On-Line Balancing of Random Inputs

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Abstract

We consider an online vector balancing game where vectors v_t, chosen uniformly at random in $\{-1, +1\}^n$, arrive over time and a sign $x_t \in \{-1, +1\}$ must be picked immediately upon the arrival of v_t. The goal is to minimize the L^∞ norm of the signed sum $\sum_t x_t v_t$. We give an online strategy for picking the signs x_t that has value $O(n^{1/2})$ with high probability. Up to constants, this is the best possible even when the vectors are given in advance.

1 Introduction

A random set of n vectors $v_1, \ldots, v_n \in \mathbb{R}^n$ is sent to our hero, Carole. The vectors are each uniform among the 2^n vectors with coordinates $-1, +1$, and they are mutually independent. Carole's mission is to balance the vectors into two nearly equal groups. To that end she assigns to each vector v_t a sign $x_t \in \{-1, +1\}$. Critically, the signs have to be determined *on-line* – Carole has seen only vectors $v_1, \ldots, v_t$ when she determines sign x_t. Set

$$P = x_1 v_1 + \ldots + x_n v_n \tag{1.1}$$

Carole's goal is to keep *all* of the coordinates of P small in absolute value. We set $V = |P|_\infty$, the L^∞ norm of P. We consider V, the value of this (solitaire) game, which Carole tries to minimize.

As our main result, we give a simple algorithm for Carole (with somewhat less simple analysis!) such that $V \le K\sqrt{n}$ with high probability. Here K is an absolute constant which we do not attempt to optimize.

To give a feeling, imagine Carole simply selected $x_j \in \{-1, 1\}$ uniformly and independently, not looking at v_j. Then each coordinate of P would have distribution S_n, roughly $\sqrt{n}N$, N standard normal. For, say, $K = 10$, the great preponderance of the coordinates would lie in $[-K\sqrt{n}, +K\sqrt{n}]$. However, there would be a small but positive proportion of *outliers*, coordinates not lying in that interval. Indeed, the largest coordinate, with high probability, would be $\Theta(\sqrt{n \ln n})$. Carole's task, from this vantagepoint, is to avoid outliers.

We more generally define the value of the T-round game $V = V(n, T)$, where there are T vectors $v_1, \ldots, v_T$ in $\mathbb{R}^n$ with T arbitrary. In particular, think of T as very large. We modify of our algorithm for $T = n$ to give an algorithm for which $V = V(n, T) \le K\sqrt{n}$ with high probability. This is considered in section 3.3.

1.1 Four Discrepancies

Paul, our villian, sends $v_1, \ldots, v_n \in \{-1, +1\}^n$ to Carole. Carole balances with signs $x_1, \ldots, x_n \in \{-1, +1\}$. The value of this now two-player game is $V = |P|_\infty$ with $P = \sum x_i v_i$ as above. There are four variants. Paul can be an adversary (trying to make V large) or can play randomly (as above). Carole can play on-line (as above) or off-line – waiting to see all $v_1, \ldots, v_n$ before deciding on the signs $x_1, \ldots, x_n$. All of the variants are interesting.

Paul adversarial, Carole offline. Here $V = \Theta(\sqrt{n})$. This was first shown by the senior author [5] and the first algorithmic strategy (for Carole) was given by the junior author [2].

Paul random, Carole offline. Here $V = \Theta(\sqrt{n})$. In recent work [1], a value c such that $V \sim c\sqrt{n}$ (with high probability) was conjectured with strong partial results.

Paul adversarial, Carole online. Here $V = \Theta(\sqrt{n \ln n})$. These results may be found in the senior author's monograph [6]. Up to constants, Carole can do no better than playing randomly. It was this result that made our current result a bit surprising.

Paul random, Carole online. $V = \Theta(\sqrt{n})$, the object of our current work.

The T round setting is also very interesting. If Paul picks vectors $v_t \in \{-1, +1\}^n$ adversarially, and Carole plays online, then no better bound is possible than exponential in n [3]. Basically, all Carole can do is alternate signs when one of the 2^n possible vectors v is repeated.

1.2 Alternate Formulations

We return to our focus, the random online case. We find it useful to consider the problem in a variety of guises.

Consider an n-round (solitaire) game with a position vector $P \in \mathbb{R}^n$. Initially $P \leftarrow 0$. On each round a random $v \in \{-1, +1\}^n$ is given. Carole must then reset either $P \leftarrow P + v$ or $P \leftarrow P - v$. The value of the game is $|P|_\infty$ with the position vector P after the n rounds have been completed.

