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SOME PROBLEMS ON LOG-CONVEX APPROXIMATION OF CERTAIN INTEGRALS

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Some problems on log-convex approximation of certain integrals

by

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ABSTRACT

In this paper we establish some convexity properties in n of the sums

$$U_n(s) = \frac{1}{n} \sum_{k=1}^{n} \left(\frac{k}{n}\right)^s \qquad (s > 0)$$

and

$$T_n(s) = U_n(s) - \frac{1}{2n} \qquad (s > 1).$$

A conjecture is formulated which implies *inter alia*

$$T_n^2(s) < T_{n-1}(s)\, T_{n+1}(s) \qquad (n \geq 2).$$

KEY WORDS & PHRASES: *Convexity, approximations.*

1. APPROXIMATION OF $\int_0^1 x^s\,dx$ BY RIEMANN UPPER SUMS.

In [2] the first named author proved that the canonical Riemann upper sums

$$U_n(s) := \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^s$$

corresponding to the integral $\int_0^1 x^s\,dx$, where s is fixed and positive, tend *decreasingly* to the limit $1/(s+1)$ as $n \to \infty$.

Doornbos [4; pp. 254-255] proved this statement very elegantly in a more direct way whereas van Lint [4; pp. 255-256] showed that the statement is a special case of a more general theorem.

In spite of all these proofs we present here two more proofs.

FIRST PROOF. We write $\sigma_n(s) = \Sigma_{k=1}^{n} k^s$ and want to show that

$$\frac{\sigma_n(s)}{n^{s+1}} > \frac{\sigma_{n+1}(s)}{(n+1)^{s+1}}, \qquad (n \geq 1).$$

After crossmultiplication we see that we may just as well show that

(1) $$(n+1)\sum_{k=1}^{n}(k(n+1))^s - n\sum_{k=1}^{n+1}(kn)^s > 0.$$

Observe that the left hand side of (1) may be written as

$$\sum_{k=1}^{n}\{k((n-k+1)(n+1))^s - n((n-k+1)n)^s + (n-k)((n-k)(n+1))^s\}$$

from which it is clear that it suffices to show that

(2) $$k((n-k+1)(n+1))^s + (n-k)((n-k)(n+1))^s > n((n-k+1)n)^s$$

for $1 \leq k \leq n-1$.

Obviously (2) may be written as

$$\frac{k((n-k+1)(n+1))^s + (n-k)((n-k)(n+1))^s}{n} > ((n-k+1)n)^s$$

so that, by the arithmetic-geometric-mean-inequality (A ≥ G), it suffices to show that

$$((n-k+1)(n+1))^{ks}((n-k)(n+1))^{(n-k)s} > ((n-k+1)n)^{ns}$$

or, equivalently

$$((n-k+1)(n+1))^{k}((n-k)(n+1))^{n-k} > ((n-k+1)n)^{n}$$

which may be simplified to

$$(n-k)^{n-k}(n+1)^n > (n-k+1)^{n-k}n^n$$

or

$$(1+\frac{1}{n})^n > (1+\frac{1}{n-k})^{n-k}, \qquad (1 \le k \le n-1).$$

Since $(1+\frac{1}{n})^n$ is increasing in n, this completes our proof.

<u>SECOND PROOF</u>. Again we shall show that

$$\sum_{k=1}^{n} \{k((n-k+1)(n+1))^s - n((n-k+1)n)^s + (n-k)((n-k)(n+1))^s\} > 0.$$

This time we will establish this inequality by showing the deeper statement that for every $k \in \{1,2,\ldots,n\}$ all coefficients $c_{n,r}$ in the power series expansion of

$$k((n-k+1)(n+1))^s - n((n-k+1)n)^s + (n-k)((n-k)(n+1))^s = \sum_{r=0}^{\infty} c_{n,r}\, s^r$$

(as an entire function of s) around the point s = 0, are non-negative. The r-th coefficient $c_{n,r}$ satisfies

$$r!\, c_{n,r} = k(\log(n-k+1)(n+1))^r - n(\log(n-k+1)n)^r + (n-k)(\log(n-k)(n+1)^r$$

so that $c_{n,0} = 0$.

We will show that if $1 \le k \le n-1$, then

$$\frac{k(\log(n-k+1)(n+1))^r + (n-k)(\log(n-k)(n+1))^r}{n} > (\log(n-k+1)n)^r$$

for all $r \ge 1$.

