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Termination for direct sums of left-linear complete term rewriting systems

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Abstract
A Term Rewriting System is called complete if it is confluent and terminating. We prove that completeness of TRSs is a 'modular' property (meaning that it stays preserved under direct sums), provided the constituent TRSs are left-linear. Here the direct sum $R_0 \oplus R_1$ is the union of TRSs R_0, R_1 with disjoint signature. The proof hinges crucially upon the (non)deterministic collapsing behaviour of terms from the sum TRS.

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Introduction

An important concern in building algebraic specifications is their hierarchical or modular structure. The same holds for term rewriting systems (see Huet & Oppen [80], Klop [89] or Dershowitz & Jouannaud [88]) which can be viewed as implementations of equational algebraic specifications. Specifically, it is of obvious interest to determine which properties of term rewriting systems (TRSs) have a 'modular' character, where we call a property 'modular' if its validity for a TRS, hierarchically composed of some smaller TRSs, can be inferred from the validity of that property for the constituent TRSs. Naturally, the first step in such an investigation considers the most basic properties of TRSs: confluence, termination, unique normal form property, and similar fundamental properties as well as combinations thereof.

As to the modular structure of TRSs, it is again natural to consider as a start the most simple way that TRSs can be combined to form a larger TRS: namely, as a disjoint sum. This means that the alphabets of the TRSs to be combined are disjoint, and that the rewrite rules of the sum TRS are the rules of the summand TRSs together. (Without the disjointness requirement the situation is even more complicated—see for some results in this direction: Dershowitz [81], Toyama [88].) A

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disjoint union of two TRSs R_0, R_1 is called in our paper a direct sum, notation $R_0 \oplus R_1$.

Another simplifying assumption that we will make, is that R_0, R_1 are homogeneous TRSs, i.e. their signature is one-sorted (as opposed to the many-sorted or heterogeneous case; for results about direct sums of heterogeneous TRSs, see Ganzinger & Giegerich [87]).

The first result in this setting is due to Toyama [87], where it is proved that confluence is a modular property. (I.e. $R_0 \oplus R_1$ is confluent $\Leftrightarrow$ R_0 and R_1 are confluent. Here "$\Rightarrow$" is trivial; "$\Leftarrow$" is what we are interested in.) To appreciate the non-triviality of this fact, it may be contrasted with the fact that another fundamental property, termination, is *not* modular, as the following simple counterexample in Toyama [87a] shows:

$$R_0 = \{F(0,1,x) \to F(x,x,x)\}$$
$$R_1 = \{G(x,y) \to x,\ G(x,y) \to y\}.$$

It is trivial that R_0 and R_1 are terminating. However, $R_0 \oplus R_1$ is not terminating, because $R_0 \oplus R_1$ has the infinite reduction sequence:

$$F(G(0,1), G(0,1), G(0,1)) \to F(0, G(0,1), G(0,1)) \to F(0,1, G(0,1)))$$
$$\to F(G(0,1), G(0,1), G(0,1)) \to \ldots$$

However, this counterexample uses a non-confluent TRS R_1. A more complicated counterexample to the modularity of 'termination', involving only confluent TRSs, was given by Barendregt and Klop (for ground terms only). For this counterexample as well as for some improved versions, holding for open terms as well, and even using TRSs which are 'irreducible', see Toyama [87a]. Rephrased, this means that the important property of 'completeness' of TRSs (a TRS is complete if it is both confluent and terminating) is not modular, i.e. there are complete TRSs R_0, R_1 such that $R_0 \oplus R_1$ is not complete (in fact, not terminating; confluence of $R_0 \oplus R_1$ is ensured by the theorem in Toyama [87]). This counterexample, however, uses non-left-linear TRSs.

The point of the present paper is that left-linearity is essential; if we restrict ourselves to left-linear TRSs, then completeness is modular. Thus we prove: If R_0, R_1 are left-linear (meaning that the rewrite rules have no repeated variables in their left-hand sides), then $R_0 \oplus R_1$ is complete iff R_0, R_1 are so. As left-linearity is a property which is so easily checked, and many equational algebraic specifications can be given by TRSs which are left-linear, we feel that this result is worth-while.

The proof, however, is rather intricate and not easily digested. A crucial element in the proof, and in general in the way that the summand TRSs interact, is how terms may 'collapse' to a subterm. The problem is that this collapsing behaviour may exhibit a 'nondeterministic' feature, which is caused by ambiguities among the rewrite rules. We hope that the present paper is of value not only because it establishes a result that in itself is simple enough, but also because of the analysis necessary for the proof which gives a kind of structure theory for disjoint combinations of TRSs and which may be of relevance in other, similar, studies.

Regarding the question of modular properties in the present simple set-up, we mention the recent results by Rusinowitch [87] and Middeldorp [89]; these papers, together, contain a complete analysis of the cases in which termination for $R_0 \oplus R_1$ may be concluded from termination of R_0,

R_1, depending on the distribution among R_0, R_1 of so-called collapsing and duplicating rules.

Another useful fact is established in Middeldorp [89a], where it is proved that the 'unique normal form property' is a modular property.

CONTENTS

1. Preliminaries

We assume that the reader is familiar with the basic concepts and notations concerning term rewriting systems (TRSs); otherwise, see the basic references mentioned in the Introduction. In this section we exhibit the notions and concepts which are specific for the present paper, and we briefly recapitulate some of the more basic concepts.

(i) A term rewriting system R has an *alphabet* consisting of a set $\mathcal{F}$ of function symbols F, G, H, ... , each having an 'arity', i.e. the number of arguments that the function symbols requires, and a set of variables x,y,z,... . So if F is n-ary, then $F(t_1,...,t_n)$ is a term, for terms $t_1,...,t_n$. Constants are 0-ary function symbols. The set of terms of R, notation Ter(R), contains the terms which are inductively generated from the constant symbols, the variables x,y,z,... and the other function symbols. Terms are denoted by t, s, ... but occasionally also by M, N, ...

(ii) Furthermore, a TRS R has a set Red(R) of *reduction* or *rewrite rules* r: $t \to s$, or $t \to_r s$. Here r is the name of the rewrite rule. A rewrite step has the form $C[t^\sigma] \to_r C[s^\sigma]$, where σ is a substitution and C[ ] a context, i.e. a term with a 'hole' $\square$. The transitive reflexive closure of $\to_r$ is $\twoheadrightarrow_r$; the transitive closure of $\to_r$ is $\to_r^+$. The reflexive closure of $\to_r$ is $\to_r^\equiv$. The convertibility (i.e. equivalence relation) generated by $\to_r$ is $=_r$. Often the subscript r is omitted. Convertibility (=) should not be confused with $\equiv$, which denotes syntactical equality. The notation $t \twoheadrightarrow^n s$ is short for $t \to ... \to s$ (n steps).

(iii) The concepts of confluence and termination are as usual. A TRS is '*complete*' if its reduction relation is confluent and terminating (this is also called in the literature: *canonical*). A TRS R is *left-linear* if R contains no rewrite rule $t \to s$ such that t contains two or more occurrences of the same variable.

(iv) We write $t \subseteq s$ to indicate that t is a subterm of s. Always we will have a specific occurrence of s in t in mind; we will however not need a more precise formalism to indicate occurrences (e.g. as sequence numbers). If $t \subseteq s$ and $t \not\equiv s$, we write $t \subset s$, and call t a *proper* subterm of s.

(v) In this paper every TRS will be terminating; hence every term has a normal form. The normal form of a term t is denoted by $t\downarrow$.

2. Underlined reduction and frozen subterms

Consider the TRS with set of reduction rules {F(x, C) → x, F(C, x) → x, H(x) → x, G(x) → x} and the term M ≡ F(H(C), G(C)). Figure 2.1(a) displays the node-labeled tree corresponding to M. The term M has the following reductions to its normal form:

(1) M → F(C, G(C)) → G(C) → C
(2) M → F(H(C), C) → H(C) → C.

Although both reductions end in C, the two C's are different with respect to their occurrence in M. This is graphically expressed in Figure 2.1(b) where the arrows indicate to which occurrence of C the term M is 'collapsed'.

(a) (b)

Figure 2.1

In the sequel we will need to be precise about such reductions to *occurrences* of subterms, rather than mere subterms. Therefore we introduce the concepts of "underlined" reductions and "frozen" subterms, as follows.

2.1. DEFINITION. (i) Let R be some TRS. Then R_e is the TRS with alphabet that of R together with a new unary function symbol 'e', not occurring in R, and with rules: those of R together with e(x) → x.
(ii) Reduction according to the rule e(x) → x is called e-reduction; notation: $\rightarrow_e$ for one e-reduction step. Thus: C[e(M)] $\rightarrow_e$ C[M] for a context C[ ] and a term M in R_e.
(iii) For terms M_1, M_2 of R_e we write $M_1 \rightarrow_f M_2$ ('f' for 'frozen') if the redex contracted is not an e-redex nor in the scope of some 'e'. So if C[e(P)] $\rightarrow_f{}^S$ N where S is the contracted redex, then it is not the case that S ⊆ e(P).

2.2. NOTATION. (i) For notational ease we will henceforth write $\underline{M}$ instead of e(M) and $\underline{R}$ instead of R_e. Terms from $\underline{R}$ are "underlined" terms (even if they contain no actual underlining).
(ii) We write ↠ for the transitive-reflexive closure of $\twoheadrightarrow_f \cup \twoheadrightarrow_e$. (This is in fact an ambiguous use of ↠, since it was already in use for not underlined terms. But the present extension of the old ↠ to the case of underlined terms will cause no confusion.)
(iii) In the sequel, $C[\underline{P}_1,...,\underline{P}_p]$ denotes a term such that all underlinings are displayed, i.e. $C[P_1,...,P_p]$ contains no underlined subterm.

2.3. EXAMPLE. (i) Let R be the TRS as in the introduction of this section. Then the $\underline{R}$-term

$F(\ \underline{H(C)},\ \underline{G(\underline{C})}\ )$ (in the e-notation: F(e(H(C)), e(G(e(C))))) has the reduction:

$$F(\ \underline{H(\underline{C})},\ \underline{G(\underline{C})}\ ) \to_e F(\ H(C),\ \underline{G(\underline{C})}\ ) \to_f F(\ C,\ \underline{G(\underline{C})}\ ) \to_f$$
$$\underline{G(\underline{C})} \to_e G(\underline{C}) \to_f \underline{C} \to_e C.$$

(ii) Note that the terms $\underline{F(\ H(C),\ G(C)\ )}$ and $F(\ \underline{H(C)},\ \underline{G(C)}\ )$ are normal forms with respect to $\to_f$ (f-normal forms).

2.4. PROPOSITION. *Let* R *be a confluent and left-linear TRS. Then:*

(i) *In* $\underline{R}$*, the reduction* $\to_f$ *is confluent. (See diagram in Figure 2.2(a).)*

(ii) *In* $\underline{R}$*, the reductions* $\to_e$ *and* $\to_f$ *commute. (See diagram in Figure 2.2(b).)*

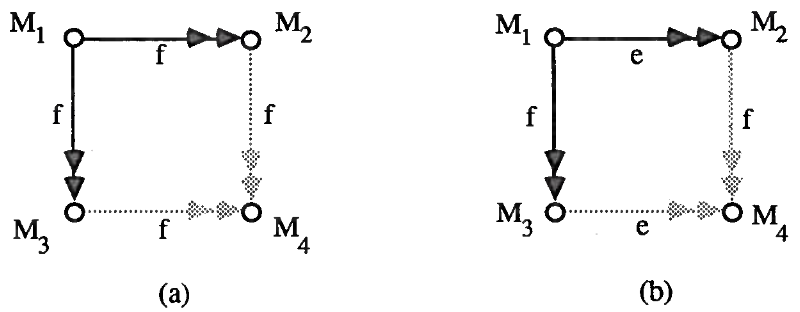


Figure 2.2

PROOF. (The shading of the arrows denotes that such arrows can be found, given the others.)
(i) Consider in M_1 the maximal underlined (occurrences of) subterms. Here 'maximal' refers to the subterm ordering $\subseteq$.) Replace these subterms by mutually different fresh variables, in order to "code" these subterms. Do this everywhere in the reductions $M_1 \twoheadrightarrow_f M_i$, i = 2,3. The resulting reductions $M_1^* \twoheadrightarrow M_i^*$, i = 2,3, are 'ordinary' (not underlined) reductions in R. Take the common reduct M_4^* according to R; and replace in $M_i^* \twoheadrightarrow M_4^*$ (i = 2,3) the coding variables by the original underlined subterms.
(ii) It suffices to prove the statement for the case that $M_1 \twoheadrightarrow_e M_2$ is one step $M_1 \to_e M_2$. Let this step be in fact $M_1 \equiv C[\underline{N}] \to_e C[N] \equiv M_2$. Then $M_3 \equiv C'[\underline{N},...,\underline{N}]$ where all descendants of $\underline{N}$ are displayed. Now take $M_4 \equiv C'[N,...,N]$. □

We will be especially interested in reductions of the form $M \equiv C[\underline{P}] \twoheadrightarrow \underline{P}$ where $\underline{P}$ is the only underlined subterm in $C[\underline{P}]$. (Here and in the sequel we will permit ourselves a slight abuse of notation by letting stand "$M \equiv C[\ \underline{P}\ ] \twoheadrightarrow \underline{P}$" for "$M \equiv C[P]$ and $C[\ \underline{P}\ ] \twoheadrightarrow \underline{P}$".) Graphically, the existence of such a reduction is indicated by an arrow as in Figure 2.3. Cf. the arrows in Figure 2.1(b). Indeed the two arrows there correspond with the $\to_f$-reductions:

$$M \equiv F(\ H(C),\ G(\underline{C})\ ) \to_f F(C,\ G(\underline{C})) \to_f G(\underline{C}) \to_f \underline{C}$$
$$M \equiv F(\ H(\underline{C}),\ G(C)\ ) \to_f F(H(\underline{C}),\ C) \to_f H(\underline{C}) \to_f \underline{C}.$$

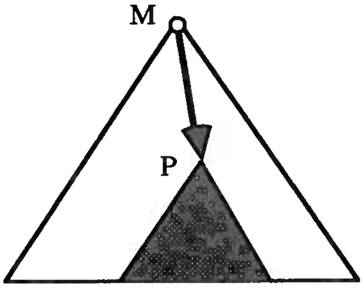


Figure 2.3

In the situation of Figure 2.3 we will sometimes say that (the displayed occurrence of) P can be "pulled up from M". We will also say that M "collapses to (the displayed occurrence of) P".