Chip game. Consider n chips on $\mathbb{Z}$, all initially at 0. Each round each chip selects a random direction. Carole then either moves all of the chips in their selected direction or moves all of the chips in the opposite of their selected direction. After n rounds the value V is the longest distance from the origin to a chip. (Here chip j at position s represents that the j-th coordinate of P is s.)

Folded chip game. Consider n chips on the nonnegative integers, initially all at 0. The rules are as above except that a chip at position 0 automatically is moved to 1. (Here the chip position is the absolute value of its position in the previous formulation.)

1.3 Erdős

Historically, discrepancy was examined for families of sets. Let $(V, \mathcal{S})$ be a set system with $V = [n]$ and $\mathcal{S} = \{S_1, \ldots, S_n\}$ a collection of subsets of V. For a two-coloring $\chi : V \to \{-1, +1\}$, the discrepancy of a set S is defined as $\chi(S) = |\sum_{i \in S} \chi(i)|$, and measures the imbalance from an even split of S. The discrepancy of the system $(V, \mathcal{S})$ is defined as

$$\text{disc}(\mathcal{S}) = \min_{\chi : V \to \{-1, +1\}} \max_{S \in \mathcal{S}} \chi(S) \tag{1.2}$$

That is, it is the minimum imbalance of all sets in $\mathcal{S}$ over all possible two-colorings χ. Erdős famously asked for the maximal possible $\text{disc}(\mathcal{S})$ over all such set systems. It was in this formulation that the senior author first showed that $\text{disc}(\mathcal{S}) \le K\sqrt{n}$.

Consider the $n \times n$ incidence matrix A for the set system $(V, \mathcal{S})$. That is, set $a_{ij} = 1$ if $j \in S_i$, otherwise $a_{ij} = 0$. Let $v_1, \ldots, v_n$ denote the column vectors of A. The coloring χ corresponds to the choice of $x_j = \chi(j)$. Then $|\sum_j x_j v_j|_\infty$ measures the maximal imbalance of the coloring. The set-system problem is then essentially the Adversarial, Off-Line Paul/Carole game. The distinction is only that the coordinates of the v_i are $0, 1$ instead of $-1, +1$.

2 Carole's Algorithm

The time will be indexed $t = 0, 1, \ldots, n$. Initially $P = 0 \in \mathbb{R}^n$. In round t random v_t arrives and Carole resets $P \leftarrow P \pm v_t$. Let P_t denote the vector P after the t-th round. Let $d_j(t)$ denote the j-th coordinate of P_t.

The algorithm depends on variables c, p. We shall want $V \le \sqrt{cn}$ with high probability. c will be a large constant as specified later. p will be a positive integer central to the algorithm. We may take $p = 4$ and $c = 10^5$ to be specific. However, we use the variables c and p in the analysis until the end to understand the various dependencies among the parameters.

Define the gap for set j as

$$g_j(t) := cn - d_j(t)^2 \tag{2.3}$$

The algorithm will, with high probability, keep all $|d_j(t)| \le \sqrt{cn}$ so that the gaps are positive. Let

$$\Phi_j(t) = c^p n^{p-1} g_j(t)^{-p} \tag{2.4}$$

and define the potential function

$$\Phi(t) = \sum_j \Phi_j(t) = c^p n^{p-1} \sum_{j=1}^n g_j(t)^{-p} \tag{2.5}$$

As $d_j(0) = 0$ for all $j \in [n]$, $\Phi(0) = nc^p n^{p-1}(cn)^{-p} = 1$. Note that potential blows up as any discrepancy $|d_j(t)|$ approaches $(cn)^{1/2}$. The $c^p n^{p-1}$ factor provides a convenient normalization. When all $g_j(t) = (1 - \kappa)\sqrt{cn}$, $\Phi = (2\kappa - \kappa^2)^{-p}$.

The algorithm is simple. On the t-th round, seeing v_t, Carole selects the sign $x_t \in \{-1, +1\}$, that minimizes the increase in the potential $\Phi(t) - \Phi(t-1)$.

