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Sequentiality in Orthogonal Term Rewriting Systems

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For orthogonal term rewriting systems G. Huet and J.-J. Lévy have introduced the property of 'strong sequentiality'. A strongly sequential orthogonal term rewriting system admits an efficiently computable normalizing one-step reduction strategy. As shown by Huet and Lévy, strong sequentiality is a decidable property. In this paper we present an alternative analysis of strongly sequential term rewriting systems, leading to two simplified proofs of the decidability of this property. We also compare some related notions of sequentiality that recently have been proposed.

1. Introduction

The analysis of term rewriting systems is of growing interest for a large number of applications having to do with computing with equations. Two main streams can be distinguished in the study of term rewriting systems: (1) theory and applications of Knuth-Bendix completion procedures—here the point of departure is a given set of equations for which one tries to generate a complete (i.e. confluent and terminating) term rewriting system—and (2) theory and applications of orthogonal term rewriting systems; here the term rewriting system is fixed but subject to the restrictions of being 'left-linear' and 'non-ambiguous', for short 'orthogonal'. (Previously, we used 'regular' instead of 'orthogonal'.) The restriction of orthogonality enables one to develop a quite sizeable amount of theory, for a large part due to the efforts of the 'French school' (Berry & Lévy, 1979; Boudol, 1985; Huet & Lévy, 1979).

The present paper is exclusively concerned with orthogonal term rewriting systems. In an admirable paper, Huet and Lévy (1979) investigated the issue of parallel versus sequential reduction in an orthogonal term rewriting system. More specifically, they formulated a criterion

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'strong sequentiality', guaranteeing the existence of an effective sequential normalizing reduction strategy, that is a strategy Φ such that its iteration on a given term t leads to a reduction sequence

$$t \to \Phi(t) \to \Phi^2(t) \to \ldots$$

which ends in the (unique) normal form of t if it exists and is infinite otherwise. The sequentiality is in the fact that the strategy indicates in each step just one redex to be rewritten, rather than a set of redexes to be rewritten in parallel. Actually, Huet and Lévy prove that every orthogonal term rewriting system possesses a sequential normalizing 'call-by-need' strategy: a deep theorem in Huet and Lévy (1979) says that every term t in an orthogonal term rewriting system contains a 'needed' redex, that is one which has to be rewritten in any reduction to normal form. A call-by-need strategy is then obtained by rewriting in each step such a needed redex, and it is proved in Huet and Lévy (1979) that such a strategy is normalizing. Unfortunately, it is undecidable in general whether a redex is needed or not. However, Huet and Lévy go on to show that in 'strongly sequential' term rewriting systems, a needed redex can be found effectively. This does not mean that in a strongly sequential term rewriting system *all* needed redexes can be determined effectively. For instance Combinatory Logic

$$\mathrm{CL} = \begin{cases} Ap(Ap(Ap(S,x),y),z) & \to \; Ap(Ap(x,z),Ap(y,z)) \\ Ap(Ap(K,x),y) & \to \; x \\ Ap(I,x) & \to \; x \end{cases}$$

is a strongly sequential term rewriting system where this is impossible; cf. the analogous statement for λ-calculus in Barendregt *et al.* (1987). In fact, a needed redex is very easy to determine in the case of CL: the leftmost redex is always needed. By contrast, consider CL ⊕ B, that is CL extended with B ('Berry's term rewriting system', also called 'Gustave's term rewriting system' in Huet (1986)):

$$\mathrm{B} = \begin{cases} F(A,B,x) & \to \; C \\ F(B,x,A) & \to \; C \\ F(x,A,B) & \to \; C. \end{cases}$$

In the term rewriting system CL ⊕ B it is not clear at all how to find a needed redex: in a term $F(t_1, t_2, t_3)$ the redexes in t_1 may be non-needed because t_2, t_3 reduce to the constants A, B respectively, and likewise for redexes in t_2 and t_3. (The presence of CL serves to make the system non-trivial; in the system B alone the needed redexes are just the outermost redexes.) Actually, we do not know whether there is an algorithm to determine a needed redex in a term of CL ⊕ B (cf. the surprising fact in Kennaway (1989) where it is shown that every orthogonal term rewriting system, including CL ⊕ B, has a computable normalizing one-step reduction strategy), but it seems safe to conjecture that if such an algorithm exists, it will not be very 'feasible'.

However, in strongly sequential term rewriting systems a needed redex can be found really effectively, as shown in Huet & Lévy (1979). Moreover, it is decidable whether a term rewriting system is strongly sequential. This brings us to the point dealt with in this paper: in Huet & Lévy (1979) a proof of the decidability of strong sequentiality is given with great ingenuity; but it is also very complicated, and in the present paper our endeavour is to analyze the notion of a strongly sequential term rewriting system in order to arrive at a simplified proof of the decidability. We present two proofs of which the first is the most direct; but the corresponding decision procedure itself is only of mathematical relevance as its computational complexity forbids a practical application. We feel however that this proof is conceptually simple and gives a good insight in the structure of a strongly sequential term rewriting system. Some of the underlying notions in Huet & Lévy (1979) are eliminated here; notably: the 'matching dag', 'directions', 'increasing indices' and 'Δ-sets' (or: 'properties Q_1, Q_2'). Also our proof is direct in the sense that it does not take the form of a correctness proof of some algorithm. The second proof is of comparable computational complexity as the one in Huet & Lévy (1979); conceptually it is harder than the first, though still simpler than the one in Huet & Lévy (1979). This proof is essentially already in Huet & Lévy (1979) and uses their notions of increasing indices and Δ-sets (the latter with a slight simplification by us). In both proofs our concepts of a 'preredex' and of a 'tower of preredexes' play a crucial role. We construct a term rewriting system which is 'inherently difficult' with respect to deciding strong sequentiality, and we make the simple but useful observation that strong sequentiality is a 'modular' property, i.e. depends on the 'disjoint pieces' of a term rewriting system. In the last section we give an overview of other notions of sequentiality proposed in the literature.

Especially in the first part of our paper we follow Huet & Lévy (1979) quite closely; also some proofs there are repeated for the sake of completeness. Although our paper is self-contained, familiarity with term rewriting systems might be helpful (Dershowitz & Jouannaud, 1990; Huet & Oppen, 1980; Klop, 1990).

2. Orthogonal Term Rewriting Systems: Preliminaries

We start with a number of definitions. A *signature* is a set $\mathcal{F}$ of *function symbols*. Associated with every $F \in \mathcal{F}$ is a natural number denoting its arity. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F},\mathcal{V})$ of *terms* built from a signature $\mathcal{F}$ and a countably infinite set of *variables* $\mathcal{V}$ with $\mathcal{F} \cap \mathcal{V} = \emptyset$ is the smallest set such that $\mathcal{V} \subset \mathcal{T}(\mathcal{F},\mathcal{V})$ and if $F \in \mathcal{F}$ has arity n and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F},\mathcal{V})$ then $F(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F},\mathcal{V})$. We write C instead of $C()$ whenever C is a constant. Terms not containing variables are called *ground* terms. Identity of terms is denoted by $\equiv$.

A *term rewriting system* (TRS for short) is a pair $(\mathcal{F}, \mathcal{R})$ consisting of a signature $\mathcal{F}$ and a finite set $\mathcal{R} \subset \mathcal{T}(\mathcal{F},\mathcal{V}) \times \mathcal{T}(\mathcal{F},\mathcal{V})$ of *rewrite rules* or *reduction rules*. Every rewrite rule (l, r) is subject to the following two constraints:

(1) the left-hand side l is not a variable,

(2) the variables which occur in the right-hand side r also occur in l.

Rewrite rules (l, r) will henceforth be written as $l \to r$. We often present a TRS as a set of rewrite rules, without making explicit its signature.

A *substitution* σ is a mapping from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F},\mathcal{V})$. Substitutions are extended to $\mathcal{T}(\mathcal{F},\mathcal{V})$ in the obvious way. The term obtained from t by applying the substitution σ is denoted by t^σ. We call t^σ an *instance* of t. An instance of a left-hand side of a rewrite rule is a *redex* (reducible expression).

Let $\Box$ be a special constant symbol. A *context* $C[\,,\ldots,\,]$ is a term in $\mathcal{T}(\mathcal{F} \cup \{\Box\},\mathcal{V})$. If $C[\,,\ldots,\,]$ is a context with n occurrences of $\Box$ and $t_1, \ldots, t_n$ are terms then $C[t_1, \ldots, t_n]$ is the result of replacing from left to right the occurrences of $\Box$ by $t_1, \ldots, t_n$. A context containing precisely one occurrence of $\Box$ is denoted by $C[\,]$. A term s is a *subterm* of a term t if there exists a context $C[\,]$ such that $t \equiv C[s]$.

The rewrite rules of a TRS $(\mathcal{F}, \mathcal{R})$ define a *rewrite relation* $\rightarrow_{\mathcal{R}}$ on $\mathcal{T}(\mathcal{F},\mathcal{V})$ as follows: $s \rightarrow_{\mathcal{R}} t$ if there exists a rewrite rule $l \rightarrow r$ in $\mathcal{R}$, a substitution σ and a context $C[\,]$ such that $s \equiv C[l^\sigma]$ and $t \equiv C[r^\sigma]$. We say that s rewrites to t by *contracting* redex l^σ and we call r^σ the *contractum* of l^σ. We call $s \rightarrow_{\mathcal{R}} t$ a *rewrite step* or *reduction step*. The transitive-reflexive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\twoheadrightarrow_{\mathcal{R}}$. If $s \twoheadrightarrow_{\mathcal{R}} t$ we say that s *reduces* to t and we call t a *reduct* of s. The transitive closure of $\rightarrow_{\mathcal{R}}$ is denoted by $\rightarrow^+_{\mathcal{R}}$. In the sequel we often omit the subscript $\mathcal{R}$.

EXAMPLE 2.1. Let

$$\mathcal{R} = \begin{cases} A(x, 0) & \rightarrow \; x \\ A(x, S(y)) & \rightarrow \; S(A(x, y)) \end{cases}$$

and consider the term $A(A(0, 0), A(S(0), 0))$. To this term we can apply the following reduction sequence (at each step the contracted redex is underlined):

$$A(\underline{A(0,0)}, A(S(0),0)) \rightarrow A(0, \underline{A(S(0),0)}) \rightarrow \underline{A(0, S(0))} \rightarrow S(\underline{A(0,0)}) \rightarrow S(S(0)).$$

A *normal form* is a term without redexes. A term s *has* a normal form if $s \twoheadrightarrow_{\mathcal{R}} t$ for some normal form t. The set of normal forms of a TRS $\mathcal{R}$ is denoted by $\mathrm{NF}_{\mathcal{R}}$ (NF for short).

A precise formalism for describing subterm occurrences is obtained through the notion of *positions*. For any term $t \in \mathcal{T}(\mathcal{F},\mathcal{V})$, the set $O(t)$ of positions in t is inductively defined as follows:

- $O(t) = \{\lambda\}$ if $t \in \mathcal{V}$,
- $O(t) = \{\lambda\} \cup \{i.u \mid 1 \le i \le n \text{ and } u \in O(t_i)\}$ if $t \equiv F(t_1, \ldots, t_n)$.

In the literature positions are often called *occurrences*. Positions are sequences of natural numbers denoting subterm occurrences. If $u \in O(t)$ then the subterm t/u and the symbol $t(u)$ of t at position u are defined by

$$t/u = \begin{cases} t & \text{if } u = \lambda, \\ t_i/v & \text{if } t \equiv F(t_1, \ldots, t_n) \text{ and } u = i.v, \end{cases}$$

$$t(u) = \begin{cases} t & \text{if } t \in \mathcal{V} \text{ and } u = \lambda, \\ F & \text{if } t \equiv F(t_1, \ldots, t_n) \text{ and } u = \lambda, \\ t_i(v) & \text{if } t \equiv F(t_1, \ldots, t_n) \text{ and } u = i.v. \end{cases}$$

If $u \in O(t)$ and $s \in \mathcal{T}(\mathcal{F},\mathcal{V})$ then the term $t[u \leftarrow s]$ is defined as follows:

- $t[u \leftarrow s] = s$ if $u = \lambda$,
- $t[u \leftarrow s] = F(t_1, \ldots, t_i[v \leftarrow s], \ldots, t_n)$ if $u = i.v$ and $t \equiv F(t_1, \ldots, t_n)$.

Positions are partially ordered by the *prefix ordering* $\le$, i.e. $u \le v$ if there exists a w such that $uw = v$ (if such a w exists, it is unique). In this case we define $v/u = w$. If $u \le v$ and $u \ne v$, we write $u < v$. Two positions u, v are *disjoint*, notation $u \perp v$, if neither $u \le v$ nor $v \le u$. If $u_1, \ldots, u_n \in O(t)$ are pairwise disjoint, we write $t[u_i \leftarrow s_i \mid 1 \le i \le n]$ as an alternative for $t[u_1 \leftarrow s_1] \ldots [u_n \leftarrow s_n]$ (the order of the u_i's is irrelevant). Sometimes we write $t[s \leftarrow s']$

instead of $t[u \leftarrow s' \mid t/u \equiv s]$. Finally, the *depth* $|u|$ of a position u is defined by

$$|u| = \begin{cases} 0 & \text{if } u = \lambda, \\ 1 + |v| & \text{if } u = i.v. \end{cases}$$

EXAMPLE 2.2. Consider again the TRS of Example 2.1. The positions in $t = S(A(S(0), 0))$ are exhibited in Figure 1. We have $t/1 \equiv A(S(0), 0)$, $t(1.1.1) \equiv 0$, $t[1.1 \leftarrow t/1.2] = S(A(0, 0))$ and $|1.1.1| = 3$.

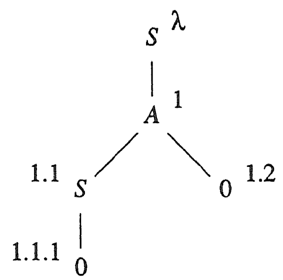


FIGURE 1.

In this paper we restrict ourselves to the subclass of *orthogonal* TRS's. A TRS is orthogonal if it satisfies the following two constraints:

(1) *left-linearity*: the left-hand side l of a rewrite rule $l \to r$ does not contain multiple occurrences of the same variable.

