Sequential Pivotal Mechanisms for Public Project Problems

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Abstract. It is well-known that for several natural decision problems no budget balanced Groves mechanisms exist. This has motivated recent research on designing variants of feasible Groves mechanisms (termed as 'redistribution of VCG (Vickrey-Clarke-Groves) payments') that generate reduced deficit. With this in mind, we study sequential mechanisms and consider optimal strategies that could reduce the deficit resulting under the simultaneous mechanism. We show that such strategies exist for the sequential pivotal mechanism of the well-known public project problem. We also exhibit an optimal strategy with the property that a maximal social welfare is generated when each player follows it. Finally, we show that these strategies can be achieved by an implementation in Nash equilibrium. All proofs can be found in the full version posted in Computing Research Repository (CoRR), `http://arxiv.org/abs/0810.1383`

1 Introduction

1.1 Motivation

Mechanism design is concerned with designing non-cooperative games in which the participating rational players achieve the desired social outcome by reporting their types. Among the most commonly studied mechanisms are the ones in the Groves family that are based on transfer payments (taxes). For the case of efficient decision functions they are *incentive compatible*, i.e., they achieve truth-telling in dominant strategies. The special case called *pivotal mechanism* (sometimes also called *VCG* (Vickrey-Clarke-Groves) mechanism) is additionally *pay only* (i.e., each player needs to pay a tax) and hence *feasible* (i.e., the generated deficit is negative or zero).

It is well-known that for several problems incentive compatible mechanisms cannot achieve *budget balance* (which states that the generated deficit is zero), see, e.g., Chapter 23 of [14]. This has motivated recent research in designing appropriate instances of Groves mechanisms that generate a reduced deficit (or equivalently higher social welfare). These modifications are termed as 'redistribution of VCG payments'. In fact, they are variants of feasible Groves mechanisms. Notably, [3] and [9] showed that a deficit reduction is possible for the case of Vickrey auctions concerned with multiple units of a single good. On the

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other hand, [1] recently showed that no such deficit reduction is possible in the well-known case of the pivotal mechanism for the public project problem.

This research direction motivates our study of sequential Groves mechanisms, in particular sequential pivotal mechanism, in which players move sequentially. We face then a new situation since each player knows the types reported by the previous players. Sequential Groves mechanisms apply to a realistic situation in which there is no central authority that computes and imposes taxes and where the players move in a randomly chosen order.

1.2 Contributions

We show here that natural strategies exist in the sequential pivotal mechanism for the public project problem that generate larger social welfare than truth-telling. We also exhibit a strategy such that the social welfare is maximized when each player follows it. Finally, we show that the resulting sequential mechanisms yield an implementation in Nash equilibrium. Moreover, the vector of the latter strategies is also Pareto optimal.

To properly describe the nature of the introduced strategies we consider two concepts. An ***optimal*** strategy guarantees a player the maximum utility under the assumption that he moves simultaneously with the players who follow him. It also guarantess the player at least the same utility as truth-telling, under the assumption that the other players are truth-telling. In turn, a ***socially optimal*** strategy yields the maximal social welfare among all optimal strategies.

These concepts allow us to analyze altruistic behaviour of the players in the framework of sequential pivotal mechanism. By altruistic behaviour we mean that the players do not only care about their own utility, but also about the utility of the others.

1.3 Related Work

Ever since the seminal paper of [5] mechanism design for public goods has received a huge attention in the literature. We mention here only some representative papers the results of which provide an appropriate background for our work.

Both the continuous and discrete case of public goods have been studied. The former situation has been in particular considered in [8], where a taxation scheme has been proposed which leads to a Pareto optimal solution that can be realized in a Nash equilibrium. Sequential mechanism design for public good problems has been considered in [6], where a "Stackelberg" mechanism was proposed that combines optimal Bayes strategies with dominant strategies.

Here we study the discrete case. The situation when the decision is binary (whether to realize a public project or not) has been studied in [11], where balanced but not incentive compatible sequential mechanisms have been proposed. These mechanisms can be realized in an undominated Nash equilibrium and in subgame perfect equilibrium. Many aspects of incentive compatible mechanisms for public goods have been surveyed in [4].

The consequences of sequentiality have also been studied in the context of private contributions to public goods and in voting theory. In particular, [16] has studied the behavior of players depending on the position in which they have to take a decision and [7] has explored the relationship between simultaneous and sequential voting games. More recently, [12] has studied the problem of determining the winner in elections in which the voting takes place sequentially.

