as inaccuracies appear when simulating beyond the training data of the snapshot matrix.

5.1. Shear-layer roll-up

We simulate the roll-up of a shear-layer, similar to [30]. The simulation domain is $[0, 2\pi] \times [0, 2\pi]$, with periodic boundary conditions and the following initial condition:

$$u_0(x, y) = 1 + \begin{cases} \tanh\left(\frac{y-\pi/2}{\delta}\right), & y \leq \pi, \\ \tanh\left(\frac{3\pi/2-y}{\delta}\right), & y > \pi, \end{cases} \quad v_0(x, y) = \epsilon \sin(x). \quad (74)$$

Compared to [30], a constant has been added to $u_0(x, y)$, in order to ensure that the global momentum of the u - and v - components differ. In the inviscid case, the energy of the flow should be exactly conserved. The FOM discretisation consists of 200×200 finite volumes, giving a total of $N_V + N_p = 1.2 \cdot 10^5$ unknowns.

Time integration of the FOM is performed with explicit RK4 with a time step of $\Delta t = 0.01$ from $t = 0$ to $t = 4$ (as mentioned in section 3.6, the FOM snapshots need not be energy-conserving). Time integration of the ROM is performed with the implicit midpoint method, with the same Δt and end time as used for the FOM.

The singular values of the velocity snapshot matrix are shown in figure 2. The ROM basis consists of the first M_V left singular vectors of the snapshot matrix, where we take $M_V = 2, 4, 8, 16$. The rapid decay in the singular values indicates that the problem is suited for dimension reduction. The effect of using the proposed momentum-conserving SVD instead of the standard SVD is a small shift in the singular values.

Independent of whether the standard SVD or the momentum-conserving SVD is used, the ROM is conserving kinetic energy. This is shown in figure 3a. In all cases it holds that K_r remains exactly equal to the initial energy $K_r(0)$, so that the energy error remains constant in time. The error shown is due to the error in approximating (projecting) the initial FOM velocity field onto the truncated snapshot basis. We observe that, especially for small M_V , the momentum conserving approach is less accurate in terms of energy error. This is because two modes have been sacrificed in order to achieve momentum conservation. For the case $M_V = 2$ this means that momentum is enforced, but that the FOM snapshots are not taken into account in the basis Φ .

In figure 3b the momentum error of the u -component is plotted as a function of time. For the standard SVD, the error in global momentum increases as a function of time, and decreases when more modes are taken. With the new momentum-conserving SVD, the error in momentum stays at machine precision, independent of the number of modes.

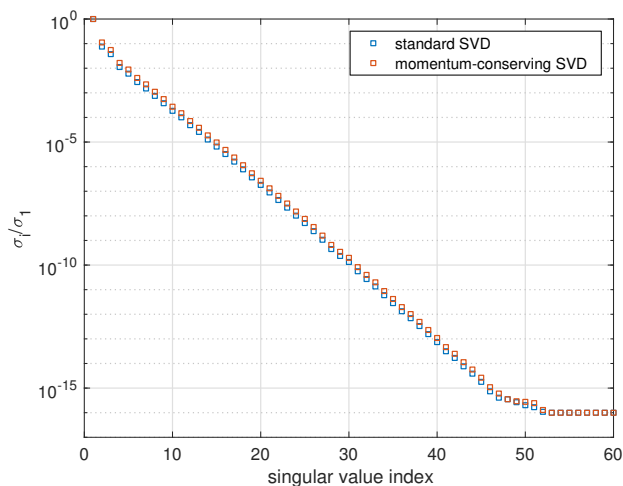


Figure 2: Singular values for inviscid shear-layer roll-up.

