Decreasing Diagrams with Two Labels Are Complete for Confluence of Countable Systems

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Abstract

Like termination, confluence is a central property of rewrite systems. Unlike for termination, however, there exists no known complexity hierarchy for confluence. In this paper we investigate whether the decreasing diagrams technique can be used to obtain such a hierarchy. The decreasing diagrams technique is one of the strongest and most versatile methods for proving confluence of abstract reduction systems, it is complete for countable systems, and it has many well-known confluence criteria as corollaries.

So what makes decreasing diagrams so powerful? In contrast to other confluence techniques, decreasing diagrams employ a labelling of the steps $\to$ with labels from a well-founded order in order to conclude confluence of the underlying unlabelled relation. Hence it is natural to ask how the size of the label set influences the strength of the technique. In particular, what class of abstract reduction systems can be proven confluent using decreasing diagrams restricted to 1 label, 2 labels, 3 labels, and so on? Surprisingly, we find that two labels suffice for proving confluence for every abstract rewrite system having the cofinality property, thus in particular for every confluent, countable system. We also show that this result stands in sharp contrast to the situation for commutation of rewrite relations, where the hierarchy does not collapse.

Finally, as a background theme, we discuss the logical issue of first-order definability of the notion of confluence.

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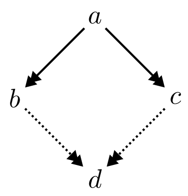

Figure 1 Confluence.

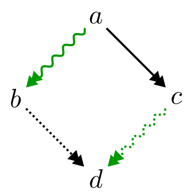

Figure 2 Commutation.

1 Introduction

A binary relation $\to$ is called *confluent* if two coinitial reductions (i.e., reductions having the same starting term) can always be extended to cofinal reductions, that is:

$$\forall abc.\ \big(b \twoheadleftarrow a \twoheadrightarrow c \Rightarrow \exists d.\ b \twoheadrightarrow d \twoheadleftarrow c\big)\ . \tag{1}$$

The confluence property is illustrated in Figure 1, in which solid and dotted lines stand for universal and existential quantification, respectively. The relation $\to$ is called *terminating* if there are no infinite sequences $a_0 \to a_1 \to a_2 \to \ldots$.

Termination and confluence are central properties of rewrite systems. For both properties there exist numerous proof techniques, and there are annual competitions for comparing the performance of automated provers. It is therefore a natural question how to measure and classify the complexity of termination and confluence problems. While there is a well-known hierarchy for termination [20], no such classification is known for confluence.[1]

The termination hierarchy [20] is based on the characterisation of termination in terms of well-founded monotone algebras. This entails an interpretation of the symbols of the signature as functions over the algebra. Then the class of the functions (or other properties of the algebra) used to establish termination can serve as a measure for the complexity of the termination problem. For instance, if polynomial functions over the natural numbers suffice to establish termination, then the rewrite system is said to be polynomially terminating.

In order to address the question of a hierarchy and complexity measure for the confluence property, our point of departure is the decreasing diagrams technique [17]. Decreasing diagrams are for confluence what well-founded interpretations are for termination. The decreasing diagrams technique is complete for systems having the cofinality property [15, p. 766]. Thus, in particular for every confluent, countable abstract reduction system, the confluence property can be proven using the decreasing diagrams technique. The power of decreasing diagrams is moreover witnessed by the fact that many well-known confluence criteria are direct consequences of decreasing diagrams [17], including the lemma of Hindley–Rosen [6, 13], Rosen's request lemma [13], Newman's lemma [12], and Huet's strong confluence lemma [7].

What makes the decreasing diagrams technique so powerful? The freedom to label the steps distinguishes decreasing diagrams from all other confluence criteria, with the exception of the weak diamond property [1, 4] by De Bruijn which has equal strength. This suggests that the power of these techniques arises from the labelling. This naturally leads to the following questions:

1. How does the size of the label set influence the strength of decreasing diagrams?

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[1] Ketema and Simonsen [8] consider peaks $t_1 \twoheadleftarrow s \twoheadrightarrow t_2$ and measure the length of joining reductions $t_1 \twoheadrightarrow \cdot \twoheadleftarrow t_2$ as a function of the size of s and the length of the reductions in the peak. The nature of this function can serve as a complexity measure for a confluence problem.

2. What class of abstract reduction systems can be proven confluent using decreasing diagrams with 1 label, 2 labels, 3 labels and so on?
3. Can the size of the label set serve as a complexity measure for a confluence problem?

Let *DCR* denote the class of abstract reduction systems (ARSs) whose confluence can be proven using decreasing diagrams. For an ordinal α, we write DCR_α for the class of ARSs whose confluence can be proven using decreasing diagrams with label set α (see Definition 15).

For every ARS $\mathcal{A}$, we have

$$DCR(\mathcal{A}) \implies DCR_\alpha(\mathcal{A}) \text{ for some ordinal } \alpha \tag{2}$$

The reason is that any partial well-founded order can be transformed into a total well-founded order (thus an ordinal). This transformation does not require the Axiom of Choice, see [4].

Clearly, we have $DCR_\alpha \subseteq DCR_\beta$ whenever $\alpha < \beta$. So

$$DCR_0 \subseteq DCR_1 \subseteq DCR_2 \subseteq DCR_3 \subseteq \ldots \subseteq DCR_\omega \subseteq \ldots \tag{3}$$

But which of these inclusions are strict? From the completeness proof in [18] it follows that all abstract reduction systems having the *cofinality property*, including all countable systems, belong to DCR_ω. In other words, for confluence of countable systems it suffices to label steps with natural numbers.

Contribution and outline

Our main result is that all systems with the cofinality property are in the class DCR_2, see Section 4. In particular, for proving confluence of countable abstract reduction systems it always suffices to label steps with 0 or 1 using the order $0 < 1$. So for countable systems, the hierarchy (3) collapses at level DCR_2. This is somewhat surprising, as one might expect that decreasing diagrams draws its strength from a rich labelling of the steps.

Interestingly, there is a stark contrast with commutation. For commutation the hierarchy does not collapse, see Section 5. We prove that, for commutation of countable systems, all inclusions are strict up to level DC_ω.

Our findings also provide new ways to approach the long-standing open problem of completeness of decreasing diagrams for uncountable systems, see Section 6.

2 Preliminaries

We repeat some of the main definitions, for the sake of self-containedness, and to fix notations. Let A be a set. For a relation $\to \subseteq A \times A$ we write $\to^*$ or $\twoheadrightarrow$ for its reflexive transitive closure. We write $\equiv$ for the empty step, that is, $\equiv = \{(a,a) \mid a \in A\}$, and we define $\to^{\equiv} = \to \cup \equiv$.

▶ **Definition 1** (Abstract Reduction System). An *abstract reduction system (ARS)* $\mathcal{A} = (A, \to)$ consists of a non-empty set A together with a binary relation $\to \subseteq A \times A$. For $B \subseteq A$ we define $\mathcal{A}|_B$, the *restriction of* $\mathcal{A}$ *to* B, by $\mathcal{A}|_B = (B, \to \cap (B \times B))$.