Our focus on maximizing social welfare is related to research on altruistic behaviour of the player. This subject has been studied in a number of papers in game theory, most recently in [13], where several references to earlier literature on this subject can be found. Finally, in a recent work, [2], we carried out an analogous analysis for two feasible Groves mechanisms used for single item auctions: the Vickrey auction and the Bailey-Cavallo mechanism.

1.4 Plan of the Paper

The paper is organized as follows. In the next section we recall Groves mechanisms and the pivotal mechanism by focusing on decision problems. Then, in Section 3, we introduce sequential decision problems, in particular sequential Groves mechanisms.

In the remaining sections we study the sequential pivotal mechanism for the public project problem. In Section 4 we exhibit an optimal strategy that in a limited sense simultaneously maximizes players' final utilities and another optimal strategy that maximizes the social welfare among all vectors of optimal strategies. Finally, in Section 5, we clarify the status of the optimal strategies introduced in Section 4 by showing that their vector is a Nash equilibrium w.r.t. appropriately defined preference relations on the strategy vectors, and by providing a corresponding revelation-type result. We conclude by mentioning some open problems in Section 6.

2 Preliminaries

We briefly recall the family of Groves mechanisms here. In this section we follow [10]. Let D be a set of ***decisions***, $\{1, \ldots, n\}$ be the set of players with $n \geq 2$, and for each player i let Θ_i be a set of his ***types*** and $v_i : D \times \Theta_i \rightarrow \mathbb{R}$ be his (***initial***) ***utility function***.

A ***decision rule*** is a function $f : \Theta \rightarrow D$, where $\Theta := \Theta_1 \times \cdots \times \Theta_n$. It is called ***efficient*** if for all $\theta \in \Theta$ and $d' \in D$

$$\sum_{i=1}^{n} v_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(d', \theta_i).$$

We call the tuple $(D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n, f)$ a ***decision problem***.

Recall that a ***direct mechanism*** is obtained by transforming the initial decision problem $(D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n, f)$ as follows:

- the set of decisions is $D \times \mathbb{R}^n$,

– the decision rule is a function $(f, t) : \Theta \rightarrow D \times \mathbb{R}^n$, where $t : \Theta \rightarrow \mathbb{R}^n$ and $(f, t)(\theta) := (f(\theta), t(\theta))$,
– each ***final utility function*** for player i is a function $u_i : D \times \mathbb{R}^n \times \Theta_i \rightarrow \mathbb{R}$ defined by $u_i(d, t_1, \ldots, t_n, \theta_i) := v_i(d, \theta_i) + t_i$.

We call then $\sum_{i=1}^{n} u_i((f, t)(\theta), \theta_i)$ the corresponding ***social welfare*** and refer to t as the ***tax function***.

A direct mechanism with tax function t is called

– (***dominant strategy***) ***incentive compatible*** if for all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta_i' \in \Theta_i$
$$u_i((f, t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f, t)(\theta_i', \theta_{-i}), \theta_i),$$
– ***budget balanced*** if $\sum_{i=1}^{n} t_i(\theta) = 0$ for all θ,
– ***feasible*** if $\sum_{i=1}^{n} t_i(\theta) \leq 0$ for all θ,
– ***pay only*** if $t_i(\theta) \leq 0$ for all θ and all $i \in \{1, \ldots, n\}$.

Each ***Groves mechanism*** is obtained by using the tax function $t := (t_1, \ldots, t_n)$, where[1] for all $i \in \{1, \ldots, n\}$
$$t_i(\theta) := \sum_{j \neq i} v_j(f(\theta), \theta_j) + h_i(\theta_{-i}),$$
with $h_i : \Theta_{-i} \rightarrow \mathbb{R}$ an arbitrary function.

Finally, we recall the following crucial result.

Groves Theorem. Consider a decision problem with an efficient decision rule f. Then each Groves mechanism is incentive compatible.

A special case of Groves mechanism is the ***pivotal mechanism***, which is a pay only mechanism obtained by $h_i(\theta_{-i}) := -\max_{d \in D} \sum_{j \neq i} v_j(d, \theta_j)$.