(2) *non-ambiguity*: the left-hand sides of the rewrite rules do not overlap. This means that whenever $l_1 \to r_1$, $l_2 \to r_2$ are rewrite rules and $u \in O(l_1)$ such that $l_1/u \notin \mathcal{V}$, there are no substitutions σ, τ such that $(l_1/u)^\sigma \equiv l_2^\tau$, except in the case where $l_1 \to r_1$, $l_2 \to r_2$ are the same rewrite rule and $u = \lambda$.

EXAMPLE 2.3. The TRS

$$\mathcal{R} = \begin{cases} IF(T, x, y) & \to & x \\ IF(F, x, y) & \to & y \\ IF(x, y, y) & \to & y \end{cases}$$

is neither left-linear (the left-hand side of the rule $IF(x, y, y) \to y$ contains two occurrences of the variable y) nor non-ambiguous (take $l_1 \equiv IF(T, x, y)$, $l_2 \equiv IF(x, y, y)$ and $u = \lambda$ in the above definition). The TRS of Example 2.1 is orthogonal.

Orthogonal TRS's have some very nice properties. Among these is the important Church-Rosser property. A TRS is *confluent* or has the *Church-Rosser* property (CR) if for all terms s, t_1, t_2 with $s \twoheadrightarrow t_1$ and $s \twoheadrightarrow t_2$ we can find a term t_3 such that $t_1 \twoheadrightarrow t_3$ and $t_2 \twoheadrightarrow t_3$, see Figure 2. Such a term t_3 is called a *common reduct* of t_1 and t_2.

THEOREM 2.4 (Huet, 1980). *Every orthogonal* TRS *has the Church-Rosser property.* □

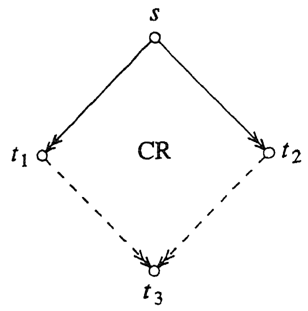


FIGURE 2.

An immediate consequence of Theorem 2.4 is the fact that in orthogonal TRS's every term has at most one normal form, i.e. if $s \twoheadrightarrow t$, $s \twoheadrightarrow t'$ and $t, t' \in \mathrm{NF}$ then $t \equiv t'$. In the next section we will encounter some more important properties of orthogonal TRS's.

3. Strongly Sequential Term Rewriting Systems

There are orthogonal TRS's in which some terms have a normal form, but also admit an infinite reduction sequence.

EXAMPLE 3.1. Let

$$\mathcal{R} = \begin{cases} F(x, A) & \to A \\ B & \to A \\ C & \to C. \end{cases}$$

The term $F(C, B)$ has the normal form A:

$$F(C, \underline{B}) \to \underline{F(C, A)} \to A,$$

but always choosing the leftmost redex results in an infinite reduction sequence:

$$F(\underline{C}, B) \to F(\underline{C}, B) \to F(\underline{C}, B) \to \dots .$$

Therefore, it is important to have a 'good' *reduction strategy*. Informally, a reduction strategy tells us, when presented a term, which redex(es) to rewrite. To be more precise, a *many-step reduction strategy* is a mapping ϕ which assigns to every term $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ a subset of its redex occurrences, i.e. $\phi(t) \subseteq O(t)$ such that t/u is a redex for all $u \in \phi(t)$. We call ϕ a *one-step reduction strategy* if $\phi(t)$ is a singleton set for every $t \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ which is not a normal form. The result of applying a reduction strategy to a term t is denoted by $\Phi(t)$, i.e.

$$\Phi(t) = t[u \leftarrow \downarrow(t/u) \mid u \in \phi(t)]$$

where $\downarrow(t/u)$ denotes the (unique) contractum of redex t/u. (This definition of $\Phi(t)$ only makes sense if the positions in $\phi(t)$ are pairwise disjoint. By means of so-called 'finite developments' it is possible to lift this disjointness requirement, but since all strategies considered in the sequel satisfy this restriction the above definition serves our purpose.) A reduction strategy ϕ is *normalizing* if for all terms t having a normal form, the sequence

$$t,\ \Phi(t),\ \Phi(\Phi(t)),\ \ldots,\ \Phi^n(t),\ \ldots$$

contains that normal form. We are only interested in effective normalizing strategies. (A reduction strategy ϕ is *effective* if $\Phi(t)$ can be computed from t.)

An important normalizing many-step reduction strategy for orthogonal TRS's is the *parallel-outermost* strategy: rewrite simultaneously all outermost redexes. (A redex s in a term t is *outermost* if s is not contained in a larger redex of t.) For a proof that the parallel-outermost strategy is normalizing for orthogonal TRS's, see O'Donnell (1977) or the appendix of Bergstra & Klop (1986). Alternatively, this fact can be obtained as a corollary of Theorem 3.4 below. The following example shows that the parallel-outermost strategy does not always give the shortest reduction sequence to normal form.

EXAMPLE 3.2. Let

$$\mathcal{R} = \begin{cases} IF(T, x, y) & \to\ x \\ IF(F, x, y) & \to\ y \\ A & \to\ B. \end{cases}$$

Consider the term $IF(IF(T, F, T), A, A)$. The parallel-outermost strategy rewrites a total of four redexes:

$$IF(\underline{IF(T, F, T)}, \underline{A}, \underline{A}) \twoheadrightarrow \underline{IF(F, B, B)} \to B,$$

The following normalizing sequence contracts only three redexes:

$$IF(\underline{IF(T, F, T)}, A, A) \to \underline{IF(F, A, A)} \to \underline{A} \to B.$$

In the example above it is not necessary to rewrite the redex A at position 2 in the term $IF(IF(T, F, T), A, A)$ in order to find the normal form. Before we make this precise, we introduce the notion of 'descendants' in reduction sequences. Consider the rewrite rule $F(x, y) \to G(F(x, x))$. When instantiated to $F(t_1, t_2) \to G(F(t_1, t_1))$ it is clear that t_1 is doubled and that t_2 has been erased. Obviously we have an intuition of the subterms of t_1 as propagating to the right. We say that a subterm s of t_1 has (two) *descendants* in $G(F(t_1, t_1))$ after the reduction step $F(t_1, t_2) \to G(F(t_1, t_1))$. A formal definition can be found in Huet & Lévy (1979). We prefer to illustrate this notion by Example 3.3 below.

EXAMPLE 3.3. Let

$$\mathcal{R} = \begin{cases} F(x, y) & \to\ G(x, x) \\ A & \to\ B \end{cases}$$

and consider the reduction sequence

$$t \equiv F(F(A, B), A) \to G(F(A, B), F(A, B)) \to G(F(B, B), F(A, B)) \equiv t'.$$

The redex A in t at position 1.1 has one descendant in t': the redex A at position 2.1. The redex

$F(A, B)$ in t at position 1 has two descendants in t': redex $F(B, B)$ at position 1 and redex $F(A, B)$ at position 2. Neither the redex A in t at position 2 nor t itself have descendants in t'.

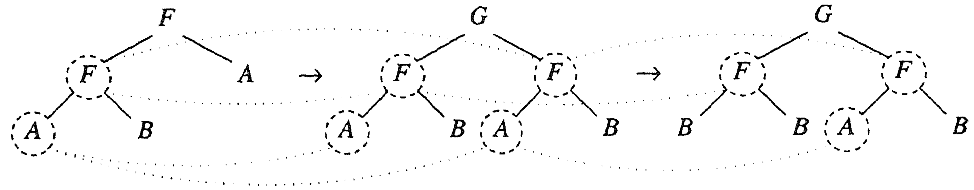


FIGURE 3.

Orthogonal TRS's have the property that descendants of redexes remain redexes. A redex s in a term t is called *needed* if in every reduction sequence from t to normal form a descendant of s is contracted. (Actually, s refers to a redex *occurrence*; likewise in the formulation of the following theorem. In the formal part of this paper we will use the precise notational formalism for redex occurrences as in Huet & Lévy (1979).) A needed redex must eventually be contracted in order to find the normal form. In Example 3.2 the underlined redex in the term $IF(IF(T, F, T), \underline{A}, A)$ is not needed. Huet and Lévy proved the following very important result.

THEOREM 3.4 (Huet & Lévy, 1979). *Let t be a term in an orthogonal* TRS.
(1) *If t is not a normal form then t contains a needed redex.*
(2) *If t has a normal form then there does not exist an infinite reduction sequence starting from t in which infinitely many needed redexes are contracted.*
□

So if a term has a normal form, repeated contraction of needed redexes leads to that normal form. Hence this theorem gives us a normalizing one-step reduction strategy: just contract some needed redex. However, the definition of 'needed' refers to all reductions to normal form, so in order to determine what the needed redexes are, we have to inspect the normalizing reductions first, which is not a very good recipe for a reduction strategy. In other words, the determination of needed redexes involves *look-ahead*, and it is this necessity for look-ahead that we wish to eliminate.

Every term t not in normal form can be written as $t \equiv C[r_1, \ldots, r_n]$ where $C[\ , \ldots, \ ]$ is a context in normal form and $r_1, \ldots, r_n$ are the outermost redexes of t. Using Theorem 3.4 and the orthogonality of the TRS under consideration, it is not difficult to see that one of the r_i is needed. An actual i such that r_i is needed may depend on the 'substitution' of the redexes $r_1, \ldots, r_n$ for the □'s in $C[\ , \ldots, \ ]$. A more pleasant state of affairs is expressed in the following definition.

DEFINITION 3.5. An orthogonal TRS is *sequential** if for every context $C[\ , \ldots, \ ]$ in normal form there exists an i such that for all redexes $r_1, \ldots, r_n$ redex r_i in the term $C[r_1, \ldots, r_n]$ is needed.

This concept is only introduced for expository purposes. It is not a satisfactory property as it is undecidable. By abstracting from the right-hand sides of the rewrite rules, the situation takes a pleasant turn.

DEFINITION 3.6. Let $\mathcal{R}$ be an orthogonal TRS.

(1) The rewrite relation $\to_?$ (*arbitrary reduction*) is defined as follows:

$$C[s] \to_? C[t]$$

for every context $C[\ ]$, redex s and *arbitrary* term t. Clearly, the set of normal forms with respect to $\to_?$ coincides with the set of $\to$-normal forms.

(2) A redex s in a term t is *strongly needed* if in every arbitrary reduction sequence from t to normal form a descendant of s is contracted. (Descendants with respect to arbitrary reduction are defined in the obvious way.)

(3) The TRS $\mathcal{R}$ is *strongly sequential** if for every context $C[\ ,\ldots,\ ]$ in normal form there exists an i such that for all redexes $r_1, \ldots, r_n$ redex r_i in the term $C[r_1, \ldots, r_n]$ is strongly needed.

Notice that the property of being strongly sequential* is determined by the left-hand sides of the rewrite rules of a TRS only. Because reduction is a special case of arbitrary reduction, every strongly needed redex is needed. Hence every strongly sequential* TRS is sequential*. The reverse is not true, as the following example of Huet and Lévy shows.

EXAMPLE 3.7. Let

$$\mathcal{R} = \begin{cases} F(G(A,x),B) & \to & x \\ F(G(x,A),C) & \to & x \\ F(D,x) & \to & x \\ G(E,E) & \to & E. \end{cases}$$

It is not difficult to see that every redex of a given term is needed. Therefore, $\mathcal{R}$ is sequential*. Consider the term $F(G(r_1,r_2),r_3)$ with arbitrary redexes r_1, r_2, r_3. The following arbitrary reductions show that none of r_1, r_2, r_3 is strongly needed:

$$F(G(r_1,\underline{r_2}),\underline{r_3}) \twoheadrightarrow_? \underline{F(G(r_1,A),C)} \to_? A,$$

$$F(G(\underline{r_1},r_2),\underline{r_3}) \twoheadrightarrow_? \underline{F(G(A,r_2),B)} \to_? A,$$

$$F(G(\underline{r_1},\underline{r_2}),r_3) \twoheadrightarrow_? F(\underline{G(E,E)},r_3) \to_? \underline{F(D,r_3)} \to_? A.$$

Hence $\mathcal{R}$ is not strongly sequential*.

Huet and Lévy defined the properties 'sequentiality' and 'strong sequentiality' in a somewhat different way. Our sequentiality* does not exactly coincide with their sequentiality, but strong sequentiality* and strong sequentiality are equivalent. In order to define these concepts we have to introduce some more formalism.

We add a fresh constant Ω to our signature, representing an unknown part of a term. The set of Ω-terms $\mathcal{T}(\mathcal{F} \cup \{\Omega\}, \mathcal{V})$ is abbreviated to $\mathcal{T}_\Omega$. If $t \in \mathcal{T}_\Omega$ then we write $O_\Omega(t)$ for the Ω-positions of t, i.e. $O_\Omega(t) = \{u \in O(t) \mid t/u \equiv \Omega\}$. The set $O(t) - O_\Omega(t)$ is denoted by $\overline{O}(t)$. An *Ω-normal form* is an Ω-term without redexes, containing at least one occurrence of Ω. We reserve the phrase *normal form* for terms containing neither redexes nor Ω's. So every Ω-term without redexes is either a normal form or an Ω-normal form. The set of all normal forms is denoted by NF and NF_Ω denotes the set of all Ω-normal forms. The *prefix ordering* $\le$ on $\mathcal{T}_\Omega$ is defined as follows:

- $x \le x$ for every $x \in \mathcal{V}$,
- $\Omega \le t$ for every $t \in \mathcal{T}_\Omega$,
- if $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{T}_\Omega$ such that $s_i \le t_i$ for $i = 1, \ldots, n$ then $F(s_1, \ldots, s_n) \le F(t_1, \ldots, t_n)$ for every n-ary $F \in \mathcal{F}$.

We write $s < t$ if $s \le t$ and $s \not\equiv t$. Clearly, $s \le t$ if and only if $s \equiv C[\Omega, \ldots, \Omega]$ and $t \equiv C[t_1, \ldots, t_n]$ for some context $C[\,, \ldots, \,]$ not containing Ω's and Ω-terms $t_1, \ldots, t_n$. The *greatest lower bound* of two Ω-terms s and t with respect to $\le$ is denoted by $s \cap t$.

DEFINITION 3.8.
(1) A predicate P on $\mathcal{T}_\Omega$ is *monotonic* if $P(t)$ implies $P(t')$ whenever $t \le t'$.
(2) We define predicates *nf* and $nf_?$ on $\mathcal{T}_\Omega$ as follows: *nf*(t) holds if t has a normal form and $nf_?(t)$ holds if there exists an arbitrary reduction sequence from t to some normal form.