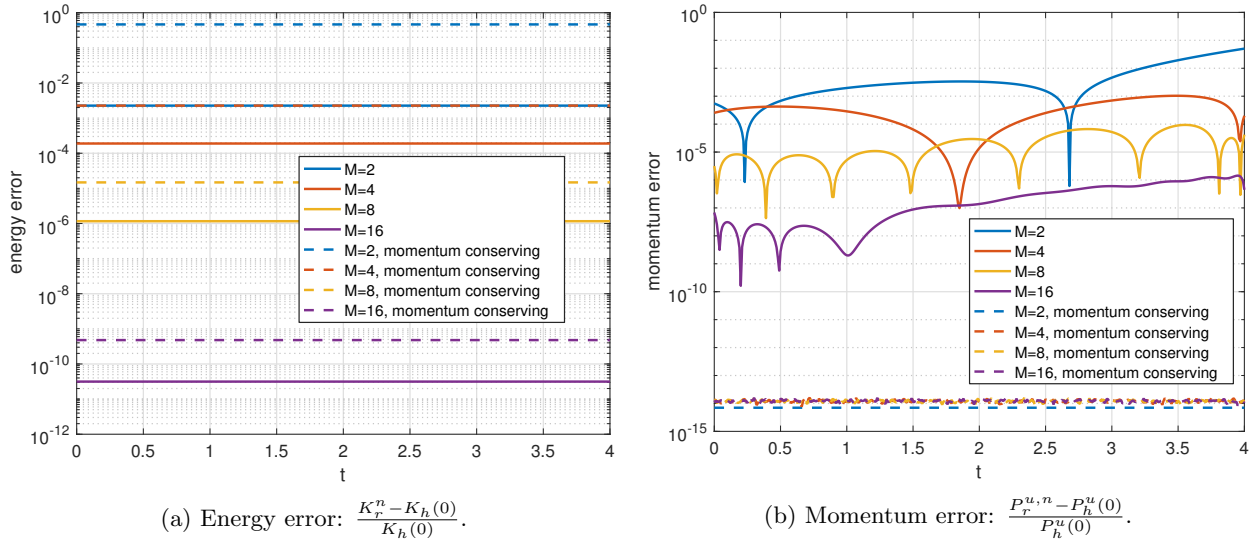


Figure 3: Energy and momentum conservation of ROM for inviscid shear-layer roll-up.

5.2. Actuator in non-uniform inflow

In this test case we consider an actuator disk in a non-uniform flow field. The actuator disk concept is typically used to model the flow through wind turbines [29]. This test case features non-homogeneous boundary conditions and we employ the method proposed in section 4.

The test case set up is as follows. We consider a simulation domain $[-4, 4] \times [-2, 2]$ with the following inflow conditions at $x = -4$:

$$u(x = -4, y) = \frac{3}{4} - \frac{3}{32}(y - 2)(y + 2). \quad (75)$$

This is a parabolic velocity profile with a mean equal to 1. At the domain boundaries $x = 4$, $y = -2$, and $y = 2$ we employ outflow conditions. The initial condition is the parabolic velocity profile. The Reynolds number is 100, and the thrust coefficient of the actuator is $C_T = \frac{1}{2}$; for details, see [27].

We first simulate the FOM with 80×40 finite volumes from $t = 0$ to $t = 10$ with $\Delta t = 0.02$ and a classic RK4 scheme. The resulting velocity field at $t = 10$ is shown in figure 4a. Based on the FOM, the velocity field due to nonhomogeneous boundary conditions, denoted V_{bc} and given by (69), is computed from equation (71). Note that V_{bc} is a vector field defined throughout the domain, and not on the boundary. The components of V_{bc} are shown in figure 5. The V_{bc} field is subtracted from the snapshot matrix. We then simulate the ROM with $M_V = 10$ modes, and the same time integration method and time step. Figure 4b shows the velocity field, which is almost identical to the one obtained by the FOM. A quantitative comparison is given in figure 6 in terms of the velocity error and the error in mass conservation. The velocity error at each time instant t^n is defined as

$$\epsilon_V^n = \|V_r^n - V_h^n\|_\infty. \quad (76)$$

As theoretically derived, the velocity field of the ROM remains exactly divergence-free, and the error in the velocity field remains smaller than 10^{-2} for almost the entire simulation.