Again, by the arithmetic-geometric-mean-inequality, it suffices to show that

$$(\log(n-k+1)(n+1))^k(\log(n-k)(n+1))^{n-k} > (\log(n-k+1)n)^n.$$

Replacing k by n-k we still have to show that

$$\left(\frac{\log(k+1) + \log(n+1)}{\log(k+1) + \log n}\right)^n > \left(\frac{\log(k+1) + \log(n+1)}{\log k + \log(n+1)}\right)^k$$

for $1 \le k \le n-1$.

One may verify that this inequality is equivalent to

$$\left\{1 - \frac{\log(1 + \frac{1}{k})}{\log(k+1) + \log(n+1)}\right\}^{-k} < \left\{1 - \frac{\log(1 + \frac{1}{n})}{\log(k+1) + \log(n+1)}\right\}^{-n}.$$

The validity of this inequality may be verified numerically for $n \le 10$, $1 \le k \le n-1$, and its validity for $n > 10$, $1 \le k \le n-1$, is an easy consequence of the following

LEMMA. *Let* T *be a constant* ≥ 3. *Then the function*

$$\phi(x) := \left\{1 - \frac{\log(1 + \frac{1}{x})}{T}\right\}^{-x}$$

is increasing for $x \ge 4$.

PROOF OF LEMMA. Define

$$\psi(x) := \log \phi(x) = -x \log\left(1 - \frac{\log(1+\frac{1}{x})}{T}\right)$$

$$= -x \log(T - \log(x+1) + \log x) + x \log T$$

so that

$$\psi'(x) = -x \frac{-\frac{1}{x+1} + \frac{1}{x}}{T - \log(1+\frac{1}{x})} - \log(T - \log(1+\frac{1}{x})) + \log T$$

$$= \frac{-1}{(x+1)(T - \log(1+\frac{1}{x}))} - \log(T - \log(1+\frac{1}{x})) - \log \frac{1}{T} .$$

Hence, it suffices to prove that $\psi'(x) > 0$:

$$-\log\left(1 - \frac{\log(1+\frac{1}{x})}{T}\right) > \frac{1}{(x+1)(T - \log(1+\frac{1}{x}))} .$$

Since $-\log(1-u) = u + \frac{u^2}{2} + \ldots > u$ for $0 < u < 1$, it suffices to show that

$$\frac{\log(1+\frac{1}{x})}{T} \geq \frac{1}{(x+1)(T - \log(1+\frac{1}{x}))}$$

or

$$\log(1 + \frac{1}{x}) \geq \frac{1}{(x+1)\left(1 - \frac{\log(1+\frac{1}{x})}{T}\right)} .$$

Hence, it suffices to show that

$$\frac{1}{x} - \frac{1}{2x^2} \geq \frac{1}{(x+1)(1 - \frac{1}{xT})}$$

or

$$\frac{x-1}{2x^2} \geq \frac{1}{xT-1} ,$$

which is equivalent to

$$T x^2 - (T+1)x + 1 \geq 2x^2.$$

It follows that it suffices to show that

$$Tx - (T+1) \geq 2x$$

or

$$x \geq 1 + \frac{3}{T-2} .$$

Since $T \geq 3$ we have $1 + \frac{3}{T-2} \leq 4$ and since, by assumption, $x \geq 4$, the proof is complete.

We conclude this section by stating the following

CONJECTURE 1.1. For any fixed $s > 0$ the sequence $\{U_n(s)\}_{n=1}^{\infty}$ is logarithmically convex.

The reasons for this conjecture will become clear in the next section.

2. APPROXIMATION OF $\int_0^1 x^s\,dx$ BY TRAPEZOIDAL SUMS.

In [2] it was shown that for any fixed $s > 1$ the canonical trapezoidal sums

$$T_n(s) := \frac{1}{2}\{\frac{1}{n}\sum_{k=0}^{n-1} (\frac{k}{n})^s + \frac{1}{n}\sum_{k=1}^{n} (\frac{k}{n})^s\},$$

corresponding to the integral $\int_0^1 x^s dx$, tend *decreasingly* to its limit $1/(s+1)$ as $n \to \infty$. In [4; p. 257] Jagers gave a different proof of this statement.

In [3] it was shown that for any fixed $s \in \mathbb{N}$ the sequence $\{T_n(s)\}_{n=1}^{\infty}$

is convex (in n) and it was stated as a conjecture that this sequence is even logarithmically convex.

In this section we will present another still deeper conjecture from which the above conjecture is a trivial consequence.