▶ **Definition 2** (Indexed ARS). An *indexed ARS* $\mathcal{A} = (A, \{\to_\alpha\}_{\alpha \in I})$ consists of a non-empty set A of *objects*, and a family $\{\to_\alpha\}_{\alpha \in I}$ of relations $\to_\alpha \subseteq A \times A$ indexed by some set I.

▶ **Definition 3** (Confluence). An ARS $(A, \to)$ is *confluent (CR)* if $\twoheadleftarrow \cdot \twoheadrightarrow \subseteq \twoheadrightarrow \cdot \twoheadleftarrow$, that is, every pair of finite, coinitial rewrite sequences can be joined to a common reduct.

▶ **Definition 4** (Commutation). Let $(A, \to, \rightsquigarrow)$ be an indexed ARS. Then the relation $\to$ *commutes with* $\rightsquigarrow$ if $\leftarrow^* \cdot \rightsquigarrow^* \subseteq \rightsquigarrow^* \cdot \leftarrow^*$; see Figure 2.

▶ **Definition 5** (Countable)**.** An ARS $(A, \to)$ is *countable (CNT)* if there exists a surjective function from the set of natural numbers $\mathbb{N}$ to A.

▶ **Definition 6** (Cofinal Reduction)**.** Let $\mathcal{A} = (A, \to)$ be an ARS. A set $B \subseteq A$ is *cofinal* in $\mathcal{A}$ if for every $a \in A$ we have $a \twoheadrightarrow b$ for some $b \in B$. A finite or infinite reduction sequence $b_0 \to b_1 \to b_2 \to \cdots$ is *cofinal* in $\mathcal{A}$ if the set $B = \{ b_i \mid i \geq 0 \}$ is cofinal in $\mathcal{A}$.

▶ **Definition 7** (Cofinality Property)**.** An ARS $\mathcal{A} = (A, \to)$ has the *cofinality property (CP)* if for every $a \in A$, there exists a reduction $a \equiv b_0 \to b_1 \to b_2 \to \cdots$ that is cofinal in $\mathcal{A}|_{\{b \,|\, a \twoheadrightarrow b\}}$.

▶ **Lemma 8.** *Let $\mathcal{A} = (A, \to)$ be a confluent ARS and $a \in A$. If a rewrite sequence is cofinal in $\mathcal{A}|_{\{b \,|\, a \twoheadrightarrow b\}}$, then it is also cofinal in $\mathcal{A}|_{\{b \,|\, a \leftrightarrow^* b\}}$.* ◀

▶ **Theorem 9** (Klop [9])**.** *Every confluent countable ARS has the cofinality property.* ◀

3 First-order Definability of Confluence

As we are investigating a confluence hierarchy, the question of first-order definability of confluence arises naturally. Namely, if confluence were definable by a set of first-order formulas, then we could obtain a confluence hierarchy by imposing syntactic restrictions on this set of formulas.

At first glance this question may appear trivial since confluence is typically defined via the first-order formula (1). However, this formula involves the transitive closure $\twoheadrightarrow$ of the one-step relation $\to$ which is itself not first-order definable. We show that confluence is not first-order definable over the one-step relation $\to$.

▶ Remark. In [16] it is shown that the first-order theory of linear one-step rewriting is undecidable. In this paper it is mentioned as a conjecture that undecidable properties like confluence and weak termination (see further [2]) cannot be expressed in the first-order logic of one-step rewriting.

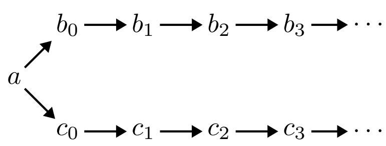
▶ **Theorem 10.** *Confluence and local confluence cannot be defined in the first-order logic with equality and the predicate $\to$ (one-step rewriting), neither by a single formula nor by a set of formulas.*

Proof. Assume, for a contradiction, that there is a set Δ of first-order formulas over the predicate $\to$ such that for every ARS $\mathcal{A} = (A, \to)$ it holds that:

$$\mathcal{A} \text{ is confluent} \quad \Longleftrightarrow \quad \mathcal{A} \models \Delta$$

Here $\mathcal{A} \models \Delta$ means that $\mathcal{A}$ is a *model* of Δ, that is, $\mathcal{A}$ satisfies all formulas in Δ. In what follows, we write $[c]$ for the interpretation of a constant c. For convenience, we write $\to$ for the predicate symbol in formulas as well as for the actual one-step rewrite relation or $\mathcal{A}$.

Our goal is to describe the following non-confluent structure using formulas:



We start by describing each single step by a formula:

$$\Lambda = \{ a \to b_0,\ a \to c_0 \} \cup \{ b_i \to b_{i+1} \mid i \in \mathbb{N} \} \cup \{ c_j \to c_{j+1} \mid j \in \mathbb{N} \}$$

We need to ensure that the interpretation of distinct constants is distinct:

$$\Lambda_{\neq} = \{\, x \neq y \mid x, y \in N \,\} \quad \text{where} \quad N = \{\, a \,\} \cup \{\, b_i \mid i \in \mathbb{N} \,\} \cup \{\, c_j \mid j \in \mathbb{N} \,\}$$

Finally, the following formula requires all elements, except for $[a]$, to be deterministic:

$$\xi = \forall xyz.\ (x \neq a \wedge x \to y \wedge x \to z) \Rightarrow y = z$$

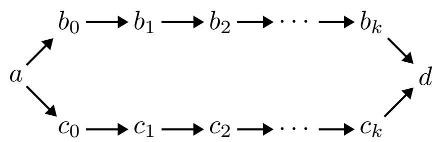
This simple trick excludes that elements $\{[b_n] \mid n \in \mathbb{N}\} \cup \{[c_n] \mid n \in \mathbb{N}\}$ admit steps other than the ones specified in Λ.

Now consider the following set of formulas:

$$\Gamma = \Delta \cup \Lambda \cup \Lambda_{\neq} \cup \{\, \xi \,\}$$

By the above construction, any model of $\Lambda \cup \Lambda_{\neq} \cup \{\, \xi \,\}$ cannot be confluent. However, any model of Δ must be confluent. Thus Γ does not have a model.

On the other hand, any finite subset Γ' of Γ has a model. This can be seen as follows. There exists a $k \in \mathbb{N}$ such that none of the constants $\{\, b_i \mid i \geq k \,\} \cup \{\, c_j \mid j \geq k \,\}$ appears in Γ'. Then the following structure is a model of Γ':



This is a contradiction! Due to the *compactness theorem*, Γ has a model if and only if every finite subset of Γ has a model. Thus confluence is not first-order definable.

Note that the same proof also shows undefinability of local confluence. ◀

▶ **Theorem 11.** *For $\alpha \geq 2$, DCR_α cannot be defined in the first-order logic with equality and the predicate $\to$ (one-step rewriting), neither by a single formula nor by a set of formulas.*

Proof. Follows by an extension of the proof for Theorem 10, noting that the model of Γ' admits a decreasing labelling with 2 labels. ◀

Note that DCR_1 is equivalent to the diamond property for the reflexive closure of the rewrite relation, and thus is first-order definable.