2.1 Rough Analysis

Lets imagine all the $d_j(t)$ as positive and near the boundary $\sqrt{cn}$. The gap basically acts like

$$g_j^*(t) = 2\sqrt{cn}[\sqrt{cn} - d_j(t)] \tag{2.6}$$

Let $\Phi_j^*(t), \Phi^*(t)$ be the potential values using this cleaner gap function. Suppose *all* $d_j(t) = \sqrt{cn}(1 - \kappa)$. Then $g_j^*(t) = 2\kappa cn$ and $\Phi^* = (2\kappa)^{-p}$. Set $f(x) = x^{-p}$ and consider the change (x large) when x is incremented or decremented by one. From Taylor Series we approximate

$$\frac{f(x \pm 1) - f(x)}{f(x)} \sim \mp px^{-1} + \frac{p(p+1)}{2}x^{-2} \tag{2.7}$$

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ignoring the higher order terms. Consider the change in Φ when a random vector v_{j+1} is added. We break it into a linear part L and a quadratic part Q. We compare their sizes using (2.7). The quadratic term is always positive, $(p(p+1)/2)(\kappa\sqrt{cn})^{-2}$ for each term, adding to $Q = (p(p+1)\kappa^{-2}/2)/c$. The linear term is $\mp p\kappa^{-1}c^{-1/2}n^{-1/2}$ for each term. As the vector (critically!) is random the signs $\mp$ are random and so add to distribution roughly $\sqrt{n}N$, N standard normal. Thus $L \sim p\kappa^{-1}c^{-1/2}N$. Carole's sign selection, effectively, replaces L with $-|L|$. The change in Φ is then proportional to $-|L| + Q$. With probability at least $1/4$, say, $|N| \geq 1$. Then, fixing p, κ, $|L|$ will be on the order of $c^{-1/2}$ while Q will be on the order of c^{-1}. Letting c be very large, say $c = 10^{100}$, the linear term $-|L|$ will be much bigger than the positive quadratic term Q.

Now lets keep the total potential $\Phi^* = (2\kappa)^{-p}$ fixed but suppose that some of the gaps $g_j(t)$ were smaller and the other gaps had zero affect on the total potential. Say, giving a good parametrization, that $d_j(t) = \sqrt{cn}(1 - 2^{-u}\kappa)$ for $m = n2^{-pu}$ values of j (As the potential takes $\sqrt{cn} - d_j(t)$ to power $-p$, the total potential will remain the same.) Again we break the change in Φ into L and Q. We think of p, κ, c as fixed and consider the effect of u. The quadratic terms are now $(p(p+1)/2)(2^{-u}\kappa\sqrt{cn})^{-2}$ for each term, an extra factor of 2^{2u}. But the number of terms is $n2^{-pu}$ so the new value is $Q = 2^{(2-p)u}(p(p+1)\kappa^{-2}/2)/c$. The linear terms are now $\mp p(2^{-u}\kappa)^{-1}c^{-1/2}n^{-1/2}$ for each term, an extra factor of 2^u. Now, however, we sum $m = n2^{-pu}$ random signs, giving $\sqrt{m}N = 2^{-pu/2}\sqrt{n}N$. Compared to the base $u = 0$ case the quadratic term Q has been multiplied by $2^{(2-p)u}$ while the linear term L has been multiplied by $2^{(2-p)u/2}$. We've taken $p = 4$ so these factors are 2^{-2u} and 2^{-u} respectively. As u gets bigger the domination of L over Q becomes stronger. This gives us "extra room" and works even if only a proportion of the potential function came from these d_j.

In the actual analysis the total potential Φ is in a prescribed moderate range. However, we cannot assume that all of the potential comes from some $n\theta$ coordinates with the same gaps. We split the coordinates into classes, those in the same class having roughly the same d value. We find some class that has so much of the total potential Φ that L will dominate over Q. Making all this precise is the object of Lemma 2.2 below.

2.2 Analysis

We will show the following result.

Theorem 2.1. *The strategy above achieves value* $V = O(n^{1/2})$, *with probability at least* $1 - \exp(-\Omega(n^\gamma))$, *where* $\gamma = 1 - 2/p$.

The potential starts initially at 1. Let $H = 4e^3$. We consider the situation when the potential Φ lies between $\frac{H}{2}$ and H. (The value H could be any sufficiently large constant.) We will show that if $\Phi(t-1) \leq H$, then at any step t the potential can increase by at most $n^{-1+(2/p)}$. More importantly, whenever $\Phi(t-1) \in [H/2, H]$, the sign x_t for the element at time t can be chosen so that there is a strong negative drift that more than offsets the increase. More formally, we can decompose the rise in potential into a linear part $L(t)x_t$ and some quadratic part $Q(t)$, satisfying the following properties.