Direct mechanisms for a given decision problem can be compared w.r.t. the social welfare they entail. More precisely, given a decision problem
$$(D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n, f)$$
and direct mechanisms (determined by the sequences of tax functions) t and t' we say that t' ***welfare dominates*** t if

– for all $\theta \in \Theta$
$$\sum_{i=1}^{n} u_i((f, t)(\theta), \theta_i) \leq \sum_{i=1}^{n} u_i((f, t')(\theta), \theta_i),$$
– for some $\theta \in \Theta$
$$\sum_{i=1}^{n} u_i((f, t)(\theta), \theta_i) < \sum_{i=1}^{n} u_i((f, t')(\theta), \theta_i).$$

In this paper we analyze the following well-known decision problem, originally due to [5], and extensively discussed in the economic literature, see, e.g. [14,10].

[1] Here and below $\sum_{j \neq i}$ is a shorthand for the summation over all $j \in \{1, \ldots, n\}$, $j \neq i$.

Public project problem
Consider $(D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n, f)$, where

- $D = \{0, 1\}$ (reflecting whether a project is cancelled or takes place),
- for all $i \in \{1, \ldots, n\}$, $\Theta_i = [0, c]$, where $c > 0$,
- for all $i \in \{1, \ldots, n\}$, $v_i(d, \theta_i) := d(\theta_i - \frac{c}{n})$,
- $f(\theta) := \begin{cases} 1 & \text{if } \sum_{i=1}^n \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$

In this setting c is the cost of the project, $\frac{c}{n}$ is the cost share of the project for each player, and θ_i is the value of the project for player i. Note that the decision rule f is efficient since $\sum_{i=1}^n v_i(d, \theta_i) = d(\sum_{i=1}^n \theta_i - c)$.

It is well-known that for no $n \geq 2$ and $c > 0$ an incentive compatible direct mechanism for the public project problem exists that is budget balanced, see, e.g. [14, page 861-862]. It is then natural to search for incentive compatible direct mechanisms that generate a smaller deficit than the one obtained by the pivotal mechanism. However, the following optimality result concerning the pivotal mechanism, recently established in [1], dashed hope.

Theorem 1. *In the public project problem there exists no feasible incentive compatible direct mechanism that welfare dominates the pivotal mechanism.* □

Our aim is to show that when the original setting of the public project problem is changed to one where all players announce their types sequentially in a random order, then the deficit can be reduced.

3 Sequential Decision Problems

In this section we introduce sequential decision problems. For notational simplicity, and without loss of generality, we assume that players sequentially report their types in the order $1, \ldots, n$. To capture this type of situations, given a decision problem $\mathcal{D} := (D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n, f)$, we assume that successively stages $1, \ldots, n$ take place, where in stage i player i announces a type θ'_i to the *other players*. After stage n this yields a joint type $\theta' := (\theta'_1, \ldots, \theta'_n)$. Then each player takes the decision $d := f(\theta')$.

We call the resulting situation a ***sequential decision problem*** or more specifically, a ***sequential version of*** $\mathcal{D}$. Note that in a sequential decision problem a central planner may not exist and decisions may be taken by the players themselves. Each player i *knows* the types announced by players $1, \ldots, i-1$, so that he can use this information to decide which type to announce. To properly describe this situation we need to specify what is a strategy in this setting. A ***strategy*** of player i in the sequential version of $\mathcal{D}$ is a function

$$s_i : \Theta_1 \times \ldots \times \Theta_i \rightarrow \Theta_i.$$

In this context ***truth-telling***, as a strategy, is represented by the projection function $\pi_i(\cdot)$, defined by $\pi_i(\theta_1, \ldots, \theta_i) := \theta_i$.

From now on, we consider a direct mechanism

$$\mathcal{D} := (D \times \mathbb{R}^n, \Theta_1, \ldots, \Theta_n, u_1, \ldots, u_n, (f, t))$$

and mainly focus on Groves mechanisms.