4 Decreasing Diagrams for Confluence with Two Labels

In this section we show that two labels suffice for proving confluence using decreasing diagrams for any abstract reduction system having the cofinality property. We start by introducing the decreasing diagrams technique.

▶ **Notation 12.** For an indexed ARS $\mathcal{A} = (A, \{\to_\alpha\}_{\alpha \in I})$ and a relation $< \subseteq I \times I$, we define

$$\to \;=\; \textstyle\bigcup_{\alpha \in I} \to_\alpha \qquad\qquad \to_{<\beta} \;=\; \textstyle\bigcup_{\alpha < \beta} \to_\alpha \qquad\qquad \to_{\leq\beta} \;=\; \textstyle\bigcup_{\alpha \leq \beta} \to_\alpha$$

Moreover, we use $\to_{<\alpha \cup <\beta}$ as shorthand for $(\to_{<\alpha} \cup \to_{<\beta})$.

▶ **Definition 13** (Decreasing Church–Rosser [17])**.** An ARS $\mathcal{A} = (A, \rightsquigarrow)$ is called *decreasing Church–Rosser (DCR)* if there exists an ARS $\mathcal{B} = (A, \{\to_\alpha\}_{\alpha \in I})$ indexed by a well-founded partial order $(I, <)$ such that $\rightsquigarrow \;=\; \to$ and every peak $c \leftarrow_\beta a \to_\alpha b$ can be joined by reductions of the form shown in Figure 3.[2]

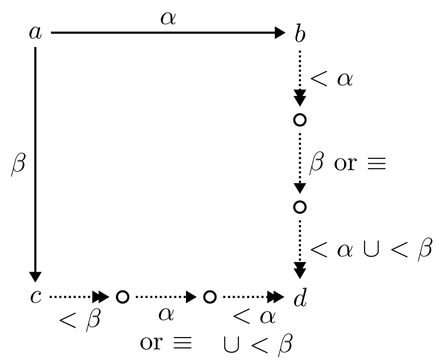


Figure 3 Decreasing elementary diagram.

The following is the main theorem of decreasing diagrams.

▶ **Theorem 14** (Decreasing Diagrams – De Bruijn [1] & Van Oostrom [17]). *If an ARS is decreasing Church–Rosser, then it is confluent.* ◀

In other words $DCR \implies CR$.

As already suggested in the introduction, we introduce classes DCR_α restricting the well-founded order $(I, <)$ in Definition 13 to the ordinal α.

▶ **Definition 15.** For ordinals α, let DCR_α denote the class of ARSs $\mathcal{A}$ that are decreasing Church–Rosser (Definition 13) with label set $\{\, \beta \mid \beta < \alpha \,\}$ ordered by the usual order $<$ on ordinals. We say that $\mathcal{A}$ has the property DCR_α, denoted $DCR_\alpha(\mathcal{A})$, if $\mathcal{A} \in DCR_\alpha$.

The remainder of this section is devoted to the proof that every system with the cofinality property is DCR_2. Put differently, it suffices to label steps with $I = \{\,0, 1\,\}$. Let $\mathcal{A} = (A, \to)$ be an ARS having the cofinality property. Note that, for defining the labelling, we can consider connected components with respect to $\leftrightarrow^*$ separately. Thus assume that $\mathcal{A}$ consists of a single connected component, that is, for every $a, b \in A$ we have $a \leftrightarrow^* b$. By the cofinality property, which implies confluence, and Lemma 8 there exists a rewrite sequence

$$m_0 \to m_1 \to m_2 \to m_3 \to \cdots$$

that is cofinal in $\mathcal{A}$; we call this rewrite sequence the *main road*. Without loss of generality we may assume that the main road is acyclic, that is, $m_i \not\equiv m_j$ whenever $i \neq j$. (We can eliminate loops without harming the cofinality property. Note that the main road is allowed to be finite.)

The idea of labelling the steps in $\mathcal{A}$ is as follows. For every node $a \in A$, we label precisely one of the outgoing edges with 0 and all others with 1. The edge labelled with 0 must be part of a shortest path from a to the main road. For the case that a lies on the main road, the step labelled 0 must be the step on the main road. This is illustrated in Figure 4.

Note that there is a choice about which edge to label with 0 whenever there are multiple outgoing edges that all start a shortest path to the main road. To resolve this choice, the

[2] Van Oostrom [19] generalises the shape of the decreasing elementary diagrams by allowing the joining reductions to be conversions. This can be helpful to find suitable elementary diagrams. However, if there are conversions then we can always obtain joining reductions by diagram tiling. So a system is locally decreasing with respect to conversions if and only if it is locally decreasing with respect to reductions (using the same labelling of the steps).

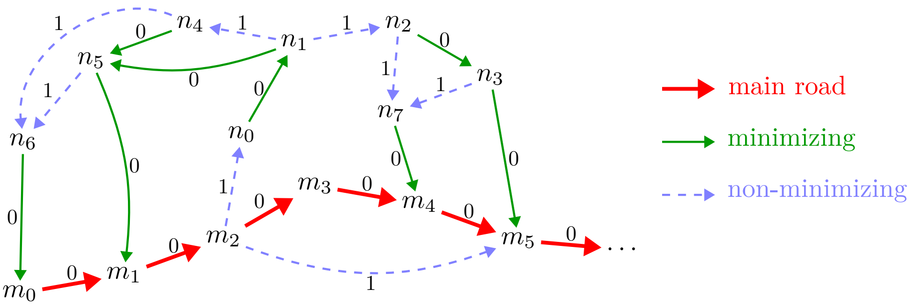


Figure 4 Example labelling.

following definition assumes a well-order $<$ on the universe A, whose existence is guaranteed by the well-ordering theorem. Then, whenever there is a choice, we choose the edge for which the target is minimal in this order.

▶ Remark. Recall that the Axiom of Choice is equivalent to the well-ordering theorem. In many practical cases, however, the existence of such a well-order does not require the Axiom of Choice. If the universe is countable, then such a well-order can be derived directly from the surjective counting function $f : \mathbb{N} \to A$.

In the following definition we follow the proof in [15, Proposition 14.2.30, p. 766], employing the notion of a cofinal sequence and the rewrite distance from a point to this sequence. While the proof in [15] labels steps by their distance to the target node, we need a more sophisticated labelling.

▶ **Definition 16.** Let $\mathcal{A} = (A, \to)$ be an ARS and $M : m_0 \to m_1 \to m_2 \to \cdots$ be a finite or infinite rewrite sequence in $\mathcal{A}$. For $a, b \in A$, we write

(i) $a \in M$ if $a \equiv m_i$ for some $i \geq 0$, and
(ii) $(a \to b) \in M$ if $a \equiv m_i$ and $b \equiv m_{i+1}$ for some $i \geq 0$.

If M is cofinal in $\mathcal{A}$, we define the *distance* $d(a, M)$ as the least natural number $n \in \mathbb{N}$ such that $a \to^n m$ for some $m \in M$. If M is clear from the context, we write $d(a)$ for $d(a, M)$.

▶ **Definition 17** (Labelling with two labels)**.** Let $\mathcal{A} = (A, \to)$ be an ARS equipped with a well-order $<$ on A such that there exists a cofinal reduction $M : m_0 \to m_1 \to m_2 \to \cdots$ that is acyclic (that is, for all $i < j$, $m_i \not\equiv m_j$).

