STICHTING
MATHEMATISCH CENTRUM
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AMSTERDAM

AFDELING ZUIVERE WISKUNDE

ZW 1963-005

Topological semigroups II

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May 1963

CHAPTER II

Semigroups with zero and identity.

§ 1. Semigroups with zero.

Let S be a mob with 0, and a an element of S. If $a^n \to 0$, i.e. if for every neighbourhood U of 0 there exists an integer n_o, such that $a^n \in U$ if $n \geq n_o$, then a is termed a nilpotent element.

We denote by N the set of all nilpotent elements of S. An ideal (right, left) A of S with the property $A^n \to 0$ is called a nilpotent ideal.

A nil-ideal A is an ideal consisting entirely of nilpotent elements.

Then it is clear that every nilpotent ideal is a nil-ideal, and that the join of a family of (right, left) nilideals is again a (right, left) nilideal of S.

Example: Let S be the unit interval with the usual multiplication. Then I = $[0,1)$ is an ideal consisting entirely of nilpotent elements.

I is not a nilpotent ideal, since $I^n = I$ for all n.

Lemma 1: Every right (left) nilideal of S is contained in some nilideal of S.

Proof: Let A be a right nilideal of S. Then SA is an ideal of S. Suppose $x = sa \in SA$, and let U be any neighbourhood of 0. Then there exists a neighbourhood V of 0 such that $sVa \subset U$.

As A is a right nilideal of S, $as \in A$, and $(as)^n \in V$ for $n \geq n_o$. Hence if $m \geq n_o+1$ we have $(sa)^m = s(as)^{m-1} a \in sVa \subset U$.

Therefore S A is a nilideal of S, and hence $A \cup SA$, is a nilideal of S containing A.

Definition 1: The join R of all nil-ideals of a mob S with zero is called the radical of S.

Lemma 1 implies that R is a nil-ideal, which contains every right and every left nil-ideal of S.
Hence R is the maximal right and the maximal left nil-ideal.
If S = R i.e if S consists only of nilpotent elements, then S is called a nil - semigroup.

Let $a \in S$, then we shall denote by $\Gamma(a)$, the closure of the set $\{a^n\}_{n=1}^{\infty}$. $\Gamma(a) = \overline{\{a^n\}_{n=1}^{\infty}}$

<u>Lemma 2:</u> Let S be a mob and let A be a compact part of S such that $Ax \subset A$, with $\Gamma(x)$ compact.
Then $\bigcap_{n=1}^{\infty} A x^n = Ae$, with $e=e^2 \in \Gamma(x)$.

<u>Proof:</u> Let $p \in \bigcap_{n=1}^{\infty} Ax^n$

Then $p = a_1x = a_2x^2 = \ldots\ldots$

Hence from §1 lemma 2 it follows that there is an element. $a \in \{a_i\}^-$ such that $p = a\,e$, where $e = e^2 \in \Gamma(x)$ (see §1 th. 4).

This implies $\bigcap_{n=1}^{\infty} A\,x^n \subset A\,e$.

Now let $a_1\,e \notin A\,x^k$. Then we can find a neighbourhood V of e such that $a_1 V \cap A\,x^k = \emptyset$.

But since $e \in \Gamma(x)$, there is a $K_o \geqslant k$ such that $x^{k_o} \in V$ and hence $a_1 x^{k_o} \notin A\,x^k$.

This is a contradiction, since $A\,x \subset A$ implies $A\,x^{k_o} \subset A\,x^k$.

Hence $A\,e \subset A\,x^k \Rightarrow \bigcap_{n=1}^{\infty} A\,x^n = A\,e$.

<u>Theorem 1:</u> Let S be an element - wise compact mob with zero (i.e for every a, $\Gamma(a)$ is compact).

Then every (right, left) ideal of S is either a nil-ideal or contains non - zero idempotents.

<u>Proof:</u> Let a be a non-nilpotent element of the ideal A.
Then the identity e of the group $D = \bigcap_{n=1}^{\infty} \{\{a^i\}_{i \geqslant n}\}^-$ is not equal to zero.

Furthermore a $D \subset D$, and a $D \subset A$ $D \subset A$, since A is an ideal. Hence $D \cap A. \neq \emptyset$, so that $D \subset A$, since no group can properly contain an ideal. Thus $e \in A$.

Theorem 2: Let e be a non-zero idempotent of the compact mob S with zero.
Then these are equivalent.

1) $e S e \setminus N$ is a group
2) e is primitive
3) $S e$ is a minimal non-nil left ideal
4) $S e S$ is a minimal non-nil ideal
5) each idempotent of $S e S$ is primitive.

Proof: $(1 \rightarrow 2)$: If $e S e \setminus N$ is a group, then e is the only idempotent in $e S e \setminus \{0\}$, since no idempotent $\neq 0$, can be nilpotent.
Hence e is primitive
$(2 \rightarrow 3)$: Let L be a non-nil left ideal $L \subset S e$.
Then by theorem 1 there is an idempotent $f \in L$, $f \neq 0$.
Since $f \in Se$, we have $fe = f$, and $(ef)(ef) = ef$.
Thus ef is an idempotent $\neq 0$ and $ef \in eSe$.
Hence since e is primitive $ef = e$.
This implies that $ef = e \in eL \subset L$. $\Longrightarrow L = Se$.
$(3 \rightarrow 4)$ Let I be a non-nil ideal $I \subset SeS$.
Then there exists an idempotent $f \in I$ $f \neq 0$, and elements $a, b \in S$, such that $aeb=f$.
We can choose b such that $bf = b$.
Let $g=bae$. Then $g^2=baebae=bfae=g$.
Furthermore $g \neq 0$, since otherwise $0=gb=baeb=bf=b$.
Now $g \in Se$ and $g \in SfS$.
Hence by (3) $Se = Sg \subset SfS$, and we conclude $SeS=SfS=I$.
$(4 \rightarrow 5)$ Let f be a non-zero idempotent of SeS, and let $g=g^2 \neq 0 \in fSf$.
Since $f, g \in SeS$, we have $SgS=SfS=SeS$ and $f \in SgS$. Hence $f=agb$,

and we may assume $ag = a$, $gb = b$.
Since $gf = fg = g$, this implies $afb = agfb = agb = f$.
Hence $f = a^n g b^n$.
It follows from §1 lemma 2 that there is an idempotent $g^* \in \Gamma(a) \in Sg$ and $b' \in \Gamma(b)$ such that $f = g^* g b'$.
We note that $g^* g = g^*$, hence $g^* f = f = g^* g f = g^*$, and $f = g^* = g^* g = fg = g$.
($5 \rightarrow 1$) Since every idempotent in SeS is primitive, e is primitive and hence $Se = L$ is a minimal non-nil left ideal.
Now let $a \in eSe \setminus N$, then $a \in \{Se \cap eS\} \setminus N$.
Since L is minimal. $a = ea \in La = L$.
Hence there is $\bar{a} \in L$ such that $\bar{a}a = e$.
Let $e\bar{a} = a'$, then $a' \in eSe$ and $a'a = e$.
$(aa')(aa') = aea' = aa'$. Hence aa' is an idempotent and $aa' \in eSe \setminus N$. Since e is primitive $aa' = e$.
So we can find for every $a \in eSe \setminus N$ an element $a' \in eSe$ such that $aa' = e = a'a$.
This implies that $eSe \setminus N$ is a group, since $a' \notin N$.
For if $a' \in N$, then $\bigcap_{n=1}^{\infty} S(a')^n = S.0 = 0$ by lemma 2.
This is in contradiction with $aa' = a^2(a')^2 = a^n(a')^n = e$.