It is easily proved that *nf* and $nf_?$ are monotonic predicates.

DEFINITION 3.9.
(1) Let P be a predicate on $\mathcal{T}_\Omega$. An Ω-position u of an Ω-term t is an *index* with respect to P if every Ω-term t' with $t' \ge t$ and $P(t')$ satisfies $t'/u \not\equiv \Omega$. (In particular, if t has an index with respect to P then $P(t)$ does not hold.) The set of indices of t with respect to P is denoted by $I_P(t)$.
(2) An orthogonal TRS $\mathcal{R}$ is *sequential* if every Ω-normal form has an index with respect to *nf* and $\mathcal{R}$ is *strongly sequential* if every Ω-normal form has an index with respect to $nf_?$.

Figure 4 exhibits the relationship between the properties introduced so far. The equivalence of strong sequentiality* and strong sequentiality is an immediate consequence of the following observation. Consider a term $t \equiv C[r_1, \ldots, r_n]$ with context $C[\,, \ldots, \,]$ in normal form and outermost redexes $r_1, \ldots, r_n$ at positions $u_1, \ldots, u_n$ respectively. Redex r_i is strongly needed if and only if position u_i is an index of $C[\Omega, \ldots, \Omega]$ with respect to $nf_?$. Notice that not

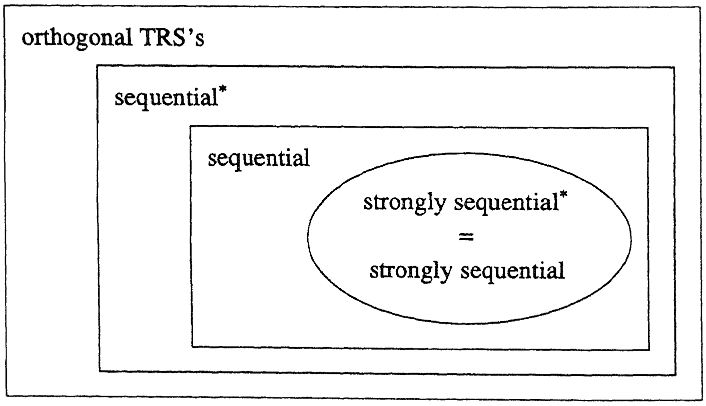


FIGURE 4.

every sequential* TRS is sequential. Consider for instance Berry's TRS

$$\mathcal{R} = \begin{cases} F(A, B, x) & \to & C \\ F(B, x, A) & \to & C \\ F(x, A, B) & \to & C. \end{cases}$$

Using the fact that redexes can only be contracted to C, one easily shows that all outermost redexes of a given term are needed. Hence $\mathcal{R}$ is sequential*. But $\mathcal{R}$ is not sequential: the Ω-term $F(\Omega, \Omega, \Omega)$ does not have an index with respect to *nf*.

4. Indices with respect to Strong Sequentiality

In this section we describe a procedure of Huet and Lévy to compute the indices of a given Ω-term with respect to $nf_?$. First we prove two useful properties of indices, not necessarily with respect to $nf_?$.

PROPOSITION 4.1. *Let P be a monotonic predicate on $\mathcal{T}_\Omega$ and let $t \in \mathcal{T}_\Omega$.*
(1) *If $u \in I_P(t)$, $t \le t'$ and $t'/u \equiv \Omega$ then $u \in I_P(t')$.*
(2) *If $uv \in I_P(t)$ then $u \in I_P(t[u \leftarrow \Omega])$.*

PROOF.
(1) If $u \notin I_P(t')$ then there exists a term $t'' \ge t'$ such that $t''/u \equiv \Omega$ and $P(t'')$ is true. Clearly $t'' \ge t$ and therefore $u \notin I_P(t)$.
(2) If $u \notin I_P(t[u \leftarrow \Omega])$ then there exists a term $t' \ge t[u \leftarrow \Omega]$ such that $t'/u \equiv \Omega$ and $P(t')$ holds. Let $t'' \equiv t'[u \leftarrow t/u]$. From $t'' \ge t'$ and the monotonicity of P we obtain $P(t'')$. Together with $t''/uv \equiv \Omega$, this implies $uv \notin I_P(t'')$.

□

These properties are depicted in Figure 5, where an arrow points to an index with respect to P. In the remainder of this paper index means index with respect to $nf_?$, unless stated otherwise. Furthermore, we abbreviate $I_{nf_?}$ to I.

DEFINITION 4.2.
(1) An Ω-term t is *redex compatible* if t can be refined to a redex (i.e. $t \le t'$ for some redex t').
(2) The reduction relation $\to_\Omega$ (*Ω-reduction*) is defined as follows:

$$C[t] \to_\Omega C[\Omega]$$

for every context $C[\,]$ and redex compatible term $t \not\equiv \Omega$.

EXAMPLE 4.3. Let

$$\mathcal{R} = \begin{cases} F(F(A, x), y) & \to & x \\ G(B, B) & \to & A \end{cases}$$

and $t \equiv F(F(\Omega, A), G(B, \Omega))$. Figure 6 shows all Ω-reductions starting from t.

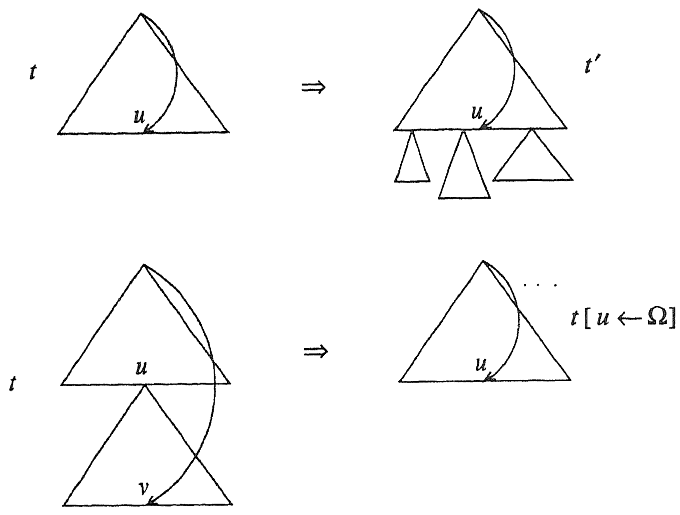


FIGURE 5.

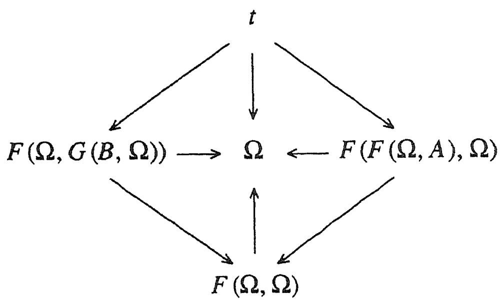


FIGURE 6.

The next proposition relates Ω-reduction to arbitrary reduction.

PROPOSITION 4.4.

(1) *If* $s \twoheadrightarrow_\Omega t$ *then* $s' \twoheadrightarrow_? t$ *for some* $s' \geq s$.

(2) *If* $s \twoheadrightarrow_? t$ *then* $s \twoheadrightarrow_\Omega t'$ *for some* $t' \leq t$.

PROOF.

(1) We use induction on the length of $s \twoheadrightarrow_\Omega t$. The case of zero length is trivial. Suppose $s \rightarrow_\Omega t_1 \twoheadrightarrow_\Omega t$. We have $s \equiv C[s_1] \rightarrow_\Omega C[\Omega] \equiv t_1$ for some redex compatible subterm $s_1 \not\equiv \Omega$ of s. From the induction hypothesis we obtain the existence of a term $t_2 \geq t_1$ such that $t_2 \twoheadrightarrow_? t$. Because $t_2 \geq t_1 \equiv C[\Omega]$ we can write $t_2 \equiv C'[t_3]$ for some context $C'[\,] \geq C[\,]$ and term $t_3 \geq \Omega$. Let r be any redex with $s_1 \leq r$. Define $s' \equiv C'[r]$. Clearly $s' \geq s$. We have the following arbitrary reduction:

$$s' \equiv C'[r] \rightarrow_? C'[t_3] \equiv t_2 \twoheadrightarrow_? t.$$

(2) Similar to (1), using the fact that if $t_1 \le t_2 \rightarrow_\Omega t_3$ then $t_1 \rightarrow_\Omega t_4 \le t_3$ for some $t_4 \in \mathcal{T}_\Omega$.

□

PROPOSITION 4.5.

(1) *Ω-reduction is confluent: $\forall s, t_1, t_2 \in \mathcal{T}_\Omega$ if $s \twoheadrightarrow_\Omega t_1$ and $s \twoheadrightarrow_\Omega t_2$ then $\exists t_3 \in \mathcal{T}_\Omega$ such that $t_1 \twoheadrightarrow_\Omega t_3$ and $t_2 \twoheadrightarrow_\Omega t_3$.*

(2) *Ω-reduction is terminating: there are no infinite reduction sequences*

$$t_0 \rightarrow_\Omega t_1 \rightarrow_\Omega t_2 \rightarrow_\Omega \cdots .$$

PROOF.

(1) Let $\rightarrow_\Omega^{=}$ be the reflexive closure of $\rightarrow_\Omega$. Suppose $s \rightarrow_\Omega t_1$ and $s \rightarrow_\Omega t_2$. By considering the relative positions of the redex compatible subterms contracted in both steps, one easily shows the existence of a term $t_3 \in \mathcal{T}_\Omega$ such that $t_1 \rightarrow_\Omega^{=} t_3$ and $t_2 \rightarrow_\Omega^{=} t_3$. From this the confluence of Ω-reduction follows by induction.

(2) This is an immediate consequence of the fact that $\bar{O}(t)$ is a proper subset of $\bar{O}(s)$ whenever $s \rightarrow_\Omega t$.

□

DEFINITION 4.6 (Huet & Lévy, 1979). The *direct approximant* $\omega(t)$ of an Ω-term t is the normal form of t with respect to Ω-reduction. Notice that $\omega(t)$ is well-defined according to the previous proposition.

The direct approximant can intuitively be viewed as the *fixed part* of the term; in the sequel we will also use this term instead of direct approximant. The following properties are heavily used in the sequel. Their simple proofs have been omitted.

PROPOSITION 4.7. *Let $s, t \in \mathcal{T}_\Omega$ and $u \in O(t)$.*

(1) $\omega(t) \le t$.

(2) $\omega(t) \equiv \omega(t[u \leftarrow \omega(t/u)])$.

(3) *If $s \le t$ then $\omega(s) \le \omega(t)$.*

(4) $\omega(\omega(t)) \equiv \omega(t)$.

(5) *If $s \rightarrow_? t$ then $\omega(s) \le \omega(t)$.*

(6) *If t is redex compatible then $\omega(t) \equiv \Omega$.*

□

Let $t \in \mathcal{T}_\Omega$ and $u \in O_\Omega(t)$. Let $\bullet$ be a fresh constant symbol. The following procedure determines whether u is an index of t:

(1)	Replace in t the Ω at position u by $\bullet$, result $t' \equiv t[u \leftarrow \bullet]$.
(2)	Compute the normal form of t' with respect to $\rightarrow_\Omega$, result $\omega(t')$.
(3)	Position u is an index of t if and only if $\bullet$ occurs in $\omega(t')$.

The procedure is illustrated in Figure 7. Intuitively, the persistence of the 'test symbol' $\bullet$ in $\omega(t')$ means that whatever the redexes in the other (Ω-)places are and whatever their reducts might be, the $\bullet$ does not vanish. So if instead of $\bullet$ an actual redex r was present, the only way to

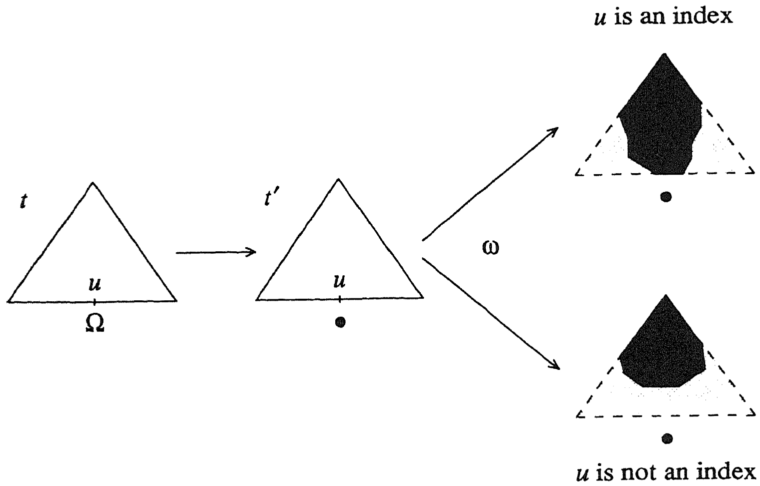


FIGURE 7.

($\rightarrow_?$-) normalize the term at hand is to reduce r itself, eventually. The formal justification of the above procedure is given by the following lemma.

LEMMA 4.8. *Let $t \in \mathcal{T}_\Omega$ and $u \in O_\Omega(t)$. The following three statements are equivalent:*
(1) $u \in I(t)$;
(2) $\omega(t[u \leftarrow \bullet]) \not\equiv \omega(t)$;
(3) $u \in O(\omega(t[u \leftarrow \bullet]))$.

PROOF.

(1) $\Rightarrow$ (2) If $\omega(t[u \leftarrow \bullet]) \equiv \omega(t)$ then $t[u \leftarrow \bullet] \twoheadrightarrow_\Omega \omega(t)$. Proposition 4.4(1) yields a term t' such that $t' \twoheadrightarrow_? \omega(t)$ and $t' \geq t[u \leftarrow \bullet]$. Let $t'' \equiv t'[\Omega \leftarrow x][u \leftarrow \Omega]$ and $\omega(t)' \equiv \omega(t)[\Omega \leftarrow x]$ for some variable x. It is not difficult to see that we can transform the reduction $t' \twoheadrightarrow_? \omega(t)$ into $t'' \twoheadrightarrow_? \omega(t)'$. Because $\omega(t)$ is an Ω-normal form, $\omega(t)'$ is a normal form and hence $nf_?(t'')$ is true. Clearly $t'' \geq t$ and $t''/u \equiv \Omega$. Therefore $u \notin I(t)$.