2.5. REMARK. Since in $C[\underline{P}] \twoheadrightarrow \underline{P}$ the subterm P initially is "frozen", it might be thought that $C[\underline{P}] \twoheadrightarrow \underline{P}$ implies $C[z] \twoheadrightarrow z$ for a fresh variable z. This is not the case as the following example shows: Let R have the reduction rules

$$F(x) \to G(x, x)$$
$$G(C, x) \to x$$
$$H(x) \to x.$$

Then $F(\,\underline{H(C)}\,) \twoheadrightarrow \underline{H(C)}$ in view of the reduction sequence

$$F(\,\underline{H(C)}\,) \to G(\,\underline{H(C)}, \underline{H(C)}\,) \to G(\,H(C), \underline{H(C)}\,) \to G(\,C, \underline{H(C)}\,) \to \underline{H(C)}.$$

However, $F(z) \twoheadrightarrow z$ does not hold. The explanation is that in a reduction $C[\underline{P}] \twoheadrightarrow \underline{P}$ not *all* descendants of the initial $\underline{P}$ need to remain frozen; only the $\underline{P}$ on the 'main line' of descendants leading to the ultimate $\underline{P}$ in the right-hand side of $C[\underline{P}] \twoheadrightarrow \underline{P}$ must be frozen. As the above reduction sequence shows, some descendants of the initial $\underline{P}$ in $C[\underline{P}]$, not in the main line of descendants, may actually play a necessary role in the collapse to the ultimate $\underline{P}$. (What does hold is the implication $C[\,\underline{P}\,] \twoheadrightarrow_f \underline{P} \;\Rightarrow\; C[z] \twoheadrightarrow z$ for a fresh variable z. The next proposition (part (i)) generalizes this obvious fact.)

2.6. PROPOSITION.

(i) $C[\underline{P}] \twoheadrightarrow_f C'[\underline{P},...,\underline{P}] \;\Leftrightarrow$
$C[z] \twoheadrightarrow C'[z,...,z]$ *for a fresh variable* z $\;\Leftrightarrow$
$C[\,\underline{Q}\,] \twoheadrightarrow_f C'[\underline{Q},...,\underline{Q}]$ *for all* Q.

(ii) *Let* $C[\underline{P}] \twoheadrightarrow_f C'[\underline{P},...,\underline{P}]$ *and* $P \equiv C''[Q]$. *Then* $C[C''[\,\underline{Q}\,]] \twoheadrightarrow_f C'[C''[\,\underline{Q}\,],...,C''[\,\underline{Q}\,]]$.

PROOF. Routine. □

2.7. PROPOSITION. *Let* $C[\underline{P},...,\underline{P}] \twoheadrightarrow^k \underline{P}$. *(I.e. a reduction of* k *steps* $\to_e$ *or* $\to_f$.*) Then for some occurrence of* $\underline{P}$ *in* $C[\underline{P},...,\underline{P}]$ *and some* $k' \le k$:

$$C[P,...,\underline{P},...,P] \twoheadrightarrow^{k'} \underline{P}.$$

PROOF. Consider a reduction $C[\underline{P},...,\underline{P}] \twoheadrightarrow^{k} \underline{P}$. Now the final $\underline{P}$ can be traced back to a unique ancestor $\underline{P}$ in $C[\underline{P},...,\underline{P}]$. Removing the underlining of the other $\underline{P}$ in $C[\underline{P},...,\underline{P}]$ we obtain $C[P,...,\underline{P},...,P]$. Clearly, there is now a reduction $C[P,...,\underline{P},...,P] \twoheadrightarrow \underline{P}$ which is the 'same' as the original reduction $C[\underline{P},...,\underline{P},...,\underline{P}] \twoheadrightarrow \underline{P}$ except that we possibly gain some e-steps (removals of underlinings). □

2.8. LEMMA. *Let* $C[\underline{P}] \twoheadrightarrow \underline{P}$ *and* $Q \twoheadrightarrow P$. *Then* $C[\,\underline{Q}\,] \twoheadrightarrow \underline{Q}$.

PROOF. Suppose $C[\underline{P}] \twoheadrightarrow^{k} \underline{P}$. We will prove the lemma by induction on k. The case $k = 0$ is trivial. Now let

$$C[\underline{P}] \rightarrow C'[\,\underline{P},...,\,\underline{P}\,] \twoheadrightarrow^{k-1} \underline{P}.$$

By Proposition 2.7 we have for some occurrence of $\underline{P}$ in $C'[\,\underline{P},...,\,\underline{P}\,]$ and some $k' \leq k - 1$:

$$C'[\,P,...,\,\underline{P},...,\,P\,] \twoheadrightarrow^{k'} \underline{P}.$$

By the induction hypothesis $C'[\,P,...,\,\underline{Q},...,\,P\,] \twoheadrightarrow \underline{Q}$. So we have

$$C[\,\underline{Q}\,] \rightarrow C'[\,\underline{Q},...,\,\underline{Q},...,\,\underline{Q}\,] \twoheadrightarrow C'[\,Q,...,\,\underline{Q},...,\,Q\,] \twoheadrightarrow C'[\,P,...,\,\underline{Q},...,\,P\,] \twoheadrightarrow \underline{Q}.$$

□

2.9. PROPOSITION. *Let* $C[\underline{P}] \twoheadrightarrow \underline{P}$ *and let* $C[\underline{P}] \twoheadrightarrow_{f} C'[\,\underline{P}\,,...,\,\underline{P}\,]$ *where all occurrences of* $\underline{P}$ *in* $C'[\,\underline{P}\,,...,\,\underline{P}\,]$ *are displayed.*

Then $C'[\,\underline{P}\,,...,\,\underline{P}\,]$ *contains at least one occurrence of* $\underline{P}$, *and* $C'[\,\underline{P}\,,...,\,\underline{P}\,] \twoheadrightarrow \underline{P}$ *(see Figure 2.4).*

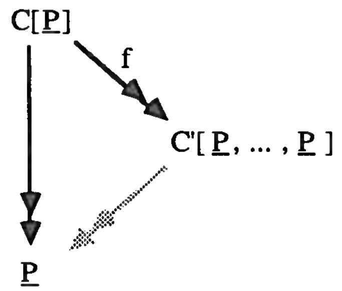


Figure 2.4

PROOF. That $C'[\,\underline{P}\,,...,\,\underline{P}\,]$ contains some occurrence of $\underline{P}$ follows immediately from $C'[\,\underline{P}\,,...,\,\underline{P}\,] \twoheadrightarrow \underline{P}$, since underlinings cannot be created during a reduction.

The proof of $C'[\,\underline{P}\,,...,\,\underline{P}\,] \twoheadrightarrow \underline{P}$ follows from the diagram in Figure 2.5. Note that the given reduction $C[\underline{P}] \twoheadrightarrow \underline{P}$ consists of some sequence of $\twoheadrightarrow_{f}$ and $\twoheadrightarrow_{e}$ reductions; it is displayed in the upper part of the diagram in Figure 2.5.

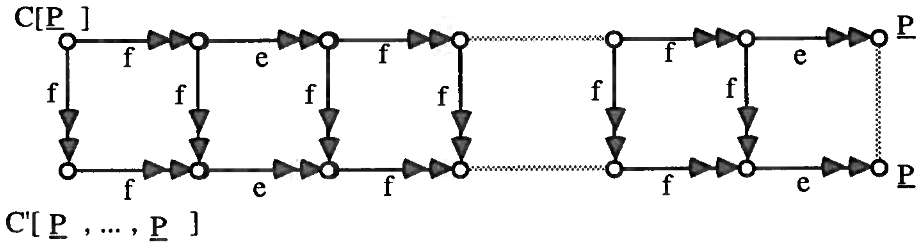


Figure 2.5

This diagram construction is possible by Proposition 2.4. Note that the right-hand side of the diagram is the *empty* reduction $\underline{P} \twoheadrightarrow_f \underline{P}$ (i.e. consisting of zero steps), since $\underline{P}$ is an f-normal form. Hence the lower side of the reduction diagram gives us a reduction $C'[\,\underline{P},\ldots,\underline{P}\,] \twoheadrightarrow \underline{P}$. □

2.10. LEMMA. *If* $C[\underline{P}] \twoheadrightarrow \underline{P}$ *and* $P \twoheadrightarrow Q$ *then* $C[\,\underline{Q}\,] \twoheadrightarrow \underline{Q}$.

PROOF. Suppose $C[\underline{P}] \twoheadrightarrow^k \underline{P}$. We will prove the proposition by induction on k. The case $k = 0$ is trivial: then $C[\underline{P}] \equiv \underline{P}$ and indeed $\underline{Q} \twoheadrightarrow \underline{Q}$. Induction hypothesis: the statement holds for $k - 1$ ($k > 0$). Now let $C[\underline{P}] \twoheadrightarrow^k \underline{P}$. So $C[\underline{P}] \to C'[\underline{P},\ldots,\underline{P}] \twoheadrightarrow^{k-1} \underline{P}$. By Proposition 2.7, we have a reduction $C'[P,\ldots,\underline{P},\ldots,P] \twoheadrightarrow^{k'} \underline{P}$ for some $k' \leq k - 1$ and for some occurrence of P. Hence, by the induction hypothesis, $C'[P,\ldots,\underline{Q},\ldots,P] \twoheadrightarrow \underline{Q}$.

By Proposition 2.9, since $C'[P,\ldots,\underline{Q},\ldots,P] \twoheadrightarrow_f C'[Q,\ldots,\underline{Q},\ldots,Q]$ we have $C'[Q,\ldots,\underline{Q},\ldots,Q] \twoheadrightarrow \underline{Q}$. Concatenating this reduction with $C[\,\underline{Q}\,] \to C'[\,\underline{Q},\ldots,\underline{Q},\ldots,\underline{Q}\,] \twoheadrightarrow C'[Q,\ldots,\underline{Q},\ldots,Q]$ we have indeed $C[\,\underline{Q}\,] \twoheadrightarrow \underline{Q}$. □

2.11. REMARK. From the preceding propositions we see that the relation $C[\underline{P}] \twoheadrightarrow \underline{P}$ is preserved under convertibility (=, the equivalence generated by $\twoheadrightarrow$, i.e. by $\to_e$, $\to_f$.). For, combining Lemma's 2.8 and 2.10 we have:

$$C[\underline{P}] \twoheadrightarrow \underline{P} \ \&\ P = Q \ \Rightarrow\ C[\underline{Q}] \twoheadrightarrow \underline{Q}.$$

Moreover, $C[\underline{P}] \twoheadrightarrow \underline{P}$ is preserved under any reduction of $C[\underline{P}]$ which leaves $\underline{P}$ unaffected, as Proposition 2.9 states ($\underline{P}$ may be multiplied, though.)

3. Mixed terms

We will now consider disjoint unions, or as we will call them, *direct sums* $R_0 \oplus R_1$ of TRSs R_0, R_1 having disjoint alphabets. These are defined as follows. Let $\mathcal{F}$ be a set of function and constant symbols, and let $\mathcal{V}$ be a countably infinite set of variables. Then $Ter(\mathcal{F}, \mathcal{V})$ is the set of terms constructed from $\mathcal{F}$ and $\mathcal{V}$. If R_i ($i = 0,1$) are TRSs with rule sets $Red(R_i)$, terms $Ter(\mathcal{F}_i, \mathcal{V})$ such that $\mathcal{F}_0$ and $\mathcal{F}_1$ are disjoint, then $R_0 \oplus R_1$ is the TRS with terms $Ter(\mathcal{F}_0 \cup \mathcal{F}_1, \mathcal{V})$ and reduction rules $Red(R_0) \cup Red(R_1)$. Instead of $Ter(\mathcal{F}_0 \cup \mathcal{F}_1, \mathcal{V})$ we will also write $Ter(R_0 \oplus R_1)$.

For mnemotechnical reasons we will call the function and constant symbols of R_0: *black* and those of R_1: *white*. To distinguish in print between them, the black symbols are capitals and the white symbols are lower case. Thus a term $M \in \text{Ter}(R_0 \oplus R_1)$, in its tree notation, is a constellation of black and white "triangles", as in Figure 3.1. Here the root of M is the leading symbol of M.