We assume that in the considered sequential decision problem each player uses a strategy $s_i(\cdot)$ to select the type he will announce. We say that strategy $s_i(\cdot)$ of player i is ***optimal*** in the sequential version of $\mathcal{D}$ if for all $\theta \in \Theta$ and $\theta'_i \in \Theta_i$

$$u_i((f, t)(s_i(\theta_1, \ldots, \theta_i), \theta_{-i}), \theta_i) \geq u_i((f, t)(\theta'_i, \theta_{-i}), \theta_i).$$

Call a strategy of player j ***memoryless*** if it does not depend on the types of players $1, \ldots, j-1$. Then a strategy $s_i(\cdot)$ of player i is optimal if for all $\theta \in \Theta$ it yields a best response to all joint strategies of players $j \neq i$ under the assumption that players $i+1, \ldots, n$ use memoryless strategies or move jointly with player i. In particular, an optimal strategy is a best response to the truth-telling by players $j \neq i$.

A particular case of sequential decision problems are sequential Groves mechanisms. The following direct consequence of Groves Theorem provides us with a simple method of determining whether a strategy is optimal in such a mechanism.

Lemma 1. *Let $(D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n, f)$ be a decision problem with efficient decision rule f. Suppose that $s_i(\cdot)$ is a strategy for player i such that for all $\theta \in \Theta$, $f(s_i(\theta_1, \ldots, \theta_i), \theta_{-i}) = f(\theta_i, \theta_{-i})$. Then $s_i(\cdot)$ is optimal in each sequential Groves mechanism $(D \times \mathbb{R}^n, \Theta_1, \ldots, \Theta_n, u_1, \ldots, u_n, (f, t))$.* □

In particular, when the decision rule is efficient, the truth-telling strategy $\pi_i(\cdot)$ is optimal in each sequential Groves mechanism.

We are interested in maximizing the social welfare. This motivates the following notion. We say that strategy $s_i(\cdot)$ of player i is ***socially optimal*** in the sequential version of $\mathcal{D}$ if it is optimal and for all optimal strategies $s'_i(\cdot)$ of player i and all $\theta \in \Theta$ we have

$$\textstyle\sum_{j=1}^n u_j((f, t)(s_i(\theta_1, \ldots, \theta_i), \theta_{-i}), \theta_j) \geq \sum_{j=1}^n u_j((f, t)(s'_i(\theta_1, \ldots, \theta_i), \theta_{-i}), \theta_j).$$

Hence a socially optimal strategy of player i yields the maximal social welfare among all optimal strategies, under the assumption that players $i+1, \ldots, n$ use memoryless strategies or move jointly.

Consider now a sequential version of a given direct mechanism

$$(D \times \mathbb{R}^n, \Theta_1, \ldots, \Theta_n, u_1, \ldots, u_n, (f, t))$$

and assume that each player i receives a type $\theta_i \in \Theta_i$ and follows a strategy $s_i(\cdot)$. The resulting social welfare is then

$$SW(\theta, s(\cdot)) := \sum_{j=1}^n u_j((f, t)([s(\cdot), \theta]), \theta_j),$$

where $s(\cdot) := (s_1(\cdot), \ldots, s_n(\cdot))$ and $[s(\cdot), \theta]$ is defined inductively by $[s(\cdot), \theta]_1 := s_1(\theta_1)$ and $[s(\cdot), \theta]_{i+1} := s_{i+1}([s(\cdot), \theta]_1, \ldots, [s(\cdot), \theta]_i, \theta_{i+1})$.

In general, if player i assumes that he moves jointly with players $i+1, \ldots, n$ he will choose an optimal strategy. And if additionally he wants to maximize the social welfare, he will choose a socially optimal strategy (if it exists). In the next section we shall see that for the public project problem a sequence of socially optimal strategies can be found for which the resulting social welfare is always maximal. In general, we only have the following limited result.

Lemma 2. *Consider a direct mechanism* $(D \times \mathbb{R}^n, \Theta_1, \ldots, \Theta_n, u_1, \ldots, u_n, (f, t))$ *and let* $s_n(\cdot)$ *be a socially optimal strategy for player* n. *Then*

$$SW(\theta, (s'_{-n}(\cdot), s_n(\cdot))) \geq SW(\theta, s'(\cdot))$$

for all $\theta \in \Theta$ *and vectors* $s'(\cdot)$ *of optimal players' strategies.*

Proof. Directly by the definition of a socially optimal strategy. □

4 Public Project Problem

In what follows, we focus on the special case of sequential pivotal mechanisms for the public project problem. First, the following theorem gives an optimal strategy for player i that may differ from truth-telling. Part (ii) shows that, under certain natural conditions, this strategy simultaneously maximizes the final utility of every other player.