We say that a step $a \to b$ is

(i) *on the main road* if $(a \to b) \in M$;
(ii) *minimizing* if $d(a) = d(b) + 1$ and $b' \geq b$ for every $a \to b'$ with $d(b') = d(b)$.

We define an indexed ARS $\mathcal{A}_{\{0,1\}} = (A, \{\to_i\}_{i \in I})$ where $I = \{\, 0, 1 \,\}$ as follows:

$$a \to_0 b \iff a \to b \text{ and this step is on the main road or minimizing}$$

$$a \to_1 b \iff a \to b \text{ and this step is not on the main road and not minimizing}$$

for every $a, b \in A$.

▶ **Lemma 18.** *Let $\mathcal{A} = (A, \to)$ be an ARS with a cofinal rewrite sequence $M : m_0 \to m_1 \to \cdots$ that is acyclic. Furthermore, let $<$ be a well-order over A. Then for $\mathcal{A}_{\{0,1\}} = (A, \to_0, \to_1)$ we have:*

(i) $\to \; = \; \to_0 \cup \to_1$;
(ii) *for every $a, b \in M$ we have $a \twoheadrightarrow_0 \cdot \twoheadleftarrow_0 b$;*
(iii) *for every $a \in A$, there is at most one $b \in A$ such that $a \to_0 b$;*

(iv) *for every $a \notin M$, there exists $b \in A$ with $a \to_0 b$ and $d(a) > d(b)$;*
(v) *for every $a \in A$, there exists $m \in M$ such that $a \twoheadrightarrow_0 m$;*
(vi) *every peak $c \leftarrow_\beta a \to_\alpha b$ can be joined as in Figure 3, and, explicitly for labels $\{0, 1\}$, as in Figure 5.*

Proof. Properties i and ii follow from the definitions.

For iii assume that $b \leftarrow_0 a \to_0 c$. We show that $b \equiv c$. The steps $a \to b$ and $a \to c$ are either minimizing or on the main road. We distinguish cases $a \in M$ and $a \notin M$:

(i) Assume that $a \in M$. Then $d(a) = 0$, and thus neither $a \to b$ nor $a \to c$ is a minimizing step. Hence $(a \to b) \in M$ and $(a \to c) \in M$. Since M is acyclic, we get $b \equiv c$.
(ii) If $a \notin M$, both steps $a \to b$ and $a \to c$ must be minimizing. If $d(b) \neq d(c)$, then we have either $d(a) \neq d(b) + 1$ or $d(a) \neq d(c) + 1$, contradicting minimization. Thus $d(b) = d(c)$. Then by minimization we have $b \geq c$ and $c \geq b$, from which we obtain $b \equiv c$.

For iv, consider an element $a \notin M$. Let $B = \{b' \mid a \to b' \wedge d(a) = d(b') + 1\}$. By definition of the distance $d(\cdot)$, $B \neq \varnothing$. Define b as the least element of B in the well-order $<$ on A. It follows that $a \to b$ is a minimization step. Hence $a \to_0 b$ and $d(a) > d(b)$. Property v follows directly from iv using induction on the distance.

For vi, consider a peak $c \leftarrow_\beta a \to_\alpha b$. If $b \equiv c$, then the joining reductions are empty steps. Thus assume that $b \not\equiv c$. By iii we have either $\alpha = 1$ or $\beta = 1$. By v there exist $m_b, m_c \in M$ such that $b \twoheadrightarrow_0 m_b$ and $c \twoheadrightarrow_0 m_c$. By ii we have $m_b \twoheadrightarrow_0 \cdot \twoheadleftarrow_0 m_c$. Hence $b \twoheadrightarrow_0 \cdot \twoheadleftarrow_0 c$. These joining reductions are of the form required by Figure 3 since $\twoheadrightarrow_0 = \twoheadrightarrow_{<\alpha \cup <\beta}$. ◀

▶ **Theorem 19.** *If an ARS $\mathcal{A} = (A, \to)$ satisfies the cofinality property, then there exists an indexed ARS $(A, (\to_\alpha)_{\alpha \in \{0,1\}})$ such that $\to = \to_0 \cup \to_1$ and every peak $c \leftarrow_\beta a \to_\alpha b$ can be joined according to the elementary decreasing diagram in Figure 3, and, explicitly for labels $\{0, 1\}$, as in Figure 5.*

Proof. It suffices to consider a connected component of $\mathcal{A}$. Let $\mathcal{B} = (B, \to)$ be a connected component of $\mathcal{A}$: we have $a \leftrightarrow^* b$ for all $a, b \in B$. By the cofinality property and Lemma 8, there exists a cofinal reduction $m_0 \to m_1 \to \cdots$ in $\mathcal{B}$. By the well-ordering theorem, there exists a well-order $<$ over B. Then $\mathcal{B}$ has the required properties by Lemma 18vi. ◀

▶ **Corollary 20.** *DCR_2 is a complete method for proving confluence of countable ARSs.*

Proof. Immediate from Theorems 9 and 19. ◀

Theorem 19 also holds for De Bruijn's weak diamond property. Note the following caveat: when restricting the index set I to a single label, the decreasing diagram technique is equivalent to $\leftarrow \cdot \to \subseteq \to^\equiv \cdot \leftarrow^\equiv$, i.e. the *diamond property* for $\to \cup \equiv$, while the weak diamond property with one label is equivalent to *strong confluence* $\leftarrow \cdot \to \subseteq \to^\equiv \cdot \twoheadleftarrow$.

The property DCR_2 is given implicitly by the decreasing diagrams as in Figure 3, but it is also instructive to give explicitly the elementary reduction diagrams making up the property DCR_2. These are shown in Figure 5. Note that the 1-steps do not split in the diagram construction, i.e. they cross over in at most one copy. This facilitates a simple proof of confluence.

Actually, from our proof it follows that the joining reductions can be required to only contain steps with label 0. Thus even the simple shape of diagrams shown in Figure 6 is complete for proving confluence of systems having the cofinality property. Here the 1-steps do not cross over at all! Note that while this set of elementary diagrams has a trivial proof of confluence, the work to prove $DCR_2 \implies CR$ from the original elementary diagrams as in Figure 5, consists in showing from our earlier construction that it actually suffices to join by using only 0's.

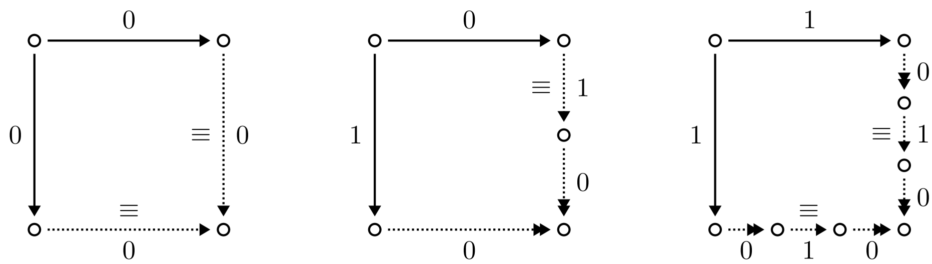


Figure 5 Decreasing diagrams with labels 0 and 1 where $0 < 1$.