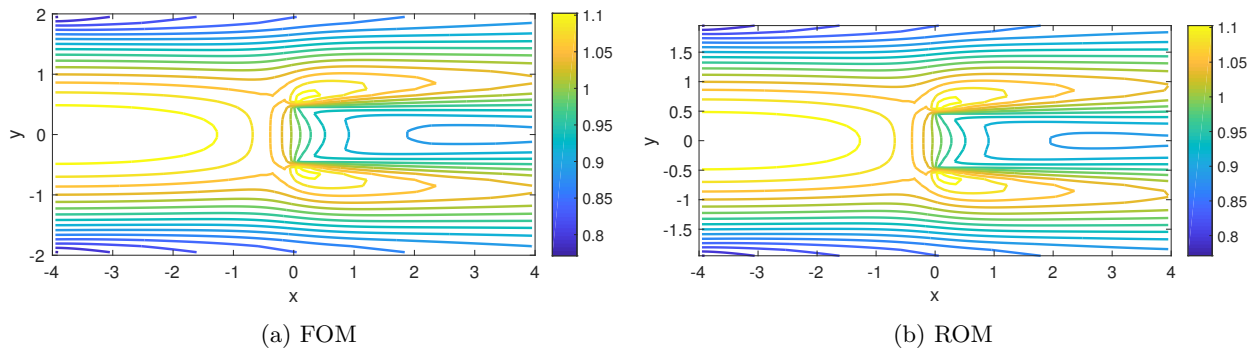


Figure 4: Velocity field through actuator disk at $t = 10$.

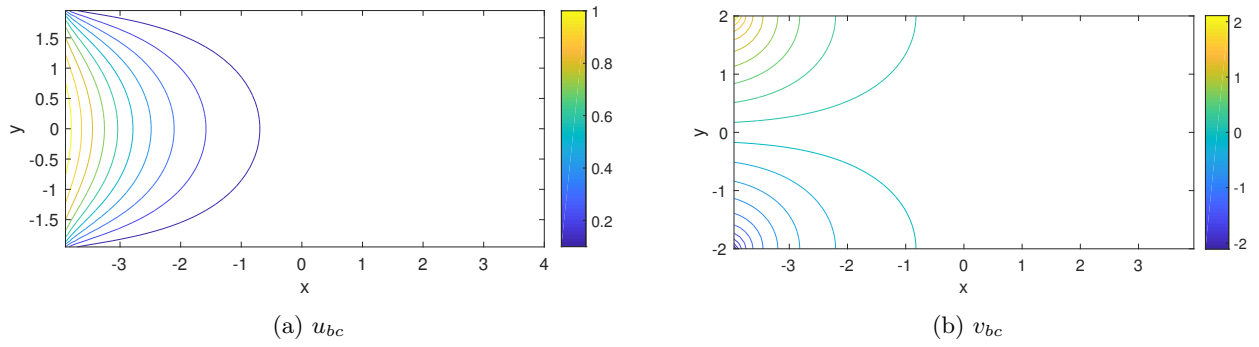


Figure 5: Components of boundary condition function V_{bc} for actuator test case.