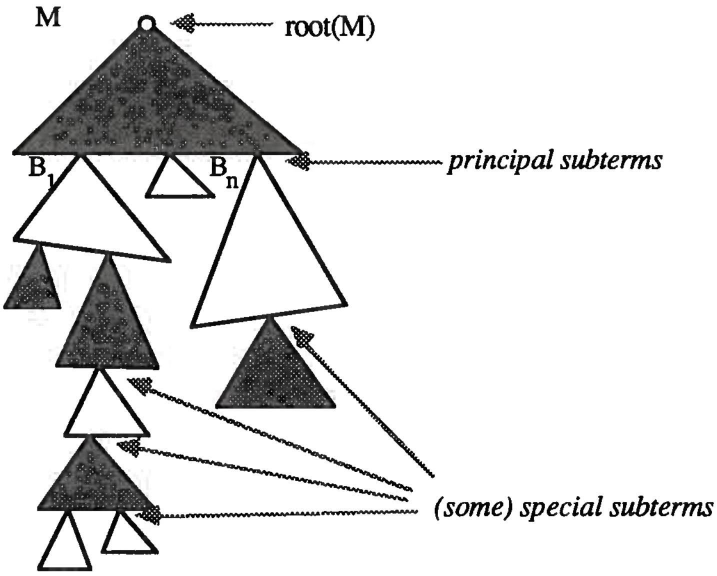


Figure 3.1

Note that if R_0 and R_1 are complete (as always assumed in this paper), every term in $\text{Ter}(R_0 \oplus R_1)$ has a normal form; this can easily be proved using innermost reductions (in which by definition only redexes are reduced containing no proper subredexes). Moreover, the normal form is unique, since $R_0 \oplus R_1$ is confluent (by the main theorem in Toyama [87]). The normal form of term t will be denoted by $t\downarrow$.

3.1. DEFINITION. (i) Let $M \equiv C[B_1,\ldots,B_n] \in \text{Ter}(R_0 \oplus R_1)$ and $C[\ ] \not\equiv \square$. Then we write $M \equiv C[\![B_1,\ldots,B_n]\!]$ if $C[\ ,\ldots,\ ]$ is a context of R_0 and $\text{root}(B_i) \in \mathcal{F}_1$ for $i = 1,\ldots,n$. (Likewise with 0,1 interchanged.) The B_i are called the *principal* subterms of M.

(ii) The set $\mathcal{S}(M)$ of *special* subterms (more precisely, subterm occurrences) is inductively defined as follows:

$$\mathcal{S}(M) = \begin{cases} \{M\} & \text{if } M \in \text{Ter}(R_d)\ \ (d = 0,1) \\ \{M\} \cup \bigcup_i \mathcal{S}(B_i) & \text{if } M \equiv C[\![B_1,\ldots,B_n]\!]\ \ (n > 0) \end{cases}$$

(iii) $\mathcal{S}_d(M) = \{N \mid N \in \mathcal{S}(M)\ \&\ \text{root}(N) \in \mathcal{F}_d\}\ \ (d = 0,1)$.

(iv) $\mathcal{G}_d(M) = \{N \mid M \twoheadrightarrow N\ \&\ \text{root}(N) \in \mathcal{F}_d\}$.

3.2. DEFINITION. Let $M \in \text{Ter}(R_0 \oplus R_1)$. Then:

$$\text{rank}(M) = \begin{cases} 1 & \text{if } M \in \text{Ter}(R_d) \ (d = 0,1) \\ \max_i \{\text{rank}(B_i)\} + 1 & \text{if } M \equiv C[B_1,\ldots,B_n] \ (n > 0) \end{cases}$$

The following fact (where $\rightarrow$ is reduction in $R_0 \oplus R_1$) has a routine proof which is omitted.

3.3. PROPOSITION. *If* $M \rightarrow N$ *then* $\text{rank}(M) \geq \text{rank}(N)$. □

3.4. PROPOSITION. *Let* $M \twoheadrightarrow N$ *where both* M, N *have a black root. Then there exists a reduction* $M \equiv M_0 \rightarrow M_1 \rightarrow M_2 \rightarrow \ldots \rightarrow M_n \equiv N$ *such that all* M_i $(i = 0,\ldots,n)$ *have a black root.*

PROOF. Let $M \twoheadrightarrow^k N$ $(k \geq 0)$. We will prove the proposition by induction on k. The case $k = 0$ is trivial. Now let $M \rightarrow M' \twoheadrightarrow^{k-1} N$. If the root of M' is black, we are through, using the induction hypothesis. If the root of M' is white, then there exists a context C[ ] with black root such that $M \equiv C[M']$ and $C[\ ] \rightarrow \Box$, the trivial context. Thus, we have a reduction $M \equiv C[M'] \twoheadrightarrow C[N] \rightarrow N$ in which all terms have black root. □

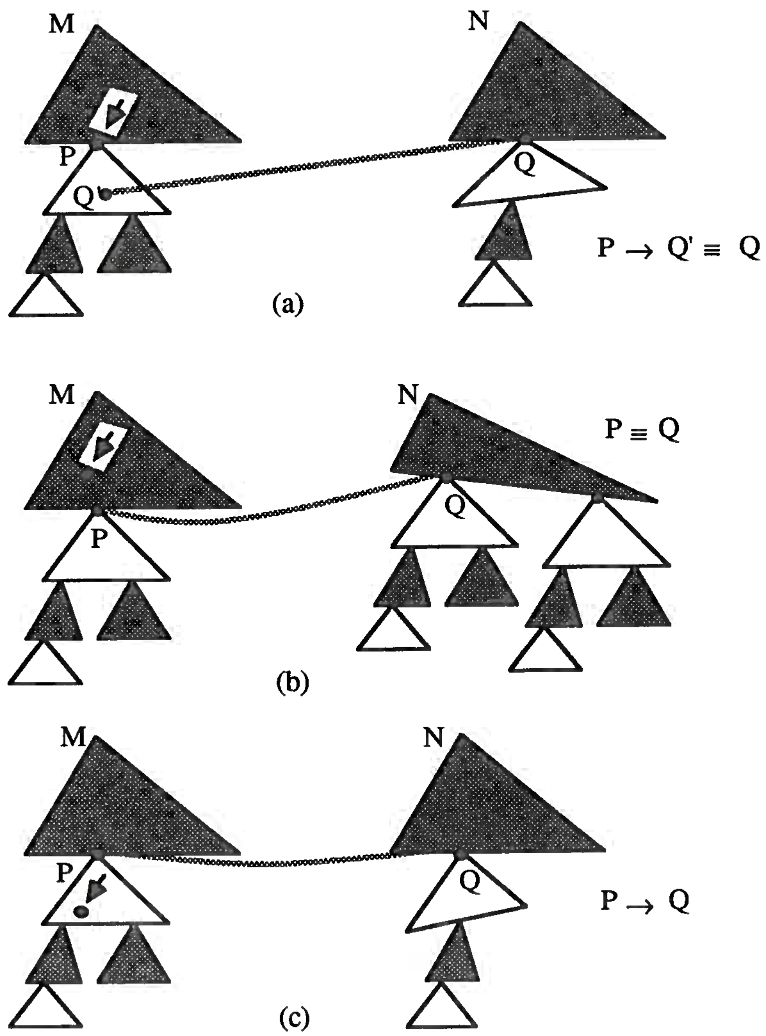


Figure 3.2

3.5. LEMMA. *Let* $M \to N$ *where* M, N *have black roots. Let* Q *be a special subterm of* N *with white root. Then there is a special subterm* P *of* M *with white root such that* $P \equiv Q$ *or* $P \to Q$.

PROOF. Consider the (white) root symbol of $Q \subset N$ and trace it back to its ancestor symbol in M. Of course the ancestor symbol is in some white 'triangle' of M. In case it is not the root of the white triangle (as in Figure 3.2(a)), which is the top triangle of the special subterm $P \subset M$, then clearly P collapses to $Q' \equiv Q$. So $P \to Q' \equiv Q$.

In case the root of Q traces back to the root of some special subterm P of N with white root, there are two possibilities. Either in the reduction step $M \to_A N$ a redex A has been contracted whose root (indicated with an arrow in the figure) is below the root of P, in which case $P \to Q$; or the root of A was above that of P or incomparable with that of P, in which case $P \equiv Q$. These cases are illustrated by Figures 3.2(c),(b) respectively. □

3.6. LEMMA. *Let* M *have a black root* ($\in \mathcal{F}_0$) *and suppose* $M \twoheadrightarrow N$ *where* N *has a white root. Then* M *has a special subterm* P *with white root such that* $M \equiv C[\underline{P}] \twoheadrightarrow \underline{P}$ *and* $P \twoheadrightarrow N$.
(See Figure 3.3.)

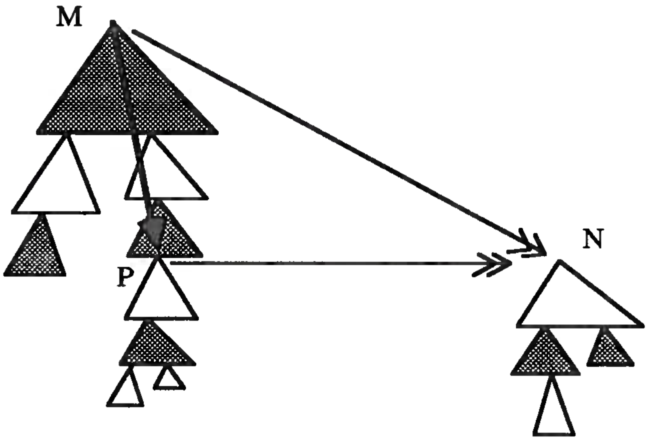


Figure 3.3

PROOF. Suppose $M \twoheadrightarrow^k N$. We will prove the proposition by induction on k. The case k = 1 is trivial; then N must be in fact one of the principal subterms M_r of $M \equiv C[\![M_1,\ldots,M_r,\ldots,M_n]\!]$ and we can take $P \equiv M_r$.

Induction hypothesis: suppose the statement is proved for k - 1. Now consider $M \twoheadrightarrow^k N$, i.e. $M \to M' \twoheadrightarrow^{k-1} N$ for some M'.

Case 1. The root of M' is white. Then $M \equiv C[\![M_1,\ldots,M_r,\ldots,M_n]\!] \to M' \equiv M_r$ for some r. Take $P \equiv M_r$.

Case 2. The root of M' is black. According to the induction hypothesis M' has a special subterm P' with white root such that $M' \equiv C[\underline{P'}] \twoheadrightarrow \underline{P'}$ and $P' \twoheadrightarrow N$. By Lemma 3.5 there is a special subterm $P \in S_1(M)$ such that $P \to P'$ or $P \equiv P'$. We distinguish two subcases:

Case 2.1. $P \to P'$. Then $M \equiv C[P] \to M' \equiv C[P']$. By Lemma 2.8 $M \equiv C[\underline{P}] \twoheadrightarrow \underline{P}$. Since $P \to P' \twoheadrightarrow N$ the statement is proved for this case.

Case 2.2. $P \equiv P'$. Then $M \equiv C[\underline{P}] \to C^*[\underline{P},\ldots,\underline{P},\ldots,\underline{P}] \twoheadrightarrow_e C^*[P,\ldots,\underline{P},\ldots,P] \equiv M' \equiv C[\underline{P}] \equiv C[\underline{P'}] \twoheadrightarrow \underline{P'} \equiv \underline{P}$. □

3.7. Essential subterms.
As the last lemma (3.6) states, if M has a black root all reductions of M to a term with white root can be 'factored through' reductions of M to its special subterms with white root. Of these special subterms with white root, some are even more special: the *essential* subterms of M. As we will see, every collapse reduction of M to a special subterm Q with white root can be factored as a collapse of M to an essential subterm P followed by a collapse of P to Q.

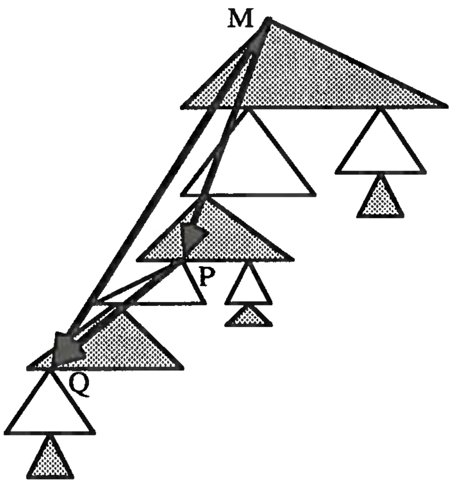


Figure 3.4

3.7.1. DEFINITION. Let M have black root. Let P be a special subterm of M with white root such that M collapses to P. Then P is an *essential* subterm (occurrence) of M if there is no special subterm P' with white root such that P ≢ P', M collapses to P', P' collapses to P. The set of essential subterms of M is $\mathfrak{E}$(M). (Likewise with colors interchanged.)

In other words: Let root(M) ∈ $\mathcal{F}_0$. Then the essential subterms of M are the *maximal* elements in the set {N ∈ $\mathcal{S}_1$(M) | M collapses to N}, partially ordered by the relation '... collapses to ...'.

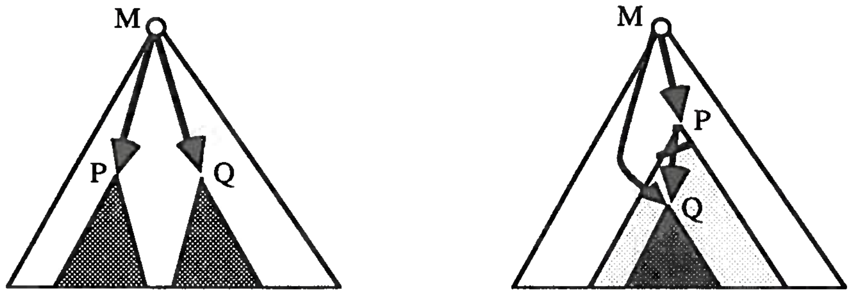
3.7.2. LEMMA. *Let* M *have black root, and suppose* M ↠ N *where* N *has white root. Then for some essential subterm* P *of* M: P ↠ N.