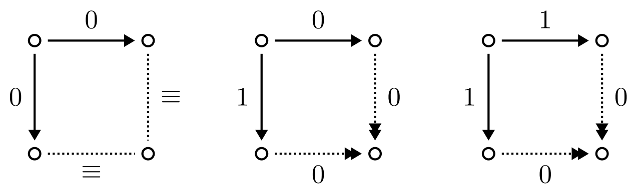
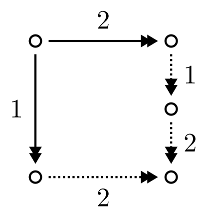


Figure 6 A simple set of diagrams that is complete for confluence of countable systems.

▶ Remark. We note a certain similarity between the notion of a decreasing diagram based on labels $\{0, 1\}$ with $0 < 1$ and the classical 'requests' lemma of J. Staples [10, 15, Exercise 2.08.5, p. 9]. In $\mathcal{A} = (A, \to_1, \to_2)$ define: $\to_1$ requests $\to_2$ if



If in addition $\to_1$ and $\to_2$ are confluent, then $\to_{1,2} = \to_1 \cup \to_2$ is confluent.

The requests lemma states that the 'dominant' reduction $\twoheadrightarrow_1$ needs the 'support' of the secondary reduction $\twoheadrightarrow_2$ for making the divergence $\twoheadleftarrow_1 \cdot \twoheadrightarrow_2$ convergent. Similarly for the property DCR_2, the dominant reduction $\to_1$ needs support by $\twoheadrightarrow_0$ for making the divergence $\leftarrow_1 \cdot \to_0$ convergent. However, the requests lemma employs $\twoheadrightarrow$, not $\to$.

5 Decreasing Diagrams for Commutation

The decreasing diagram technique can also be used for proving commutation, see [17]. It turns out that the situation for commutation stands in sharp contrast to that for confluence. For commutation the hierarchy does not collapse. In particular, we show that, for every $n \leq \omega$, decreasing diagrams for commutation with n labels is *strictly* stronger than decreasing diagrams with less than n labels.

The elementary decreasing diagram for commutation is shown in Figure 7, which is very similar to Figure 3, but now refers to two 'basis' relations $\to$, $\rightsquigarrow$.

▶ **Definition 21** (Decreasing Commutation). An ARS $\mathcal{A} = (A, \to, \rightsquigarrow)$ is called *decreasing commuting (DC)* if there is an ARS $\mathcal{B} = (A, \{\to_\alpha\}_{\alpha \in I}, \{\rightsquigarrow_\alpha\}_{\alpha \in I})$ indexed by a well-founded partial order $(I, <)$ such that $\to_\mathcal{A} = \to_\mathcal{B}$ and $\rightsquigarrow_\mathcal{A} = \rightsquigarrow_\mathcal{B}$, and every peak $c \leftarrow_\beta a \rightsquigarrow_\alpha b$ in $\mathcal{B}$ can be joined by reductions of the form shown in Figure 7.

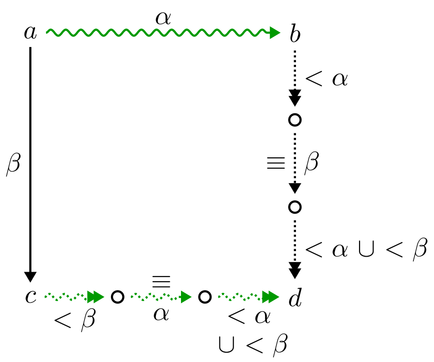


Figure 7 Decreasing elementary diagram for proving commutation.

If all conditions are fulfilled, we call $\mathcal{B}$ a *decreasing labelling* of $\mathcal{A}$.

▶ **Theorem 22** (Decreasing Diagrams for Commutation – Van Oostrom [17]). *If an ARS $\mathcal{A} = (A, \to, \rightsquigarrow)$ is decreasing commuting, then $\to$ commutes with $\rightsquigarrow$.* ◀

Analogous to the classes DCR_α for confluence, we introduce classes DC_α for commutation.

▶ **Definition 23.** For ordinals α, let DC_α denote the class of ARSs $\mathcal{A} = (A, \to, \rightsquigarrow)$ that are decreasing commuting (Definition 21) with label set $\{ \beta \mid \beta < \alpha \}$ ordered by the usual order $<$ on ordinals. We say that $\mathcal{A}$ has the property DC_α, denoted $DC_\alpha(\mathcal{A})$, if $\mathcal{A} \in DC_\alpha$.

In Definition 23 it suffices to consider total orders since every partial well-founded order can be transformed into a total well-founded order. This transformation [4] preserves the decreasing elementary diagrams and does not need the Axiom of Choice.

In order to show that the hierarchy for commutation does not collapse, we inductively construct, for every $n \in \mathbb{N}$, an ARS $\mathcal{A}_n$ that is DC_{5n+1}, but not DC_n.

▶ **Definition 24.** For every $n \in \mathbb{N}$ we define a tuple $\Phi_n = (\mathcal{A}_n, a_1, a, c, b, b_1)$ consisting of an ARS $\mathcal{A}_n = (A_n, \to_n, \rightsquigarrow_n)$ and distinguished elements $a_1, a, c, b, b_1 \in A_n$ by induction on n:

1. Let $\Phi_0 = (\mathcal{A}_0, a_1, c, c, c, b_1)$ where $\mathcal{A}_0$ is the ARS displayed in Figure 8.
2. Let $\Phi_n = (\mathcal{A}_n, a, a', c, b', b)$. We obtain $\mathcal{A}_{n+1}$ as an extension of $\mathcal{A}_n$ as shown in Figure 9. The inner dark part with the darker background is $\mathcal{A}_n$. The extension consists of the addition of fresh elements $a_1, \ldots, a_7$ and $b_1, \ldots, b_7$ and rewrite steps as shown in the figure. We define $\Phi_{n+1} = (\mathcal{A}_{n+1}, a_1, a, c, b, b_1)$.

We start with a few important properties of the construction.

▶ **Lemma 25.** *For every $n \in \mathbb{N}$ and $\Phi_n = (\mathcal{A}_n, a_1, a, c, b, b_1)$ with $\mathcal{A}_n = (A_n, \to, \rightsquigarrow)$ we have the following properties:*

(i) *The relations $\to$ and $\rightsquigarrow$ are deterministic.*
(ii) *For every element $x \in A_n$ we have $x \to^* c$ and $x \rightsquigarrow^* c$.*
(iii) *For $x \in A_n$, we have $a_1 \rightsquigarrow^* x \leftarrow^* b_1$ if and only if $a \rightsquigarrow^* x$ and $a \to^* x$.*
(iv) *For $x \in A_n$, we have $a_1 \to^* x \leftsquigarrow^* b_1$ if and only if $b \rightsquigarrow^* x$ and $b \to^* x$.*

Proof. We use induction on $n \in \mathbb{N}$. For the base case $n = 0$, we have $\Phi_0 = (\mathcal{A}_0, a_1, c, c, c, b_1)$ where $\mathcal{A}_0$ is given in Figure 8. The properties follow from an inspection of the figure.