Lemma 2.2. *Consider time* t. *The increase in potential is a random variable (depending on the randomness in column* t*) that can be written as* $\Phi(t) - \Phi(t-1) \leq L(t)x_t + Q(t)$, *where*

1. $Q(t) \leq Q_{\max} := O(n^{-1+(2/p)})$ *with probability 1, whenever* $\Phi(t-1) \leq H$.

2. $|L(t)| \geq 10Q_{\max}$ *with probability at least* $1/4$, *whenever* $\Phi(t-1) \leq [H/2, H]$.

Lemma 2.2 directly implies Theorem 2.1 as follows. Whenever $\Phi \in [H/2, H]$, the potential has a strong negative drift as $\mathbb{E}[\Delta \Phi] \leq -Q_{\max}$. As the interval $[H/2, H]$ has size $\Omega(1)$, and the positive increment is $Q_{\max} = O(n^{-\gamma})$, standard probabilistic tail bounds give that whenever Φ reaches $H/2$ it will have an exponentially small $\exp(-\Omega(n^\gamma))$ probability of reach H. Are there are only n rounds of play, the result follows.

We now try to compute some relevant quantities. Let $x_t \in \{-1, 1\}$ denote the color of element t, and let $y(j, t)$ be indicator random variable if element t lies in j. Then,

$$\Delta d_j(t) := d_j(t) - d_j(t-1) = x_t y(j, t) \tag{2.8}$$

and note that $|\Delta d_j(t)| \leq 1$.

Let us condition on the event that $\Phi(t-1) \leq H$. Then $\Phi_j(t-1) \leq H$ for each j and hence $g_j(t) = \Omega(n^{1-(1/p)})$, which implies that $d_j(t) \leq (cn)^{1/2} - \omega(1)$.

We now upper bound the increase in potential, $\Phi(t) - \Phi(t-1)$ as follows.

Let $f(x) = (cn - x^2)^{-p}$ with $x \in (-(cn)^{1/2}, (cn)^{1/2})$. Then $f'(x) = 2px(cn - x^2)^{-p-1}$, and

$$
\begin{aligned}
f''(x) &= 2p(cn - x^2)^{-p-1} + 4p(p+1)x^2(cn - x^2)^{-p-2} \\
&= \left(2p(cn - x^2) + 4p(p+1)x^2\right)(cn - x^2)^{-p-2} \\
&\leq 4p(p+1)cn(cn - x^2)^{-p-2} \qquad (\text{as } x^2 < cn)
\end{aligned} \tag{2.9}
$$

As $f(x + \eta) - f(x) \leq f'(x)\eta + \frac{1}{2}\max_{z \in [x, x+\eta]} f''(z)\eta^2$, for any smooth function f, and as $cn - x^2 = \omega(1)$ we have $f''(z) \leq 2f''(x)$ for $z \in [x-1, x+1]$. Using the expression for $f'(x)$ and the upper bound on $f''(x)$ in (2.9), we have that for $|\eta| \leq 1$

$$f(x + \eta) - f(x) \leq 2p\frac{x}{(cn - x^2)^{p+1}}\eta + 4p(p+1)\frac{cn}{(cn - x^2)^{p+2}} \tag{2.10}$$

Setting $f = g_j$, $x = d_j(t-1)$ and $\eta = d_j(t) - d_j(t-1) = x_t y(j, t)$ gives

$$\Phi_j(t) - \Phi_j(t-1) \leq L_j(t)x_t + Q_j(t) \tag{2.11}$$

where

$$L_j := c^p n^{p-1} 2p\frac{d_j(t-1)y(j,t)}{g_j(t-1)^{p+1}} \quad \text{and} \quad Q_j := c^p n^{p-1} 4p(p+1)\frac{cn}{(g_j(t-1))^{p+2}} \tag{2.12}$$

As we only be interested in time t, henceforth we drop t. Let $L = \sum_j L_j$ and $Q = \sum_j Q_j$. Then,

$$L = \sum_j c^p n^{p-1} 2p\frac{d_j y_j}{g_j^{p+1}} \quad \text{and} \quad Q = \sum_j c^p n^{p-1} 4p(p+1)\frac{cn}{g_j^{p+2}} \tag{2.13}$$

and $\Phi(t) - \Phi(t-1) \leq L + Q$.