We would like to show that for any fixed $s > 1$ the sequence $\{T_n(s)\}_{n=1}^{\infty}$ is strictly log-convex, i.e.

$$T_n^2(s) < T_{n-1}(s)T_{n+1}(s), \qquad (n \geq 2).$$

In terms of $\sigma_n(s)$, defined in section 1, this inequality may also be written as

$$\left\{\frac{2\sigma_n(s) - n^s}{n^{s+1}}\right\}^2 < \left\{\frac{2\sigma_{n-1}(s) - (n-1)^s}{(n-1)^{s+1}}\right\}\left\{\frac{2\sigma_{n+1}(s) - (n+1)^s}{(n+1)^{s+1}}\right\}$$

or, equivalently

$$D_n(s) := n^{2s+2}\left\{2\sigma_{n-1}(s) - (n-1)^s\right\}\left\{2\sigma_{n+1}(s) - (n+1)^s\right\}$$
$$- (n^2-1)\left\{2\sigma_n(s) - n^s\right\} > 0.$$

so that we would like to show that $D_n(s) > 0$ for $s > 1$ and $n \geq 2$. Clearly $D_n(s)$ is an entire function of s and its power series expansion around the point $s = 1$ may be written as

$$D_n(s) = \sum_{k=1}^{\infty} c_{n,k}(s-1)^k.$$

Since $D_n(s)$ is an exponential polynomial in s one may easily write down an explicit formula for the coefficients $c_{n,k}$, i.e.

$$k!\, c_{n,k} = D_n^{(k)}(1),$$

where

$$D_n(s) = n^2\Big\{4\sum_{k=1}^{n-1}\sum_{\ell=1}^{n+1}(k\ell n^2)^s - 2\sum_{k=1}^{n-1}(kn^2(n+1))^s +$$

$$- 2\sum_{\ell=1}^{n+1}(\ell(n-1)n^2)^s + ((n-1)n^2(n+1))^s\Big\} +$$

$$-(n^2-1)\Big\{4\sum_{k=1}^{n}\sum_{\ell=1}^{n}(k\ell(n^2-1))^s - 4\sum_{k=1}^{n}(k(n-1)n(n+1))^s +$$

$$+ ((n-1)n(n+1))^s\Big\}.$$

Numerical computations indicate that $D_n^{(k)}(1) > 0$ for $k \geq 1$ and $n \geq 2$. A closer look at our numerical experiments leads us even to the still stronger

CONJECTURE 2.1. (i) $D_n'(1) > 0$ for all $n \geq 2$

(ii) for every fixed $n \geq 2$ the sequence $\{D_n^{(k)}(1)\}_{k=1}^{\infty}$ is increasing.

Similar observations led us to the conjecture that for any fixed $s > 0$ the sequence $\{U_n(s)\}_{n=1}^{\infty}$ is log-convex.

More positively we have the following

THEOREM 2.1. *For any fixed* $a \in (0,1)$ *the sequence*

$$\{\sum_{m=1}^{\infty} U_n(m)a^m\}_{m=1}^{\infty}$$

is log-convex (in n*).*

PROOF. Consider the sum

$$S_n(a) := \sum_{k=1}^{n} \frac{1}{n-ka}$$

and observe that

$$S_n(a) = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1-a\frac{k}{n}} = \sum_{k=1}^{n} \frac{1}{n} \sum_{m=0}^{\infty} (a\,\frac{k}{n})^m =$$

$$= \sum_{m=0}^{\infty} a^m \frac{1}{n} \sum_{k=1}^{n} (\frac{k}{n})^m = \sum_{m=0}^{\infty} U_n(m)a^m,$$

whereas on the other hand we have

$$S_n(a) = \sum_{k=1}^{n} \int_0^\infty e^{-(n-ka)u}du = \int_0^\infty e^{-nu}e^{au}\{1 + e^{au} + \ldots + e^{(n-1)au}\}du =$$

$$= \int_0^\infty e^{-nu}e^{au}\,\frac{e^{nau}-1}{e^{au}-1}\,du = \int_0^\infty e^{-u}\,\frac{e^{au}-1}{au}\,\frac{\frac{au}{n}}{1-e^{-\frac{au}{n}}}\,du.$$

Since $e^{-u}\frac{e^{au}-1}{au} > 0$ and the function $\dfrac{1}{x(1-e^{-\frac{1}{x}})}$ is log-convex on $\mathbb{R}^+$, it follows from the general theory of log-convex functions (see ARTIN [1]) that the sequence $\{S_n(a)\}_{n=1}^{\infty}$ is log-convex (in n).