Theorem 2. *Let* $\mathcal{D}$ *be a public project problem. Let*

$$s_i(\theta_1, \ldots, \theta_i) := \begin{cases} \theta_i & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i < n, \\ 0 & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i = n, \\ c & \text{if } \sum_{j=1}^{i} \theta_j \geq c \end{cases}$$

be strategy for player i. *Then*

(i) $s_i(\cdot)$ *is optimal for player* i *in the sequential pivotal mechanism,*
(ii) for all $\theta \in \Theta$ *and* $\theta'_i \in \Theta_i$ *such that* $s_i(\theta_1, \ldots, \theta_i) \neq \theta_i$ *and* $f(\theta'_i, \theta_{-i}) = f(\theta_i, \theta_{-i})$ *we have for all* $j \neq i$

$$u_j((f, t)(s_i(\theta_1, \ldots, \theta_i), \theta_{-i}), \theta_j) \geq u_j((f, t)(\theta'_i, \theta_{-i}), \theta_j).$$

In part (ii) θ_{-i} are the types submitted by players $j \neq i$ and θ_i is the type received by player i. So part (ii) states that if strategy $s_i(\cdot)$ of player i deviates from truth-telling ($s_i(\theta_1, \ldots, \theta_i) \neq \theta_i$) and the players who follow i use memoryless strategies (so in particular, the types they submit do not depend on the type submitted by player i), then player i simultaneously maximizes the final utility of the other players (and hence the social welfare). This happens under the assumption that player i submits a type that does not alter the decision to be taken.

Table 1. Pivotal mechanism

player	type	submitted type	tax	u_i
A	110	110	-10	0
B	80	80	0	-20
C	110	110	-10	0

Table 2. Sequential pivotal mechanism

player	type	submitted type	tax	u_i
A	110	110	0	10
B	80	80	0	-20
C	110	300	-10	0

When each player follows strategy $s_i(\cdot)$, always the same decision is taken as when each player is truthful, independently on the players' order. Additionally, by part (ii) of Theorem 2 with $\theta'_i = \theta_i$, social welfare weakly increases. The following example shows that sometimes a strictly larger social welfare can be achieved.

Example 1. Assume that $c = 300$ and that there are three players, A, B and C. Table 1 illustrates the situation in the case of pivotal mechanism. In Table 2 we assume that the players submit their types in the order A, B, C. Here the social welfare increases from -20 to -10. $\square$

However, as Table 2 shows, budget balance does not need to be achieved. The following result shows that an order can always be found that yields budget balancedness.

Theorem 3. *Let $\mathcal{D}$ be a public project problem with the sequential pivotal mechanism. For all $c > 0, n \geq 2$ and $\theta \in \Theta$ there exists a permutation of players such that when each player i follows strategy $s_i(\cdot)$ of Theorem 2, budget balance is achieved.*

Proof. (Sketch). Recall that in the pivotal mechanism, given the sequence of types θ, a player i is called ***pivotal*** if $t_i(\theta) \neq 0$. First we show that not all players can be pivotal. Then we show that the desired permutation is the one in which the last player is not pivotal. $\square$

For instance, in Example 1 when the order is A, C, B or C, A, B, the decision is taken with no taxes incurred, i.e., budget balance is then achieved.

In Theorem 2(ii) we seem to be maximizing the social welfare. However, this is not the case because we assume there that each player submits a type that does not alter the decision to be taken. In fact, strategy $s_i(\cdot)$ of Theorem 2 is not socially optimal.

The following theorem provides a socially optimal strategy that in some circumstances yields a higher social welfare than the above strategy.

Theorem 4. *Let $\mathcal{D}$ be a public project problem. Let*

$$s_i(\theta_1, \ldots, \theta_i) := \begin{cases} \theta_i & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i < n, \\ 0 & \text{if } \sum_{j=1}^{i} \theta_j < c \text{ and } i = n, \\ 0 & \text{if } \sum_{j=1}^{i} \theta_j = c,\ \theta_i > \frac{c}{n} \text{ and } i = n, \\ c & \text{otherwise} \end{cases}$$

be a strategy for player i. Then

(i) $s_i(\cdot)$ is socially optimal for player i in the sequential pivotal mechanism,
(ii) for all $\theta \in \Theta$ and vectors $s'(\cdot)$ of optimal players' strategies,

$$SW(\theta, s(\cdot)) \geq SW(\theta, s'(\cdot)),$$

where $s(\cdot)$ is the vector of strategies $s_i(\cdot)$.