Notation. Let $\beta = 1 + 1/p$. For $k = 0, 1, 2, \dots$ we say that a set j lies in class k if $d_j^2 \in [cn(1 - \beta^{-k}), cn(1 - \beta^{-k-1}))$, or equivalently $g_j \in (cn\beta^{-k-1}, cn\beta^{-k}]$. Let n_k denote the number of sets of class k. As $g_j \geq cn\beta^{-k-1}$ for a set j of class k, $1/g_j^{(p+2)} \leq \beta^{(k+1)(p+2)}/(c^{p+2}n^{p+2})$, by (2.13) Q can be upper bounded as,

$$Q \leq \frac{4p(p+1)}{cn^2} \sum_{k \geq 0} \beta^{(k+1)(p+2)} n_k. \tag{2.14}$$

We also have the following useful bounds.

Lemma 2.3. *If $\Phi \leq H$, then*

1. *For each class $k \geq 0$, $n_k \leq \min(n, n\beta^{-kp}H)$.*

2. *$Q = O(n^{-1+2/p})$.*

Proof. As $\Phi = \sum_j c^p n^{p-1} g_j^{-p}$ and $g_j \leq cn\beta^{-k}$ for a set j of class k, we have that

$$\Phi \geq c^p n^{p-1} \sum_{k \geq 0} \frac{\beta^{kp} n_k}{(cn)^p} = \sum_{k \geq 0} \beta^{kp} \frac{n_k}{n}. \tag{2.15}$$

As $\Phi \leq H$, each class k contributes at most H, which gives $n_k \leq \beta^{-kp} nH$.

We now bound Q. Let $k_{\max}$ be the maximum class index for which $n_k \geq 1$. As $1 \leq n_{k_{\max}} \leq nH\beta^{-pk_{\max}}$, we have $\beta^{k_{\max}} \leq (nH)^{1/p} = O(n^{1/p})$.

Plugging $n_k \leq n\beta^{-pk}H$ in the bound for Q in (2.14) gives

$$Q \leq \sum_{k=0}^{k_{\max}} \frac{4p(p+1)H}{cn} \beta^{p+2k+2} = O\left(\frac{\beta^{2k_{\max}}}{n}\right) = O(n^{-1+2/p}), \tag{2.16}$$

where we use that $c, p, H, \beta^{p+2} = O(1)$ and $\sum_{i=0}^{k_{\max}} \beta^{2k} = O(p)\beta^{2k_{\max}}$. $\qquad\square$

We now focus on lower bounding $|L|$, when $\Phi \geq H/2$. Recall that $L = 2pc^p n^{p-1} \sum_j d_j g_j^{-p-1} y_j$, and hence is a weighted sum of $\pm$ random variables y_j. We will call $a_j := 2pc^p n^{p-1} d_j g_j^{-p-1}$ the weight of y_j.

We will use the following fact from [4]

Lemma 2.4. *Let $a_1, \ldots, a_m$ all have absolute value at least 1. Consider the 2^m signed sums $\sum_{i=1}^m y_i a_i$ for $y_i \in \{-1, +1\}$. The number of sums that lie in any interval of length $2S$ is maximized when all the $a_i = 1$ and the interval is $[-S, +S]$. In particular, taking $S = d\sqrt{m}$ for a small constant d, the sums lie in $[-S, +S]$ only a small fraction of the time.*

We use this as follows to show that the probability that $L \in [-S, S]$ is small. Consider the indices j where the weights a_j lies in (suitably chosen) weight class, and fix the signs outside that class. Then for any values of signs outside that class, the signs in the class that will put the total sum in $[-S, +S]$ is bounded by the probability in the lemma above.

We now do the computations.

Claim 2.5. *For a set j of class $k \geq 1$, the weight $|a_j|$ is at least $2p^{1/2}\beta^{k(p+1)}/(cn^3)^{1/2}$.*

Proof. This follows as $a_j = 2p^{1/2}c^p n^{p-1} d_j g_j^{-p-1}$, and for any class $k \geq 1$, $d_j \geq (cn(1-\beta^{-1}))^{1/2} \geq (cn/p)^{1/2}$ and $g_j^{-p-1} \geq (cn)^{-p-1}\beta^{k(p+1)}$. $\qquad\square$

By Lemma 2.4 and Claim 2.5, to show that $L \gg Q$ with a constant probability, it would suffice to show that there is some class $k^* \geq 1$ such that

$$\frac{2p^{1/2}\beta^{k^*(p+1)}}{(cn^3)^{1/2}} n_{k^*}^{1/2} \gg Q \tag{2.17}$$

Note that only classes $k \geq 1$ are considered in Claim 2.5, while Q also has terms from class 0, so we need the a final technical lemma to show that this contribution from class 0 can be ignored.