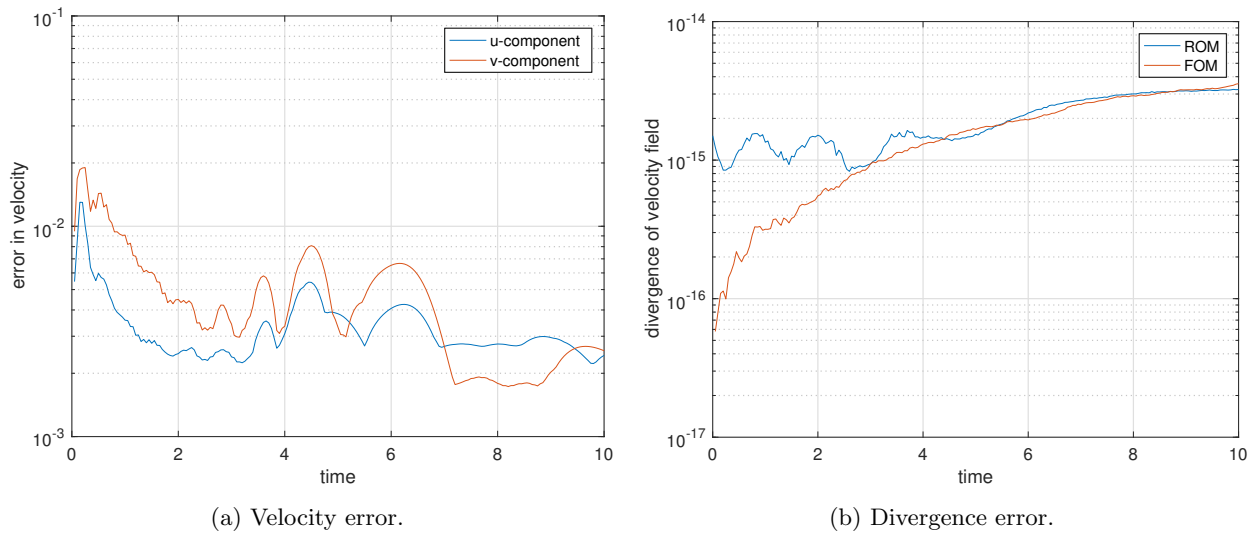


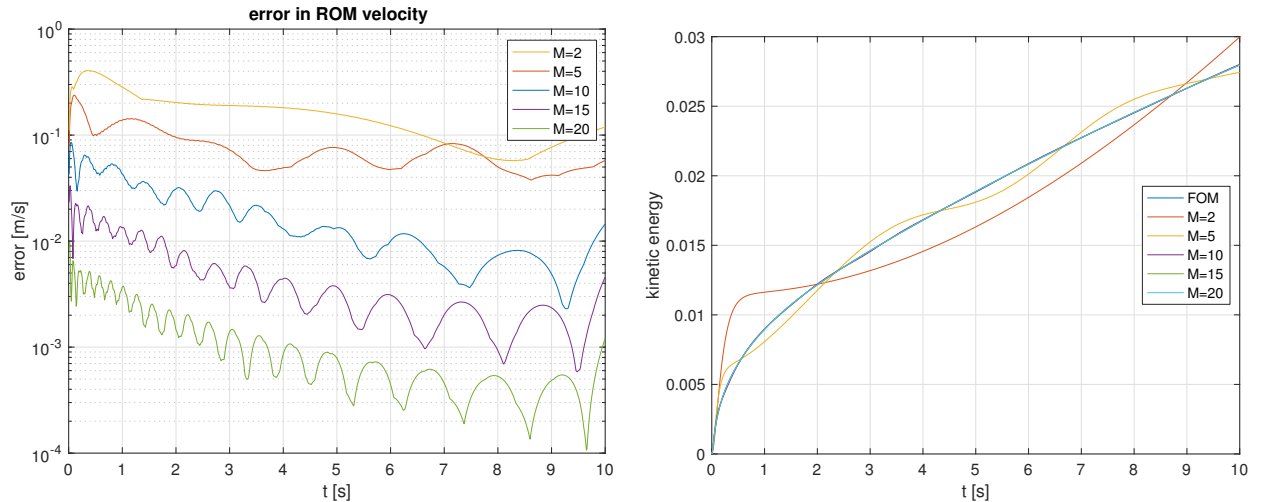
Figure 6: Error in velocity field and in mass conservation.

5.3. Lid-driven cavity

We repeat the test case performed in [32]: a lid-driven cavity flow at $Re = 1,000$. The velocity of the lid enters as boundary contribution in r_h , but does not appear in b_h , since b_h only contains velocity components normal to the boundary. Consequently, in this test case the procedures described in section 4 are not required. In contrast to [32], no measures need to be taken to ensure stability of the ROM.

The full-order model is run on a grid with 100×100 volumes, resulting in $N_V + N_p = 3 \cdot 10^4$ unknowns, at a time step of $\Delta t = 0.01$, until a final time $T = 10$. This results in $K = 1,000$ snapshots. The error with respect to the FOM is given by equation (76).

First, we study the effect of increasing the number of modes on the accuracy of the velocity field, while using the full snapshot set as basis for the SVD; see figure 7a. We clearly see how the accuracy increases when increasing the number of modes. The kinetic energy evolution as a function of number of modes is shown in figure 7b. Note that the kinetic energy increases as a function of time in this problem, as kinetic energy is added to the flow through the moving lid, which is initially larger than the energy dissipation in the interior (when the flow reaches a steady state, the two effects balance each other).