The remarkable thing about the above strategy $s_i(\cdot)$ is that when $\sum_{j=1}^{n} \theta_j = c$ and $\theta_n > \frac{c}{n}$, player n submits type 0, as a result of which the project does not take place. To illustrate this situation reconsider Example 1. When the players submit their types sequentially in order A, B, C following the above strategy $s_i(\cdot)$, then player C submits 0. The resulting social welfare is 0 as opposed to -10 which results when all players follow strategy $s_i(\cdot)$ of Theorem 2 (see Table 2). This also shows that the latter strategy is not socially optimal.

However, in general strategy $s_i(\cdot)$ of Theorem 4 does not need to ensure budget balance.

Example 2. Suppose that there are three players, A, B, and C, whose true types are 60, 70, and 250, respectively, while c remains 300. When the players submit their types following strategy $s_i(\cdot)$ of Theorem 4, we get the situation summarized in Table 3.

Table 3. Sequential pivotal mechanism

player	type	submitted type	tax	u_i
A	60	60	0	-40
B	70	70	0	-30
C	250	300	-70	80

Here the same decision is taken as when each player is truthful and in both situations the deficit is -70. □

On other other hand, part (*ii*) shows that when we limit ourselves to optimal strategies and each player follows the introduced strategy $s_i(\cdot)$, then a maximal social welfare results. The restriction to the vectors of optimal strategies is necessary. Indeed, Table 3 of Example 2 shows that when the order is A, B, C and each player follows the strategy $s_i(\cdot)$ of Theorem 4, then the resulting social welfare is $380 - 300 - 70 = 10$. However, when player B submits 300, then player C pays no tax and the resulting social welfare is higher, namely $380 - 300 = 80$.

5 Comments on a Nash Implementation

The sequential mechanisms here considered circumvent the limitations of the customary, simultaneous, Groves mechanisms. This and the fact that we maximize social welfare by using strategies that deviate from truth-telling requires some clarification. First of all, we can explain these sequential mechanisms by turning them into simultaneous ones as follows.

We assume that each player i receives a type $\theta_i \in \Theta_i$ and subsequently submits a function $r_i : \Theta_1 \times \ldots \times \Theta_{i-1} \to \Theta_i$ instead of a type $\theta'_i \in \Theta_i$. (In particular, player 1 submits a type, i.e., $r_1(\cdot) \in \Theta_1$.) The submissions are simultaneous. Then the behaviour of player i can be described by a strategy $s_i : \Theta_1 \times \ldots \times \Theta_i \to \Theta_i$ which when applied to the received type θ_i yields the function $s_i(\cdot, \theta_i) : \Theta_1 \times \ldots \times \Theta_{i-1} \to \Theta_i$ that player i submits. Then θ and the vector $s(\cdot) := (s_1(\cdot), \ldots, s_n(\cdot))$ of strategies that the players follow yield an element $[s(\cdot), \theta]$ of Θ, where, recall, $[s(\cdot), \theta]_1 := s_1(\theta_1)$ and $[s(\cdot), \theta]_{i+1} := s_{i+1}([s(\cdot), \theta]_1, \ldots, [s(\cdot), \theta]_i, \theta_{i+1})$.

Given a decision problem $\mathcal{D} := (D, \Theta_1, \ldots, \Theta_n, v_1, \ldots, v_n, f)$ and two strategies $s_i(\cdot)$ and $s'_i(\cdot)$ of player i in the sequential version of $\mathcal{D}$, we write

$$
\begin{aligned}
s_i(\cdot) \geq_d s'_i(\cdot) \text{ iff for all } \theta \in \Theta \\
&v_i(f(s_i(\theta_1, \ldots, \theta_i), \theta_{-i}), \theta_i) \geq v_i(f(s'_i(\theta_1, \ldots, \theta_i), \theta_{-i}), \theta_i).
\end{aligned}
$$

We write $s_i(\cdot) >_d s'_i(\cdot)$ if additionally one of these inequalities is strict, and we write $s_i(\cdot) =_d s'_i(\cdot)$ if all these inequalities are equalities.

Note that $s_i(\cdot) \geq_d s'_i(\cdot)$ for all strategies $s'_i(\cdot)$ of player i iff strategy $s_i(\cdot)$ of player i is optimal in the sequential version of $\mathcal{D}$.