PROOF. Immediately by Lemma 3.6 and Definition 3.7.1. □

4. Deterministic terms

In the preceding section we have already set up some notions to discuss the 'collapsing behaviour' of mixed terms. We will now introduce an important property of this collapsing behaviour—first for the case of a single TRS.

4.1. DEFINITION. Let R be a TRS and M ∈ Ter(R). Then M is a *nondeterministic* term if



(a) (b)

Figure 4.1

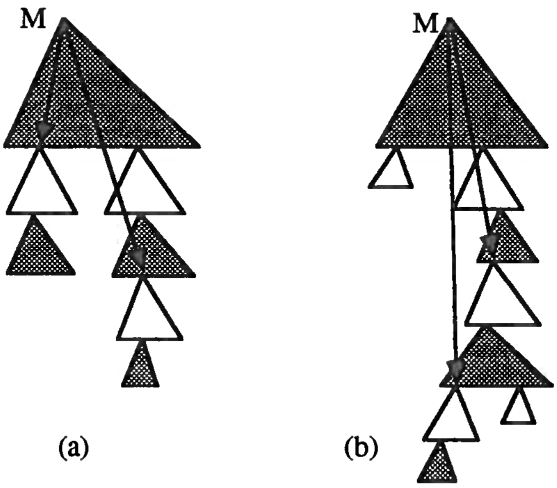
(i) $M \equiv C[P, Q]$ and $C[\underline{P}, Q] \twoheadrightarrow \underline{P}$, $C[P, \underline{Q}\,] \twoheadrightarrow \underline{Q}$, *or*

(ii) $M \equiv C[P]$, $P \equiv C'[Q]$, $C[\underline{P}] \twoheadrightarrow \underline{P}$, $C[C'[\,\underline{Q}\,]] \twoheadrightarrow \underline{Q}$ but not $C'[\,\underline{Q}\,] \twoheadrightarrow \underline{Q}$.

An example of a nondeterministic term was given in the introduction of Section 2, for nondeterminism of type (i). As an example of nondeterminism of type (ii) consider
$R = \{F(x) \to G(x,x), G(D,x) \to x, G(H(y),D) \to y, H(D) \to D, C \to D\}$. This TRS is left-linear and complete. Now take $M \equiv F(H(C))$; then $F(\underline{H(C)}) \twoheadrightarrow \underline{H(C)}$, $F(H(\,\underline{C}\,)) \twoheadrightarrow \underline{C}$, but not $H(\,\underline{C}\,) \twoheadrightarrow \underline{C}$.

4.2. REMARK. The phenomenon of nondeterministic terms is caused by ambiguities between the rewrite rules (i.e. the presence of 'critical pairs'). Indeed, one can prove: In a left-linear, non-ambiguous TRS (called 'regular' in Klop [89]) all terms are deterministic. The proof is rather lengthy and since we have no need for this fact here, not included in this paper.

4.3. DEFINITION. Let R_0, R_1 be arbitrary TRSs and let $M \in Ter(R_0 \oplus R_1)$. Then M is a *mixed nondeterministic term* if M has at least two essential subterm occurrences. (See Figure 4.2.)



(a) (b)

Figure 4.2

4.3.1. REMARK. There are terms M, M' with $M \to M'$ such that M' is mixed nondeterministic, but M is not. Example: consider $R_0 = \{G(x) \to F(x,x), F(x,C) \to x, F(C,x) \to x\}$, $R_1 = \{g(x) \to x\}$ and $M \equiv G(g(C)) \to F(g(C), g(C)) \equiv M'$.

Clearly, a mixed nondeterministic term is nondeterministic in the sense of Definition 4.1.

4.4. PROPOSITION. *Let* $C[\![M_1,\ldots,\underline{M}_p,\ldots,M_m]\!] \twoheadrightarrow \underline{M}_p$ *where all* M_i $(i = 1,\ldots,m)$ *are normal forms. Then* $C[z_1,\ldots,z_p,\ldots,z_m] \twoheadrightarrow z_p$ *(for fresh variables* $z_1,\ldots,z_m$*).*

PROOF. An obvious consequence of the definition of direct sum. □

In the sequel we will say that a term *has colour change* if root(M) is black and root(M↓) is white, or vice versa.

4.5. PROPOSITION. *Let the root of* M *be black and suppose* M *has color change (i.e. the root of* M↓ *is white). Let* $M \equiv C[\![M_1,\ldots,M_p,\ldots,M_m]\!]$ *where* M_p *is an essential subterm of* M.

(i) *Then* M *cannot have an essential subterm* $Q \subset M_p$.

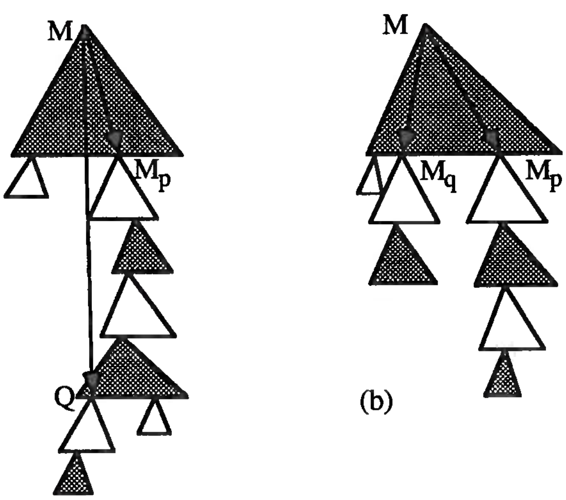
(ii) *No* M_q *with* $q \neq p$ *is an essential subterm of* M.

PROOF. (i) Since by confluence $M_p{\downarrow} \equiv M{\downarrow}$, the root of $M_p{\downarrow}$ is white. Thus we can write $C[M_1{\downarrow},\ldots,M_p{\downarrow},\ldots,M_m{\downarrow}] \equiv C'[\![N_1,\ldots,N_{k-1},M_p{\downarrow},N_{k+1},\ldots,N_n]\!]$ where N_i is a normal form for all i. (Note the brackets $[\![\ ]\!]$ in the last context.)

By Proposition 2.9 and Lemma 2.10 we have

$$C'[\![N_1,\ldots,N_{k-1},\underline{M}_p{\downarrow},N_{k+1},\ldots,N_n]\!] \twoheadrightarrow \underline{M}_p{\downarrow}.$$

By Proposition 4.4: $C'[z_1,\ldots,z_{k-1},z_k,z_{k+1},\ldots,z_n] \twoheadrightarrow z_k$. So:

(a) 

(b)

Figure 4.3

$$C[M_1,\ldots,M_{p-1},z_k,M_{p+1},\ldots,M_m] \twoheadrightarrow C'[N_1,\ldots,N_{k-1},z_k,N_{k+1},\ldots,N_n] \twoheadrightarrow z_k.$$

Hence $C[\![M_1,\ldots,M_{p-1},\underline{M_p},M_{p+1},\ldots,M_m]\!] \twoheadrightarrow_f \underline{M_p}$. (1)
Now put $M_p \equiv C''[Q]$, where $C''[\;] \not\equiv \Box$, and suppose Q is an essential subterm of M. Then

$$C[\![M_1,\ldots,M_{p-1},C''[\;\underline{Q}\;],M_{p+1},\ldots,M_m]\!] \twoheadrightarrow \underline{Q}. \qquad (2)$$

Now from (1) we have

$$C[\![M_1,\ldots,M_{p-1},C''[\;\underline{Q}\;],M_{p+1},\ldots,M_m]\!] \twoheadrightarrow_f C''[\;\underline{Q}\;]. \quad (3)$$

From (2) and (3) it follows by Proposition 2.9 that $C''[\;\underline{Q}\;] \twoheadrightarrow \underline{Q}$. But this contradicts the fact that Q is an essential subterm of M. This ends the proof of (i).

(ii). Suppose both M_p, M_q ($p \neq q$) are essential subterms of M. Since $M\!\downarrow \equiv M_p\!\downarrow \equiv M_q\!\downarrow$, it follows that the roots of M_p, M_q are white. Thus we can write

$$C[M_1\!\downarrow,\ldots, M_p\!\downarrow,\ldots, M_q\!\downarrow,\ldots, M_m\!\downarrow] \equiv$$
$$C'[\![N_1,\ldots,N_{k-1}, M_p\!\downarrow,N_{k+1},\ldots,N_{s-1}, M_q\!\downarrow,N_{s+1},\ldots,N_n]\!]$$

where N_i is a normal form for all i. By a similar argument as in (i), we have

$$C'[z_1,\ldots,z_{k-1},z_k,z_{k+1},\ldots,z_{s-1},z_s,z_{s+1},\ldots,z_n] \twoheadrightarrow z_k$$

and also $\twoheadrightarrow z_s$. But this contradicts the confluence property of $\rightarrow$. □

4.6. PROPOSITION. *Let* $M \equiv C[\![M_1,\ldots,M_m]\!]$ *where* M_i $(i = 1,\ldots,m)$ *is in normal form. Let* $M' \equiv C[z_1,\ldots,z_m]$ *where* $z_1,\ldots,z_m$ *are fresh variables not in* M. *If* M *has an infinite reduction* M $\rightarrow\rightarrow\rightarrow\ldots$, *then* M' *has an infinite reduction* M' $\rightarrow\rightarrow\rightarrow\ldots$.

PROOF. An obvious consequence of the definition of direct sum. □

4.7. PROPOSITION. *Let* $M \equiv C[\![M_1,\ldots,M_p,\ldots,M_m]\!]$ *have color change. Let* $Q \subset M_p$ *be an essential but not principal subterm of* M. *Then:*

(i) M_p *has color change;*

(ii) M_p *has an essential subterm* $P \supset Q$ *such that* Q *is again an essential subterm in* $M' \equiv C[M_1,\ldots,P,\ldots,M_m] \equiv [M_p{\rightarrow}P]\,M$, *i.e.* M *after collapsing* M_p *to* P.

(See Figure 4.4.)

PROOF. Suppose M has a black root, as in Figure 4.4. First we note that M_p is not an essential subterm of M, by Proposition 4.5(i).

Given is that $M \twoheadrightarrow Q$. We will now define an actual reduction from M to Q where M_p is used 'as late as possible', as if one were reluctant to actually use M_p. First M_p is frozen; result

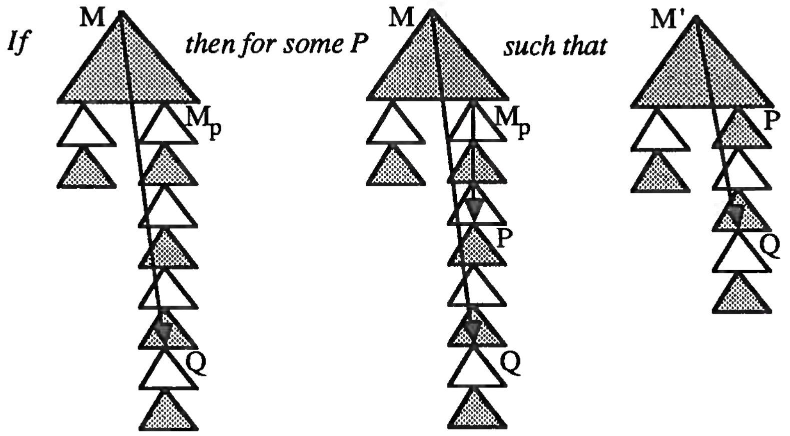


Figure 4.4

$C[M_1,\ldots,\underline{M}_p,\ldots,M_m]$. This term is reduced as far as possible in the sense of $\twoheadrightarrow_f$; i.e. it is reduced to its f-normal form $A_0 \equiv C_0[\underline{M}_p,\ldots,\underline{M}_p]$ where all occurrences of $\underline{M}_p$ are displayed. Note that there is at least one such occurrence, and that $C_0[M_p,\ldots,M_p] \twoheadrightarrow Q$. Exactly one of the M_p occurrences will contain an ancestor of Q. This occurrence of M_p we underline: $A_0' \equiv C_0[M_p,\ldots,\underline{M}_p,\ldots,M_p]$ and again we take the f-normal form, result $A_1 \equiv C_1[\underline{M}_p,\ldots,\underline{M}_p]$. This procedure is repeated, leading to a sequence $A_0, A_0', A_1, A_1', A_2 \ldots$.

Claim. *The procedure generating* $A_0, A_1, A_2, \ldots$ *stops at some* n *such that* $A_n \equiv C_n[\underline{M}_p]$, *containing exactly one occurrence of* $\underline{M}_p$.

Proof of the claim. Suppose an infinite sequence $A_0, A_1, \ldots, A_i, \ldots$ is generated. Then in $A_i \equiv C_i[\,\underline{M}_p,\ldots,\underline{M}_p]$ there are at least two occurrences of $\underline{M}_p$. This means that we have an infinite reduction

$$A_0 \twoheadrightarrow_e A_0' \rightarrow_f^+ A_1 \twoheadrightarrow_e A_1' \rightarrow_f^+ A_2 \twoheadrightarrow_e \ldots .$$

We want to prove that this gives rise to an infinite reduction $B_0 \rightarrow^+ B_1 \rightarrow^+ B_2 \rightarrow^+ \ldots$ where $B_i \equiv C_i[M_p{\downarrow},\ldots,M_p{\downarrow}]$. Since $B_0 \equiv C_0[M_p{\downarrow},\ldots,M_p{\downarrow}]$ has all its principal subterms in normal form, this contradicts termination of R_d (d = 0,1), using Proposition 4.6. Hence the sequence $A_0, A_1, \ldots$ must stop. Clearly, the sequence stops in some $A_n \equiv C_n[\underline{M}_p]$ where $\underline{M}_p$ occurs just once.