<u>Definition:</u> A mob S with zero is said to be an N-semigroup if its nilpotent elements form an open set.

<u>Lemma 3:</u> Let S be a mob with zero, and let $a \in S$.
If a^n is nilpotent for some $n \geq 0$, then a itself is a nilpotent element.

<u>Proof:</u> Let U be an arbitrary neighbourhood of 0, then since $a^j 0 = 0$, there is a neighbourhood V of 0, such that $a^j V \subset U$ $(j = 1, 2, \ldots, n)$.
Since a^n is nilpotent there exists an integer $k_o \geq 0$ such that $(a^n)^k \in V$ for $k \geq k_o$.

Thus $a^j a^{nk} = a^{nk+j} \in U$. $j = 1,2,\ldots,n$ $k \geqslant k_o$.
This implies that for $N \geqslant nk_o$ $a^N \in U$.
Hence a is nilpotent.

Theorem 3: If a mob S with O has a neighbourhood U of O, which consists entirely of nilpotent elements, then S is an N-semigroup.

Proof: Let $p \in N$, then there is an n such that $p^n \in U$.
Therefore there is a neighbourhood V of p, such that $V^n \subset U$.
Hence every point of V^n is nilpotent.
Lemma 3 then implies that $V \subset N$.

Theorem 4: A locally compact mob S with O having a neighbourhood U of O which contains no non-zero idempotents is an N-semigroup.

Proof: Since S is locally compact and Hausdorff. S is regular, and we can find a neighbourhood W of O, such that $\overline{W} \subset U$, and $\overline{W}$ is compact.
The continuity of multiplication and the compactness of $\overline{W}$ imply, that there is a neighbourhood V of O, with $V\,\overline{W} \subset W$
$V \subset W$.
Hence $V^2 \subset V$. $\overline{W} \subset W$, and $V^n \subset W$.
The set $A = \bigcup_{i=0}^{\infty} V^i$ is a mob contained in W.
Therefore $\overline{A}$ is a compact mob contained in U.
Since $\overline{A}$ contains no non-zero idempotents $\overline{A}$ is a nil-semigroup (theorem 1).
Hence V consists entirely of nilpotent elements, and by theorem 3 S is an N-semigroup.

Corollary: A locally compact semigroup with O, which is not an N-semigroup contains a set of non-zero idempotents with clusterpoint O.

<u>Theorem 5</u>: The radical of a compact N-semigroup is open.

<u>Proof</u>: Let $a \in R$, then for every $s \in S$ $sa \in R \subset N$.
Since N is open and S compact, there exists a neighbourhood V of a such that $SV \subset N$, $V \subset N$.
Since $V \cup SV$ is a left nil-ideal, $V \cup SV \subset R$.
Hence $V \subset R$ and R is open.

<u>Theorem 6</u>: Let S be a compact N-semigroup which is not a nil-semigroup.

Then any non-nilideal I of S contains a minimal non-nil ideal I^*, such that I^* / R^* is completely simple, where $R^* = I^* \cap R$ is the radical of I^*.

<u>Proof</u>: Since I is a non-nilideal of S, I contains non-zero idempotents.
Let $E^* = E - \{0\}$. Then E^* is closed, since N is open and E is closed.
Let $E_\lambda = E^* \cap Se_\lambda S$, $e_\lambda \in E^* \cap I$.
Then E_λ is closed and non-empty.
Suppose E_ν is a minimal member of $\{E_\lambda\}$. E_ν exists since S is compact.
We shall now prove that e_ν is a primitive idempotent.
Suppose $0 \neq f = f^2 \in e_\nu Se_\nu \Rightarrow f \in I$. Then $SfS \subset Se_\nu S$.
Since E_ν is minimal;
$E^* \cap SfS = E^* \cap Se_\nu S$. Hence $e_\nu = s_1 f s_2$, with $e_\nu s_1 = s_1, s_1 f = s_1$.

$$s_1^n f s_2^n = s_1^{n-1} s_1 f s_2 s_2^{n-1} = s_1^{n-1} f s_1 f s_2 s_2^{n-1} = s_1^{n-1} f e_\nu s_2^{n-1} =$$

$$= s_1^{n-1} f s_2^{n-1}$$

Hence $s_1^n e_\nu s_2^n = e_\nu$.
Thus there is an idempotent $g \in \Gamma(s_1)$ and an element $s \in \Gamma(s_2)$ so that $g e_\nu s = e_\nu$.
We note that since $\Gamma(s_1) \in Sf$, $gf = g$.
Hence $e_\nu = g e_\nu = g f e_\nu = gf = g \Rightarrow f = e_\nu f = gf = g = e_\nu$.

Thus e_ν is a non-zero primitive idempotent.
Theorem 2 then implies that $Se_\nu S = I^* \subset I$ is a minimal non nil-ideal.
Now we shall prove that $R^* = I^* \cap R$.
Since $I^* \cap R$ is a nil-ideal of I^* we have $I^* \cap R \subset R^*$.
Furthermore $SR^*S \subset SI^*S \subset I^*$.
If $SR^*S = I^*$, then $I^*SR^*SI^* = I^{*3} = I^*$, and so
$I^* = I^*SR^*SI^* \subset I^*R^*I^* \subset R^*$. This contradicts the fact that I^* is a non nil-ideal.
Hence SR^*S is an ideal of S properly contained in I^*.
This implies that SR^*S must be a nil-ideal i.e $SR^*S \subset R^* \Rightarrow$
R^* is a nil-ideal of $S \Rightarrow R^* \subset I^* \cap R$.
Since R^* is a maximal proper ideal of I^*, §3 th.3 implies that I^* / R^* is completely simple.

<u>Corollary:</u> Let S be a compact mob with zero; then S contains a non-zero primitive idempotent if and only if there is a non-zero idempotent e with $(eSe) \setminus N$ closed.