For the induction step, let $n \in \mathbb{N}$ and assume that $\Phi_n = (\mathcal{A}_n, a, a', c, b', b)$ satisfies the properties. By construction, $\mathcal{A}_{n+1}$ is an extension of $\mathcal{A}_n$ as shown in Figure 9, and we have $\Phi_{n+1} = (\mathcal{A}_{n+1}, a_1, a, c, b, b_1)$. The fresh elements introduced by the extension are $X = \{ a_1, \ldots, a_7, b_1, \ldots, b_7 \}$. We check the validity of each property for $\mathcal{A}_{n+1}$:

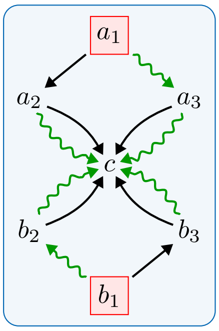


Figure 8 Base case: one label suffices.

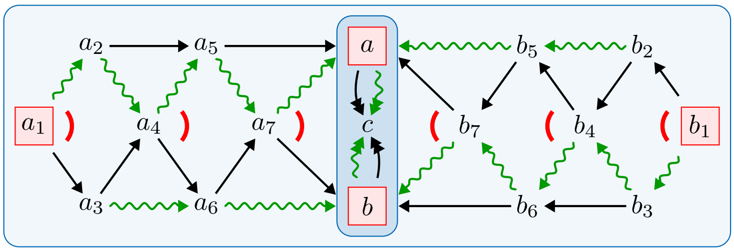


Figure 9 From n to $n+1$ labels for commutation. Rough proof sketch: Assume that at least one of the reductions $a \to^* c$, $b \rightsquigarrow^* c$, $a \rightsquigarrow^* c$ or $b \to^* c$ contains two steps labelled with n. Then each of the peaks at a_1, a_4 and a_7, or each of the peaks at b_1, b_4 and b_7 must contain a step labelled with $n+1$. As a consequence, one of the reductions $a_1 \to^* c$, $b_1 \rightsquigarrow^* c$, $a_1 \rightsquigarrow^* c$ or $b_1 \to^* c$ contains two steps labelled with $n+1$.

(i) There are no fresh steps with sources in $\mathcal{A}_n$. Every element $x \in X$ admits precisely one outgoing step $\to$ and one outgoing step $\rightsquigarrow$. So both rewrite relations remain deterministic, establishing property i.

(ii) For every element $x \in X$ we have $x \to^* a$ or $x \to^* b$, and $x \rightsquigarrow^* a$ or $x \rightsquigarrow^* b$. Together with the induction hypothesis ii for n, this yields property ii for $n+1$.

(iii) From Figure 9 it follows immediately that any reduction $a_1 \rightsquigarrow^* x \leftarrow^* b_1$ must be of the form $a_1 \rightsquigarrow^* a \rightsquigarrow^* x \leftarrow^* a \leftarrow^* b_1$. The reductions from both sides are deterministic and the first joining element is a.

(iv) Analogous to property iii. ◀

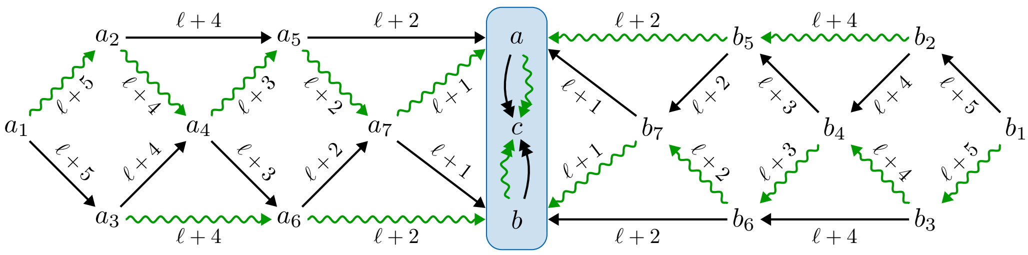
From Lemma 25 ii it follows that $\to$ and $\rightsquigarrow$ commute in $\mathcal{A}_n$. However, commutation is not sufficient to conclude that $\mathcal{A}_n$ is decreasing commuting. Decreasing diagrams are not complete for proving commutation as shown in [4].

We prove that $\mathcal{A}_n$ is decreasing commuting by constructing a labelling with $5n$ labels. This bound is by no means optimal, but easy to verify and sufficient for our purpose.

▶ **Lemma 26.** *For every $n \in \mathbb{N}$, $\mathcal{A}_n$ is DC_{5n+1}.*

Proof. We use induction on $n \in \mathbb{N}$. For the base case $n = 0$, consider $\mathcal{A}_0$ shown in Figure 8. For this system a single label suffices since the joining reductions in the elementary diagrams have length at most 1.

For the induction step, assume that $\mathcal{A}_n$ has the property DC_{5n+1}. So $\mathcal{A}_n$ is decreasing commuting with labels $\{0, \dots, \ell\}$ where $\ell = 5n$. By construction, $\mathcal{A}_{n+1}$ is an extension of $\mathcal{A}_n$ as shown in Figure 9. We extend the labelling of $\mathcal{A}_n$ with labels $\{0, \dots, \ell\}$ to a labelling of $\mathcal{A}_{n+1}$ with labels $\{0, \dots, \ell+5\}$ as follows:



Here $\mathcal{A}_n$ is the darker inner part. From the picture it is easy to verify that every peak $\leftarrow \cdot \rightsquigarrow$ in the extension can be joined by reductions that only contain labels strictly smaller than labels of the peak. As a consequence, $\mathcal{A}_{n+1}$ is $DC_{5(n+1)+1}$. ◀

Next, we show that $\mathcal{A}_n$ does not admit a decreasing labelling with n labels.

▶ **Lemma 27.** *For every $n \in \mathbb{N}$, $\mathcal{A}_n$ is not DC_n.*

Proof. We prove the following stronger claim: for every $n \in \mathbb{N}$ and $\Phi_n = (\mathcal{A}_n, a_1, a, c, b, b_1)$, and every decreasing labelling of $\mathcal{A}_n$ with labels from $\mathbb{N}$ it holds that at least one of the four paths $a_1 \to^* b$, $a_1 \rightsquigarrow^* a$, $b_1 \to^* a$ or $b_1 \rightsquigarrow^* b$ contains two labels $\geq n$. Note that these paths exist by Lemma 25. We prove this claim by induction on $n \in \mathbb{N}$.

For the base case $n = 0$, we have $\Phi_0 = (\mathcal{A}_0, a_1, c, c, c, b_1)$ where $\mathcal{A}_0$ is given in Figure 8. It suffices to consider one of the four paths. For instance, the rewrite sequence $a_1 \to^* c$ has length 2 and both steps must have a label ≥ 0.