Next, we define for all $i \in \{1, \ldots, n\}$ a preference relation $\succeq_i$ on the vectors of players' strategies by writing

$$
\begin{aligned}
s(\cdot) \succeq_i s'(\cdot) \text{ iff } & s_i(\cdot) >_d s'_i(\cdot) \text{ or} \\
& (s_i(\cdot) =_d s'_i(\cdot) \text{ and} \\
& \text{ for all } \theta \in \Theta,\ v_i(f([s(\cdot), \theta]), \theta_i) \geq v_i(f([s'(\cdot), \theta]), \theta_i)).
\end{aligned}
$$

We now say that a joint strategy $s(\cdot)$ is a ***Nash equilibrium*** in the sequential version of $\mathcal{D}$ if for all $i \in \{1, \ldots, n\}$ and all strategies $s'_i(\cdot)$ of player i we have

$$(s_i(\cdot), s_{-i}(\cdot)) \succeq_i (s'_i(\cdot), s_{-i}(\cdot)).$$

The following result clarifies the status of the strategies introduced in Theorems 2 and 4.

Theorem 5. *Let $\mathcal{D}$ be a public project problem.*

(i) Each of the vectors $s(\cdot)$ of strategies defined in Theorems 2 and 4, respectively, is a Nash equilibrium in the corresponding sequential version of the pivotal mechanism.
(ii) The vector $s(\cdot)$ of Theorem 2 is Pareto optimal in the universe of optimal strategies, in the sense that for all $\theta \in \Theta$ the resulting social welfare $SW(\theta, s(\cdot))$ is maximal among all vectors of optimal players' strategies.

This result shows that the improvement in terms of the maximization of the social welfare over the Groves mechanism is achieved by weakening the implementation in dominant strategies (see Groves Theorem) to an implementation in Nash equilibrium (in the universe of optimal strategies).

The above definition of the $\succeq_i$ relation uses the $>_d$ relation to ensure that in the definition of Nash equilibrium the deviations to non-optimal strategies are trivially discarded. This ruling out of non-optimal strategies is necessary. Indeed, when $\theta_i > \frac{c}{n}$, with $i < n$, and $\sum_{j=1}^{n} \theta_j < c$, then player's i final utility increases from 0 to $\theta_i - \frac{c}{n}$ when he deviates from any of the two strategies considered in Theorem 5 to the strategy

$$s_i(\theta_1, \ldots, \theta_i) := \begin{cases} 0 & \text{if } \theta_i \leq \frac{c}{n} \\ c & \text{otherwise.} \end{cases}$$

Recall now that the well-known revelation principle (see, e.g., [15]) states that every mechanism can be realized as a (simultaneous) direct mechanism in which truth-telling is the optimal strategy. We now show that using any Nash equilibrium $(s_1(\cdot), \ldots, s_n(\cdot))$ of Theorem 5 we can construct a revelation-type simultaneous mechanism in which the vector $(\pi_1(\cdot), \ldots, \pi_n(\cdot))$ of the projection functions forms a Nash equilibrium. (Recall that the $\pi_i(\cdot)$ function corresponds in the sequential setting to truth-telling by player i.) This mechanism is constructed using the following preference relations $\succeq_i^*$ on the vectors of players' strategies:

$$\begin{aligned} & s'(\cdot) \succeq_i^* s''(\cdot) \text{ iff} \\ & (s_1(\cdot) \circ s'_1(\cdot), \ldots, s_n(\cdot) \circ s'_n(\cdot)) \succeq_i (s_1(\cdot) \circ s''_1(\cdot), \ldots, s_n(\cdot) \circ s''_n(\cdot)), \end{aligned}$$

where strategy $s_i(\cdot) \circ s'_i(\cdot)$ of player i is defined by

$$(s_i(\cdot) \circ s'_i(\cdot))(\theta_1, \ldots, \theta_i) := s_i(\theta_1, \ldots, \theta_{i-1}, s'_i(\theta_1, \ldots, \theta_i)).$$

Theorem 6. *Let $\mathcal{D}$ be a public project problem. The vector $(\pi_1(\cdot), \ldots, \pi_n(\cdot))$ of projection strategies is a Nash equilibrium in the corresponding sequential version of the pivotal mechanism, where we use the preference relations $\succeq_1^*, \ldots, \succeq_n^*$.*