<u>Proof:</u> If $e = e^2 \neq 0$, e primitive $eSe \setminus N$ is a maximal subgroup. (th. 2). On the other hand if $(eSe) \setminus N$ is closed and $e \neq 0$, then $eSe \setminus N$ is the set of nilpotent elements of eSe, and $eSe \cap N$ is open in eSe.
We conclude from theorem 6 that eSe contains a non-zero primitive idempotent. Hence so does S.

<u>Theorem 7:</u> Let e be a non-zero primitive idempotent of the compact mob S with zero. Then $Se \setminus N$ and $(Se) \cap N$ are submobs and $Se \setminus N$ is the disjoint union of the maximal groups $e_\alpha S e_\alpha \setminus N$ where e_α runs over the non-zero idempotents of Se.

<u>Proof:</u> Suppose $a, b \in Se \setminus N$, then $a^n, b^n \in Se \setminus N$. Let $ab \in N$. Then since Se is a minimal non-nil left ideal, we know that
$Sa = Sb = Se \Rightarrow Sa^n = Sb^n = Se$.
Hence $Sab = Sb^2 = Se \Rightarrow S(ab)^n = Se$.

Thus $Se = \bigcap_n S(ab)^n = SO = O$ (lemma 2).
This is a contradiction with $e \neq 0$.
Suppose now $a, b \in Se \cap N$ and $ab \notin N$.
Then $(ab)^2 \notin N$ and hence $Sab = Se$, since Se is a minimal non-nil left ideal.
Since $a \in Se$, we have $Sa \subset Se = Sab$.
Hence $Sa \subset Sab \subset Sab^2 \subset Sab^3 \subset \ldots\ldots$
But since $ab^n \in Se$, $Sab^n = Se$.
This implies that $Se = \bigcap_n Sab^n = Sa.0 = 0$, a contradiction.

Finally let $a \in Se \setminus N$. Then $Sa = Se$.
Choose an idempotent f in $\Gamma(a)$; then $Sf = Se = Sa$, and f is a right unit for Se.
Let D be the subgroup of S contained in $\Gamma(a)$. Then D is an ideal of $\Gamma(a)$ (§ 1 th.4). Hence $\Gamma(a)\, f \subset D. \Rightarrow \Gamma(a) = D$ and $\Gamma(a)$ is a group. Thus $Se \setminus N$ is the union of groups.
For any $e_\alpha = e_\alpha^2 \neq 0$, $e_\alpha \in Se$, $Se_\alpha = Se$, so that e_α is primitive and $e_\alpha Se_\alpha \setminus N$ is a group.
Now the maximal group containing e_α is contained in $e_\alpha Se_\alpha$, moreover since any group which meets N must be zero, we conclude that $e_\alpha Se_\alpha \setminus N$ is a maximal group.

§ 2. 0 - simple semigroups.

As in Ch. 1 § 3 we call a semigroup S simple if it does not contain a proper non-zero ideal.
By a 0-simple semigroup we mean a simple semigroup containing a zero element.
A completely 0-simple semigroup is a completely simple semigroup with a zero element.
If S is completely 0-simple then S contains a non-zero idempotent and this implies that S cannot be a nil-semigroup.
On the other hand if S is not a nil-semigroup and S is 0-simple, then every right or left nilideal of S is the zero ideal $\{0\}$, since (§ 1 lemma 1) every right (left) nilideal of S is contained in some nilideal of S.
We shall call a (left, right) ideal I of a mob S with zero 0-minimal if $I \neq \{0\}$ and $\{0\}$ is the only (left, right) ideal of S properly contained in I.
Hence every minimal non-nil left ideal of a 0-simple mob is a 0-minimal left ideal.

<u>Lemma 1:</u> Let L be a 0-minimal left ideal of a 0-simple mob S and let $a \in L \setminus 0$. Then $Sa = L$.

<u>Proof:</u> Since Sa is a left ideal of S contained in L, it follows that $Sa = 0$ or $Sa = L$.
If $Sa = 0$, then $SaS = 0$, in contradiction with $SaS = S$.

If S is an element-wise compact mob with zero, then every non-nil (left, right) ideal of S, contains a non-zero idempotent.
So in this case if L is a minimal non-nil left ideal of S, then there is an idempotent $e \in L$, with $Se = L$.

<u>Lemma 2:</u> Let L be a 0-minimal left ideal of a 0-simple mob S, and let $s \in S$. Then Ls is either $\{0\}$ or a 0-minimal left ideal of S.

Proof: Assume $Ls \neq 0$. Evidently Ls is a left ideal of S.
Now let L_o be a left ideal of S contained in Ls. $L_o \subset Ls$.
Let A be the set of all $a \in L$ with $as \in L_o$.
Then $As = L_o$, and $A \subset L$.
Furthermore $SAs \subset SL_o \subset L_o$ and $SA \subset SL \subset L$.
Hence $SA \subset A$ and A is a left ideal of S.
From the minimality of L, either $A = 0$ or $A = L$, and we have correspondingly $L_o = 0$ or $L_o = Ls$.

Theorem 1: Let S be a compact 0-simple mob. Then S is the union of all minimal (i.e minimal non-nil) left ideals of S.

Proof: Since S is compact, S is completely 0-simple and hence contains a non-zero primitive idempotent e.
From §1 th.2 it then follows that Se is a minimal non-nil left ideal, and hence a 0-minimal left ideal.
Now let A be the union of all the 0-minimal left ideals of S.
Clearly A is a left ideal of S and $A \neq \{0\}$.
Now we show that A is, also a right ideal.
Let $a \in A$ and $s \in S$. Then $a \in L$ for some 0-minimal left ideal L of S.
By lemma 2 $Ls = 0$ or Ls is a 0-minimal left ideal.
Hence $Ls \subset A$ and $as \in A$.
Thus A is a non-zero ideal of S, whence $A = S$.

An analoguous result holds for 0-minimal right ideals.

Lemma 3: Let L and R be 0-minimal left and right ideals of a 0-simple mob, such that $LR \neq 0$.
Then $RL = R \cap L$ is a group with zero and the identity e of $RL \setminus \{0\}$ is a primitive idempotent of S.

Proof: Since LR is a non-zero ideal of S, we must have $LR = S$. Furthermore $RL \neq 0$, since $S = S^2 = LRLR$.
Now let $a \in RL \setminus 0$, then $a \in L \setminus 0$ and $a \in R \setminus 0$, and hence $Sa = L$ (lemma 1), and $aR = 0$ or $aR = R$.

Since $S = LR = SaR$, it follows that $aR \neq 0$.
Consequently $aRL = RL$.
In the same way we can prove that $RLa = RL$.
From this we conclude that RL is a group with zero.
Now let e be the identity of RL.
Then since $R = eS$ and $L = Se$, we have $R \cap L = eS \cap Se = eSe$ and $RL = eSSe = eSe$.
Since eSe is a group with zero, e is primitive.