Now we construct the following diagram (see Figure 4.5):

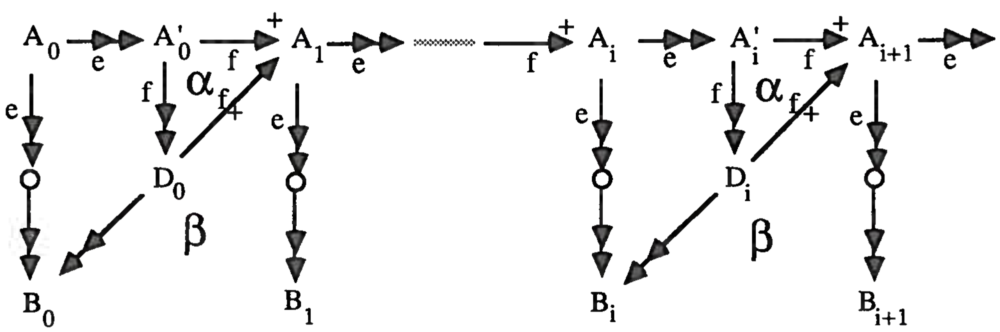


Figure 4.5

Here

$$\begin{aligned} A_i &\equiv C_i[\underline{M}_p,\ldots, \underline{M}_p,\ldots, \underline{M}_p],\\ A_i' &\equiv C_i[M_p,\ldots, \underline{M}_p,\ldots, M_p],\\ B_i &\equiv C_i[M_p\!\downarrow,\ldots, M_p\!\downarrow,\ldots, M_p\!\downarrow],\\ D_i &\equiv C_i[M_p\!\downarrow,\ldots, \underline{M}_p,\ldots, M_p\!\downarrow]. \end{aligned}$$

In the diagram of Figure 4.5 the subdiagrams (α) 'follow' from confluence of $\twoheadrightarrow_f$ (Proposition 2.4(i)) and the fact that A_i is an f-normal form. Further, the '+' in $A_i' \to^+ A_{i+1}$ follows since an underlined subterm is (at least) doubled.

We wish to show $B_0 \to^+ B_1 \to^+ B_2 \to^+ \ldots$. Consider to this end the subdiagrams (β):

$$\begin{array}{lcr} D_i \equiv C_i[M_p\!\downarrow,\ldots,\underline{M}_p,\ldots,M_p\!\downarrow] \to_f^+ C_{i+1}[\underline{M}_p,\ldots, \underline{M}_p] & & \equiv A_{i+1}\\ \downarrow & & \downarrow\\ \downarrow & & \downarrow\\ B_i \equiv C_i[M_p\!\downarrow,\ldots,M_p\!\downarrow,\ldots,M_p\!\downarrow] \to^+ C_{i+1}[M_p\!\downarrow,\ldots, M_p\!\downarrow] & & \equiv B_{i+1} \end{array}$$

and in particular the reduction $D_i \to^+ A_{i+1}$. Copy this reduction, now replacing each $\underline{M}_p$ by $M_p\!\downarrow$. Clearly, this is just the reduction $B_i \to^+ B_{i+1}$ we are looking for. $\square_{\text{claim}}$

Now consider $A_n \equiv C_n[\underline{M}_p]$. Observe that $C_n[\ ]$ is not the trivial context, i.e. $A_n \not\equiv \underline{M}_p$; otherwise we would have established that M_p is an essential subterm of M, which is not the case as we remarked earlier. Also observe that $C_n[\ ]$ is in normal form.

Clearly, M_p is still a principal subterm of $C_n[M_p]$—no white symbol can have settled 'above' the white root of M_p. Let $C_n[M_p]$ be in fact $C'[\![P_1,\ldots,P_k,M_p,P_{k+1},\ldots,P_r]\!]$. Now what is the color of the root of $M_p\!\downarrow$? If it is white, then $C_n[M_p\!\downarrow]$ is a normal form, with black root. But $C_n[M_p\!\downarrow]$ is in fact $M\!\downarrow$—contradicting the assumption that M has color change. Hence M_p has color change, and we have proved (i) of the lemma.

(ii) So we have $C_n[M_p] \equiv C'[\![P_1,\ldots,P_k,M_p,P_{k+1},\ldots,P_r]\!] \equiv C'[\![P_1,\ldots,P_k,C°[Q],P_{k+1},\ldots,P_r]\!]$ and $C'[\![P_1,\ldots,P_k,C°[\,\underline{Q}\,],P_{k+1},\ldots,P_r]\!] \twoheadrightarrow \underline{Q}$ (by Proposition 2.9). The root of $C_n[M_p]$ is black, the root of Q is white and $C_n[\ ]$ is in normal form. Therefore the only way to "get at" Q is that M_p reduces to a term with a black top, say P', where P' contains an ancestor of Q.

Since M_p reduces to P' with black root, by Lemma 3.7.2 there is an essential subterm P of M_p such that $P \twoheadrightarrow P'$ and also containing an ancestor of Q. (To see that P also contains an ancestor of Q: underline Q in M, or equivalently, replace Q by e(Q). Clearly, P' contains e(Q), since $P' \twoheadrightarrow$ e(Q) by a slight abuse of notation. Now since $P \twoheadrightarrow P'$, P must contain the symbol e. Hence P contains an ancestor of Q.)

It remains to prove that in $[M_p \to P]\, M \equiv M'$ the subterm Q is still essential. By Proposition 2.9 we have indeed $M' \equiv C''[\,\underline{Q}\,] \twoheadrightarrow \underline{Q}$. It might be however that Q is not an essential subterm of M', because of a situation as in Figure 4.6(a). But then it is evident that "similar" arrows as in Figure 4.6(b) would have existed before the collaps, in contradiction with the fact that Q is an essential subterm of M. $\square$

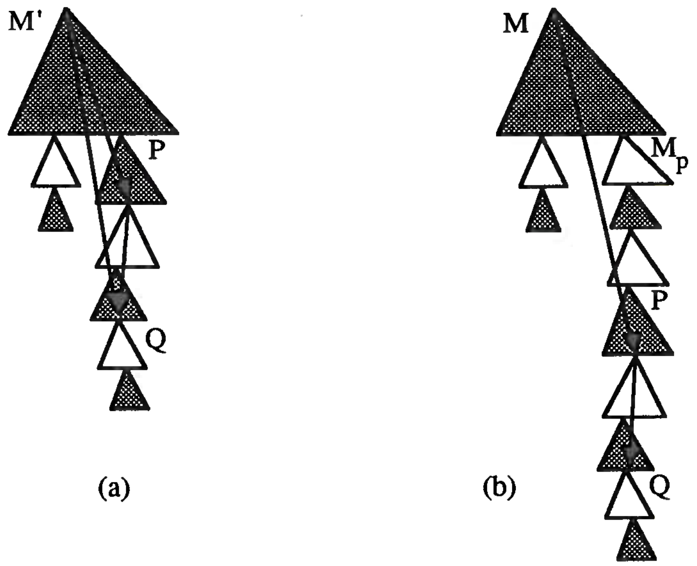


Figure 4.6

4.8. MAIN LEMMA. *Let* M *be a term with color change. Then* M *has exactly one essential subterm.*

PROOF. For a proof by contradiction, suppose that there exists a term with color change but having more than one essential subterm. Let M be such a term with minimal length (i.e. the total number of symbols in M).

By Proposition 4.5(ii), M must have an essential subterm Q which is not principal. (See Figure 4.7.) Let M_p be the principal subterm such that $M_p \supset Q$.

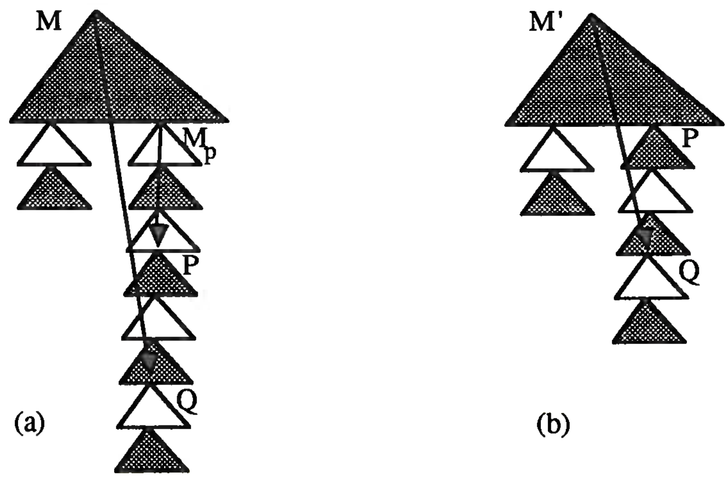


Figure 4.7

By Proposition 4.7, M_p has color change, and moreover M_p has an essential subterm $P \supset Q$. Because of the minimality property of M, P is also the *unique* essential subterm of M_p.

<u>Claim</u>. M', *originating from* M *by collapsing* M_p *to* Q *(see Figure 4.7(b)), has at least as many essential subterms as* M.

If this Claim is true we are done, because it yields a contradiction with the minimality property of M. Actually, M' has just as many essential subterms as M, but we will not need that.

Proof of the claim. As indicated in Figure 4.7(b), Q in M' is still an essential subterm. We have to show that none of the essential subterms of M is 'lost' in the collapse to M'. So let us inspect all essential subterms of M and see that they are preserved as essential subterms in M'. Figure 4.8(a) gives a catalogue of possible and impossible positions of the essential subterms of M.

Arrows of type 1, leading to an essential subterm of M not in M_p, stay preserved by Proposition 2.9 (see Figure 4.8(b)).The same holds for arrows of type 2,3 leading to subterms of P, the unique essential subterm of M_p.

However, what about possible arrows of type 4, to a subterm intermediate between M_p and P, or arrow 5, to M_p itself? Or arrows of type 6? Such arrows seem to get lost in the collapse to M'. Fortunately, they do not exist: arrow 5 is forbidden by Proposition 4.5(i), and arrow 4 cannot exist by the unicity of P. (More explicitly: suppose arrow 4 to Q" exists. Then by Proposition 4.7, there is an essential subterm P' of M_p with $P' \supset Q''$. This contradicts the unicity of P and $P \subset Q''$.) Finally, an arrow of type 6 cannot exist by the same reasoning as for type 4. □

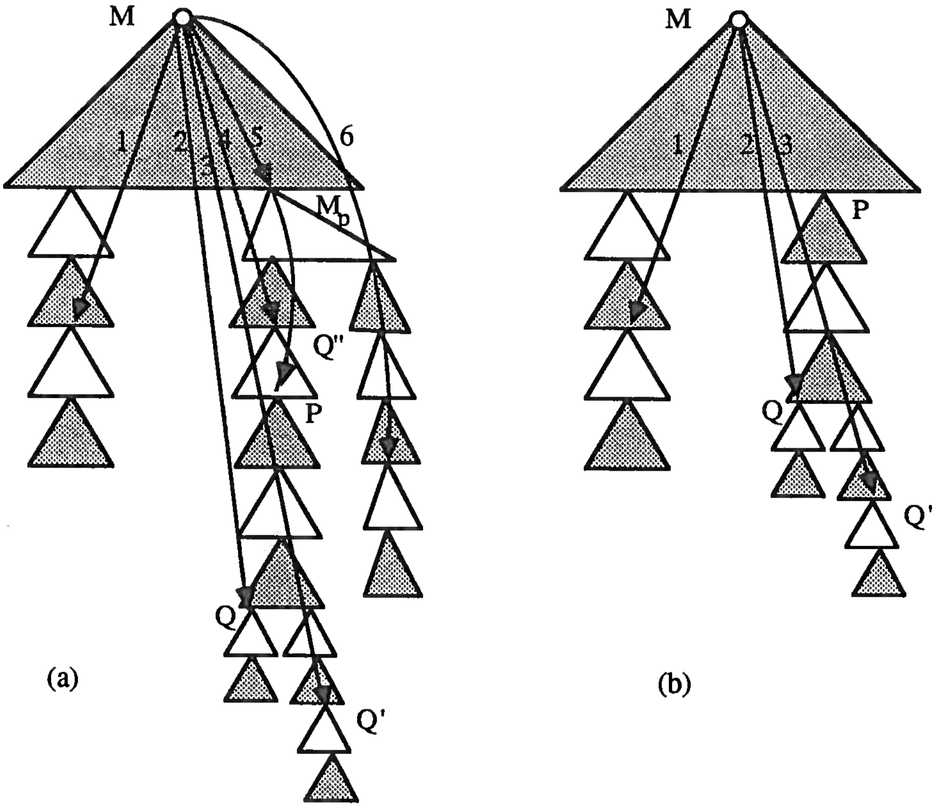


Figure 4.8

As an example, Figure 4.9 shows how M can be collapsed to yield the impossible situation as in Proposition 4.5(ii).