(2) $\Rightarrow$ (3) If $u \notin O(\omega(t[u \leftarrow \bullet]))$ then $\omega(t[u \leftarrow \bullet]) \leq t$ and thus $\omega(t[u \leftarrow \bullet]) \leq \omega(t)$. Because $t \leq t[u \leftarrow \bullet]$ we also have $\omega(t) \leq \omega(t[u \leftarrow \bullet])$. Combining these two facts, we obtain $\omega(t[u \leftarrow \bullet]) \equiv \omega(t)$.

(3) $\Rightarrow$ (1) If $u \notin I(t)$ then there exists a term $t' \geq t$ such that $t'/u \equiv \Omega$ and $nf_?(t')$ is true. Thus we have an arbitrary reduction $t' \twoheadrightarrow_? n$ from t' to some normal form n. Because n does not contain any occurrences of Ω, we can transform this reduction into $t'[u \leftarrow \bullet] \twoheadrightarrow_? n$. Using Proposition 4.7 or the second part of Proposition 4.4, we obtain $\omega(t'[u \leftarrow \bullet]) \leq n$. Now suppose $u \in O(\omega(t[u \leftarrow \bullet]))$. As $\bullet$ is not redex compatible, $\omega(t[u \leftarrow \bullet])/u \equiv \bullet$. But this is contradictory to $\omega(t'[u \leftarrow \bullet]) \leq n$ and therefore $u \notin O(\omega(t[u \leftarrow \bullet]))$.

□

The decision procedure for strong sequentiality is much more difficult. The main problem is that we do *not* have the following transitivity property for indices, which at first sight one might expect to hold: if $u \in I(s)$ and $v \in I(t)$ then $uv \in I(s[u \leftarrow t])$.

EXAMPLE 4.9. Consider the TRS $\mathcal{R} = \{F(G(x)) \to x\}$. Position 1 is an index of $F(\Omega)$, as is easily seen by applying the '$\bullet$-test': $\omega(F(\bullet)) \equiv F(\bullet)$. Similarly, position 1 is an index of $G(\Omega)$. However, position 1.1 is not an index of $F(G(\Omega))$ because $\omega(F(G(\bullet))) \equiv \Omega$.

The next two propositions express properties of indices which are used in the proof of the decidability of strong sequentiality. They originate from Huet & Lévy (1979).

PROPOSITION 4.10. *If $uv \in I(t)$ then $v \in I(t/u)$.*

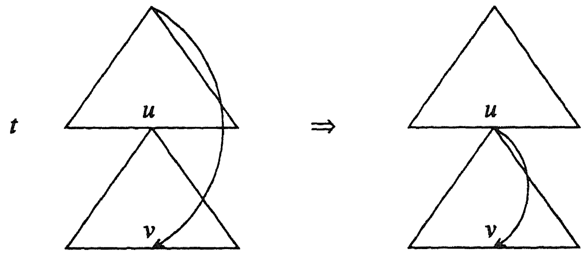


FIGURE 8.

PROOF. If $v \notin I(t/u)$ then $\omega((t/u)[v \leftarrow \bullet]) \equiv \omega(t/u)$ by Lemma 4.8. Therefore $\omega(t[uv \leftarrow \bullet]) \equiv \omega(t[u \leftarrow \omega((t/u)[v \leftarrow \bullet])]) \equiv \omega(t[u \leftarrow \omega(t/u)]) \equiv \omega(t)$ and from Lemma 4.8 we obtain $uv \notin I(t)$. □

PROPOSITION 4.11. *If $u \in I(t)$, $u \perp v$ and $\omega(t/v) \equiv \Omega$ then $u \in I(t[v \leftarrow \Omega])$.*

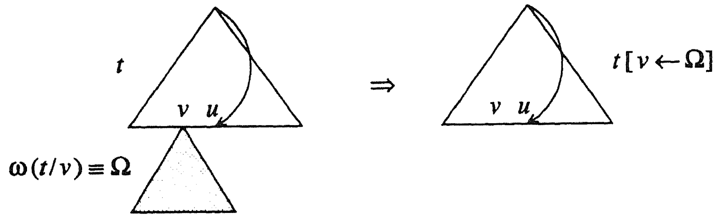


FIGURE 9.

PROOF. If $u \notin I(t[v \leftarrow \Omega])$ then $\omega(t[v \leftarrow \Omega][u \leftarrow \bullet]) \equiv \omega(t[v \leftarrow \Omega])$ by Lemma 4.8. Proposition 4.7 yields $\omega(t) \equiv \omega(t[v \leftarrow \omega(t/v)]) \equiv \omega(t[v \leftarrow \Omega])$ and likewise $\omega(t[u \leftarrow \bullet]) \equiv \omega(t[u \leftarrow \bullet][v \leftarrow \Omega])$. Hence $\omega(t[u \leftarrow \bullet]) \equiv \omega(t)$. Another application of Lemma 4.8 gives $u \notin I(t)$. □

The next example shows that the condition $\omega(t/v) \equiv \Omega$ in Proposition 4.11 is necessary.

EXAMPLE 4.12. Consider the TRS of Example 3.7. We have $1.1 \in I(F(G(\Omega, \Omega), B))$, $1.1 \perp 2$ and $\omega(B) \equiv B$, but position 1.1 is not an index of $F(G(\Omega, \Omega), \Omega)$.

5. Decidability of Strong Sequentiality

DEFINITION 5.1. A term $t \in \mathrm{NF}_\Omega$ is called *free of indices* (or *free* for short) if $I(t) = \emptyset$.

By definition, a TRS $\mathcal{R}$ is strongly sequential if and only if $\mathcal{R}$ does not have free terms. In an attempt to decide whether a given orthogonal TRS is strongly sequential, we will try to construct a free term. We are particularly interested in a *minimal* free term, minimal with respect to the number of non-Ω-positions (so $F(\Omega, \Omega)$ is considered to be smaller than $F(A, \Omega)$). We first prove that a minimal free term t, if it exists, satisfies $\omega(t) \equiv \Omega$.

DEFINITION 5.2. Let $t \in \mathcal{T}_\Omega$. We call t *rigid* if $\omega(t) \equiv t$ and t is called *soft* if $\omega(t) \equiv \Omega$. The subset of soft terms of NF_Ω is denoted by NF_s.

Notice that Ω is the only Ω-term which is both rigid and soft. Soft terms 'melt away' completely by Ω-reduction. Because $\omega(t) \le t$, every Ω-term t can be written as $t \equiv \omega(t)[u_i \leftarrow t_i \mid 1 \le i \le n]$ where $\{u_1, \dots, u_n\} = O_\Omega(\omega(t))$ and $t_i \equiv t/u_i$ $(i = 1, \dots, n)$. Notice that $\omega(t)$ is rigid and $t_1, \dots, t_n$ are soft.

PROPOSITION 5.3. *Let $t \equiv \omega(t)[u_i \leftarrow t_i \mid 1 \le i \le n]$ with $O_\Omega(\omega(t)) = \{u_1, \dots, u_n\}$. If $v \in I(t_i)$ then $u_i v \in I(t)$.*

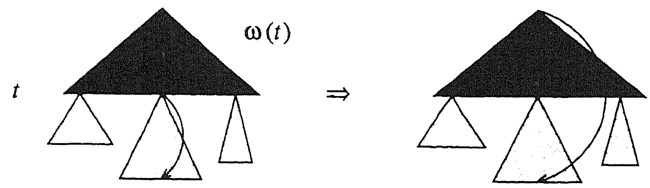


FIGURE 10.

PROOF. By Lemma 4.8 it is sufficient to show that $\omega(t[u_i v \leftarrow \bullet])$ and $\omega(t)$ are different. We have

$$\omega(t[u_i v \leftarrow \bullet]) \equiv \omega(t[u_i \leftarrow \omega(t_i[v \leftarrow \bullet])]) \equiv \omega(t)[u_i \leftarrow \omega(t_i[v \leftarrow \bullet])]$$

where the first identity follows from Proposition 4.7 and the second identity is due to the fact that $u_i \in O_\Omega(\omega(t))$ and $\omega(t)$, $\omega(t_i[v \leftarrow \bullet])$ are rigid terms. Because $v \in I(t_i)$ and t_i is a soft term, $\omega(t_i[v \leftarrow \bullet]) \not\equiv \Omega$. Therefore $\omega(t[u_i v \leftarrow \bullet]) \not\equiv \omega(t)$. □

COROLLARY 5.4. *A* TRS *is strongly sequential if and only if* NF_s *does not contain free terms.* □

Let t be a soft term. The next example shows that every Ω-reduction $t \twoheadrightarrow_\Omega \Omega$ induces a partition of t into redex compatible subterms. This idea is formalized in Definition 5.6.

EXAMPLE 5.5. Let

$$\mathcal{R} = \begin{cases} F(x, G(y, A)) & \to x \\ G(A, B) & \to A \end{cases}$$

and $t \equiv F(F(A, G(\Omega, \Omega)), F(\Omega, G(B, \Omega)))$. Figure 11(i) shows the decomposition of t into redex compatible terms with respect to the Ω-reduction

$$F(F(A, \underline{G(\Omega, \Omega)}), F(\Omega, G(B, \Omega))) \to_\Omega F(F(A, \Omega), \underline{F(\Omega, G(B, \Omega))})$$
$$\to_\Omega F(\underline{F(A, \Omega)}, \Omega) \to_\Omega \underline{F(\Omega, \Omega)} \to_\Omega \Omega$$

and Figure 11(ii) shows the decomposition corresponding to the Ω-reduction

$$F(F(A, G(\Omega, \Omega)), \underline{F(\Omega, G(B, \Omega))}) \to_\Omega \underline{F(F(A, G(\Omega, \Omega)), \Omega)} \to_\Omega \Omega.$$

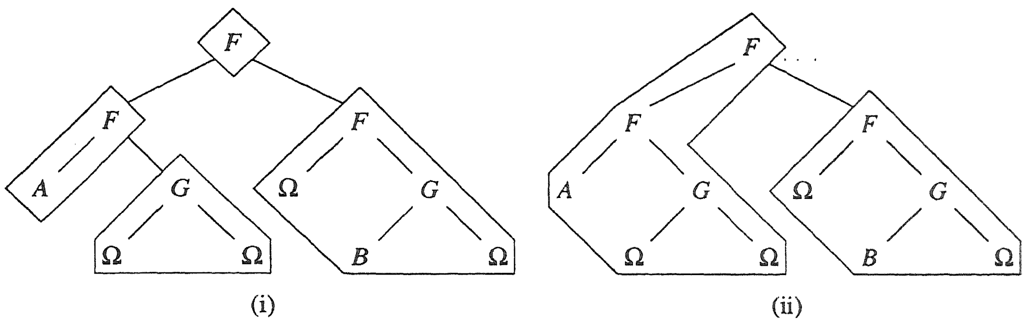


FIGURE 11.

DEFINITION 5.6. Let $t \in \mathcal{T}_\Omega$ be a soft term. Let

$$t \equiv t_0 \to_\Omega t_1 \to_\Omega \cdots \to_\Omega t_n \equiv \Omega$$

be any Ω-reduction from t to Ω and suppose that in step $t_i \to_\Omega t_{i+1}$ the redex compatible term at position u_i is replaced by Ω. Then the set $\{\langle u_i, t_i/u_i\rangle \mid 0 \le i \le n-1\}$ is a *decomposition* of t.

EXAMPLE 5.7. The Ω-reductions of the previous example correspond to the following two decompositions of $F(F(A, G(\Omega, \Omega)), F(\Omega, G(B, \Omega)))$:

$$\{\langle \lambda, F(\Omega, \Omega)\rangle, \langle 1, F(A, \Omega)\rangle, \langle 1.2, G(\Omega, \Omega)\rangle, \langle 2, F(\Omega, G(B, \Omega))\rangle\},$$
$$\{\langle \lambda, F(F(A, G(\Omega, \Omega)), \Omega)\rangle, \langle 2, F(\Omega, G(B, \Omega))\rangle\}.$$

A minimal free term is soft and hence built from redex compatible terms. However, this observation is not yet sufficient for a sensible attempt to construct a minimal free term, for there are in general infinitely many redex compatible terms. Fortunately, we may even suppose that a minimal free term is built from a special kind of redex compatible terms, the so-called *preredexes*, of which only finitely many exist.

DEFINITION 5.8.

(1) A *redex scheme* is a left-hand side of a rewrite rules in which all variables are replaced by Ω.

(2) A *preredex* is a term which can be refined to a redex scheme. A preredex is *proper* if it is neither a redex scheme nor equal to Ω.

(3) Two Ω-terms t_1, t_2 are *compatible* if there exists an Ω-term t_3 such that $t_1 \le t_3$ and $t_2 \le t_3$.

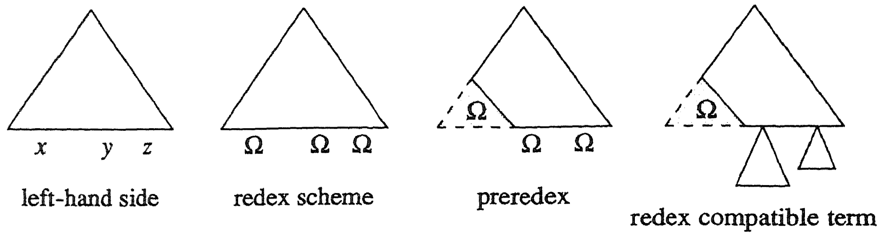


FIGURE 12.

Clearly, t is redex compatible if and only if t is compatible with a redex scheme. Notice that every preredex is redex compatible and every redex scheme is a preredex. Because we consider only TRS's with a finite number of rewrite rules, there are only finitely many preredexes.

EXAMPLE 5.9. Let

$$\mathcal{R} = \begin{cases} F(A, F(B, x)) & \to x \\ F(C, x) & \to x. \end{cases}$$

The preredexes of $\mathcal{R}$ are listed below:

Ω,

$F(\Omega, \Omega)$, $F(A, \Omega)$, $F(\Omega, F(\Omega, \Omega))$, $F(A, F(\Omega, \Omega))$, $F(\Omega, F(B, \Omega))$,

$F(A, F(B, \Omega,))$, $F(C, \Omega)$.

The second row contain^c all proper preredexes and the last two preredexes are redex schemes.

We now associate with every redex compatible term a preredex. According to Proposition 5.12 below, this transformation preserves the property of being free.