(a) Error in ROM with respect to the FOM as a function of time for different number of modes.

(b) Kinetic energy as a function of time for different number of modes.

Figure 7: Effect of number of modes on the solution of the lid-driven cavity flow.

Secondly, we study how well the ROM generalizes: we use only a part of the full snapshot set as basis for the SVD. We run the simulation until $T = 20$, and investigate the accuracy of the ROM when limiting the snapshot to the interval $[0, T]$, with $T = 5$ and $T = 10$. In all cases we take 10 modes and $\Delta t = 0.01$ for both the FOM and the ROM. Figure 8 shows that the ROM is poor in extrapolating outside the interval in which it was ‘trained’. In case of $T = 5$, the error starts to grow significantly for $t > 5$, and similar behaviour is seen for the case $T = 10$. On the other hand, in the time interval in which the models are trained, the models that are trained based on a subset of the snapshot data perform better. This can be explained by the fact that, given a fixed number of modes, it is easier for the reduced basis to lead to accurate results when a smaller time interval is considered.

Lastly, we study the effect of the Reynolds number on the error behaviour. We choose a lower Reynolds number, $Re = 100$, at which diffusive effects are more important. It is known that when the effect of diffusion becomes more important, ROMs are typically more accurate due to the faster decay of the singular values of the SVD, as indicated in figure 9b. Figure 9a indeed confirms that the ROM is more accurate for the lower Reynolds number case, and that the error decreases faster upon increasing the number of modes.

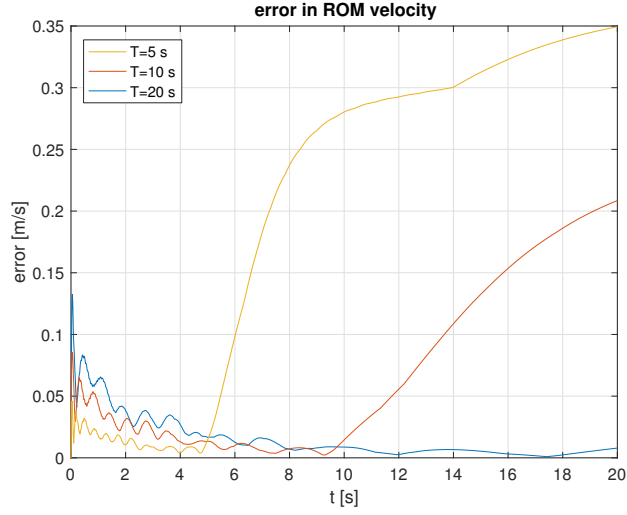
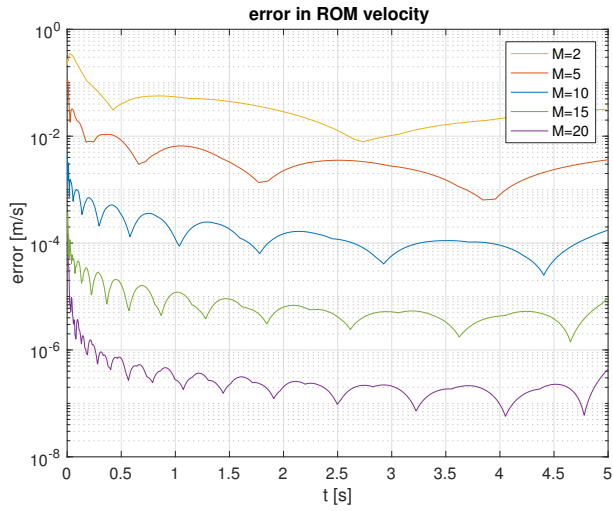
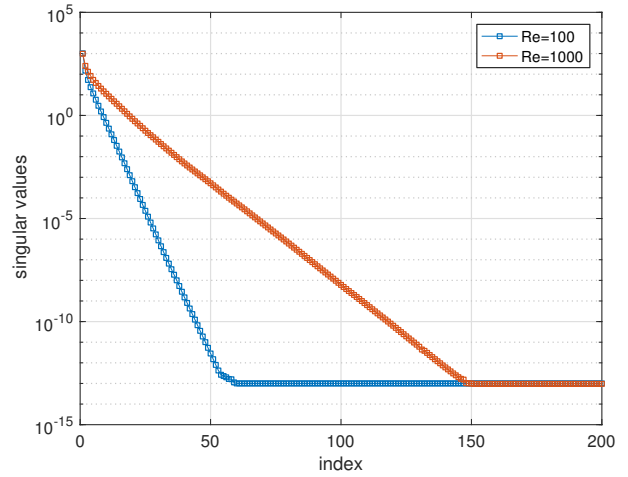