THEOREM 2.2. *For any fixed* $a \in (0,1)$ *the sequence*

$$\{\sum_{m=0}^{\infty} T_n(m)a^m\}_{n=1}^{\infty}$$

is log-convex (in n*).*

PROOF. Similarly as in the proof of theorem 2.1 it may be shown that

$$(3) \qquad \sum_{m=0}^{\infty} T_n(m)a^m = \frac{1}{2}\int_0^\infty e^{-u}\,\frac{e^{au}-1}{au}\,\frac{au}{n}\,\frac{e^{\frac{au}{n}}+1}{e^{\frac{au}{n}}-1}\,du.$$

In [3] it was shown that the function $\frac{1}{x}\,\dfrac{e^{\frac{1}{x}}+1}{e^{\frac{1}{x}}-1}$ is log-convex on $\mathbb{R}^+$ and the theorem follows as above.

REMARK. Since a positive linear combination of log-convex sequences is again log-convex, the previous two theorems would immediately follow from our conjectures 1.1 and 2.1.

We conclude this note by proving some formulas which relate the sequence $\{T_n(m)\}_{m=1}^{\infty}$ in some sense to Euler's gamma-function.

First recall that Euler's gamma-function may be represented as

$$\Gamma(s+1) = s^s e^{-s}\sqrt{2\pi s}\, e^{\mu(s)}, \qquad (s > 0)$$

where

$$\mu(s) = \int_0^\infty \frac{e^{-st}}{t}\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}dt, \qquad (s > 0)$$

is Binet's function (c.f. [5; p. 216]).
Hence,

$$\mu''(s) = \int_0^\infty e^{-st}\, t\{\frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2}\}dt, \qquad (s > 0).$$

By letting n tend to infinity in (3) we find that

$$\sum_{m=0}^{\infty} \frac{1}{m+1} a^m = \int_0^\infty e^{-u}\,\frac{e^{au}-1}{au}\,du \quad (= \frac{-\log(1-a)}{a})$$

and subtracting this from (3) it follows that

$$\sum_{m=0}^{\infty} (T_n(m) - \frac{1}{m+1})a^m = \int_0^\infty \frac{e^{-\frac{t}{a}}}{a}\,\frac{e^t-1}{t}\,\{\frac{1}{2}\,\frac{e^{\frac{t}{n}}+1}{e^{\frac{t}{n}}-1}\,\frac{t}{n} - 1\}dt.$$

Observing that

$$\frac{1}{2}\,\frac{e^x+1}{e^x-1}\,x - 1 = \frac{x}{e^x-1} + \frac{x}{2} - 1$$

it follows that

$$\sum_{m=0}^{\infty} (T_n(m) - \frac{1}{m+1})a^{m+1} = \int_0^\infty e^{-\frac{t}{a}} \int_0^1 e^{ut}du\{\frac{1}{e^{\frac{t}{n}}-1} - \frac{1}{\frac{t}{n}} + \frac{1}{2}\}\frac{t}{n}\,dt =$$

$$= \int_0^1 du \int_0^\infty e^{-(\frac{1}{a}-u)t}\{\frac{1}{e^{\frac{t}{n}}-1} - \frac{1}{\frac{t}{n}} + \frac{1}{2}\}\frac{t}{n}\,dt =$$

$$= \int_0^1 du \int_0^\infty e^{-(\frac{1}{a}-u)nx} \{\frac{1}{e^x - 1} - \frac{1}{x} + \frac{1}{2}\} xndx =$$

$$= n \int_0^1 \mu''(\frac{n}{a} - nu)du = - \int_0^1 \{-n\mu''(\frac{n}{a} - nu)\}du =$$

$$= -\mu'(\frac{n}{a} - nu)\Big|_{u=0}^{u=1} = \mu'(\frac{n}{a}) - \mu'(\frac{n}{a} - n).$$

Hence, for $x > 1$ we have the remarkable formula

$$\sum_{m=2}^{\infty} (T_n(m) - \frac{1}{m+1})x^{-m-1} = \mu'(nx) - \mu'(nx-n),$$

from which it is easily seen that

$$\sum_{m=2}^{\infty} \frac{1}{m} (T_n(m) - \frac{1}{m+1})x^{-m} = \frac{\mu(nx-n) - \mu(nx)}{n}, \qquad (x > 1).$$

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