Proof. Note that for all $j \in \{1, \ldots, n\}$, $s_j(\cdot) \circ \pi_j(\cdot) = s_j(\cdot)$. Then

$$(\pi_i(\cdot), \pi_{-i}(\cdot)) \succeq_i^* (s'_i(\cdot), \pi_{-i}(\cdot)) \text{ iff } (s_i(\cdot), s_{-i}(\cdot)) \succeq_i (s_i(\cdot) \circ s'_i(\cdot), s_{-i}(\cdot)),$$

so the result holds by Theorem 5(i). □

6 Concluding Remarks

As already mentioned, no budget balanced Groves mechanisms exist for the public project. We have investigated here to what extent the unavoidable deficit can be reduced when players move sequentially. By focusing on socially optimal strategies we have incorporated into our analysis altruistic behaviour of the players.

The results here established hold for the sequential pivotal mechanism. Some of them, but not all, can be generalized to sequential Groves mechanisms. More specifically, the strategies introduced in Theorems 2 and 4 are also optimal in arbitrary sequential Groves mechanisms. The reason is the following observation.

Note 1. Fix an initial decision problem and consider two Groves mechanisms (with tax functions) t and t'. A strategy of player i is optimal in the sequential version of t iff it is optimal in the sequential version of t'.

How to generalize the remaining claims of Theorems 2 and 4 to other sequential Groves mechanisms remains an interesting open problem.

References

1. Apt, K.R., Conitzer, V., Guo, M., Markakis, V.: Welfare undominated groves mechanisms. In: Papadimitriou, C., Zhang, S. (eds.) WINE 2008. LNCS, vol. 5385, pp. 426–437. Springer, Heidelberg (2008)
2. Apt, K.R., Markakis, V.: Optimal strategies in sequential bidding. In: Proceedings AAMAS 2009, pp. 1189–1190. IFAAMAS (2009)
3. Cavallo, R.: Optimal decision-making with minimal waste: Strategyproof redistribution of VCG payments. In: Proceedings AAMAS 2006, pp. 882–889. ACM Press, New York (2006)
4. Chen, Y.: Incentive-compatible mechanisms for pure public goods: A survey of experimental research. In: Plott, C., Smith, V. (eds.) Handbook of Experimental Economics Results. Elsevier, Amsterdam (2008) (forthcoming)
5. Clarke, E.H.: Multipart pricing of public goods. Public Choice 11, 17–33 (1971)
6. Crémer, J., Riordan, M.H.: A sequential solution to the public goods problem. Econometrica 53(1), 77–84 (1985)
7. Dekel, E., Piccione, M.: Sequential voting procedures in symmetric binary elections. Journal of Political Economy 108(1), 34–55 (2000)
8. Groves, T., Ledyard, J.: Optimal allocation of public goods: A solution to the "free rider" problem. Econometrica 45(4), 783–809 (1977)
9. Guo, M., Conitzer, V.: Worst-case optimal redistribution of VCG payments. In: EC 2007: Proceedings of the 8th ACM conference on Electronic commerce, pp. 30–39. ACM Press, New York (2007)
10. Jackson, M.: Mechanism theory. In: Derigs, U. (ed.) Encyclopedia of Life Support Systems. EOLSS Publishers, Oxford (2003)
11. Jackson, M., Moulin, H.: Implementing a public project and distributing its costs. Journal of Economic Theory (57), 125–140 (1992)
12. Lang, J., Pini, M.S., Rossi, F., Venable, K.B., Walsh, T.: Winner determination in sequential majority voting. In: Proceedings IJCAI 2007, pp. 1372–1377 (2007)
13. De Marco, G., Morgan, J.: Slightly altruistic equilibria in normal form games. In: Working paper No 185, Center for Studies in Economics and Finance, University of Salerno, Italy (2007), `http://www.csef.it/WP/wp185.pdf`
14. Mas-Collel, A., Whinston, M.D., Green, J.R.: Microeconomic Theory. Oxford University Press, Oxford (1995)
15. Myerson, R.B.: Game Theory: Analysis of Conflict. Harvard Univ. Press, Cambridge (1991)
16. Varian, H.: Sequential provision of public goods. Journal of Public Economics 53(2), 165–186 (1994)