For the induction step, assume that the claim holds for n and $\Phi_n = (\mathcal{A}_n, a, a', c, b', b)$. Accordingly, the induction hypothesis is that, for every decreasing labelling of $\mathcal{A}_n$ with labels from $\mathbb{N}$, one of the four paths $a \to^* b'$, $a \rightsquigarrow^* a'$, $b \to^* a'$ or $b \rightsquigarrow^* b'$ contains two labels $\geq n$. We prove the claim for $n+1$. Let $\Phi_{n+1} = (\mathcal{A}_{n+1}, a_1, a, c, b, b_1)$ where $\mathcal{A}_{n+1}$ is an extension of $\mathcal{A}_n$ as shown in Figure 9. Let $\mathcal{B}$ be a decreasing labelling of the steps in $\mathcal{A}_{n+1}$ with labels from $\mathbb{N}$. We show that at least one of the paths $a_1 \to^* b$, $a_1 \rightsquigarrow^* a$, $b_1 \to^* a$ or $b_1 \rightsquigarrow^* b$ contains two labels $\geq n+1$.

By construction, the systems $\mathcal{A}_{n+1}$ and $\mathcal{A}_n$ contain the same steps with sources in $\mathcal{A}_n$. Thus the restriction of the labelling $\mathcal{B}$ to $\mathcal{A}_n$ is a decreasing labelling for $\mathcal{A}_n$. By the induction hypothesis, at least one of the paths (i) $a \to^* b'$, (ii) $a \rightsquigarrow^* a'$, (iii) $b \to^* a'$ or (iv) $b \rightsquigarrow^* b'$ contains two labels $\geq n$. Without loss of generality, by symmetry, assume that the path (i) or (iv) contain two labels $\geq n$.

Consider the peak $a_3 \leftarrow a_1 \rightsquigarrow a_2$. As visible in Figure 9, every elementary diagram for this peak must have joining reductions of the form $a_3 \rightsquigarrow^* b \rightsquigarrow^* x \leftarrow^* a \leftarrow^* a_2$ for some $x \in \mathcal{A}_n$. From Lemma 25 iv we conclude that the joining reductions must be of the form

$$a_3 \rightsquigarrow^* b \rightsquigarrow^* b' \rightsquigarrow^* x \leftarrow^* b' \leftarrow^* a \leftarrow^* a_2$$

The path (i) $a \to^* b'$ or (iv) $b \rightsquigarrow^* b'$ contains two labels $\geq n$. Thus, for the elementary diagram to be decreasing, one of the steps in the peak $a_3 \leftarrow a_1 \rightsquigarrow a_2$ must have label $\geq n+1$.

The same argument can be applied to the peaks $a_6 \leftarrow a_4 \rightsquigarrow a_5$ and $b \leftarrow a_7 \rightsquigarrow a$. As a consequence, each of the peaks $a_3 \leftarrow a_1 \rightsquigarrow a_2$, $a_6 \leftarrow a_4 \rightsquigarrow a_5$ and $b \leftarrow a_7 \rightsquigarrow a$ contains one step with a label $\geq n+1$. Hence at least one of the paths

1. $a_1 \to a_3 \to a_4 \to a_6 \to a_7 \to b$, or
2. $a_1 \rightsquigarrow a_2 \rightsquigarrow a_4 \rightsquigarrow a_5 \rightsquigarrow a_7 \rightsquigarrow a$

contains two steps with labels $\geq n+1$.

If path (ii) $a \rightsquigarrow^* a'$ or (iii) $b \to^* a'$ contains two labels $\geq n$, then an analogous argument can be applied to the peaks $b_2 \leftarrow b_1 \rightsquigarrow b_3$, $b_5 \leftarrow b_4 \rightsquigarrow b_6$ and $a \leftarrow b_7 \rightsquigarrow b$, yielding that at least one of the paths $b_1 \to^* a$ or $b_1 \rightsquigarrow^* b$ contains two steps with labels $\geq n+1$.

This proves the claim and concludes the proof. ◀

We have seen that, for every $n \in \mathbb{N}$, $\mathcal{A}_n$ that is DC_{5n+1}, but not DC_n (Lemmas 26 & 27). From this we can conclude that an infinite number of the inclusions $DC_0 \subseteq DC_1 \subseteq DC_2 \subseteq \cdots$ are strict. The following proposition allows us to infer that all of them are strict.

Roughly speaking, the following proposition states that if a level $\alpha+1$ of the hierarchy does not collapse, then also the level α does not collapse. We state the proposition for the commutation hierarchy, but it also holds for the confluence hierarchy.

▶ **Proposition 28.** *If $DC_\alpha \subsetneq DC_{\alpha+1}$ for an ordinal α, then $DC_\beta \subsetneq DC_\alpha$ for every $\beta < \alpha$. This also holds when the classes are restricted to countable systems.*

Proof. Let $\mathcal{A} = (A, \to, \rightsquigarrow)$ be in $DC_{\alpha+1} \setminus DC_\alpha$. Then there exists a decreasing labelling $\mathcal{B}$ of $\mathcal{A}$ with labels $\{\, \beta \mid \beta \leq \alpha \,\}$. As $\mathcal{A}$ is not DC_α some steps must have the maximum label α. Note that

$\star$ If the joining reductions in a decreasing elementary diagram contain a step with label α, then the corresponding peak must also contain a step with label α.

Let $\mathcal{B}'$ be obtained from $\mathcal{B}$ by dropping all steps with label α, and let $\mathcal{A}'$ be obtained from $\mathcal{B}'$ by dropping the labels. By ($\star$), $\mathcal{B}'$ is a decreasing labelling of $\mathcal{A}'$, and hence $\mathcal{A}'$ is DC_α.

For a contradiction, assume that $DC_\beta = DC_\alpha$ for some $\beta < \alpha$. Then $\mathcal{A}'$ is DC_β. Let $\mathcal{B}''$ be obtained from $\mathcal{B}'$ by adding all steps that we had previously removed from $\mathcal{B}$, but we now relabel the steps from α to β. It is straightforward to check that $\mathcal{B}''$ is a decreasing labelling of $\mathcal{A}$. Hence, $\mathcal{A}$ is in $DC_{\beta+1} \subseteq DC_\alpha$. This is a contradiction. ◀

▶ **Example 29.** Assume that α is a limit ordinal and $DC_{\alpha+3} \subsetneq DC_{\alpha+4}$. By Proposition 28 we conclude $DC_{\alpha+2} \subsetneq DC_{\alpha+3}$. By repeated application of Proposition 28 we conclude

$$DC_\beta \subsetneq DC_\alpha \subsetneq DC_{\alpha+1} \subsetneq DC_{\alpha+2} \subsetneq DC_{\alpha+3} \subsetneq DC_{\alpha+4}$$

for every $\beta < \alpha$. However, the proposition does not help to conclude that $DC_\beta \subsetneq DC_{\beta'}$ for every $\beta < \beta' \leq \alpha$.

▶ **Theorem 30.** *We have*

(i) $DC_n \subsetneq DC_{n+1}$ *for every* $n \in \mathbb{N}$, *and*

(ii) $\bigcup_{n \in \mathbb{N}} DC_n \subsetneq DC_\omega$.

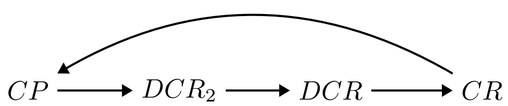
These inclusions are strict also when the classes are restricted to countable systems.