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Lemma 2.6. *If $\Phi > H/2$, the contribution of class 0 sets to Q is at most $Q/2$.*

Proof. As $g_j \geq cn/\beta$ for a class 0 set, and there at most n such sets, the contribution of class 0 to Q is at most $4p(p+1)\beta^{p+2}/(cn)$. So to prove the claim, it suffices to show that $Q > 8p(p+1)\beta^{p+2}/(cn)$.

As $g_j \geq cn\beta^{-k-1}$ for a set of class k, we have

$$\frac{H}{2} \leq \Phi \leq c^p n^{p-1} \sum_{k \geq 0} \frac{n_k}{(\beta^{-k-1}cn)^p} = \frac{1}{n} \sum_{k \geq 0} \beta^{(k+1)p} n_k, \tag{2.18}$$

which gives $\sum_{k \geq 0} \beta^{kp} n_k \geq \beta^{-p} Hn/2$. Using this together with $g_j \leq cn\beta^{-k}$ for class k sets and $\beta^{k(p+2)} \geq \beta^{kp}$ in the expression for Q in (2.13), we get

$$Q \geq \sum_{k \geq 0} \frac{4p(p+1)}{cn^2} n_k \beta^{k(p+2)} \geq \frac{2p(p+1)\beta^{-p}H}{cn} \geq \frac{8p(p+1)\beta^{p+2}}{cn}, \tag{2.19}$$

where the last equality uses our choice of $H = 4e^3 \geq 4\beta^{2p+2}$. $\qquad\square$

By (2.14) and the lemma above, to prove (2.17) it suffices to show that

Lemma 2.7. *There is a $k^* \geq 1$ such that*

$$\beta^{(p+1)k^*} n_{k^*}^{1/2} \gg O(p^{3/2}) \sum_{k \geq 1} \beta^{(k+1)(p+2)} \frac{n_k}{(cn)^{1/2}} \tag{2.20}$$

Proof. Let $\ell_k = n_k \beta^{kp}/(nH)$, and note that by Lemma (2.3), $\ell_k \leq 1$ for all k. Writing n_k is terms of ℓ_k, we need to show that there is some k^* satisfying

$$(\ell_{k^*} \beta^{k^*(p+2)})^{1/2} \gg O((Hp^3/c)^{1/2}) \sum_{k \geq 1} \ell_k \beta^{2k+p+2} \tag{2.21}$$

Let $k^* = \text{argmax}_{k \geq 1} \ell_k \beta^{3k}$, and let $v = \ell_{k^*} \beta^{3k^*}$. Then $\ell_k \beta^{3k} \leq v$ for all $k \geq 1$, and hence $\ell_k \beta^{2k} \leq v\beta^{-k}$. So the term $\sum_{k \geq 1} \ell_k \beta^{2k+p+2}$ on the right hand side of (2.21) is at most

$$\sum_{k \geq 1} \beta^{p+2} v \beta^{-k} \leq \frac{\beta^{p+2}v}{\beta - 1} = O(pv).$$

Next, as $p \geq 4$, the left hand side of (2.21) is at least $(\ell_{k^*} \beta^{6k^*})^{1/2} = (v^2/\ell_{k^*})^{1/2} \geq v$, where the inequality follows as $\ell_k \leq 1$ for all k. So by (2.21), choosing $c \gg Hp^5$ finishes the proof. $\qquad\square$

3 Arbitrary time horizon

We now consider the T round setting, where T can be arbitrarily large compared to n. In particular, a uniformly chosen vector $v_t \in \{-1, +1\}^n$ arrives at time t, and Carole then selects a sign $x_t \in \{-1, +1\}$. As previously, $P_t = \sum_{j=1}^t x_j v_j$, and the value $V = V(n, T)$ after T rounds is $|P_T|_\infty$.

We will assume that T is fixed in advance by Paul (and is not known to Carole). In particular, if T can be chosen adaptively by Paul depending on the Carole's play, then the problem is

not very interesting and the exponential in n lower bound [3] for adversarial input vectors still holds. This is because even if the input vectors are random, after sufficiently long time, some worst case adversarial sequence (against any online strategy) will eventually arrive.