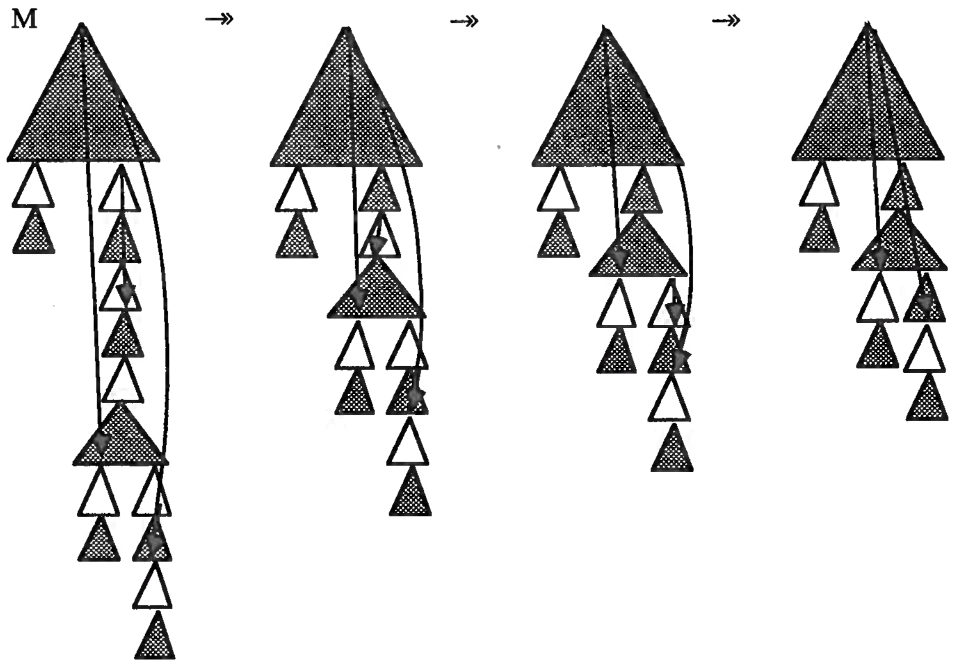


Figure 4.9

5. Termination for the direct sum

In this section we will finally prove the main result, i.e. the termination property for the direct sum $R_0 \oplus R_1$ of left-linear and complete R_0, R_1. To this end, we define for each term $M \in Ter(R_0 \oplus R_1)$ two terms: the *black projection* $M^0 \in Ter(R_0)$ of M, and the *white projection* $M^1 \in Ter(R_1)$ of M. Roughly, the black/white projections of M contain precisely the 'information' in the black, respectively white, part of M. In fact we will prove that if M is a supposed minimal (with respect to length) term with white root, admitting an infinite reduction, then the white projection M^1 has already an infinite reduction. As M^1 is in $Ter(R_1)$, this is in contradiction with the termination property of R_1 and we will have proved termination for $R_0 \oplus R_1$.

The definition of the projections is rather subtle and rests heavily upon the Main Lemma 4.8. We will prepare the way by an example. Suppose M is structured as in Figure 5.1(a); a concrete example is: $M \equiv F(g(C), h(C))$ as in Figure 5.1(b) where $R_0 = \{F(x, C) \rightarrow x, F(C, x) \rightarrow x\}$ and $R_1 = \{g(x) \rightarrow x, h(x) \rightarrow x\}$. So $P_1 \equiv g(C)$, $P_2 \equiv h(C)$ are the essential subterms of M. Now suppose we wish to determine the white projection M^1. As M can collapse to P_1 as well as to P_2, the projection M^1 should convey the information in both P_1, P_2. The problem is that these subterms are disjoint (in this case). Yet, there is a way to combine them into one term: namely by *piling* them with result as in Figure 5.1(c), respectively 5.1(d). Throughout this section the variable x will play a special role.

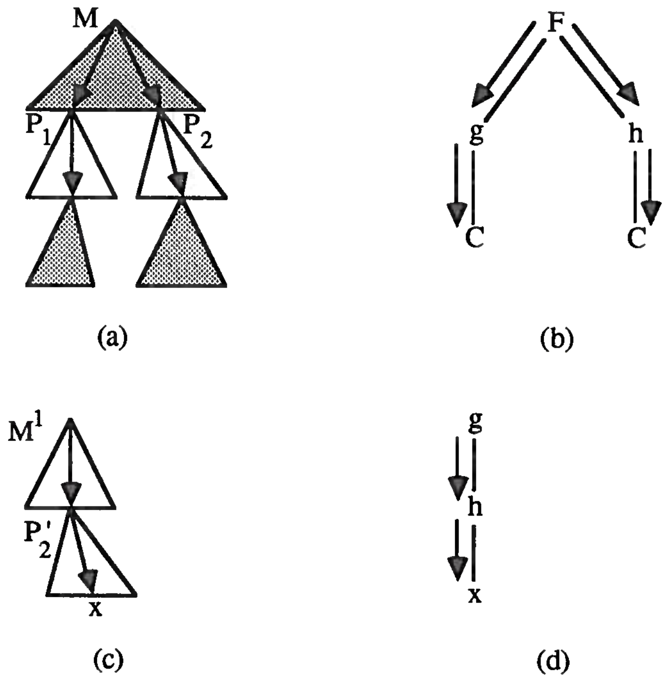


Figure 5.1

Of course, we were lucky in this example, since the white top triangles of P_1, P_2 which we wanted to pile, were indeed 'pileable'. In the situation of Figure 5.2, where P_1 is supposed to be again nondeterministic, the piling would not have succeeded, because triangles 1, 2 can be taken such that they cannot be piled. However, our Main Lemma 4.8 says that such a situation does not exist and, therefore, piling succeeds as will be proved in more detail below.

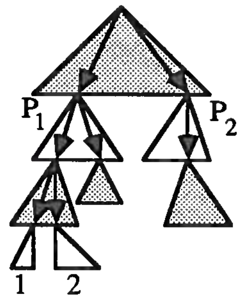


Figure 5.2

5.0.1. VARIABLE CONVENTION. *From now on we will assume that every term* $M \in Ter(R_0 \oplus R_1)$ *has only* 'x' *as variable occurrences*, unless other variables are explicitly displayed. Since $R_0 \oplus R_1$ is left-linear, this variable convention may be assumed in the sequel without loss of generality.

5.1. DEFINITION. Let R be a confluent and left-linear TRS. Let $P_1,...,P_p$ be a sequence of terms of R ($p \geq 2$). Then the term pile($P_1,...,P_p$) is defined as follows:
CASE 1. $P_i \twoheadrightarrow x$ for $i = 1,...,p$. So, $P_i \equiv C_i[x]$ such that $C_i[\underline{x}] \twoheadrightarrow \underline{x}$ (there may be other occurrences of not underlined x's in $C_i[\underline{x}]$).

Then pile($P_1,...,P_p$) $\equiv C_1[C_2[...C_{p-1}[C_p[x]]...]]$.
CASE 2. Not case 1: then pile($P_1,...,P_p$) is undefined.

5.1.1. EXAMPLE. Note that pile($P_1,...,P_p$) does not merely depend on $P_1,...,P_p$ but also on R. If $R = \{F(x,y) \to x, I(x) \to x\}$ and $P_1 \equiv F(x,x)$, $P_2 \equiv I(x)$, then pile(P_1,P_2) $\equiv F(I(x), x)$. If in R the first rule is replaced by $F(x,y) \to y$, then pile(P_1,P_2) $\equiv F(x, I(x))$.

5.1.2. REMARK. The condition in Definition 5.1, that R is confluent and left-linear, is necessary to ensure that pile is a (partial) *function*. Otherwise, taking $R = \{F(x,y) \to x, F(x,y) \to y, I(x) \to x\}$ and $P_1 \equiv F(x,x)$, $P_2 \equiv I(x)$, we would have (see the previous example) pile(P_1,P_2) $\equiv F(x, I(x))$ as well as $F(I(x), x)$. That confluence and left-linearity of R is sufficient to make pile into a function, is easily seen. For, it is impossible that then $C[\ \underline{x}, x] \twoheadrightarrow \underline{x}$ as well as $C[\ x, \underline{x}\ ] \twoheadrightarrow \underline{x}$, since this implies (by left-linearity) that $C[x,y] \twoheadrightarrow x$ as well as y, contradicting confluence.

In the sequel, we will use pile for terms of $R_0 \oplus R_1$, where R_0, R_1 are complete and left-linear. Indeed the direct sum is then confluent (and, trivially, left-linear), as guaranteed by the theorem in Toyama [87] stating that the direct sum of confluent TRSs is again confluent. Thus the operation pile is well-defined.

5.2. DEFINITION. Let $M \in Ter(R_0 \oplus R_1)$. Then the *white projection* M^1 of M is defined by induction on rank(M):

(1) $x^1 \equiv x$
(2) root(M) is white:
  (2.1) $M \in Ter(R_1)$, then $M^1 \equiv M$
  (2.2) $M \equiv C[\![M_1,...,M_m]\!]$ $(m > 0)$, then $M^1 \equiv C[M_1{}^1,...,M_m{}^1]$

(3) root(M) is black:
  (3.1) M has no essential subterm. Then $M^1 \equiv x$.
  (3.2) M has precisely one essential subterm P. Then $M^1 \equiv P^1$.
  (3.3) M is mixed nondeterministic, with sequence of essential subterms $P_1,...,P_p$. Then $M^1 \equiv$ pile($P_1{}^1,...,P_p{}^1$).

(The black projection M^0 is defined by interchanging 0,1 and black, white.) In case (3.3), the essential subterm occurrences $P_1,...,P_p$ may be ordered by precedence of their head symbol. (The precise ordering is irrelevant.) Note that M^1 may be undefined, due to the possible undefinedness of pile($P_1{}^1,...,P_p{}^1$). We will however show that in the present situation, where R_0, R_1 are left-linear and complete, pile($P_1{}^1,...,P_p{}^1$) and hence M^1 (and likewise M^0) is defined for all M. Note further that (3.2) is not a special case of (3.3) since in general pile(N) $\not\equiv$ N. (In fact: pile(N) $\equiv$ N $\Leftrightarrow$ pile(N) is defined.) Finally, note that in (3.2), (3.3) we have rank(P^1) < rank(M) and rank($P_i{}^1$) < rank(M) respectively.

5.3. EXAMPLE.

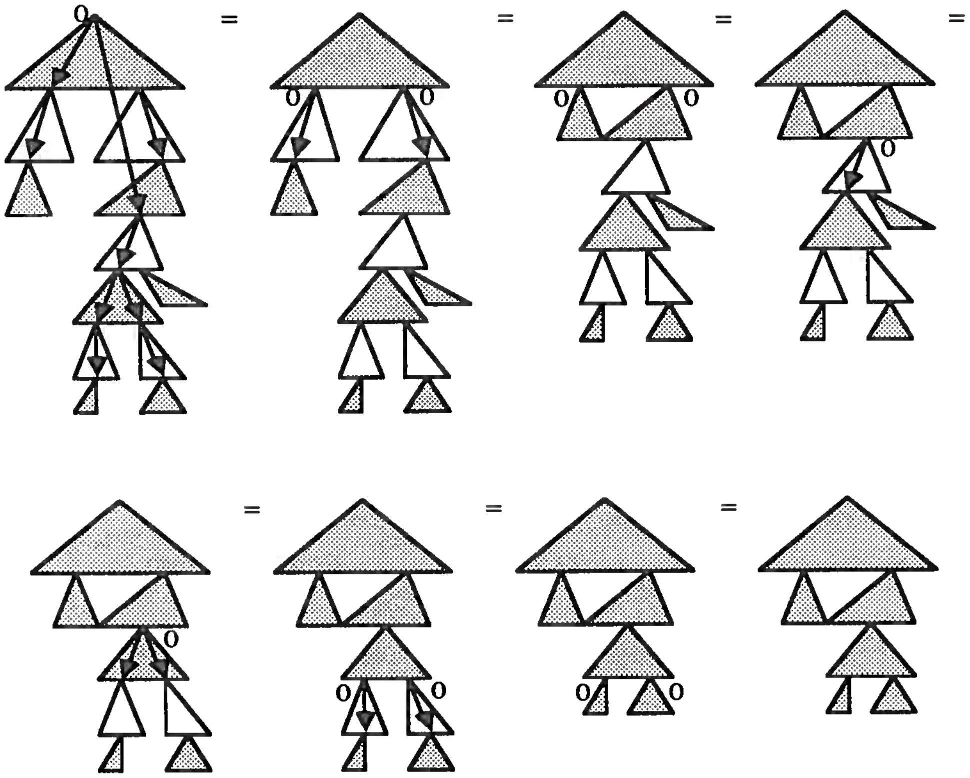


Figure 5.3

5.4. EXAMPLE.

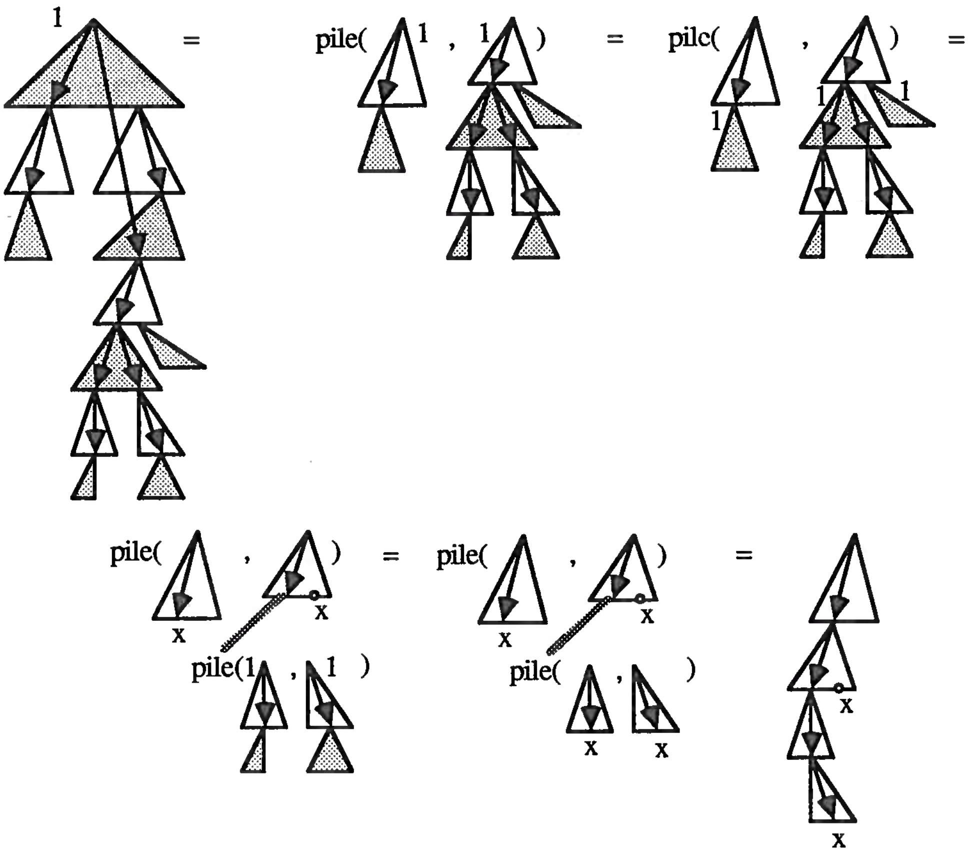


Figure 5.4

5.4.1. REMARK. (Cf. Example 5.3.) This remark will be needed in the proof of Proposition 5.13. Let M have black root, $M \equiv C[\![M_1,...,M_m]\!]$. Then, by Definition 5.2, $M^0 \equiv C[M_1{}^0,...,M_m{}^0]$, i.e. the 'projection symbol' 0 is pushed down until it reaches the principal subterms. From this it follows that if $M \equiv C'[N_1,...,N_n]$ where C'[,...,] is all black, then we have also that $M^0 \equiv C'[N_1{}^0,...,N_n{}^0]$. (See Figure 5.5.)