Figure 8: Error in ROM for lid-driven cavity flow with snapshot data gathered in the interval $[0, T]$.



(a) Error in ROM at $Re = 100$ for different number of modes.



(b) Singular value decay.

Figure 9: Effect of Reynolds number.

6. Conclusions

In this paper we have proposed a novel approach to arrive at unconditionally stable reduced-order models (ROM) for the incompressible Navier-Stokes equations. The approach hinges on the following four ingredients. First, we have expressed non-linear stability through kinetic energy conservation. Second, we have used a spatially energy-conserving discretization method as the full order model (FOM). Third, we have performed the projection of the full order model *after* spatial discretization, giving an unconditionally stable semi-discrete ROM (*first discretize, then project*). Last, we have used an energy-conserving time integration method that keeps the ROM solution stable when marching in time. The stability of the method has been shown for the roll-up of an inviscid shear-layer, for which exact energy conservation was obtained with both the FOM and the ROM.

In addition, we have derived a new constrained SVD approach that guarantees momentum conservation on periodic domains. Enforcing momentum conservation comes at the cost of losing a few modes (2 or 3) in the projection matrix, which can be accounted for by taking a few extra modes at a slight increase in computational effort. The constrained SVD approach can be easily extended to include other constraints apart from global momentum.

Furthermore, we have derived a boundary condition treatment for non-homogeneous boundary conditions. The adage of *first discretizing, then projecting* simplifies the boundary condition treatment considerably compared to conventional approaches, as the boundary conditions are built into the discretization operators. To avoid the solution of a Poisson equation at each time step, the velocity field is written in terms of a field with homogeneous boundary conditions and a non-homogeneous term, where the latter is obtained by solving (only once) a Poisson equation at the FOM level.

As mentioned, this paper has focused mainly on the stability of ROMs, and less so on the issue accuracy. Our view is that the proposed framework can serve as stepping stone to further improve the accuracy of ROMs obtained with the POD-Galerkin approach through the development of techniques such as closure models.

Acknowledgements

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Appendix A. Alternative forms of the convective operator

Appendix A.1. Continuous

The convective operator in divergence form can be written such that the role of the advective velocity $\mathbf{c} = \mathbf{u}$ becomes more clear:

$$C_{\text{div}}(\mathbf{c}, \mathbf{u}) = \nabla \cdot (\mathbf{c} \otimes \mathbf{u}), \quad (\text{A.1})$$

This distinction allows us to write the divergence form in terms of the advective form C_{adv} as follows:

$$C_{\text{div}}(\mathbf{c}, \mathbf{u}) = C_{\text{adv}}(\mathbf{c}, \mathbf{u}) + \mathbf{u} (\nabla \cdot \mathbf{c}), \quad (\text{A.2})$$

where

$$C_{\text{adv}}(\mathbf{c}, \mathbf{u}) = (\mathbf{c} \cdot \nabla) \mathbf{u}. \quad (\text{A.3})$$

Another commonly used form is the so-called skew-symmetric form,

$$C_{\text{skew}}(\mathbf{c}, \mathbf{u}) := \frac{1}{2} C_{\text{div}}(\mathbf{c}, \mathbf{u}) + \frac{1}{2} C_{\text{adv}}(\mathbf{c}, \mathbf{u}) = \frac{1}{2} \nabla \cdot (\mathbf{c} \otimes \mathbf{u}) + \frac{1}{2} (\mathbf{c} \cdot \nabla) \mathbf{u}. \quad (\text{A.4})$$