Our main result is a strategy for Carole, described in section 3.3, that achieves $V(n, T) = \Theta(\sqrt{n})$ with high probability. Before proving this result, we describe two strategies that achieve a weaker (but still independent of T) bound of $O(\sqrt{n} \log n)$. These are very natural and interesting on their own with simple analysis and are discussed in sections 3.1 and 3.2.

3.1 Strategy 1

The first strategy is based on a potential function approach as before, but with an exponential penalty function. This has the drawback of losing an extra $\log n$ factor, but has the advantage that the potential has a negative drift whenever it exceeds a certain threshold (without requiring an upper bound on Φ as in Lemma 2.2). This allows us to bound the discrepancy for an arbitrary time horizon, as whenever the potential exceeds the thresholds the negative drift will bring it back quickly.

Strategy. Consider a time step t. Let $d_i(t) = \sum_{j=1}^{t} x_j v_j(i)$ be the discrepancy of the i-th coordinate at the end of time t. Consider the potential

$$\Phi(t) = \sum_{i=1}^{n} \cosh(\lambda d_i(t)),$$

where $\lambda = 1/(cn^{1/2})$ and c is a large constant greater than 1.

As before, when presented with the vector v_t, Carole chooses the sign $x_t \in \{-1, +1\}$ that minimizes the increase in potential, $\Phi(t) - \Phi(t-1)$.

Analysis. Let $v_t(i)$ denote the i-th coordinate of v_t. As we will only consider the time t, let us denote $\Delta \Phi = \Phi(t) - \Phi(t-1)$, $\Phi = \Phi(t-1)$, $d_j = d_j(t-1)$ and $v(i) = v_t(i)$.

By the Taylor expansion and as $\cosh'(x) = \sinh(x)$ and $\sinh'(x) = \cosh(x)$, the increase in potential $\Delta \Phi$ can be written as

$$
\begin{aligned}
\Delta \Phi &= \sum_i \left(\lambda \sinh(\lambda d_i) x_t v(i) + \frac{\lambda^2}{2!} \cosh(\lambda d_i)(x_t v(i))^2 + \frac{\lambda^3}{3!} \sinh(\lambda d_i)(x_t v(i))^3 + \dots \right) \\
&\leq \sum_i \lambda \sinh(\lambda d_i) x_t v(i) + \sum_i \lambda^2 \cosh(\lambda d_i)(x_t v(i))^2
\end{aligned}
\tag{3.22}
$$

where the second step follows as $|\sinh(x)| \leq \cosh(x)$ for all $x \in \mathbb{R}$ and $|x_t v_t(i)| = 1$, $\lambda = o(1)$, and so the higher order terms are negligible compared to the second order term.

Let $L := \lambda \sum_i \sinh(\lambda d_i) v(i) x_t$ be the linear term, and $Q := \lambda^2 \sum_i \cosh(\lambda d_i)$ be the second term in (3.22) (note that $(x_t v(i))^2 = 1$). Conveniently, Q is exactly $\lambda^2 \Phi$.

As the algorithm chooses x_t to have $\Delta \Phi \leq -|L| + Q$, it suffices to show the following key lemma.

Lemma 3.1. *If $\Phi \geq 2n$, then $|L| \geq (c/2)Q$ with probability at least $1/4$.*

Before proving the lemma we need the following anti-concentration estimate.

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Lemma 3.2. *If $Y = \sum_i a_i Y_i$, with Y_i independent and uniform in $\{-1, +1\}$, and $a_i \in \mathbb{R}$. Then*

$$\Pr\left[|Y| \geq \frac{1}{\sqrt{2}}(\sum_i a_i^2)^{1/2}\right] \geq \frac{1}{4}.$$

Proof. The Paley-Zygmund theorem states that for any random variable $X \geq 0$ with finite variance and any $0 \leq s \leq 1$,

$$\Pr[X > s\mathbb{E}[X]] \geq (1 - s^2)\mathbb{E}[X]^2/\mathbb{E}[X^2]. \tag{3.23}$$

We apply this to $X = Y^2$, and note $\mathbb{E}[Y^2] = \sum_i a_i^2 \mathbb{E}[Y_i^2] = \sum_i a_i^2$ and

$$\mathbb{E}[Y^4] = \sum_i a_i^4 \mathbb{E}[Y_i^4] + 3\sum_j \sum_{i \neq j} a_i^2 a_j^2 \mathbb{E}[Y_i^2]\mathbb{E}[Y_j^2] \leq 3(\sum_i a_i^2)^2 = 3(\mathbb{E}[Y^2])^2. \tag{3.24}$$