DEFINITION 5.10. Let $t \in \mathcal{T}_\Omega$ be redex compatible. Like Procrustes, we cut off all parts of t that stick out:

$$cut(t) = t \cap r_1 \cap \ldots \cap r_n,$$

$$O_{cut}(t) = \bar{O}(t) \cap O_\Omega(cut(t)),$$

where $\{r_1, \ldots, r_n\}$ is the set of all redex schemes compatible with t. Notice that $O_{cut}(t)$ is the set of Ω-positions that are created in cutting down t to $cut(t)$.

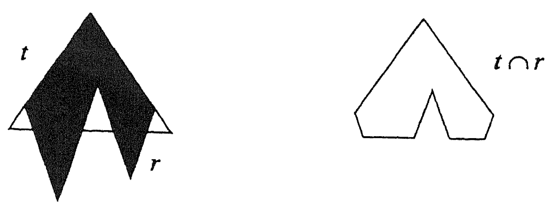


FIGURE 13.

PROPOSITION 5.11. *Let $t \in \mathcal{T}_\Omega$ be redex compatible. If $u \in O_{cut}(t)$ then $u \notin I(cut(t))$.*

PROOF. Suppose $u \in O_{cut}(t)$. Let R be the non-empty set of redex schemes compatible with t. It is easy to show that there exists a $r \in R$ such that $u \in O_\Omega(r)$. Because $r \geq cut(t)$ and $I(r) = \emptyset$ we obtain $u \notin I(cut(t))$ from Proposition 4.1. □

PROPOSITION 5.12. *If $t \in \mathcal{T}_\Omega$ is redex compatible then $I(cut(t)) \subseteq I(t)$.*

PROOF. If $u \in I(cut(t))$ then $u \in O_\Omega(cut(t))$. According to the previous proposition we cannot have $u \in O_{cut}(t)$, hence $u \in O_\Omega(t)$. Proposition 4.1 yields $u \in I(t)$. □

So the 'Procrustes procedure' does not create new indices. We may however loose some indices.

EXAMPLE 5.13. Let

$$\mathcal{R} = \begin{cases} F(A, F(x, A, A), A) & \to x \\ F(B, x, B) & \to x. \end{cases}$$

The term $t \equiv F(A, F(A, \Omega, \Omega), A)$ is redex compatible. We have $I(t) = \{2.2, 2.3\}$, $cut(t) \equiv F(A, F(\Omega, \Omega, \Omega), A)$ and $I(cut(t)) = \{2.3\}$.

The following example shows how to extend the 'Procrustes procedure' to soft terms.

EXAMPLE 5.14. Let

$$\mathcal{R} = \begin{cases} F(G(A, x), y) & \to x \\ F(G(B, x), G(B, x)) & \to x \\ G(C, C) & \to C \end{cases}$$

and $t \equiv F(F(G(F(\Omega, A), \Omega), F(\Omega, G(C, \Omega))), G(B, \Omega))$. Figure 14(i) shows a decomposition of t. If we replace the redex compatible term $t' \equiv F(G(\Omega, \Omega), F(\Omega, \Omega))$ at position 1 by $cut(t') = F(G(\Omega, \Omega), \Omega)$ we obtain Figure 14(ii). Notice that we have lost one redex compatible term, viz. $G(C, \Omega)$ at position 1.2.2.

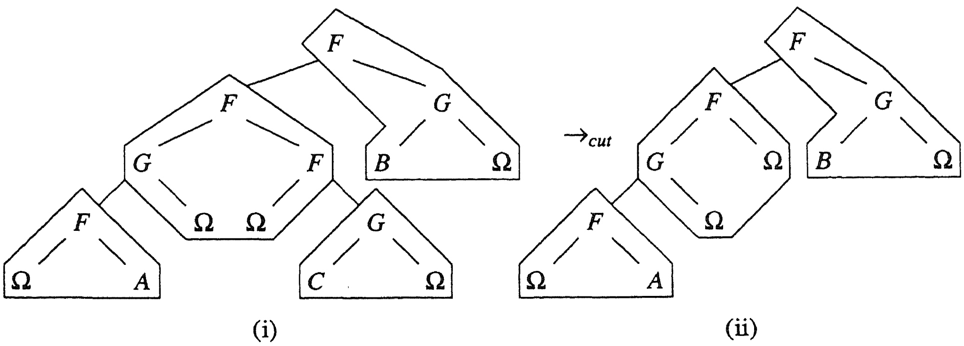


FIGURE 14.

DEFINITION 5.15. Let D be a decomposition of a soft term t. We write $t \rightarrow_{cut} t'$ if $t' \equiv t[uv \leftarrow \Omega \mid v \in O_{cut}(s)]$ for some $\langle u, s\rangle \in D$ such that $cut(s) \not\equiv s$.

PROPOSITION 5.16. *If* $t \rightarrow_{cut} t'$ *then* $t' < t$ *and* $I(t') \subseteq I(t)$.

PROOF. The first part is obvious. Suppose $w \in I(t')$. If $w \in O_\Omega(t)$ then $w \in I(t)$ by Proposition 4.1. So let us assume $w \notin O_\Omega(t)$. We know that $t' \equiv t[uv \leftarrow \Omega \mid v \in O_{cut}(s)]$ for some $\langle u, s\rangle$ in some decomposition of t, and hence $w = uv$ for some $v \in O_{cut}(s)$. From Proposition 4.10 we obtain $v \in I(t'/u)$. Together with $cut(s) \leq t'/u$ and $v \in O_{cut}(s)$ this gives us $v \in I(cut(s))$, by repeated application of Proposition 4.11. This is contradictory to Proposition 5.11. □

PROPOSITION 5.17. *Let t be a soft term. If* $t \rightarrow_{cut} t'$ *and* t' *is a* $\rightarrow_{cut}$*-normal form, then* $t' \leq t$, $I(t') \subseteq I(t)$ *and every decomposition of* t' *contains only proper preredexes.*

PROOF. This is an immediate consequence of Definition 5.15 and Proposition 5.16. □

The subset of $NF_s - \{\Omega\}$ consisting of all normal forms with respect to $\rightarrow_{cut}$ is denoted by NF_{cut}. The reason for excluding Ω is only a matter of convenience. Notice that $I(\Omega) = \{\lambda\}$ because the left-hand side of a rewrite rule is not a variable.

COROLLARY 5.18. *A* TRS *is strongly sequential if and only if* NF_{cut} *does not contain free terms.* □

We will now show that we only have to consider terms of NF_{cut} with a bounded depth, in order to decide whether a TRS is strongly sequential.

DEFINITION 5.19. The *depth* $\rho(t)$ of an Ω-term t is defined by

$$\rho(t) = \begin{cases} 1 + \max\{\rho(t_1), \ldots, \rho(t_n)\} & \text{if } t \equiv F(t_1, \ldots, t_n) \text{ and } n \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\rho(t) = \max\{|u| \mid u \in O(t)\}$. The maximum depth of the left-hand sides of the rewrite rules of a given TRS $\mathcal{R}$ is denoted by $\rho_{\mathcal{R}}$. When $\mathcal{R}$ can be inferred from the context we simply write ρ.

The following lemma states a partial transitivity result for index propagation. It plays a crucial role in our first proof of the decidability of strong sequentiality, because it enables us to restrict the search for a free term to a finite set of Ω-terms which are entirely built from preredexes.

LEMMA 5.20. *Let* $t \in \mathcal{T}_\Omega$, $u, v \in O(t)$ *and* $w \in O_\Omega(t)$ *such that* $u \leq v < w$. *If* $v \in I(t[v \leftarrow \Omega])$, $w/u \in I(t/u)$ *and* $|v/u| \geq \rho - 1$, *then* $w \in I(t)$.

PROOF. Suppose $w \notin I(t)$. According to Lemma 4.8 $w \notin O(\omega(t[w \leftarrow \bullet]))$ and hence there exists an Ω-reduction

$$t[w \leftarrow \bullet] \twoheadrightarrow_\Omega t_1 \rightarrow_\Omega t_2 \twoheadrightarrow_\Omega \omega(t[w \leftarrow \bullet])$$

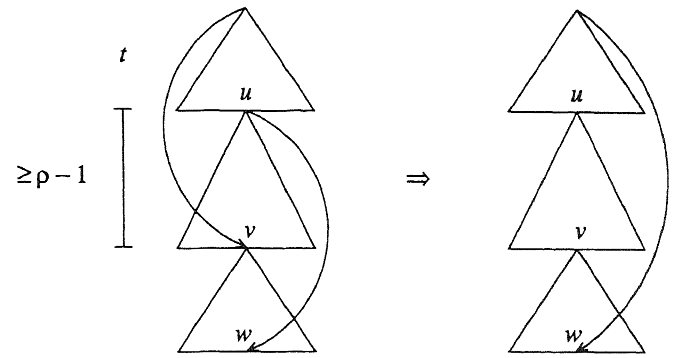


FIGURE 15.

such that $t_1/w \equiv \bullet$ and $w \notin O(t_2)$. Let t_1/u' be the redex compatible subterm contracted in the step $t_1 \to_\Omega t_2$. We have $u' < w$. We distinguish two cases: (1) $u \le u' < w$ and (2) $u' < u$.

(1) Because $u \in O(t_2)$ we can transform the Ω-reduction $t[w \leftarrow \bullet] \twoheadrightarrow_\Omega t_1 \to_\Omega t_2$ into

$$t[w \leftarrow \bullet]/u \equiv t/u[w/u \leftarrow \bullet] \twoheadrightarrow_\Omega t_1/u \to_\Omega t_2/u.$$

Clearly $w/u \notin O(t_2/u)$ and therefore $w/u \notin O(\omega(t_2/u)) = O(\omega(t/u[w/u \leftarrow \bullet]))$. This contradicts the assumption $w/u \in I(t/u)$.

(2) Let r be a redex scheme compatible with t_1/u'. Consider the term $t'_1 \equiv t_1[v \leftarrow \bullet]$. We have $|v/u'| > |v/u| \ge \rho - 1$, so if t'_1/u' is not compatible with r, then $v/u' \in \bar{O}(r)$. Because t_1/v is not a constant, $r(v/u')$ must be a function symbol of arity greater than zero. But then $\rho(r) \ge \rho + 1$, which is impossible. So t'_1/u' is redex compatible. Noting that position v is preserved in $t[w \leftarrow \bullet] \twoheadrightarrow_\Omega t_1$, we now transform the Ω-reduction $t[w \leftarrow \bullet] \twoheadrightarrow_\Omega t_1 \to_\Omega t_2$ into

$$t[v \leftarrow \bullet] \twoheadrightarrow_\Omega t_1[v \leftarrow \bullet] \equiv t'_1 \to_\Omega t'_1[u' \leftarrow \Omega] \equiv t_2.$$

A similar argument as in the previous case shows the impossible $v \notin I(t[v \leftarrow \Omega])$.

□

The bound $\rho - 1$ in Lemma 5.20 cannot be relaxed, as the following example shows.

EXAMPLE 5.21. Let $\mathcal{R} = \{F(G(H(x))) \to x\}$ and $t \equiv F(G(H(\Omega)))$. Take $u = 1$, $v = 1.1$ and $w = 1.1.1$. We have $v \in I(t[v \leftarrow \Omega]) = I(F(G(\Omega))) = \{1.1\}$, $w/u \in I(t/u) = I(G(H(\Omega))) = \{1.1\}$ and $|v/u| = 1 = \rho - 2$, but $w \notin I(t) = \varnothing$.

PROPOSITION 5.22. *If t is a minimal free term then $I(t[u \leftarrow \Omega]) = \{u\}$ for all $u \in \bar{O}(t)$.*

PROOF. Because $\bar{O}(t[u \leftarrow \Omega])$ is a proper subset of $\bar{O}(t)$ we have $I(t[u \leftarrow \Omega]) \ne \varnothing$. Let $v \in I(t[u \leftarrow \Omega])$. According to Proposition 4.1 v cannot be disjoint from u, hence $I(t[u \leftarrow \Omega]) = \{u\}$. □

PROPOSITION 5.23. *If $t \in \mathrm{NF}_\Omega$, $u \in O_\Omega(t)$ and $s \in \mathrm{NF}_s$ then $t[u \leftarrow s] \in \mathrm{NF}_\Omega$.*

PROOF. Let $D = \{\langle u_i, s_i \rangle \mid 1 \le i \le n\}$ be a decomposition of s. Without loss of generality we may assume that $i < j$ whenever $u_i < u_j$. Define a sequence of terms $t_0 < t_1 < \ldots < t_n$ as follows:

$$t_i = \begin{cases} t & \text{if } i=0, \\ t_{i-1}[uu_i \leftarrow s_i] & \text{if } 1 \leq i \leq n. \end{cases}$$

Clearly $t_n \equiv t[u \leftarrow s]$. We will show that $t_i \in \mathrm{NF}_\Omega$ by induction on i. The case $i=0$ is trivial. Suppose $i \geq 1$. If $t_i \notin \mathrm{NF}_\Omega$ then there exists a position $v \in O(t_i)$ and a redex scheme r_1 such that $t_i/v \geq r_1$. The cases $u \perp v$ and $v \geq u$ are easily shown to be contradictory to the assumptions $t \in \mathrm{NF}_\Omega$ and $s \in \mathrm{NF}_s$. Hence $v < u$ and thus $t_i/v \equiv t_{i-1}/v[uu_i/v \leftarrow s_i]$. Notice that $uu_i/v \in O_\Omega(t_{i-1}/v)$. Using the induction hypothesis we obtain $t_{i-1}/v \in \mathrm{NF}_\Omega$ and so $uu_i/v \in \bar{O}(r_1)$. Because s_i is redex compatible there exists a redex r_2 with $s_i \leq r_2$. But now the term $t_{i-1}/v[uu_i/v \leftarrow r_2]$ contains overlapping redex schemes, which is impossible in an orthogonal TRS. We conclude that $t[u \leftarrow s] \in \mathrm{NF}_\Omega$. □

We will now try to construct a minimal free term t in a tree-like procedure, as suggested in Figure 16. We start with the finitely many proper preredexes. In the next construction step we attach at every index position again a proper preredex, such that the resulting term is in Ω-normal form. (According to Propositions 4.11 and 5.22 there is no need to attach proper

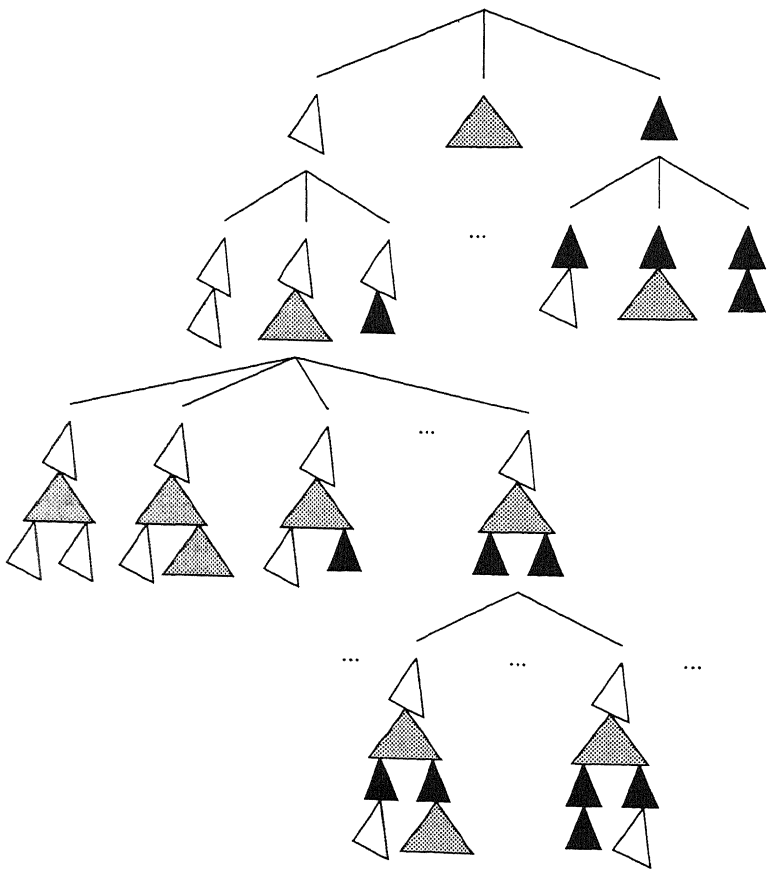


FIGURE 16.

preredexes at non-index positions.) A branch in the thus originating tree of construction terminates 'successfully' if a free term is reached. In that case the term rewriting system under consideration is not strongly sequential.