In case that the advective velocity field is divergence-free ($\nabla \cdot \mathbf{c} = 0$) and the velocity field is sufficiently smooth, the concepts of divergence, advective and skew-symmetric form are equivalent:

$$C(\mathbf{c}, \mathbf{u}) = C_{\text{div}}(\mathbf{c}, \mathbf{u}) = C_{\text{adv}}(\mathbf{c}, \mathbf{u}) = C_{\text{skew}}(\mathbf{c}, \mathbf{u}). \quad (\text{A.5})$$

The notion of skew-symmetry is related to the following property, independent of the divergence-freeness of \mathbf{c} :

$$\begin{aligned} C_{\text{skew}}(\mathbf{c}, \mathbf{u}) \cdot \mathbf{v} &= \frac{1}{2}(\nabla \cdot (\mathbf{c} \otimes \mathbf{u})) \cdot \mathbf{v} + \frac{1}{2}((\mathbf{c} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v} \\ &= \frac{1}{2} [((\mathbf{c} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v} + (\mathbf{u} \cdot \mathbf{v})\nabla \cdot \mathbf{c}] + \frac{1}{2} [\nabla \cdot ((\mathbf{u} \cdot \mathbf{v})\mathbf{c}) - (\mathbf{u} \cdot \mathbf{v})\nabla \cdot \mathbf{c} - ((\mathbf{c} \cdot \nabla)\mathbf{v}) \cdot \mathbf{u}] \quad (\text{A.6}) \\ &= \frac{1}{2}((\mathbf{c} \cdot \nabla)\mathbf{u}) \cdot \mathbf{v} - \frac{1}{2}((\mathbf{c} \cdot \nabla)\mathbf{v}) \cdot \mathbf{u} + \frac{1}{2}\nabla \cdot ((\mathbf{u} \cdot \mathbf{v})\mathbf{c}) \end{aligned}$$

Upon integration over the entire domain, the contribution of the last term cancels in case of periodic or no-slip boundary conditions, and we obtain

$$(C_{\text{skew}}(\mathbf{c}, \mathbf{u}), \mathbf{v}) = \frac{1}{2}((\mathbf{c} \cdot \nabla)\mathbf{u}, \mathbf{v}) - \frac{1}{2}((\mathbf{c} \cdot \nabla)\mathbf{v}, \mathbf{u}). \quad (\text{A.7})$$

The convective operator in skew-symmetric form is skew-symmetric ‘a priori’ (i.e. without the assumption that $\nabla \cdot \mathbf{c} = 0$),

$$(C_{\text{skew}}(\mathbf{c}, \mathbf{u}), \mathbf{v}) = -(\mathbf{u}, C_{\text{skew}}(\mathbf{c}, \mathbf{v})). \quad (\text{A.8})$$

The convective operators in advective or divergence form are skew-symmetric provided that $\nabla \cdot \mathbf{c} = 0$. In that case, we have for example

$$(C_{\text{div}}(\mathbf{c}, \mathbf{u}), \mathbf{v}) = -(\mathbf{u}, C_{\text{div}}(\mathbf{c}, \mathbf{v})). \quad (\text{A.9})$$

Appendix A.2. Discrete

In two dimensions, the integral of $C_{\text{div}} = \frac{\partial u^2}{\partial x} + \frac{\partial uv}{\partial y}$ over a finite volume surrounding $u_{i+1/2,j}$ is approximated by

$$C_{\text{div}}^u(V_h, u_h)_{i+1/2,j} := \bar{u}_{i+1,j}u_{i+1,j} - \bar{u}_{i,j}u_{i,j} + \bar{v}_{i+1/2,j+1/2}u_{i+1/2,j+1/2} - \bar{v}_{i+1/2,j-1/2}u_{i+1/2,j-1/2}. \quad (\text{A.10})$$