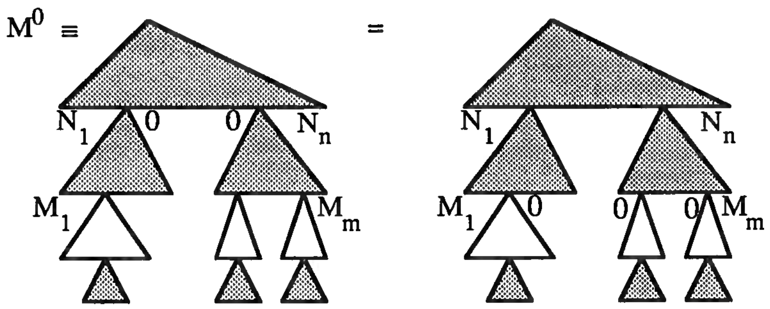


Figure 5.5

5.5. EXAMPLE. Consider the TRSs $R_0 = \{F(C(y), x) \to x, F(x, C(y)) \to x, C(y) \to D\}$, $R_1 = \{g(x) \to x, h(x) \to x\}$ with R_1 containing also a constant 'a'. Then

$$(F(g(C(a), h(C(a))))^1 = \text{pile}((g(C(a)))^1, (h(C(a)))^1) =$$
$$\text{pile}(g((C(a))^1), h((C(a))^1)) = \text{pile}(g(x), h(x)) = g(h(x)).$$

5.6. EXAMPLE. The black projection of the following term (in Figure 5.6) is undefined; however, by the Main Lemma (4.8) such terms cannot exist (when R_0, R_1 are left-linear and complete).

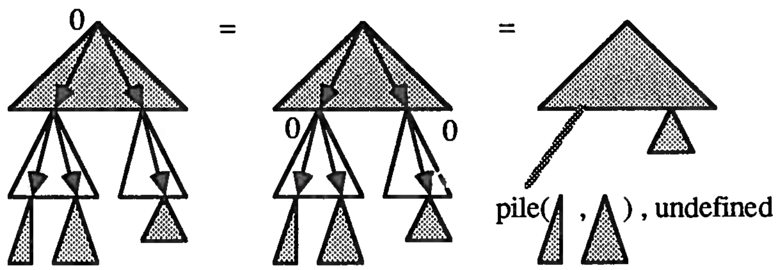


Figure 5.6

In many cases, the result of projecting M to M^0 or M^1 will be a term collapsing to the special variable x (I.e. $M^0 \twoheadrightarrow x$, respectively $M^1 \twoheadrightarrow x$.) See e.g. Example 5.5. We will prove this fact now.

5.7. LEMMA. $M^d \twoheadrightarrow x \iff \text{root}(M\downarrow) \notin \mathcal{F}_d$ $(d = 0,1)$

PROOF. We will prove a slightly stronger statement, namely (i) & (ii):

(i) If $\text{root}(M\downarrow) \notin \mathcal{F}_d$, then $M^d \twoheadrightarrow x$

(ii) If $\text{root}(M\downarrow) \in \mathcal{F}_d$ and $M\downarrow \equiv C^*[\![M_1,...,M_m]\!]$ $(m \geq 0)$, then $M^d\downarrow \equiv C^*[x,...,x]$. (Hence: not $M^d \twoheadrightarrow x$.)

We will prove (i) & (ii) by induction on rank(M).

Basis. rank(M) = 1.

Case 1. $M \in Ter(R_d)$. Then $M^d \equiv M$, by (1) or (2.1) of Definition 5.2. If $M\downarrow \equiv x$, then $M^d \equiv M \twoheadrightarrow x$, so (i) holds; (ii) holds vacuously. If $root(M\downarrow) \in \mathcal{F}_d$, then (i) holds vacuously; (ii) holds since $M^d\downarrow \equiv M\downarrow$.

Case 2. $M \in Ter(R_{1-d})$. We may suppose $M \not\equiv x$, since the case $M \equiv x$ was covered in case 1. By (3.1) of Definition 5.2, $M^d \equiv x$. So (i) holds. Statement (ii) holds vacuously.

Induction hypothesis. Assume (i) & (ii) hold for rank(M) < k ($k \geq 2$).
Now consider M with rank(M) = k.

Case 3. $root(M) \in \mathcal{F}_d$. Let $M \equiv C[\![M_1,...,M_m]\!]$ ($m \geq 1$), so $M^d \equiv C[M_1{}^d,...,M_m{}^d]$. Without loss of generality we may assume that $root(M_i\downarrow) \notin \mathcal{F}_d$ for $1 \leq i < p$ and $root(M_j\downarrow) \in \mathcal{F}_d$ for $p \leq j \leq m$. So, by the induction hypothesis: $M_i{}^d \twoheadrightarrow x$ ($1 \leq i < p$), and writing $M_j\downarrow \equiv C_j{}^*[\![N_{j,1},...,N_{j,nj}]\!]$ ($n_j \geq 0$, $p \leq j \leq m$): $M_j{}^d\downarrow \equiv C_j{}^*[x,...x]$. Thus

$$M\downarrow \equiv C[M_1\downarrow,...,M_m\downarrow]\downarrow$$
$$\equiv C[M_1\downarrow,...,M_{p-1}\downarrow,\ C_p{}^*[\![N_{p,1},...,N_{p,np}]\!],...,C_m{}^*[\![N_{m,1},...,N_{m,nm}]\!]\ ]\downarrow$$

and

$$M^d\downarrow \equiv C[M_1{}^d\downarrow,...,M_m{}^d\downarrow]\downarrow$$
$$\equiv C[x,...,x,\ C_p{}^*[x,...,x],...,\ C_m{}^*[x,...,x]\ ]\downarrow.$$

Note that $M_1\downarrow,...,M_{p-1}\downarrow, N_{p,1},...,N_{m,nm}$ are normal forms having roots not in $\mathcal{F}_d$. Therefore, if $root(M\downarrow) \notin \mathcal{F}_d$, then

$$C[x,...,x,\ C_p{}^*[x,...,x],...,C_m{}^*[x,...,x]\ ]\downarrow \equiv x$$

and if $root(M\downarrow) \in \mathcal{F}_d$, then we have a context $C^*[\ ,...,\ ] \equiv C[\ ,...,\ ,\ C_p{}^*[\ ,...,\ ],...,\ C_m{}^*[\ ,...,\ ]]\downarrow$ such that $M\downarrow \equiv C^*[N_1,...,N_n]$ where $N_i \in \{M_1\downarrow,...,M_{p-1}\downarrow, N_{p,1},...,N_{m,nm}\}$ and $M^d\downarrow \equiv C^*[x,...,x] \not\equiv x$ (using $N_{p,1}{}^d \equiv ... \equiv N_{n,nm}{}^d \equiv x$ by (3.1) of Definition 5.2).

Case 4. $root(M) \notin \mathcal{F}_d$. Distinguish the subcases:

Case 4.1. M has no essential subterm. Then $M^d \equiv x$, either by (1) of Definition 5.2 or (3.1). Hence $M^d\downarrow \equiv x$, and (i) & (ii) hold.

Case 4.2. M has precisely one essential subterm P. Then $M^d \equiv P^d$. Note that rank(P) < k. Since $M\downarrow \equiv P\downarrow$ and $M^d\downarrow \equiv P^d\downarrow$, the claim follows by using the induction hypothesis.

Case 4.3. M has essential subterms $P_1,...,P_p$ ($p > 1$). Note that $rank(P_i) < k$ for all i. By the Main Lemma, $root(M\downarrow) \notin \mathcal{F}_d$. Since $M\downarrow \equiv P_i\downarrow$, also $root(P_i\downarrow) \notin \mathcal{F}_d$ for all i. So, by the induction hypothesis, $P_i{}^d \twoheadrightarrow x$ for all i. Now $M^d \equiv pile(P_1{}^d,...,P_p{}^d)$ and since $P_i{}^d \twoheadrightarrow x$ ($i = 1,...,p$), M^d is defined. Obviously, $M^d \equiv pile(P_1{}^d,...,P_p{}^d) \twoheadrightarrow x$. Hence (i) is true and (ii) holds vacuously. □

5.7.1. REMARK. Note that the formulation of Lemma 5.7 entails:

$$M \twoheadrightarrow x \;\Rightarrow\; M^d \twoheadrightarrow x.$$

5.8. REMARK. From Lemma 5.7 and the Main Lemma 4.8 it follows that the projections M^0, M^1 are always defined. For, consider case (3.3) in Definition 5.2 of M^1. So, root(M) is black. Since M is nondeterministic, it cannot have color change, i.e. root($M\downarrow$) is black or $M\downarrow \equiv x$. Now $M\downarrow \equiv P_1\downarrow \equiv ... \equiv P_p\downarrow$ where the P_i ($i = 1,...,p$) are the essential subterms of M. By Lemma 5.7:

$$\text{root}(P_i\downarrow) \text{ is black } \Leftrightarrow P_i^{\,1} \twoheadrightarrow x$$

($i = 1,...,p$). Hence pile($P_1^{\,1},...,P_p^{\,1}$) is defined.

5.9. PROPOSITION. *Let* M *have a black root and suppose* P *is an essential subterm of* M. *Then* $M^1 \twoheadrightarrow P^1$.

PROOF. See Definition 5.2 of M^1. The only possible cases are (3.2) and (3.3). In case (3.2), $M^1 \equiv P^1$. In case (3.3), $M^1 \equiv \text{pile}(P_1^{\,1},...,P_p^{\,1})$ where $P \equiv P_k$ for some $k \in \{1,...,p\}$. From Remark 5.8 we know that $P_i^{\,1} \equiv C_i[x]$ such that $C_i[\,\underline{z}\,] \twoheadrightarrow \underline{z}$. Hence by definition of 'pile':

$$M^1 \equiv C_1[...[C_p[x]]...]$$

which yields $M^1 \twoheadrightarrow C_k[x] \equiv P_k^{\,1}$. □

Now we would like to project a supposed infinite reduction $M_0 \to M_1 \to M_2 \to ...$ of some $M_0 \in \text{Ter}(R_0 \oplus R_1)$ directly into a reduction $M_0^{\,d} \twoheadrightarrow M_1^{\,d} \twoheadrightarrow M_2^{\,d} \twoheadrightarrow ...$ containing infinitely many proper steps. Unfortunately, a step $M \to N$ in general does not project into a reduction $M^d \twoheadrightarrow N^d$, as the following example shows.

5.10. EXAMPLE. Let R_0, R_1 be as in Remark 4.3.1:

$$R_0 = \{G(x) \to F(x,x),\ F(x,C) \to x,\ F(C,x) \to x\}$$
$$R_1 = \{g(x) \to x\}.$$

Consider $M \equiv G(g(C)) \to F(g(C), g(C)) \equiv N$. Then $M^1 \equiv g(x)$ and $N^1 \equiv g(g(x))$. So: not $M^1 \twoheadrightarrow N^1$.

However, we can translate an infinite reduction in $R_0 \oplus R_1$ into an infinite reduction in one of the components in an indirect way.

5.11. NOTATION. (i) We write $M \equiv_o N$ when M, N have the same outermost-layer context, i.e. $M \equiv C[\![M_1,...,M_m]\!]$ and $N \equiv C[\![N_1,...,N_m]\!]$ for some M_i, N_i ($i = 1,...,m$).

(ii) Let $M \equiv C[\![M_1,...,M_m]\!]$ and suppose $M \to^R N$. If the redex occurrence R occurs in some M_i, we write $M \to_i N$ ('inner reduction'); otherwise we write $M \to_o N$ ('outer reduction').

Note that $M_1 \twoheadrightarrow_i M_2$, $M_2 \twoheadrightarrow_i M_3$ implies $M_1 \twoheadrightarrow_i M_3$.