DEFINITION 5.24. Let D be a decomposition of a term $t \in \mathrm{NF}_{cut}$.

(1) A non-empty subset D' of D is a *tower of preredexes* if the following two conditions are satisfied:
- if $\langle u_1, s_1 \rangle$ and $\langle u_2, s_2 \rangle$ are different elements of D' then either $u_1 < u_2$ or $u_2 < u_1$;
- if $\langle u_1, s_1 \rangle, \langle u_2, s_2 \rangle \in D'$ and $\langle u, s \rangle \in D$ such that $u_1 < u < u_2$ then $\langle u, s \rangle \in D'$.

For convenience we will assume that $u_1 < u_2 < \ldots < u_n$ whenever $\{\langle u_i, s_i \rangle \mid 1 \le i \le n\}$ is a tower of preredexes. A *main tower* is a tower of preredexes $\{\langle u_i, s_i \rangle \mid 1 \le i \le n\}$ satisfying the additional requirements that $u_1 = \lambda$ and there is no element $\langle u, s \rangle \in D$ with $u_n < u$.

(2) Let $D' = \{\langle u_i, s_i \rangle \mid 1 \le i \le n\}$ be a tower of preredexes. The term $\pi(D')$ is defined as follows:

$$\pi(D') = \begin{cases} s_1 & \text{if } n = 1, \\ \pi(\{\langle u_i, s_i \rangle \mid 1 \le i \le n-1\})[u_n/u_1 \leftarrow s_n] & \text{if } n > 1. \end{cases}$$

(3) A tower of preredexes $\{\langle u_i, s_i \rangle \mid 1 \le i \le n\}$ is *special* if $|u_n/u_1| \ge \rho - 1$.

EXAMPLE 5.25. Let

$$\mathcal{R} = \begin{cases} F(G(x, F(y, A)), A) & \to x \\ G(x, B) & \to x \end{cases}$$

and consider the term $F(F(G(\Omega, \Omega), G(\Omega, \Omega)), G(\Omega, \Omega))$ with decomposition $\{\langle \lambda, F(\Omega, \Omega) \rangle, \langle 1, F(G(\Omega, \Omega), \Omega) \rangle, \langle 1.2, G(\Omega, \Omega) \rangle, \langle 2, G(\Omega, \Omega) \rangle\}$, see Figure 17. Table 1 lists all towers of preredexes containing at least two elements.

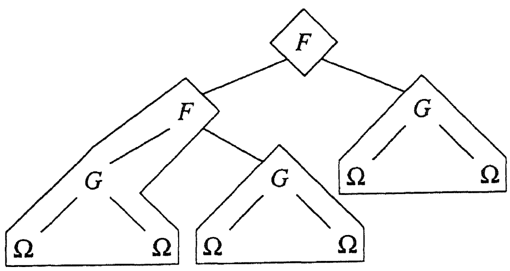


FIGURE 17.

If we observe at some branch in the construction tree the arising of a term which has a main tower containing two occurrences of a special tower of preredexes, that branch is stopped unsuccessfully. This is justified in the next lemma.

LEMMA 5.26. *Suppose t is a minimal free term and let D be a decomposition of t. If a main tower $D' \subseteq D$ contains two distinct special towers of preredexes D_1, D_2 then $\pi(D_1) \not\equiv \pi(D_2)$.*

tower of preredexes	main	special
$\{\langle\lambda, F(\Omega,\Omega)\rangle, \langle 1, F(G(\Omega,\Omega),\Omega)\rangle\}$	×	
$\{\langle\lambda, F(\Omega,\Omega)\rangle, \langle 2, G(\Omega,\Omega)\rangle\}$	×	
$\{\langle 1, F(G(\Omega,\Omega),\Omega)\rangle, \langle 1.2, G(\Omega,\Omega)\rangle\}$		
$\{\langle\lambda, F(\Omega,\Omega)\rangle, \langle 1, F(G(\Omega,\Omega),\Omega)\rangle, \langle 1.2, G(\Omega,\Omega)\rangle\}$	×	×

TABLE 1.

PROOF. Suppose a main tower $D' = \{\langle u_i, s_i\rangle \mid 1 \le i \le n\}$ in a decomposition of t contains two special towers of preredexes $D_1 = \{\langle u_i, s_i\rangle \mid j \le i \le k\}$ and $D_2 = \{\langle u_i, s_i\rangle \mid l \le i \le m\}$ such that $j < l$ and $\pi(D_1) \equiv \pi(D_2)$. Let

$$t' \equiv t[u_{k+1} \leftarrow t/u_l v]$$

with $v = u_{k+1}/u_j$, see Figure 18. Using Proposition 5.23 we easily obtain $t' \in \mathrm{NF}_\Omega$. In order to

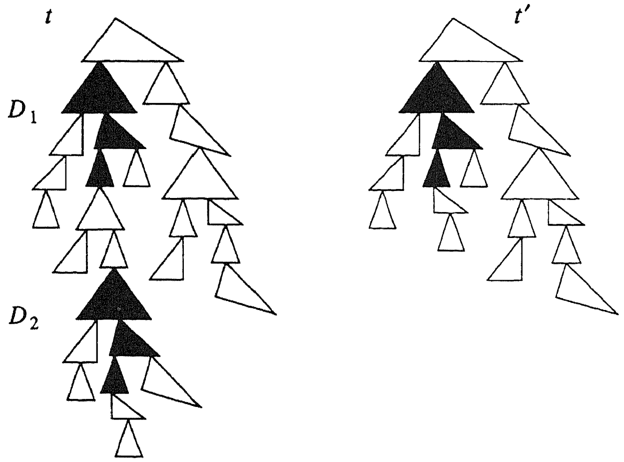


FIGURE 18.

arrive at a contradiction, we will show that t' is a free term. Suppose $w \in I(t')$. If $w \perp u_{k+1}$ then $w \in I(t'[u_{k+1} \leftarrow \Omega]) = I(t[u_{k+1} \leftarrow \Omega])$ by Proposition 4.11 and therefore $w \in I(t)$ using Proposition 4.1. This is impossible because t is free. So if $w \in I(t')$ then $w \ge u_{k+1}$. From Proposition 4.10 we obtain $w/u_j \in I(t'/u_j)$. Repeated application of Proposition 4.11 and a single application of Proposition 4.1 yields $w/u_j \in I(t/u_l)$. From Proposition 5.22 we obtain $u_m \in I(t[u_m \leftarrow \Omega])$. We have $|u_m/u_l| \ge \rho - 1$ since D_2 is special. Applying Lemma 5.20 yields the impossible $u_l(w/u_j) \in I(t)$. Hence t' is a free term and we are done. □

It is not difficult to see that every branch of the construction tree terminates, either successfully in a free term or unsuccessfully in a term containing a repetition of a special tower of preredexes along a main tower. Because the construction is finitely branching, we obtain a finite construction tree. A TRS is strongly sequential if and only if all branches in its construction tree terminate unsuccessfully. Hence we obtain the following result.

COROLLARY 5.27. *Strong sequentiality is a decidable property of orthogonal* TRS'*s*. □

6. Δ-sets and Increasing Indices

Huet and Lévy proved the decidability of strong sequentiality by showing the equivalence of strong sequentiality and the existence of so-called Δ-*sets*:

For every proper preredex t, $\Delta(t)$ is a non-empty subset of $I(t)$ subject to the following constraint: for all $u \in \Delta(t)$, if s is a proper preredex such that $t[u \leftarrow s]$ is again a proper preredex, then $\{v \mid uv \in \Delta(t[u \leftarrow s])\}$ is a non-empty subset of $\Delta(s)$.

Assuming the existence of Δ-sets, Huet and Lévy constructed a 'matching dag', a special kind of graph on which they defined an efficient algorithm to find a strongly needed redex in a given term. (In Huet & Lévy (1979) it is proved that strong sequentiality is equivalent to the existence of a function Q satisfying two constraints Q_1 and Q_2. The equivalent notion of Δ-sets stems from Huet (1986).) Actually, the notion of Δ-sets in Huet & Lévy (1979), Huet (1986) is more complicated than the one we use, since in Huet & Lévy (1979), Huet (1986) it involves so-called 'directions', not introduced in the present paper.

The second part of the equivalence proof (existence of Δ-sets ⇒ strong sequentiality) is in essence a correctness proof of their algorithm. In this section we will give a direct proof of this implication. For the other implication (strong sequentiality ⇒ existence of Δ-sets) we use the increasing indices of Huet & Lévy (1979).

DEFINITION 6.1. Let $t \in \mathcal{T}_\Omega$. A position $u \in I(t)$ is an *increasing index* if for every term $s \in \mathrm{NF}_s$ there exists an index $v \in I(t[u \leftarrow s])$ such that $u \leq v$. The set of all increasing indices of t is denoted by $J(t)$.

The following proposition shows that every term $t \in \mathrm{NF}_\Omega$ has at least one increasing index, provided $\mathcal{R}$ is strongly sequential.

PROPOSITION 6.2. *If $\mathcal{R}$ is strongly sequential then for any term $t \in \mathrm{NF}_\Omega$ we have $J(t) \neq \emptyset$.*

PROOF. Suppose $\mathcal{R}$ is strongly sequential and let $t \in \mathrm{NF}_\Omega$. We have $I(t) \neq \emptyset$, say $I(t) = \{u_1, \ldots, u_n\}$. If $J(t) = \emptyset$ then for every $i \in \{1, \ldots, n\}$ there exists a term $s_i \in \mathrm{NF}_s$ such that $\{v \in I(t[u_i \leftarrow s_i]) \mid v \geq u_i\} = \emptyset$. Consider

$$t' \equiv t[u_i \leftarrow s_i \mid 1 \leq i \leq n].$$

Repeated application of Proposition 5.23 yields $t' \in \mathrm{NF}_\Omega$. Hence $I(t') \neq \emptyset$. Let $v \in I(t')$. If $v \geq u_i$ for some $i \in \{1, \ldots, n\}$ then $v \in I(t[u_i \leftarrow s_i])$ by $n-1$ applications of Proposition 4.11. This is impossible, so $v \perp u_i$ for all $i \in \{1, \ldots, n\}$. Now we have $v \in I(t)$, again by applications of Proposition 4.11. But $v \notin \{u_1, \ldots, u_n\}$. We conclude that $J(t) \neq \emptyset$. □

The 'suffix property' (Proposition 4.10) also holds for increasing indices.

PROPOSITION 6.3. *If $uv \in J(t)$ then $v \in J(t/u)$.*

PROOF. If $v \notin J(t/u)$ then there exists a term $s \in \mathrm{NF}_s$ such that

$$\{w \in I(t/u[v \leftarrow s]) \mid w \geq v\} = \emptyset.$$

Let $t' \equiv t[uv \leftarrow s]$. We have $\{w \in I(t') \mid w \geq uv\} = \emptyset$ by Proposition 4.10 and therefore $uv \notin J(t)$. □

PROPOSITION 6.4. *Suppose $\mathcal{R}$ is strongly sequential. Let $t \in \mathrm{NF}_\Omega$ and $s \in \mathrm{NF}_s$. If $u \in J(t)$ then there exists a $v \in J(t[u \leftarrow s])$ with $u \leq v$.*

PROOF. By definition the set $\{v \in I(t[u \leftarrow s]) \mid v \geq u\}$ is non-empty, say

$$\{v \in I(t[u \leftarrow s]) \mid v \geq u\} = \{u_1, \ldots, u_n\}.$$

Suppose $\{v \in J(t[u \leftarrow s]) \mid v \geq u\} = \emptyset$. For every $i \in \{1, \ldots, n\}$ there exists a term $s_i \in \mathrm{NF}_s$ such that

$$\{v \in I(t[u \leftarrow s][u_i \leftarrow s_i]) \mid v \geq u_i\} = \emptyset.$$

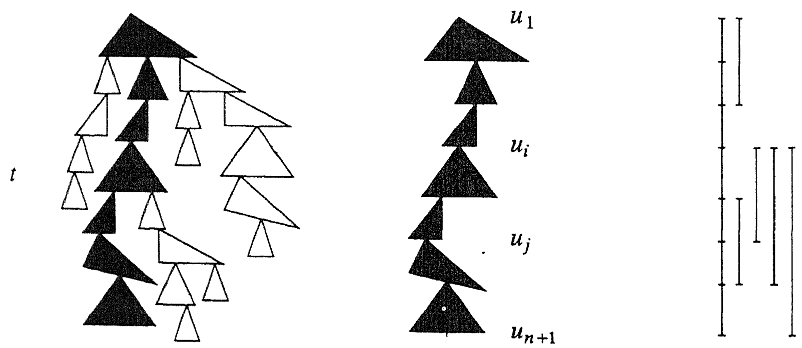
Let $t' \equiv t[u \leftarrow s']$ with $s' \equiv s[u_i/u \leftarrow s_i \mid 1 \leq i \leq n]$. By definition there exists an index $v \in I(t')$ such that $u \leq v$. We obtain a contradiction like in the proof of Proposition 6.2. □

DEFINITION 6.5.