<u>Theorem 2:</u> Let S be a compact 0-simple mob and let e and f be non-zero primitive idempotents of S.
Then the maximal subgroups H(e) and H(f) containing e and f respectively are topological isomorphic compact groups.

<u>Proof:</u> Since Se and Sf are 0-minimal left ideals and eS and fS 0-minimal right ideals (§1 th. 2) it follows from lemma 3 that $eSe \setminus \{0\}$ and $fSf \setminus \{0\}$ are groups.
Since $H(e) \subset eSe \setminus \{0\}$ we have $H(e) = eSe \setminus \{0\}$, $H(f) = fSf \setminus \{0\}$.
Now $eSSf \neq 0$, since $eSSfS = eS^2 = eS$.
Hence $eS \cap Sf \neq 0$.
Let $a \neq 0 \in eS \cap Sf$. Then $ea = a = af$.
Since $eS = aS$ and $Sf = Sa$ (lemma 1), there exists a_1 and $a_2 \in S$ such that $e = aa_1$ $f = a_2a$.
Now let $b = fa_1e$, then $b \neq 0$ and
$ab = afa_1e = aa_1e = ee = e$; $ba = fba = a_2aba = a_2ea = f$.
Furthermore $bS = fS$, $Sb = Se$.
We now proof that the mappings $\varphi : x \to bxa$ and $\psi : y \to ayb$ are mutually inverse one-to-one mappings of H(e) and H(f) upon each other.
For let $x \in H(e)$ then $bxa \in bS \cap Sa = fS \cap SF = H(f) \cup \{0\}$.
Simularly $y \in H(f)$ implies $ayb \in aS \cap Sb = eS \cap Se = H(e) \cup \{0\}$.
And if $x \in H(e)$ $a(bxa)b = exe = x$.
φ is an isomorphism since $(bx_1a)(bx_2a) = bx_1ex_2a = bx_1x_2a$.
Since φ is continuous and one-to-one φ is topological.

Corollary: Let S be a compact 0-simple mob.
Then S is the disjoint union of isomorphic compact groups $H(e)$ and of sets A_α with the property $A_\alpha^2 = 0$.

Corollary: Let S be a commutative compact 0-simple mob.
Then S is a group with zero.

Proof: By lemma 3 we have $S^2 = S \cap S = S$ is a group with zero since S is both a 0-minimal left and right ideal.

Theorem 3: Let J be a maximal proper ideal of the compact mob S.
Then the following are equivalent.
1^o) S-J is the disjoint union of groups.
2^o) for each element of S-J, there exists a unit element
3^o) $a \in S-J$ implies $a^2 \in S-J$
4^o) J is a completely prime ideal
5^o) S-J contains an idempotent, and the product of two idempotents of S-J lies in S-J.

Proof:

(1) clearly inplies (2).
(2) $\rightarrow$ (3). Let $a \in S-J$ and $ax = xa = a$.
Then $ae = ea = a$, with $e = e^2 \in \Gamma(x)$ and $e \in S-J$.
Hence since $S-J \cup \{0\} = \cup H(e_\alpha) \cup \cup A_\alpha$, we have $a \in H(e)$ which implies $a^2 \in H(e) \Rightarrow a^2 \in S-J$.
(3) $\rightarrow$ (4) Let $a, b \in S-J$ and suppose $ab \in J$.
Then $I = \{ x | x \in S \quad xb \in J \}$ is a left ideal with $I \supset J$.
Now let $x \in I$, $xs \notin I$, then $xsb \notin J$, and hence $xsbxsb \notin J \Rightarrow bx \notin J \Rightarrow bxbx \notin J \Rightarrow xb \notin J$ a contradiction.
Since I is an ideal containing J, we have $I = S \Rightarrow b^2 \in J$ a contradiction.
(4) $\rightarrow$ (5) This follows from the fact that $J = J_o(S-e)$.
(5) $\rightarrow$ (1) Since $e \in S-J$, we have S/J completely simple and $S/J = \cup H(e_\alpha) \cup \cup A_\beta$.

Now let $a \neq 0 \in A_\beta$, then $a \in Se$ and $a \in fS$. with $SefS = 0$, or else it would follow from lemma 3 that $a \in Se \cap fS = H(e_\alpha) \cup \{0\}$. Since $ef \notin J$, we have however $SefS \neq 0$, a contradiction. Hence $A_\beta = \emptyset$ and $S-J = \cup H(e_\alpha)$.

From theorem 3 it follows that S-J is a group if and only S-J contains a unique idempotent.

§ 3. Connected semigroups

Lemma 1: If S is connected, then each minimal (left,right) ideal of S is connected.

Proof:

Let L be a minimal left ideal of S, then for any $a \in L$, $Sa = L$ and hence L is connected.

If K is the minimal ideal of S, then $K = SaS$ for each $a \in K$. Hence $K = \bigcup_{s_\alpha \in S} Sas_\alpha$: Since each Sas_α is connected and meets the connected set aaS it follows that K is connected.

Lemma 2: If S is connected, then each ideal of S is connected, provided S has a left or right unit.

Proof:

Let I be an ideal of S. Then $I = \bigcup_{x \in I} Sx$ if e is a left unit of S. Since each Sx meets aS with $a \in I$ we have that I is connected.

Example: Let $S = \{(x,y) \mid 0 \leq x \leq 1 \quad 0 \leq y \leq 1\}$.

For (x_1,y_1) and $(x_2,y_2) \in S$ define the product $(x_1,y_1) \cdot (x_2,y_2)$ to be $(0, y_1 y_2)$.

Then S is a compact connected commutative mob.

Let $I = \{(x,y) \mid x = 0,1 \quad 0 \leq y \leq 1\}$.

And $I^* = \{(x,y) \mid 0 \leq x < \frac{1}{4} \quad \frac{3}{4} < x \leq 1 \quad 0 \leq y < 1\}$.

Then I is a disconnected closed ideal, and I^* is a disconnected open ideal.

Theorem 1: If S is connected and I an ideal of S, then one and only one component of I is an ideal of S.

Proof:

Let $I^* = SI \cup IS$. Then I^* is connected and the component of I which contains I^* is an ideal of S.

Furthermore it is readily seen that this is the only component of S which is an ideal. This ideal will be called the component ideal of I.

<u>Lemma 3:</u> Let S be a compact connected mob and U a proper open subset of S with $J_o(U) \neq \emptyset$.
Let C_o be the component ideal of $J_o(U)$, then C_o intersects $\overline{U} \setminus U$.