When interpolating the velocities by mesh-independent weighting of the neighbouring velocities (e.g. $u_{i+1,j} = \frac{1}{2}(u_{i+1/2,j} + u_{i+3/2,j})$), leaving the interpolation of the fluxes ($\bar{\cdot}$) still unspecified, this convective term can be expressed in terms of a matrix-vector product as

$$C_{\text{div}}^u(V_h, u_h) = \tilde{C}_{\text{div}}^u(V_h)u_h, \quad (\text{A.11})$$

where (focusing on the u -velocities)

$$\tilde{C}_{\text{div}}^u(V_h) = \frac{1}{2} \begin{pmatrix} \ddots & & & & & & \\ & \ddots & & & & & \\ & & -\bar{u}_{i,j} & \bar{u}_{i+1,j} - \bar{u}_{i,j} & & & \\ & & & -\bar{u}_{i+1,j} & \bar{u}_{i+2,j} - \bar{u}_{i+1,j} & \bar{u}_{i+2,j} & \\ & & & & & \ddots & \\ & & & & & & \ddots & \end{pmatrix} \quad (\text{A.12})$$

This is possible because the nonlinearity in the convective terms is only quadratic. We note that, apart from the diagonal elements, the matrix is skew-symmetric, independent of the interpolation method for the fluxes \bar{u} , \bar{v} . Subsequently, the fluxes are computed via mesh-independent weighting of the neighbouring velocities (e.g. $\bar{u}_{i+1,j} = \frac{1}{2}(\bar{u}_{i+1/2,j} + \bar{u}_{i+3/2,j})$). The convective terms can then be rewritten as

$$\begin{aligned} \frac{1}{2}u_{i+1/2,j} &\left[\frac{1}{2}(\bar{u}_{i+1/2,j} + \bar{u}_{i+3/2,j}) - \frac{1}{2}(\bar{u}_{i-1/2,j} + \bar{u}_{i+1/2,j}) + \frac{1}{2}(\bar{v}_{i,j+1/2} + \bar{v}_{i+1,j+1/2}) - \frac{1}{2}(\bar{v}_{i,j-1/2} + \bar{v}_{i+1,j-1/2}) \right] \\ &+ \frac{1}{2}u_{i+3/2,j} \frac{1}{2}(\bar{u}_{i+1/2,j} + \bar{u}_{i+3/2,j}) - \frac{1}{2}u_{i-1/2,j} \frac{1}{2}(\bar{u}_{i-1/2,j} + \bar{u}_{i+1/2,j}) \\ &+ \frac{1}{2}u_{i+1/2,j+1} \frac{1}{2}(\bar{v}_{i,j+1/2} + \bar{v}_{i+1,j+1/2}) - \frac{1}{2}u_{i+1/2,j-1} \frac{1}{2}(\bar{v}_{i,j-1/2} + \bar{v}_{i+1,j-1/2}), \quad (\text{A.13}) \end{aligned}$$

where the term between brackets $[\cdot]$ is zero as long as the continuity equation is satisfied.

Appendix B. Conservation of momentum

Integration of the incompressible Navier-Stokes equations over a stationary domain Ω with boundary $\partial\Omega$ yields

$$\frac{d}{dt} \int_{\Omega} \mathbf{u} \, d\Omega + \int_{\partial\Omega} \mathbf{u} \mathbf{u} \cdot \mathbf{n} \, dS = - \int_{\partial\Omega} p \mathbf{n} \, dS + \int_{\partial\Omega} \nu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \cdot \mathbf{n} \, dS. \quad (\text{B.1})$$

We define total momentum as

$$\mathbf{P}(t) = \int_{\Omega} \mathbf{u} \, d\Omega. \quad (\text{B.2})$$

On periodic domains, all boundary integrals vanish and we obtain

$$\frac{d\mathbf{P}(t)}{dt} = 0, \quad (\text{B.3})$$

so that momentum is exactly conserved. In case of no-penetration (or no-slip) boundary conditions ($\mathbf{u} \cdot \mathbf{n} = 0$), only the convective contribution vanishes, and we are left with

$$\frac{d\mathbf{P}(t)}{dt} = - \int_{\partial\Omega} p \mathbf{n} \, dS + \int_{\partial\Omega} \nu (\nabla \mathbf{u} + (\nabla \mathbf{u})^T) \cdot \mathbf{n} \, dS. \quad (\text{B.4})$$

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