(1) A proper preredex t is called *atomic* if t does not contain other proper preredexes, i.e. t/u is not a proper preredex for all $u \in O(t) - \{\lambda\}$.

(2) An *atomic* decomposition D of a term $t \in \mathrm{NF}_{cut}$ consists only of atomic preredexes, i.e. s is an atomic preredex whenever $\langle u, s \rangle \in D$. Clearly every decomposition of a term $t \in \mathrm{NF}_{cut}$ can be refined to an atomic decomposition.

We are now ready for the main theorem of this section. First we will give an intuitive description of the proof idea. As noted before, the problem with indices is that they are not 'transitive'. However, 'partial transitivity' properties do hold; in our first proof of the decidability of strong sequentiality this was embodied by Lemma 5.20, in the following proof this is embodied by the Δ-sets. To show that the existence of Δ-sets guarantees the existence of an index in a term $t \in \mathrm{NF}_{cut}$, we consider an atomic decomposition of t and we select a main tower as in Figure 19(i) which has the property that Δ-indices are transmitted along the tower, in the following sense. The main tower in Figure 19(ii) may contain next to the atomic preredexes, larger preredexes formed by some consecutive atomic pieces of the tower, e.g. as indicated in Figure 19(iii) where every line segment denotes a preredex between some u_i, u_j. Now for every such preredex between u_i, u_j we have that u_j/u_i is a Δ-index of that preredex. The result is that



(i) (ii) (iii)

FIGURE 19.

the main tower leads indeed to a position u_{n+1} which is an index of that tower, and hence of the whole term t. This can be seen as follows: if the test symbol $\bullet$ is inserted at u_{n+1}, then the tower is perfectly rigid, no chunk can be melted away. First by our use of atomic preredexes, so no chunk away from the main path $u_1 - u_2 - ... - u_{n+1}$ of the main tower can be melted away, and second by the arrangement that all preredexes in the tower 'looking at' the test symbol $\bullet$ at position u_{n+1} have an index at that point. We will now give the formal proof.

THEOREM 6.6. *$\mathcal{R}$ is strongly sequential if and only if there exist Δ-sets for $\mathcal{R}$.*

PROOF.

$\Rightarrow$ If $\mathcal{R}$ is strongly sequential then the increasing indices satisfy the conditions for being Δ-sets, by Propositions 6.2, 6.3 and 6.4.

$\Leftarrow$ We have to prove that every term $t \in \mathrm{NF}_\Omega$ has an index. By previous results (Corollary 5.18) it is sufficient to prove that every term $t \in \mathrm{NF}_{cut}$ has an index. Let $t \in \mathrm{NF}_{cut}$ and suppose D is an atomic decomposition of t. We will construct a sequence of towers of preredexes $D_1 \subseteq D_2 \subseteq ... \subseteq D_n \subseteq D$ and a position u_{n+1} such that $D_n = \{\langle u_i, s_i \rangle \mid 1 \leq i \leq n\}$ is a main tower and the following property (*) holds:

if $D_l^k = \{\langle u_i, s_i \rangle \mid k \leq i \leq l\}$ is a tower of preredexes such that $\pi(D_l^k)$ is a preredex, then $u_{l+1}/u_k \in \Delta(\pi(D_l^k))$.

D_1 is the singleton set $\{\langle u_1, s_1 \rangle\}$ where $u_1 = \lambda$ and $\langle \lambda, s_1 \rangle \in D$. Because s_1 is a proper preredex, $\Delta(s_1)$ is non-empty, and hence we can take $u_2 \in \Delta(s_1)$. Suppose we have defined $D_1, ..., D_{j-1}$ and position u_j. If D_{j-1} is a main tower then we end the construction and set $n = j-1$. Otherwise we extend D_{j-1} with the unique element $\langle u_j, s_j \rangle \in D$ to obtain D_j. Let $k \in \{1, ..., j\}$ be minimal under the restriction that $\pi(D_j)/u_k$ is a preredex. In order to define u_{j+1} we consider two cases: (1) $k = j$ and (2) $k < j$.

(1) If $k = j$ then we choose some $v \in \Delta(s_j)$ and define $u_{j+1} = u_j v$. In this case the hypothesis (*) is clearly satisfied.

(2) If $k < j$ then $\pi(D_{j-1})/u_k \equiv \pi(D_{j-1}^k)$ also is a preredex. From the induction hypothesis we obtain $u_j/u_k \in \Delta(\pi(D_{j-1}^k))$ and the existence of Δ-sets implies the existence of a position $u' > u_j/u_k$ such that $u' \in \Delta(\pi(D_j^k))$ and $u'/(u_j/u_k) \in \Delta(\pi(D_j^j)) = \Delta(s_j)$. Now we define $u_{j+1} = u_k u'$. We still have to show that the hypothesis (*) is satisfied. Suppose $\pi(D_m^l)$ is a preredex. If $m < j$ the result follows by induction. So assume $m = j$. We have $k \leq l$ by the definition of k. If $k = l$ then we already know that $u_{m+1}/u_l = u' \in \Delta(\pi(D_m^l))$. If $k < l$ then $u_l/u_k \in \Delta(\pi(D_{l-1}^k))$ by the induction hypothesis. Because $\pi(D_j^k) \equiv \pi(D_{l-1}^k)[u_l/u_k \leftarrow \pi(D_j^l)]$ and $u_{j+1}/u_k \in \Delta(\pi(D_j^k))$, we obtain $u_{j+1}/u_l = (u_{j+1}/u_k)/(u_l/u_k) \in \Delta(D_j^l)$ from the definition of Δ-sets.

We will now show that $u_{n+1} \in I(\pi(D_n))$. Suppose $\pi(D_n)[u_{n+1} \leftarrow \bullet]$ contains a redex compatible subterm $s \not\equiv \Omega$ at position v. Because $\pi(D_n)[u_{n+1} \leftarrow \bullet]$ is a normal form with respect to $\rightarrow_{cut}$, s must be a preredex. If v is disjoint from u_{n+1} then s is a proper subterm of an atomic preredex, which is impossible. For similar reasons v cannot be distinct from $u_1, ..., u_n$. So $v = u_i$ for some $i \leq n$. Clearly $s[u_{n+1}/u_i \leftarrow \Omega] \equiv \pi(D_n^i)$ is also a preredex. From (*) we obtain $u_{n+1}/u_i \in \Delta(\pi(D_n^i)) \subseteq I(\pi(D_n^i))$ and hence

$$\omega(s) \equiv \omega(\pi(D_n^i)[u_{n+1}/u_i \leftarrow \bullet]) \not\equiv \omega(\pi(D_n^i)) \equiv \Omega.$$

This contradicts the assumption that s is redex compatible. Therefore $\pi(D_n)[u_{n+1} \leftarrow \bullet]$ does not contain redex compatible subterms different from Ω and thus

$\omega(\pi(D_n)[u_{n+1} \leftarrow \bullet]) \equiv \pi(D_n)[u_{n+1} \leftarrow \bullet]$. We conclude that $u_{n+1} \in I(\pi(D_n))$. Finally, Proposition 4.1 yields $u_{n+1} \in I(t)$.

□

Because it is straightforward to give an (inefficient) algorithm for finding Δ-sets, Theorem 6.6 gives a decision procedure for strong sequentiality.

7. Further Remarks on Deciding Strong Sequentiality

In this section we present some new observations on deciding strong sequentiality. We conjectured for some time that, with the help of Lemma 5.20, it should be possible to prove that the depth of a minimal free term is bounded by 2ρ or perhaps 3ρ (where ρ is the maximum depth of the redex schemes as defined in Section 5), which would imply a very simple decision procedure for strong sequentiality: just check all terms with depth up to 2ρ (3ρ). Unfortunately, this is not the case.

DEFINITION 7.1. The TRS's $\mathcal{R}_n$ ($n \geq 2$) and $\mathcal{S}_n$ ($n \geq 3$) are defined as follows:

$$\mathcal{R}_2 = \begin{cases} F_0(A, B, x) & \to x \\ F_1(F_0(x, A, B), A) & \to x \\ F_2(F_1(F_0(B, x, A), B), A) & \to x \end{cases}$$

and if $n \geq 2$ then

$$\mathcal{R}_{n+1} = \mathcal{R}_n \cup \{F_{n+1}(F_n(F_{n-1}(A, x), B), A) \to x\},$$

$$\mathcal{S}_{n+1} = \mathcal{R}_n \cup \{F_{n+1}(F_n(F_{n-1}(A, x), y), z) \to x\}.$$

PROPOSITION 7.2. *The TRS's $\mathcal{R}_n$ are strongly sequential for all $n \geq 2$.*

PROOF. We will inductively define collections Δ_i for $i \geq 2$, satisfying the conditions for being Δ-sets with respect to $\mathcal{R}_i$. The collection Δ_2 is defined as follows (the underlined Ω's denote the Δ-indices):

$$F_1(\underline{\Omega}, \Omega),\ F_2(\underline{\Omega}, \Omega),\ F_2(F_1(\underline{\Omega}, \underline{\Omega}), \Omega),\ F_2(F_1(\Omega, \Omega), \underline{A})$$

and $\Delta_2(t) = I(t)$ for all other proper preredexes t of $\mathcal{R}_2$. It is straightforward to show that Δ_2 satisfies the conditions for being Δ-sets with respect to $\mathcal{R}_2$. Suppose we have defined $\Delta_2, \ldots, \Delta_i$. Let t be a proper preredex of $\mathcal{R}_{i+1}$. If t is a proper preredex of $\mathcal{R}_i$ then we define

$$\Delta_{i+1}(t) = \begin{cases} \{1,2\} & \text{if } t \equiv F_i(\Omega, \Omega), \\ \Delta_i(t) & \text{otherwise,} \end{cases}$$

and if t is not a proper preredex of $\mathcal{R}_i$ then $\Delta_{i+1}(t)$ is given below:

$$F_{i+1}(\underline{\Omega}, \Omega),\ F_{i+1}(F_i(\underline{\Omega}, \underline{\Omega}), \Omega),\ F_{i+1}(F_i(\Omega, \Omega), \underline{A}),$$

$$F_{i+1}(F_i(F_{i-1}(\Omega,\Omega),\Omega),\Omega),\ F_{i+1}(F_i(F_{i-1}(\Omega,\Omega),\Omega),A)$$

and $\Delta_{i+1}(t) = I(t)$ if t is not listed above. Although very tedious, it is not difficult to verify that Δ_{i+1} indeed satisfies the conditions for being Δ-sets with respect to $\mathcal{R}_{i+1}$. Theorem 6.6 yields the strong sequentiality of $\mathcal{R}_n$, for every $n \geq 2$. □

PROPOSITION 7.3. *Let $n \geq 3$. The* TRS S_n *is not strongly sequential; its minimal free term is* $t_n \equiv F_n(F_{n-1}(\ldots(F_1(F_0(\Omega,\Omega,\Omega),\Omega)\ldots),\Omega)$.

PROOF. Because $I(t_n) = \emptyset$, S_n is not strongly sequential. Let t be a minimal free term of S_n. The following observation is easily proved:

$$\text{if } t(u) \equiv F_j \text{ and } t(u.i) \equiv F_k \text{ then } i=1 \text{ and } j=k+1.$$

From this one obtains $t \equiv t_n$ by a sequence of routine arguments. □

COROLLARY 7.4. *For every $n \geq 1$ there exists a* TRS $\mathcal{R}$ *which is not strongly sequential such that every free term t of $\mathcal{R}$ has depth $\rho(t) > n\rho_{\mathcal{R}}$.*

PROOF. Choose $n \geq 1$ and let $\mathcal{R} = S_{3n}$. Suppose t is a free term of $\mathcal{R}$. From Proposition 7.3 we obtain $\rho(t) \geq \rho(t_{3n}) = 3n+1$ and since $\rho_{\mathcal{R}} = 3$ we are done. □

The above gives evidence that deciding strong sequentiality is not a trivial matter. Indeed, there is no known efficient method for finding Δ-sets. (We conjecture that deciding strong sequentiality is NP-complete.) Huet and Lévy pointed out that for the practically relevant case of constructor systems, deciding strong sequentiality is easy. Laville (1987) showed the close connection between strong sequentiality of constructor systems and the existence of lazy pattern matching algorithms for functional programming languages.

DEFINITION 7.5. A *constructor system* is a TRS $(\mathcal{F}, \mathcal{R})$ whose signature $\mathcal{F}$ can be partitioned into a set $\mathcal{D}$ of *defined* function symbols and a set $\mathcal{C}$ of *constructors* such that every left-hand side of a rewrite rule of $\mathcal{R}$ has the form $F(t_1, \ldots, t_n)$ with $F \in \mathcal{D}$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{C}, \mathcal{V})$.

The nice thing about constructor systems is the transitivity of index propagation for terms starting with a defined function symbols.

PROPOSITION 7.6. *Let $\mathcal{R}$ be a constructor system. Let $s, t \in \mathcal{T}_\Omega$ such that $t(\lambda) \in \mathcal{D}$. If $u \in I(s)$ and $v \in I(t)$ then $uv \in I(s[u \leftarrow t])$.*

PROOF. If $uv \notin I(s[u \leftarrow t])$ then $uv \notin O(\omega(s[u \leftarrow t][uv \leftarrow \bullet]))$ and hence there exists an Ω-reduction

$$s[u \leftarrow t][uv \leftarrow \bullet] \twoheadrightarrow_\Omega t_1 \rightarrow_\Omega t_2$$

such that $t_1/uv \equiv \bullet$ and $uv \notin O(t_2)$. Let t_1/u' be the redex compatible subterm contracted in the step $t_1 \rightarrow_\Omega t_2$. Clearly $u' < uv$. We distinguish two cases: (1) $u \leq u' < uv$ and (2) $u' < u$.