<u>Proof:</u>
If $\overline{C}_o \cap \overline{U} \setminus U = \emptyset$, then $\overline{C}_o \subset U$, and since $\overline{C}_o$ is an ideal, we have $\overline{C}_o \subset J_o(U)$ and $C_o = \overline{C}_o$.
Furthermore $J_o(U)$ is open and hence we can find an open set V, with $C_o \subset V \subset \overline{V} \subset J_o(U)$.
Since C_o is a component of the compact set $\overline{V}$ of the connected set S, we have $C_o \cap \overline{V} \setminus V \neq \emptyset$ a contradiction.

<u>Corollary 1:</u>
Let S be a compact connected mob and F a closed subset of $S \setminus K$, with the property that if $F \cap I \neq \emptyset$, then $F \subset I$ for any ideal I of S.
Then if C is the component of $S \setminus F$ which contains K then $F = \overline{C} \setminus C$.

<u>Proof:</u>
Since C is closed in $S \setminus F$ we have $\overline{C} \cap S \setminus F = C \Rightarrow F \supset \overline{C} \setminus C$.
Furthermore it follows from lemma 3 that if C_o is the component ideal of $J_o(S \setminus F)$, then $K \subset C_o$ and $\overline{C}_o$ intersects $\overline{S \setminus F} \setminus S \setminus F \subset F$
Hence $F \subset \overline{C}_o \subset \overline{C}$.
Since $F \cap C = \emptyset$ we have $F \subset \overline{C} \setminus C$.

If we take in corollary 1 $F = H(e)$ with $e \in E \setminus K$ and if C is the component of $S \setminus H(e)$ which contains K, then $H(e) = \overline{C} \setminus C$.

This follows immediately from corollary 1, since if $H(e)\ I \neq \emptyset$, then $H(e) \subset I$ for any ideal I of S. Furthermore it follows that $H(e)$ with $e \in E \setminus K$ can contain no innerpoints.

<u>Theorem 2:</u> Let S be a compact connected mob. If K is not the cartesian product of two non-degenerate connected sets, then

either K is a group or the multiplication in K is of type (a) or (b).

(a) $xy = x$ all $x, y \in S$

(b) $xy = y$ all $x, y \in S$.

Proof:

From Ch I. §2 lemma 4 we know that

$K = \{Se \cap E\} . eSe . \{eS \cap E\}$ $e \in E \cap K$.

Now let $K^* = (Se \cap E) \times (eSe) \times (eS \cap E)$ and $\phi : K^* \rightarrow K$

$\varphi(x, y, z) = xyz$.

Then φ is clearly a continuous mapping of K^* onto K.

Now let $x_1 y_1 z_1 = x_2 y_2 z_2$ with $x_1, x_2 \in Se \cap E$, $z_1, z_2 \in eS \cap E$ $y_1, y_2 \in eSe$.

Then since $x_1 S$ and $x_2 S$ are minimal ideals with $x_1 S \cap x_2 S \neq \emptyset$ we have $x_1 x_2 = x_2$.

Furthermore since

$x_1, x_2 \in Se$, $Se = Sx_1 = Sx_2 \Rightarrow x_1 e = x_1$, $x_2 e = x_2$, $ex_1 = ex_2 = e$.

Hence $x_2 = x_1 x_2 = x_1(ex_2) = x_1 e = x_1$.

In the same way we can prove $z_1 = z_2$.

Since $x_1 y_1 z_1 = x_2 y_2 z_2$ we have $ex_1 y_1 z_1 e = ex_2 y_2 z_2 e \Rightarrow ey_1 e =$ $= ey_2 e \Rightarrow y_1 = y_2$.

Hence φ is one to one and K is homeomorphic to K^*.

Since K is connected, each of eSe, $Se \cap E$ and $eS \cap E$ must be connected.

Hence at least two of the factors must consist of single elements.

If $eS \cap E = Se \cap E = e$, then $K = eSe$ and hence a group.

If $eS \cap E = eSe = e$, then $K = Se$, and if $x, y \in K$ we have $xy =$ $= (xe)(ye) = x(eye) = xe = x$.

If $Se \cap E$ and eSe are both e, then the multiplication is of type (b).

Corollary: Let S be a compact connected mob. If K contains a cutpoint, then the multiplication in K is of type (a) or (b).

Proof:

If K contains a cutpoint, then K is not the cartesian product of two non-degenerate connected sets.

Hence from theorem 2 it follows that K is a group or the multiplication is of type (a) or (b).

Since a compact connected group, contains no cutpoints, the corollary follows.

Definition 1: A clan is a compact connected mob with a unit element.

Lemma 4: Let B be the solid unit ball in Euclidean n-space and let f be a map of B into itself, such that $|x - f(x)| < \frac{1}{2}$ for all $x \in B$. Then $0 \in f(B)$.

Proof:

Let $x = (x_1, \ldots, x_n)$ $f(x) = (f_1(x), \ldots f_n(x))$.

We now consider the mapping $h(x) = (x_1, \ldots, x_n) - (f_1(x), \ldots f_n(x))$.

This mapping transforms the ball $|x| \leq \frac{1}{2}$ into itself and hence by Brouwers fixed point theorem there is a point x^* for which $h(x^*) = x^*$.

i.e. $(x_1^*, \ldots, x_n^*) = (x_1^*, \ldots, x_n^*) - (f_1(x^*) \ldots f_n(x^*))$ $f(x\ ) = 0$.

Theorem 3: Let S be a mob with unit element u having an Euclidean neighbourhood U of u.

Then H(u) is an open subset of S and is a Lie group.

Proof:

We identify U with E^n and let $F_\varepsilon = \{ x \mid , |u-x| \leq \varepsilon \}$.

Since the multiplication on F is uniformly continuous there is a δ such that $|x-xy| < \varepsilon/2$, $|x-yx| < \varepsilon/2$ whenever $|u-y| < \delta$.

By lemma 4 $u \in F_\varepsilon y$ and $u \in yF_\varepsilon$, hence y has an inverse y^{-1} in F_ε and the mapping $y \rightarrow y^{-1}$ is continuous.

Therefore H(u) is a topological group, and since it contains an open set it must be open in S.

Furthermore H(u) is locally Euclidean and hence a Lie group.

Corollary 1: If S is a clan having a Euclidean neighbourhood of the identity then S is a Lie group.

Proof:
By theorem 3 $H(u)$ is open. $H(u)$ is closed since S is compact, and hence $H(u)$, must be all of S. Thus if S is a clan and S is an n-sphere, then S is a topological group, and hence $n = 0, 1$ or 3.
In general a compact manifold which admits a continuous associative multiplication with identity must be a group.

Corollary 2: Let S be a clan and F a closed subset of S such that S-F is locally Euclidean.
Then either S is a group or $H(u) \subset F$.

Proof:
Let $h \in H(u)$ and $h \notin F$. Then h has a Euclidean neighbourhood V. Since $h^{-1} V$ is a Euclidean neighbourhood of u, it follows from corollary 1 that S is a group.