Proof. By Lemmas 26 and 27 we know that $DC_n \subsetneq DC_{n+1}$ for infinitely many $n \in \mathbb{N}$. Then repeated application of Proposition 28 yields $DC_n \subsetneq DC_{n+1}$ for every $n \in \mathbb{N}$.

Let $\mathcal{A}$ be the infinite disjoint union $\mathcal{A}_0 \uplus \mathcal{A}_1 \uplus \mathcal{A}_2 \uplus \cdots$. As a consequence of Lemmas 26 and 27 the ARS $\mathcal{A}$ is DC_ω but not DC_n for any $n \in \mathbb{N}$. ◀

6 Conclusion

We study how the strength of decreasing diagrams is influenced by the size of the label set. We find that all abstract reduction systems with the cofinality property (in particular, all confluent, countable systems) can be proven confluent using the decreasing diagrams technique with the almost trivial label set $I = \{\, 0, 1 \,\}$. So for confluence of *countable* ARSs, we have the following implications:



This is in sharp contrast to the situation for commutation for which we prove

$$DC_0 \subsetneq DC_1 \subsetneq DC_2 \subsetneq DC_3 \subsetneq \cdots \subsetneq DC_\omega$$

even for countable systems. So for commutation, for every $n \leq \omega$, there exists a system that requires n labels. The structure of this hierarchy above level DC_ω remains open.

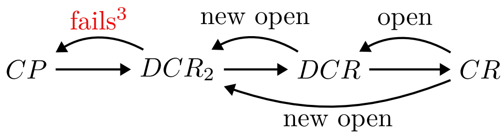
▶ **Open Problem 31.** *What inclusions* $DC_\alpha \subseteq DC_\beta$ *are strict for* $\omega \leq \alpha < \beta$*?*

Decreasing diagrams are complete for confluence of countable systems. However, it is a long-standing open problem whether the method of decreasing diagrams is also complete for proving confluence of uncountable systems [17]. Our observations provide new ways for approaching this problem. In particular, it may be helpful to investigate the following:

▶ **Open Problem 32.** *Is there a confluent, uncountable system that is CR but not* DCR_2*?*

▶ **Open Problem 33.** *Is there a confluent, uncountable system that needs more than* 2 *labels to establish confluence using decreasing diagrams? In other words, is there an uncountable system that is DCR but not* DCR_2*? Is there an uncountable system that is* DCR_3 *but not* DCR_2*?*

So we have the following situation for uncountable systems:



For a better understanding of this hierarchy, it would be interesting to investigate whether Proposition 28 can be generalised as follows.

▶ **Open Problem 34.** *Assume that* $DC_\alpha \subsetneq DC_\beta$ *for ordinals* $\alpha < \beta$*. Does this imply that none of the lower levels of the hierarchy collapse? That is, does it imply that* $DC_{\alpha'} \subsetneq DC_{\beta'}$ *for every* $\alpha' < \beta' \leq \alpha$*?*

Our findings indicate that the size of the label set in decreasing diagrams is not a suitable measure for the complexity of a confluence problem. So the complexity arises rather from the distribution of the labels, and the proof that every peak has suitable joining reductions. The complexity of the label distribution can be measured in terms of the complexity of machine required for computing the labels. For this purpose, one can consider Turing machines, finite automata or finite state transducers. The complexity of Turing machines can be measured in terms of time or space complexity, Kolmogorov Complexity [11] or degrees of unsolvability [14]. For finite state transducers the complexity can be classified by degrees of transducibility [5, 3].

References

1 N.G. de Bruijn. A Note on Weak Diamond Properties. Memorandum 78–08, Eindhoven Uninversity of Technology, 1978.

2 J. Endrullis, H. Geuvers, J.G. Simonsen, and H. Zantema. Levels of Undecidability in Rewriting. *Information and Computation*, 209(2):227–245, 2011.

3 J. Endrullis, J. Karhumäki, J.W. Klop, and A. Saarela. Degrees of infinite words, polynomials and atoms. In *Proc. Conf. Developments in Language Theory (DLT 2016)*, LNCS, pages 164–176. Springer, 2016.

4 J. Endrullis and J.W. Klop. De Bruijn's weak diamond property revisited. *Indagationes Mathematicae*, 24(4):1050–1072, 2013. In memory of N.G. (Dick) de Bruijn (1918–2012).

5 J. Endrullis, J.W. Klop, A. Saarela, and M. Whiteland. Degrees of transducibility. In *Proc. Conf. on Combinatorics on Words (WORDS 2015)*, volume 9304 of *LNCS*, pages 1–13. Springer, 2015.

[3] Already the implication $DCR_1 \implies CP$ fails. To see this, consider the ARS $(2^{\mathbb{R}}, \to)$ where the steps are of the form $X \to X \cup \{ y \}$ for $X \subseteq \mathbb{R}$ and $y \in \mathbb{R}$.

6 J.R. Hindley. *The Church–Rosser Property and a Result in Combinatory Logic*. PhD thesis, University of Newcastle-upon-Tyne, 1964.

7 G.P. Huet. Confluent Reductions: Abstract Properties and Applications to Term Rewriting Systems. *Journal of the ACM*, 27(4):797–821, 1980.

8 J. Ketema and J.G. Simonsen. Least upper bounds on the size of confluence and church-rosser diagrams in term rewriting and λ-calculus. *ACM Trans. Comput. Log.*, 14(4):31:1–31:28, 2013.

9 J.W. Klop. *Combinatory Reduction Systems*, volume 127 of *Mathematical centre tracts*. Mathematisch Centrum, 1980.

10 J.W. Klop. Term Rewriting Systems. In *Handbook of Logic in Computer Science*, volume II, pages 1–116. Oxford University Press, 1992.

11 M. Li and P.M.B. Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications*. Springer, 2nd edition, 2008.

12 M.H.A. Newman. On Theories with a Combinatorial Definition of "Equivalence". *Annals of Mathematics*, 42(2):223–243, 1942.

13 B.K. Rosen. Tree-manipulating systems and Church-Rosser theorems. *Journal of the ACM*, 20:160–187, 1973.

14 J.R. Shoenfield. *Degrees of Unsolvability*. North-Holland, Elsevier, 1971.

15 Terese. *Term Rewriting Systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.

16 R. Treinen. The first-order theory of one-step rewriting is undecidable. In *Proc. Conf. on Rewriting Techniques and Applications (RTA)*, volume 1103 of *LNCS*, pages 276–286. Springer, 1996.

17 V. van Oostrom. Confluence by Decreasing Diagrams. *Theoretical Computer Science*, 126(2):259–280, 1994.

18 V. van Oostrom. *Confluence for Abstract and Higher-Order Rewriting*. PhD thesis, Vrije Universiteit Amsterdam, 1994.

19 V. van Oostrom. Confluence by Decreasing Diagrams, Converted. In *Proc. Conf. on Rewriting Techniques and Applications (RTA 2008)*, volume 5117 of *LNCS*, pages 306–320. Springer, 2008.

20 H. Zantema. The Termination Hierarchy for Term Rewriting. *Applicable Algebra in Engineering, Communication and Computing*, 12(1):3–19, 2001.