5.12. PROPOSITION. *Let* $M \rightarrow_o N$ *where* M, N *have white roots. Suppose* $M \equiv_o A$ *and* $A \twoheadrightarrow_i M$ *(internal reduction). Then there exists a term* B *such that* $N \equiv_o B$, $A \rightarrow_o B$, $B \twoheadrightarrow_i N$ *and* $A^1 \rightarrow B^1$. (See diagram in Figure 5.7.)

PROOF. Let $A \equiv C[\![A_1,\ldots,A_m]\!]$, $M \equiv C[\![M_1,\ldots,M_m]\!]$ and $N \equiv C'[\![M_{i1},\ldots,M_{in}]\!]$ ($i_j \in \{1,\ldots,m\}$). Take $B \equiv C'[\![A_{i1},\ldots,A_{in}]\!]$. Then $A \rightarrow_o B$ and $B \twoheadrightarrow_i N$. From $A^1 \equiv C[A_1{}^1,\ldots,A_m{}^1]$ and $B^1 \equiv C'[A_{i1}{}^1,\ldots,A_{in}{}^1]$ it follows that $A^1 \rightarrow B^1$. □

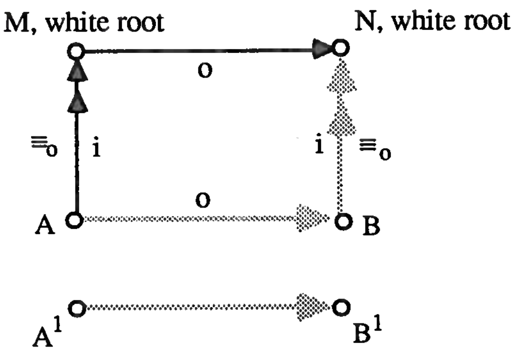


Figure 5.7

5.13. PROPOSITION. *Let* $M \twoheadrightarrow N$ *where* root(N) *is white. Then there exists a term* A *such that* $N \equiv_o A$, $A \twoheadrightarrow_i N$, $M \twoheadrightarrow A$, *and* $M^1 \twoheadrightarrow A^1$.
(See diagram in Figure 5.8.)

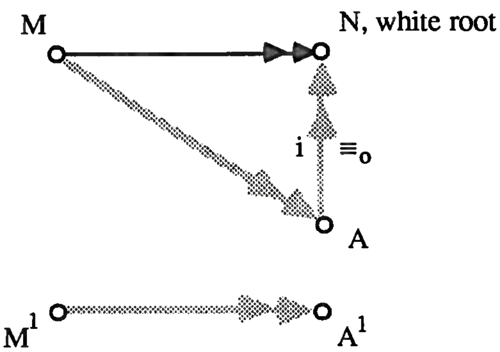


Figure 5.8

PROOF. We will prove the proposition by induction on rank(M).
Basis: rank(M) = 1. This case is trivial: take $A \equiv N$.
Induction hypothesis: the proposition holds for M with rank(M) < k. Now let M have rank k.
CLAIM. *The proposition holds if* $M \twoheadrightarrow_i N$.
PROOF OF THE CLAIM. Let $M \equiv C[\![M_1,\ldots,M_m]\!] \twoheadrightarrow_i N \equiv C[N_1,\ldots,N_m]$ where $M_i \twoheadrightarrow N_i$ for $i = 1,\ldots,m$. Without loss of generality we may assume that $N_1 \equiv x, \ldots, N_{p-1} \equiv x$, root($N_i$) is white for $p \leq i < q$, and root(N_j) is black for $q \leq j \leq m$. Thus

$$N \equiv C[x,\ldots,x,N_p,\ldots,N_{q-1},N_q,\ldots,N_m].$$

By the induction hypothesis, for every M_i ($p \le i < q$) there is a term A_i such that we have the diagram in Figure 5.9.

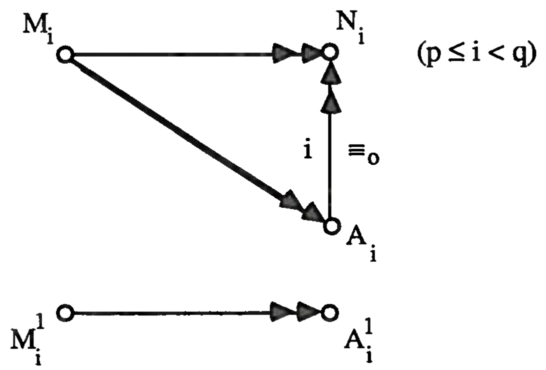


Figure 5.9

Now take $A \equiv C[x,\ldots,x,A_p,\ldots,A_{q-1},M_q,\ldots,M_m]$. Clearly, $M \twoheadrightarrow A$. Since $A_i \equiv_o N_i$ ($p \le i < q$) and both M_j, N_j ($q \le j \le m$) have black root, we have $A \equiv_o N$. Furthermore, $A \twoheadrightarrow_i N$ since $A_i \twoheadrightarrow_i N_i$ ($p \le i < q$) and by Proposition 3.4 the reductions $M_j \twoheadrightarrow N_j$ ($q \le j \le m$) can be taken such that every term in them has a black root. Now

$$M^1 \equiv C[M_1{}^1,\ldots,M_{p-1}{}^1,M_p{}^1,\ldots,M_{q-1}{}^1, M_q{}^1,\ldots,M_m{}^1]$$
$$A^1 \equiv C[x,\ldots\ldots,x, \quad A_p{}^1,\ldots, A_{q-1}{}^1, M_q{}^1,\ldots,M_m{}^1]$$

(for A^1, see Remark 5.4.1). By Remark 5.7.1 we have $M_i{}^1 \twoheadrightarrow x$ ($1 \le i < p$), since $M_i \twoheadrightarrow x$. We had already $M_i{}^1 \twoheadrightarrow A_i{}^1$ ($p \le i < q$). Hence $M^1 \twoheadrightarrow A^1$. (See Figure 5.10.) $\square_{claim}$

Now we will prove the full proposition (without the additional assumption $M \twoheadrightarrow_i N$ as in the Claim) for rank(M) = k. We distinguish two cases.

Case 1. The root of M is white.
So M, N have both white roots. Hence there is, by Proposition 3.4, a reduction $M \twoheadrightarrow N$ in which every term has white root. This reduction can be splitted into

$$M \twoheadrightarrow_i \to_o \twoheadrightarrow_i \to_o \cdots \twoheadrightarrow_i N.$$

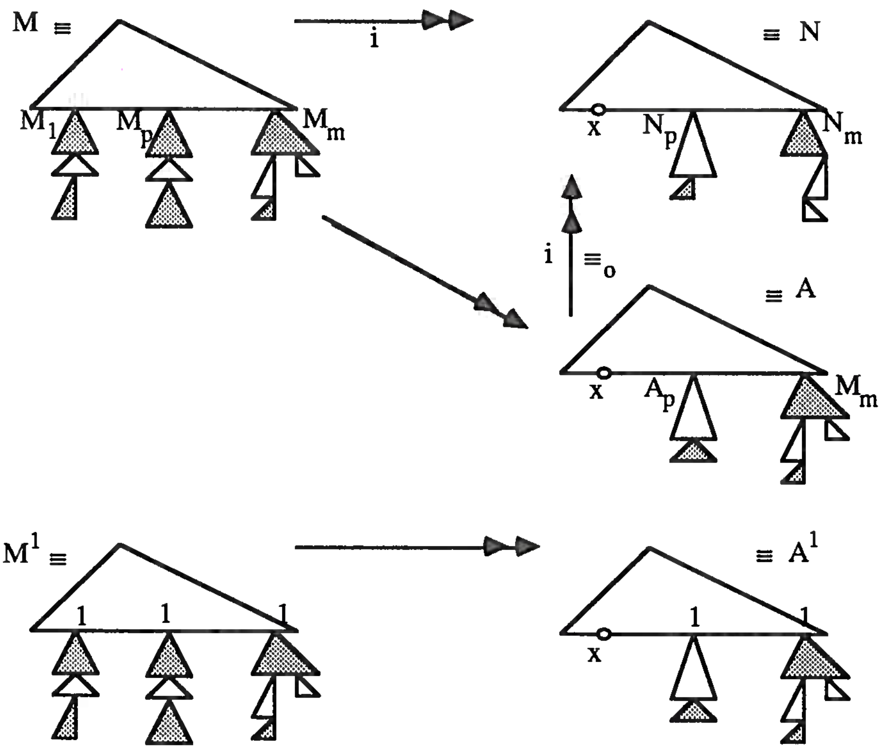


Figure 5.10

Now we can construct the diagram as in Figure 5.11.

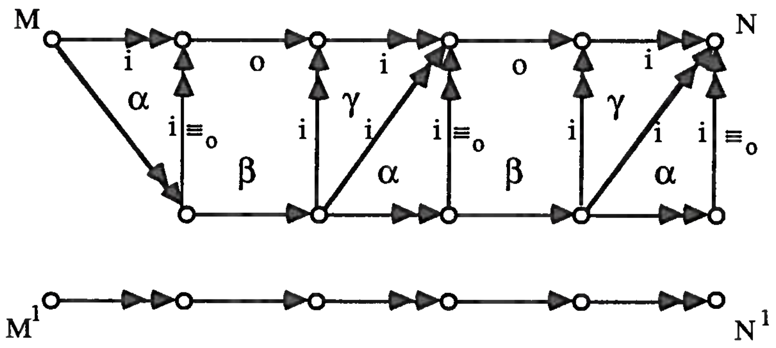


Figure 5.11

Here subdiagrams α are justified by the Claim, subdiagrams β by Proposition 5.12 and subdiagrams γ follow by transitivity of $\twoheadrightarrow_i$.

Case 2. The root of M is black.

By Lemma 3.7.2 there is an essential subterm Q of M such that M $\twoheadrightarrow$ Q $\twoheadrightarrow$ N. By Proposition 5.9, $M^1 \twoheadrightarrow Q^1$. Obviously, rank(Q) < rank(M) = k. Hence we can construct the diagram in Figure 5.12, where the triangular subdiagram is obtained by the induction hypothesis applied on Q.

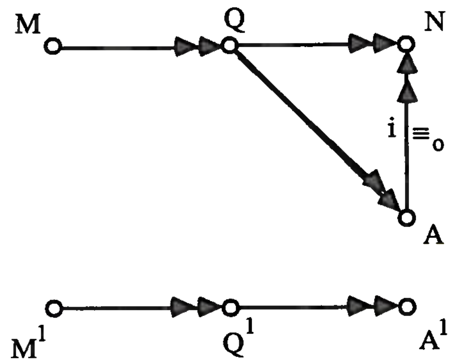


Figure 5.12

□

We are now able to state and prove the main result of our paper:

5.14. THEOREM. *Let* R_0, R_1 *be left-linear and complete. Then* $R_0 \oplus R_1$ *is a terminating TRS.*

PROOF. Let $M \in Ter(R_0 \oplus R_1)$. We will prove by induction on rank(M) that M does not have an infinite reduction.

The case rank(M) = 1 is trivial, by assumption. Induction hypothesis: if rank(M) < k, M cannot have an infinite reduction. Without loss of generality, we may assume that M has a white root. Now suppose for a proof by contradiction that there is a term M with rank(M) = k having an infinite reduction $M \equiv M_0 \to M_1 \to M_2 \to \ldots$. Now $rank(M_0) \geq rank(M_1) \geq \ldots$; by the induction hypothesis it follows that $rank(M_0) = rank(M_1) = \ldots$. Hence the roots of all M_i are white.

Now infinitely many steps $M_i \to M_{i+1}$ must be in fact $M_i \to_o M_{i+1}$; otherwise we would have an infinite internal reduction

$$M_k \equiv C_k[\![M_{k,1},\ldots,M_{k,r}]\!] \to_i \to_i \to_i \ldots$$

which would yield an infinite reduction of some $M_{k,p}$, in contradiction with the induction hypothesis.

So, we can apply the following diagram construction, using Propositions 5.12, 5.13.

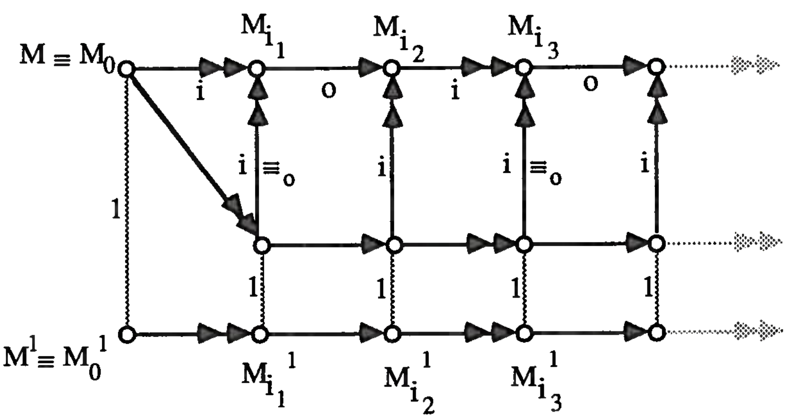


Figure 5.13

But this means that M^1 has already an infinite reduction, in contradiction with the termination property of R_1. □

5.15. COROLLARY. *Let* R_0, R_1 *be left-linear. Then:*

$$R_0 \oplus R_1 \text{ is complete} \Leftrightarrow R_0 \text{ and } R_1 \text{ are complete.}$$

PROOF. (⇒) is trivial. (⇐) follows from Theorem 5.14 and the theorem in Toyama [87] stating that for all TRSs, $R_0 \oplus R_1$ is confluent iff R_0, R_1 are confluent. □

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