In case S is a subset of Euclidean space, then it follows from corollary 2 that $H(u) \subset$ boundary of S or S a top.group.
If S contains interior points, then it cannot be a group and we have $H(u) \subset Bd(S)$.

Definition 2: A subset C of a space X is a C-set provided that $C \neq X$ and if M is a continuum with $C \cap M \neq \emptyset$ then $M \subset C$ or $C \subset M$.
It can easely be shown that if C is a C- set of a compact connected Hausd. space, then the interior of C is empty and C is connected.
For let x be an interior point of C, then there is an open set V with $x \in V \subset \overline{V} \subset C$.
Now let $y \in X-C$. Then the component M of y in X-V has a non-empty intersection with the boundary of $X-V \subset \overline{V}$.
Hence M is a continuum with $M \cap C \neq \emptyset$ and $C \not\subset M$, $M \not\subset C$.

Theorem 4: (Gleason).
Let G be a compact Lie group which acts on a completely regular space X. Let $p \in X$ such that $g(p) \neq p$ unless g is the identity; $g \in G$.
Then there exists a closed neighbourhood N of p and a closed subset C of N, such that the orbit of every point of N has exactly one point in common with C.

Proof: See Gleason Pr A.M.S. 1 1950.

Lemma 5: Let G be a compact group and let U be an open neighbourhood of the identity.
Then U contains an invariant subgroup H of G such that G/H is a Lie group.

Proof: See Montgomery Zippin: Topological transformation groups.

Theorem 5: Let S be a clan, S no group G a compact invariant subgroup of H(u) = H, such that H/G is a Lie group.
Then S contains a continuum M such that M meets H and the complement of H, and such that $u \in M \cap H \subseteq G$.

Proof:
We can consider H as transformation group acting on S. Let H' = H/G and S' the space of orbits of G. Then H' is a compact Lie group acting on S'.
By theorem 4 there exists a closed neighbourhood N of $u' = u.G$ and a closed set $C \subset N$ such that $n H' \cap C$ is a single point for each $n \in N$.
Now let S'' be the space of orbits under H.
Then we have the following canonical mappings $\alpha: S \to S'$, $\beta: S' \to S''$ $\gamma: S \to S''$, with $\gamma = \alpha.\beta$.
Since α and γ are open maps, β is also open.
Let N^0 be the interior of N then βN^0 is open and $\beta(u') \in \beta(N^0)$.
Let P be the component of $\beta(N)$ which contains $\beta(u')$.
Then P meets the boundary of $\beta(N)$ and hence P is non-degenerate.

Now let $\beta^* = \beta | C$.

Then since $n H' \cap C$ is a single point for each $n \in N$ it follows that β^* is a homeomorphism between C and $N\beta$.

$\beta^{*-1}(P)$ is a continuum which meets H' only at $C \cap H'$, and hence $\beta^{*-1}(P)$ also meets the complement of H'.

Now let K be a component of $\alpha^{-1} \beta^{*-1}(P)$.

Since α is an open mapping we have $\alpha(K) = \beta^{*-1}(P)$.

Hence K is a continuum which meets H and the complement of H and $K \cap H \subset \alpha^{-1}(c)$, where $c = C \cap M$.

Let $h \in K \cap H$, then $K \cap H \subset hG$.

Suppose now $M = h^{-1} K$, then $u \in M \cap H$ and $M \cap H \subset G$ and if $k \in K$, $k \notin H$, then $h^{-1}k \in M$; $h^{-1} k \notin H$, since $S-H$ is an ideal of S. q.e.d.

<u>Theorem 6:</u> Let S be a clan which is no group.
Then the identity u of S belongs to no non-trivial C-set.

<u>Proof:</u>

Let $u \in C$, with C a C-set. We first prove that $C \subset H(u)$.

If $x \in C$, then since xS is a continuum which meets C, we have $C \subset xS$ or $xS \subset C$.

If $u \in xS$, then x has an inverse and is thus included in $H(u)$.

Now let $u \notin xS$, then $xS \subset C$; $xS \neq C$, and there is an open set V with $xS \subset V$; $C \setminus V \neq \emptyset$. Since $xK \subset K$ we have $K \cap C \neq \emptyset$.

If $u \in K$ then S is a group, hence $u \notin K \Rightarrow K \subset C$.

We can find now an open set W with $x \in W$ $WS \subset V$.

Since C contains no interior points there exists a $y \in W \setminus C$ with $yS \subset V$.

Clearly yS is a continuum which meets both C and $S \setminus C$ and $C \not\subset yS$ a contradiction.

Hence $u \in xS$ and thus $x \in H(u) \Rightarrow C \subset H(u)$.

Now let U be a neighbourhood of u such that $C \not\subset U$.

Then by lemma 5 there is a subgroup $G \subset U$ such that H/G is a Lie group and $C \not\subset G$.

Theorem 5 implies that we can find a continuum M such that $u \in M \cap H \subset G$ and such that M meets the complement of H.

Hence $M \cap C \neq \emptyset$ and since $C \subset H$ M meets the complement of C.
$\Longrightarrow C \subset M$.
Since $M \cap H \subset G$ and $C \not\subset G \Longrightarrow C \not\subset M$ a contradiction. q.e.d.

Example: Let $A = \{(x,y) \mid y = \sin \frac{1}{x} \; 0 < x \leq 1\}$

$B = \{(2-x,y) \mid (x,y) \in A\}$.

$C = \{(0,y) \cup (2,y) \mid -1 \leq y \leq 1\}$.

and let $S = A \cup B \cup C$.
We will show that S does not admits the structure of a clan.
For suppose that S is a clan.
Since S is not homogeneous, S cannot be a topological group and hence $S \neq H(u)$.
Then $S \setminus H(u) = J \neq \emptyset$ is the maximal proper ideal of S. Since J is open, dense and connected we have $A \cup B \subset J$ and hence $u \in C$.
But since C is the union of two C- sets, u cannot be in C.

Lemma 6: Let S be a clan and C a C-set of S. If g is an idempotent with $g \notin K$, then $g \notin C$.

Proof:
Suppose $g \in C$. Since gSg is a continuum we have $C \subset gSg$ or $gSg \subset C$.
g is the identity of the clan gSg and gSg is not a group since $g \notin K$ (Ch I. § 3 th.6). Hence theorem 6 implies that $C \not\subset gSg$.
Now suppose $gSg \subset C \Longrightarrow K \cap C \neq \emptyset$ and since $g \in C$ $C \setminus K \neq \emptyset$.
Let U and V be neighbourhoods of K with $SK = K \subset U \subset \overline{U} \subset V$.
while $g \notin V$.
Since S is compact there is a neighbourhood W of K such that $SW \subset U$
$\overline{SW}$ is a continuum and hence $\overline{SW} \subset C$.
Furthermore $W \subset \overline{SW}$ and this would imply that C contains inner points; a contradiction.