(1) The proof is the same as the first case of the proof of Lemma 5.20.

(2) Let r be a redex scheme compatible with t_1/u'. Because $t_1(u) \in \mathcal{D}$ we have either $u/u' \notin O(r)$ or $r(u/u') \equiv \Omega$. In both cases the term $t_1[u \leftarrow \bullet]/u'$ also is compatible with r. We obtain a contradiction as in the second case of the proof of Lemma 5.20.

□

COROLLARY 7.7. *A constructor system is strongly sequential if and only if every proper preredex has an index.*

PROOF.

$\Rightarrow$ Trivial.

$\Leftarrow$ According to previous results it suffices to show that there are no free terms in NF_{cut}. Because every $t \in NF_{cut}$ can be partitioned into proper preredexes, this follows from Proposition 7.6.

□

Alternatively, this fact can be obtained from Theorem 6.6 and the definition of Δ-sets, noting that if s, t are proper preredexes and $u \in \Delta(t)$ then $t[u \leftarrow s]$ can never be a proper preredex. In order to decide whether a constructor system $\mathcal{R}$ is strongly sequential, we only have to compute the indices of its proper preredexes. According to the next proposition, this is very easy.

PROPOSITION 7.8. *Let t be a proper preredex in a constructor system. An Ω-position u of t is an index if and only if $t[u \leftarrow \bullet]$ is not redex compatible.*

PROOF. Easy. □

We conclude this section with the observation that strong sequentiality is a modular property, i.e. depends on the disjoint pieces of a term rewriting system.

DEFINITION 7.9.

(1) The *disjoint union* of two TRS's $\mathcal{R}_1$, $\mathcal{R}_2$ is denoted by $\mathcal{R}_1 \oplus \mathcal{R}_2$. That is, if the signatures of $\mathcal{R}_1$ and $\mathcal{R}_2$ are disjoint, then $\mathcal{R}_1 \oplus \mathcal{R}_2$ is the union of $\mathcal{R}_1$ and $\mathcal{R}_2$; otherwise we take renamed copies $\mathcal{R}'_1, \mathcal{R}'_2$ of $\mathcal{R}_1, \mathcal{R}_2$ such that $\mathcal{R}'_1$ and $\mathcal{R}'_2$ have disjoint signatures and define $\mathcal{R}_1 \oplus \mathcal{R}_2 = \mathcal{R}'_1 \cup \mathcal{R}'_2$.

(2) A property $\mathcal{P}$ of TRS's is called *modular* if the following holds for all $\mathcal{R}_1, \mathcal{R}_2$:

$$\mathcal{R}_1 \oplus \mathcal{R}_2 \text{ has the property } \mathcal{P} \iff \text{both } \mathcal{R}_1 \text{ and } \mathcal{R}_2 \text{ have the property } \mathcal{P}.$$

A well-known example of a modular property is the Church-Rosser property (Toyama, 1987). A comprehensive survey of modularity can be found in Middeldorp (1990).

THEOREM 7.10. *Strong sequentiality is a modular property of orthogonal* TRS's.

PROOF. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be orthogonal TRS's with disjoint signatures. We have to show that $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly sequential if and only if both $\mathcal{R}_1$ and $\mathcal{R}_2$ are strongly sequential.

$\Leftarrow$ If $\mathcal{R}_i$ is strongly sequential then there exists Δ-sets Δ_i for proper preredexes of $\mathcal{R}_i$ for $i = 1, 2$. Define $\Delta_{1,2}$ by

$$\Delta_{1,2}(t) = \begin{cases} \Delta_1(t) & \text{if } t \text{ is a proper preredex of } \mathcal{R}_1, \\ \Delta_2(t) & \text{if } t \text{ is a proper preredex of } \mathcal{R}_2. \end{cases}$$

It is very easy to show that $\Delta_{1,2}$ satisfies the conditions for being Δ-sets with respect to $\mathcal{R}_1 \oplus \mathcal{R}_2$. Therefore $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly sequential.

$\Rightarrow$ If $\mathcal{R}_1 \oplus \mathcal{R}_2$ is strongly sequential then, according to Theorem 6.6, we can find Δ-sets for preredexes of $\mathcal{R}_1 \oplus \mathcal{R}_2$, say $\Delta_{1,2}$. The restriction of $\Delta_{1,2}$ to preredexes of $\mathcal{R}_i$ clearly satisfies the conditions for being Δ-sets with respect to $\mathcal{R}_i$ for $i = 1, 2$. Theorem 6.6 yields the strong sequentiality of $\mathcal{R}_1$ and $\mathcal{R}_2$.

□

It should be noted that in order to apply the previous proposition for deciding the strong sequentiality of a TRS $\mathcal{R}$, it is sufficient that $\mathcal{R}$ can be partitioned into $\mathcal{R}_1 \cup \mathcal{R}_2$ such that the left-hand sides of $\mathcal{R}_1$ and $\mathcal{R}_2$ do not have function symbols in common.

REMARK. Sequentiality*, as defined in Definition 3.5, is not a modular property. For instance, the trivial TRS $\mathrm{I} = \{I(x) \to x\}$ is strongly sequential (and hence sequential*, cf. Figure 4). We already observed that Berry's TRS $\mathrm{B} = \{F(A, B, x) \to C,\ F(B, x, A) \to C,\ F(x, A, B) \to C\}$ is sequential*, but $\mathrm{I} \oplus \mathrm{B}$ is not sequential*:

$$F(\underline{I(A)}, \underline{I(B)}, r) \twoheadrightarrow \underline{F(A, B, r)} \to C$$

$$F(\underline{I(B)}, r, \underline{I(A)}) \twoheadrightarrow \underline{F(B, r, A)} \to C$$

$$F(r, \underline{I(A)}, \underline{I(B)}) \twoheadrightarrow \underline{F(r, A, B)} \to C.$$

8. Different Notions of Sequentiality

In this last section we discuss two different notions of sequentiality. The first one is *left sequentiality* introduced by Thatte (1987) (not to be confused with the notion of left sequentiality by Hoffmann & O'Donnell (1984)). Left sequentiality is intuitively more satisfactory than strong sequentiality, but Thatte showed that the notions coincide for the subclass of constructor systems. We will give a simple proof of this fact. Thatte also showed that left sequentiality is necessary for safe computation based on the analysis of left-hand sides alone, again for the subclass of constructor systems. The second notion of sequentiality we discuss is *sufficient sequentiality* introduced by Oyamaguchi (1987). Sufficient sequentiality is not only based on the analysis of the left-hand sides of the rewrite rules of TRS's (as is the case for strong and left sequentiality) but also on the non-variable parts of the right-hand sides. Oyamaguchi showed that the class of sufficiently sequential TRS's properly includes the class of strongly sequential systems. Furthermore, he established the decidability of sufficient sequentiality.

The following example from Thatte motivates the introduction of left sequentiality.

EXAMPLE 8.1. Let

$$\mathcal{R} = \begin{cases} F(A, B, x) & \to & x \\ F(B, x, A) & \to & x \\ F(x, A, B) & \to & x \\ G(A) & \to & A. \end{cases}$$

Consider the term $t \equiv F(G(\Omega), G(\Omega), \Omega)$. The third occurrence of Ω in t is not an index with respect to strong sequentiality (r_1, r_2 and r_3 are arbitrary redexes):

$$F(G(r_1), G(r_2), r_3) \rightarrow_? F(G(A), G(r_2), r_3) \rightarrow_? F(A, G(r_2), r_3)$$
$$\rightarrow_? F(A, G(A), r_3) \rightarrow_? F(A, B, r_3) \rightarrow_? A.$$

In the second step we replaced the redex $G(A)$ by A and in the fourth step we replaced the same redex by B. However, using Theorem 2.4 one easily shows that there does not exist a TRS $\mathcal{R}'$ with the same left-hand sides as $\mathcal{R}$ such that $G(r_1) \twoheadrightarrow_{\mathcal{R}'} A$ and $G(r_2) \twoheadrightarrow_{\mathcal{R}'} B$. Therefore, the above arbitrary reduction sequence is impossible for any system based on the left-hand sides of $\mathcal{R}$.

. . .

DEFINITION 8.2.

(1) Two TRS's $\mathcal{R}_1, \mathcal{R}_2$ are *left equivalent*, notation $\mathcal{R}_1 \sim_l \mathcal{R}_2$, if they have the same left-hand sides, i.e. $\mathcal{R}_1 = \{l_i \rightarrow r_i^1 \mid 1 \le i \le n\}$ and $\mathcal{R}_2 = \{l_i \rightarrow r_i^2 \mid 1 \le i \le n\}$ for some terms l_i, r_i^1, r_i^2 $(i = 1, \ldots, n)$.

(2) The monotonic predicate *lnf* is defined on $\mathcal{T}_\Omega$ by

$$lnf(t) \text{ holds } \Leftrightarrow t \twoheadrightarrow_{\mathcal{R}'} t' \text{ for some } \mathcal{R}' \sim_l \mathcal{R} \text{ and } t' \in \mathrm{NF}.$$

(3) An orthogonal TRS is *left sequential* if every $t \in \mathrm{NF}_\Omega$ has an index with respect to *lnf*.

EXAMPLE 8.3. The term t in Example 8.1 does not have an index with respect to strong sequentiality, but $I_{lnf}(t) = \{3\}$ because $t_1 \ge t$ and $t_1/3 \equiv \Omega$ imply that there does not exist a TRS $\mathcal{R}' \sim_l \mathcal{R}$ such that $t_1 \twoheadrightarrow_{\mathcal{R}'} t_2$ for some normal form t_2. Notice that $\mathcal{R}$ is not left sequential: $I_{lnf}(F(\Omega, \Omega, \Omega)) = \varnothing$.

PROPOSITION 8.4.

(1) *Every strongly sequential* TRS *is left sequential.*

(2) *Every left sequential* TRS *is sequential.*

PROOF.

(1) Suppose $\mathcal{R}$ is strongly sequential. Take $t \in \mathrm{NF}_\Omega$ and $u \in I_{nf_?}(t)$. We will show that $u \in I_{lnf}(t)$. Let $t' \ge t$ such that $lnf(t')$ holds. Then $nf_?(t')$ also holds and we obtain $t'/u \not\equiv \Omega$ from the assumption $u \in I_{nf_?}(t)$.

(2) Similar to (1), using the implication $nf(t') \Rightarrow lnf(t')$.

□

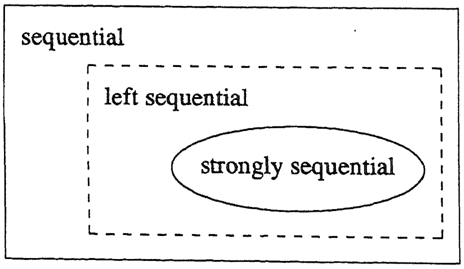


FIGURE 20.

PROPOSITION 8.5. *Every left sequential constructor system is strongly sequential.*

PROOF. Let $\mathcal{R}$ be a left sequential constructor system. According to Corollary 7.7 we have to show that every proper preredex of $\mathcal{R}$ has an index with respect to strong sequentiality. Let t be a proper preredex of $\mathcal{R}$ and take some $u \in I_{Inf}(t)$. Suppose u is not an index with respect to strong sequentiality. Then $t[u \leftarrow \bullet]$ is redex compatible by Proposition 7.8 and hence there exists a redex $t' \geq t[u \leftarrow \bullet]$. Clearly $t'' \equiv t'[u \leftarrow \Omega]$ also is a redex. Let $l \rightarrow r$ be the rewrite rule of $\mathcal{R}$ such that t'' is an instance of l. Choose some ground normal form r' and let $\mathcal{R}' = \mathcal{R} - \{l \rightarrow r\} \cup \{l \rightarrow r'\}$. Now we have $t'' \rightarrow_{\mathcal{R}'} r'$, $t'' \geq t$ and $t''/u \equiv \Omega$ which contradicts the assumption $u \in I_{Inf}(t)$. We conclude that $\mathcal{R}$ is strongly sequential. □

Thatte writes: "It is less obvious that our results apply to the full class of orthogonal systems." We conjecture that left sequentiality does not coincide with strong sequentiality: the non-constructor system

$$\mathcal{R} = \begin{cases} F(G(A, x), F(A, A)) & \rightarrow x \\ F(G(x, A), F(B, B)) & \rightarrow x \\ F(C_1, F(D_1, G(A, x))) & \rightarrow x \\ F(C_2, F(D_2, G(x, A))) & \rightarrow x \\ G(E, E) & \rightarrow E \end{cases}$$

is not strongly sequential (the term $F(G(\Omega, \Omega), F(G(\Omega, \Omega), G(\Omega, \Omega)))$ does not have an index with respect to $nf_?$) but we think that $\mathcal{R}$ is left sequential. At present it is open whether left sequentiality is a decidable property of orthogonal TRS's.

This concludes our discussion of left sequentiality. We now turn our attention to sufficient sequentiality.

DEFINITION 8.6.

(1) The reduction relation $\rightarrow_!$ is defined as follows:

$$t_1 \rightarrow_! t_2$$

if there exists a context $C[\,]$, a reduction rule $l \rightarrow r$ and a substitution σ such that $t_1 \equiv C[l^\sigma]$, $t_2 \equiv C[t]$ for some term $t \geq r_\Omega$ where $r_\Omega \equiv r[u \leftarrow \Omega \mid r/u \in \mathcal{V}]$.

(2) The predicate $term_!$ is defined on $\mathcal{T}_\Omega$ as follows:

$$term_!(t) \text{ holds } \Leftrightarrow t \twoheadrightarrow_! t' \text{ for some } t' \in \mathcal{T}(\mathcal{F}, \mathcal{V}).$$

(3) An orthogonal TRS is *sufficiently sequential* if every $t \in \mathrm{NF}_\Omega$ has an index with respect to $term_!$.

It would be more natural to define sufficient sequentiality in terms of a predicate $nf_!$: $nf_!(t)$ holds if $t \twoheadrightarrow_! t'$ for some normal form t', but Oyamaguchi argued that it will be very difficult to obtain an (efficient) algorithm for finding indices with respect to $nf_!$. Oyamaguchi showed that the computation of indices with respect to $term_!$ can be done in polynomial time.

PROPOSITION 8.7.

(1) *Every strongly sequential* TRS *is sufficiently sequential.*

(2) *Every sufficiently sequential* TRS *is sequential.*