Theorem 7: Let S be a clan and C a C-set of S, then $C \subseteq K$.

Proof:

From the proof of lemma 6 it follows that if $K \cap C \neq \emptyset$, then $C \subseteq K$.

Suppose now $C \cap K = \emptyset$ and let $x \in C$ and U a neighbourhood of x with $C \setminus U \neq \emptyset$.

Let e be a minimal member of the partial ordered set E with $xe = x$.

e exists since $E_x = \{ e \mid e^2 = e \quad xe = x \} \neq \emptyset$ and compact.

Furthermore $e \notin K$ since $x \notin K$.

Hence $H(e) \neq eSe$ and we can find a neighbourhood V of e such that $xV \subset U$ and a continuum $M \subset eSe$ such that $e \in M \subset V$ and $M \cap \{ eSe \setminus H(e) \} \neq \emptyset$.

Since $x \in xM$ we have $xM \subset C$.

Let $m \in M \cap \{ eSe \setminus H(e) \}$, then $C \subset xSm$.

This implies that $x = xsm = xesem = xp$ with $p \in \{ eSe \setminus H(e) \}$. since $\{ eSe \setminus H(e) \}$ is an ideal of eSe.

Hence $x = xf$ with $f = f^2 \in \Gamma(p) \subset eSe$, and thus $ef = fe = f \Rightarrow f \leq e$.

But since e is minimal we have $f = e$.

Furthermore $pe = p = ep \Rightarrow pf = p = fp \Rightarrow p \in H(f) = H(e)$; a contradiction.

Theorem 8: If S is a clan and if K is a C-set, then K is a maximal subgroup of S.

Proof:

If $S = K$, then S is a group and the result follows.

If $S \neq K$, then K has no interior points since K is a C-set.

Let $\{ a_\lambda \mid \lambda \in \lambda \}$ be a directed set of points of $S \setminus K$ with $a_\lambda \to e$ where $e = e^2 \in K$.

Since $K \cap a_\lambda S \neq \emptyset$ $K \cap Sa_\lambda \neq \emptyset$ and $a_\lambda \in a_\lambda S \cap Sa_\lambda$ we have $K \subset a_\lambda S \cap Sa_\lambda \Rightarrow K \subset eS \cap Se = eSe$.

Now $e \in K$ gives $H(e) = eSe$ and thus $K = H(e)$.

Theorem 9: If a clan is an indecomposable continuum it is a group.

Proof:

If $S = K$, then S is a group.

Suppose now $K \neq S$. Then there exists an open set V with $K \subset V \subset \overline{V} \neq S$. Let $J_o(V)$ be the union of all ideals of S contained in V, then $J_o(V)$ is open and connected and $K \subset J_o(V) \subset J_o(\overline{V}) \neq S$.

Since $S = \overline{J_o(V)} \cup S - \overline{J_o(V)}$ and S is indecomposable we have $S - \overline{J_o(V)}$ not connected.

Let $S - \overline{J_o(V)} = A \cup B$ $A \cap B = \emptyset$ A,B open.

Then we have $\overline{J_o(V)} \cup A$ connected and $\overline{J_o(V)} \cup B$ connected and hence S not indecomposable; a contradiction.

§ 4. I-semigroups

Definition 1:

Let $J = [a,b]$ denote a closed interval on the real line. If J is a mob such that a acts as a zero-element and b as an identity then J will be called an I-semigroup.

We will identify J usually with $[0,1]$, so that $0x = x0 = 0$ and $1x = x1 = x$ for all $x \in I$.

Example: $J_1 = [0,1]$ under the usual multiplication
$J_2 = [\frac{1}{2},1]$ with multiplication defined by $x \circ y = \max(\frac{1}{2}, xy)$ where xy denotes the usual multiplication of real numbers.
$J_3 = [0,1]$ with multiplication defined by $x \circ y = \min(x,y)$.

J_1 and J_2 have just the two idempotents zero and identity, but in J_3 every element is an idempotent.
Furthermore every non-idempotent element in J_2 is algebraically nil-potent i.e. for every $x \in J_2$ there exists an n such that x^n is equal to zero.

Lemma 1: If J is an I-semigroup, then $xJ = Jx = [0,x]$ for all $x \in J$.

Proof:
Since xJ is connected and $0, x \in xJ$ we have $[0,x] \subset xJ$ and by the same argument $Jx \supset [0,x]$.
$J_0([0,x)) = J_0$ is open and connected and hence $x \in \overline{J}_0$ and $\overline{J}_0$ an ideal of J.
Hence $Jx \subset J\overline{J}_0 \subset \overline{J}_0 \subset [0,x]$ and $xJ \subset [0,x]$.
Thus $xJ = Jx = [0,x]$.

Corollary: If J is an I-semigroup, then $x \leq y$ and $w \leq v \Rightarrow xw \leq yv$.

<u>Proof:</u> Since $x \leq y$ there is a z such that $x = zy$.
Hence $xw = z(yw) \leq yw$.
In the same way we can prove $yw \leq yv \Rightarrow xw \leq yv$.

<u>Theorem 1:</u> If J is an I-semigroup with just the two idempotents 0 and 1 and with no nilpotent elements, then J is isomorphic to J_1.

<u>Proof:</u> We first show that if $xy = xz \neq 0$ then $y = z$.
Assume $y < z$. Then by lemma 1 there is a w such that $y = zw$.
Hence $xy = x(zw) = xyw \Rightarrow xy = (xy)w^n$ for every $n > 0$.
Thus $xy = (xy)e$, with $e = e^2 \in \Gamma(w)$.
Since $1 \notin \Gamma(w)$, we have $e = 0 \Rightarrow xy = 0$ a contradiction.
We now prove that if $x \neq 0$, then x has a unique square root.
The function $f: J \to J$ defined by $f(x) = x^2$ is continuous and leaves 0 and 1 fixed. Hence f is a map of J onto J so that square roots exist for every element.
Assume $a^2 = b^2 \neq 0$ and let $a \leq b$.
Then by lemma 1 $a^2 \leq ab \leq b^2$. Hence $ab = a^2 \Rightarrow b = a$.
This establishes that for $x \neq 0$, x has a unique square root and by induction that x has unique 2^n-th roots.
Let x_n be the 2^n-th root of $x \neq 0$ and for $r = p/2^n$ define $x^r = x_n{}^p$.
Then it is easy to prove that $x^r . x^s = x^{r+s}$, where r,s are positive dyadic rationals.
Furthermore if $r < s$, then $x^r > x^s$. For by lemma 1 $x^r \geq x^s$, and if $x^r = x^s$, then $x^{r-s} = 1$. a contradiction.
This implies that $\lim x_n = 1$.
Since $x_n < x_{n+1}$ $\lim x_n$ exist. Assume $\lim x_n = y \neq 1$.
Then since $y \to 0$ there is an n_o such that $y^{n_o} < x$.
Hence $y < x_{n_o}$ a contradiction.
Now let $D = \{ x^r \mid r \text{ a positive dyadic rational} \}$.
Then D is a commutative submob of J and $\overline{D} = J$.
Assume $\overline{D} \neq J$. Then there is an open interval $P \subset J \setminus \overline{D}$.

$P = (a,b)$ and $b \in \overline{D}$.

Now since $x_n \to 1$, $x_n b \to b$, and $x_n b \leq b$ by lemma 1.

If $x_n b = b$, then $x_n = 1$ a contradiction.

Hence $x_n b < b$ and $x_n b \in P$ for n sufficiently large.

Since $b \notin \overline{D}$ and $x_n \in \overline{D}$, we have $x_n b \in \overline{D}$ a contradiction

And thus $\overline{D} = J$.

Now let $g: D \to J_1$ be defined by $g(x^r) = \frac{1}{2}^r$.

$g(D)$ is dense in J_1 and g is one- to one continuous and order preserving.

Hence g can be extended to an iseomorphism of J onto J_1.

Theorem 2: If J is an I-semigroup with just the two idempotents 0 and 1 and with at least one nilpotent, then J is iseomorphic to J_1.

Proof:

Let $d = \sup \{ x \mid x^2 = 0 \}$. Then $d \neq 0$.

For let $y \neq 0$ be nilpotent, then $y^n = 0$, $y^{n-1} \neq 0$ for some $n > 1$.

Clearly $(y^{n-1})^2 = 0$. Hence $d \geq y^{n-1}$.

As shown in theorem 1, d has a unique 2^nth root and if r and s are positive dyadic rationals, then $d^r < d^s$ if $r > s$ and $d^s \neq 0$, and $d^r d^s = d^{r+s}$.

Now let $D = \{ d^r \mid r$ a positive dyadic rational$\}$. Then by the same type of argument used in the proof of theorem 1, $\overline{D} = J$.

We define $g: D \to J_2$ by $g(d^r) = (\sqrt{\frac{1}{2}})^r$. Then g is one to one and continuous and $g(D)$ is dense in J_2.

Since g is order preserving it can be extended to an iseomorphism of J onto J_2.

Theorem 3: Let J be an I-semigroup. Then E is closed and if $e, f \in E$, then $e.f = \min(e,f)$.

The complement of E is the union of disjoint intervals.

Let P be the closure of one of these. Then P is iseomorphic to either J_1 or J_2. Furthermore if $x \in P$, $y \notin P$ then $xy = \min(x,y)$.

Proof:

Let $e, f \in E$ $e < f$. Then by lemma 1 $e.e \leq ef \Rightarrow e \leq ef$.

Since $ef \leq e$, we have $e = ef$.

Now let $Q = [e,f]$.

Then for any $x,y \in [e,f]$ we have $e.e \leq x.y \leq f.f$.

Hence Q is a submob of J.

Furthermore if $e \leq x$, then $e \geq ex \geq e.e = e \Rightarrow ex = e$.

In other words e acts as a zero for $[e,1]$.

If $x \leq f$, then by lemma 1 $x = fy$ and thus $fx = x$.

f acts as an identity for $[0,f]$.

So we have in particular P an I-semigroup with only two idempotents and hence P is iseomorphic either to J_1 or J_2.

If $x \in P$, $y \notin P$, $x \leq y$ then there is an $e \in E$, with $x \leq e \leq y$.

Hence $xy = (xe)y = x(ey) = xe = x$.

It follows from theorem 3, that every I-semigroup is commutative.

Theorem 4: Let S be the closed interval $[a,b]$. If S is a mob such that a and b are idempotents and S contains no other idempotents, then S is abelian.

Proof:

Let $e \in E \cap k$. Then $H(e) = eSe$.

Since S has the fixed point property and H(e) is a retract of S, H(e) has the fixed point property and hence $H(e) = e$.

Consequently every element of K is idempotent.

Since K is connected, $K = a$ or $K = b$.

If $K = a$, then a is a zero for S and g an identity since $gS = Sg = S$.

Hence S is an I-semigroup and abelian.

Theorem 5: Let S be the closed interval $[a,b]$. If S is a clan such that both a and b are idempotents, then S is abelian if and only if S has a zero.

Proof:

Let S be commutative, then K is a group and by the same argument used in the proof of th. 4, the maximal subgroups in K are single elements, hence K consists of only one element, a zero.

Now let S have a zero. If either a or b is the zero element, then the other is obviously a unit and the result follows by theorem 3.

Now let $a < 0 < b$. Then $S' = [a,0]$ is a submob of S.

For suppose there exists $x,y \in S'$ with $xy \in (0,b]$.

Then since a acts as a unit on S', we have $x, xy \in x[a,y]$.

Hence there is an $s^* \in [a,y]$ with $xs^* = 0$.

Since $0, s^* \in s^*S'$, we have $y = s^*q$.

Hence $xy = xs^*q = 0q = 0$ a contradiction.

In the same way we can prove that $S'' = [0,b]$ is a submob of S and both S' and S'' are commutative since they are I-semigroups. It also follows that the unit of S is either a or b.

Suppose b is the unit element. Then in the same way as above we can prove that $aS'' = S''a = [0,a]$.

Hence if $x'' \in S''$ then $ax'' = y''a = (y''a)a = a(x''a) = a(az'') =$
$= az'' = x''a$.

Furthermore if $x' \in S'$ and $x'' \in S''$, then $x'x'' = (x'a)x'' =$
$x'(ax'') = (ax'')x' = (x''a)x' = x''x'$.

Theorem 6: Let S be the closed interval $[a,b]$. If S is a mob such that a and b are idempotents, then S is abelian if and only if S has a zero and $ab = ba$.

Proof:

If S is commutative, S has a zero by the same argument as in theorem 5, and obviously $ab = ba$.

Now let S have a zero and let $ab = ba$.

Then again the result follows if either a or b is a zero.

If $a < 0 < b$, then $S' = [a,0]$ and $S'' = [0,b]$ are abelian submobs of S.

Suppose now $ab \in S'$, then $bS' = baS' = abS' = [ab,0]$ by lemma 1.
Hence $bS = Sb = [ab,b]$, and $[ab,b]$ is an abelian submob by theorem 5.
To prove the theorem it suffices to show that if $x \in [0,ab]$ and $y \in [ab,b]$ then $xy = yx$.
Now $xy = (xa)(by) = (xab)y$, and $xab \in [ab,0]$.
Hence $(xab)y = y(xab) = y(xb) = (yb)\,xb = y(bxb) = ybbx = yx$.