Setting $s = 1/2$ in (3.23), and using the upper bound on $\mathbb{E}[Y^4]$ in (3.24) gives

$$\Pr\left[|Y| \geq \frac{1}{\sqrt{2}}(\mathbb{E}[Y^2])^{1/2}\right] = \Pr\left[Y^2 \geq \frac{1}{2}\mathbb{E}[Y^2]\right] \geq \frac{3}{4}\frac{(\mathbb{E}[Y^2])^2}{\mathbb{E}[Y^4]} \geq \frac{1}{4},$$

which implies the claimed result. $\qquad\square$

Proof (Lemma 3.1). By Lemma 3.2, with probability at least $1/4$,

$$|L| \geq \frac{\lambda}{\sqrt{2}}(\sum_i \sinh^2(\lambda d_i))^{1/2} = \frac{\lambda}{\sqrt{2}}\left(\sum_i (\cosh^2(\lambda d_i) - 1)\right)^{1/2} = \frac{\lambda}{\sqrt{2}}\left((\sum_i \cosh^2 \lambda d_i) - n\right)^{1/2}. \tag{3.25}$$

As $\cosh(x) \geq 1$ for all $x \in \mathbb{R}$, $\sum_i \cosh^2(\lambda d_i) \geq \sum_i \cosh^2(\lambda d_i) = \Phi$. So for $\Phi \geq 2n$, we get

$$\left(\sum_i \cosh^2(\lambda d_i)\right) - n \geq \frac{1}{2}\sum_i \cosh^2(\lambda d_i) \underbrace{\geq}_{\text{Cauchy-Schwarz}} \frac{1}{2}\frac{(\sum_i \cosh(\lambda d_i))^2}{n} = \frac{1}{2}\frac{\Phi^2}{n} \tag{3.26}$$

Together (3.26) and (3.25) give that

$$\Pr[|L| \geq \frac{\lambda}{2\sqrt{n}}\Phi] \geq \frac{1}{4}. \tag{3.27}$$

Using $Q = \lambda^2 \Phi$ and plugging $\lambda = 1/(c\sqrt{n})$, gives that $\Pr[|L| > (c/2)Q] \geq 1/4$. $\qquad\square$

As $\Delta \Phi = -|L| + Q$, we have (i) $\Delta \Phi \leq Q = \lambda^2 \Phi$ and setting $c = 12$, by Lemma 3.1 we have that (ii) if $\Phi \geq 2n$, then $\Delta \Phi \leq -5Q$ with probability at least $1/4$. Setting $\Psi = \log \Phi$, then this gives that $\Delta \Psi \leq \log(1 + \lambda^2) = \frac{1+o(1)}{c^2 n}$ and whenever $\Psi \geq \log(2n)$, then $\Delta \Psi \leq \log(1 - 5\lambda^2) = -5\frac{(1-o(1))}{c^2 n}$ with probability at least $1/4$.

So Ψ performs a random walk over time with positive increments bounded by $\delta \approx 1/c^2 n$, and if $\Psi \geq \log(2n)$, it has an expected negative drift of $-\Omega(\delta)$. By standard arguments, this implies that any time t, $\Pr[\psi \geq \log(2n) + z] \leq \exp(-\Omega(nz))$. As $\Psi = \log \Phi \approx \lambda|d_i|$ and $\lambda = 1/(cn^{1/2})$, this gives that $V(n, T) = O(n^{1/2} \log n)$ with high probability.

3.2 Strategy 2

Our second strategy is even simpler, and we call it the *majority rule*. For convenience, it is useful to think of the folded chip view of the game, as described in section 1.2. In particular, there are n chips, originally all at 0, the position of the i-th chip being the absolute value of $P_t(i)$. From 0, a chip must go to 1. Each chip not at 0 picks a random direction, and Carole then either moves all of the chips in their selected direction or all in their opposite directions. So from a position $y \neq 0$, a chip can go to $y \pm 1$.

Majority rule strategy. Consider the directions $v_t(i)$ of the chips not at position zero. If there is a direction with strict majority, Carole chooses the sign x_t that makes the majority of the chips not at zero move towards zero. Otherwise, in case of a tie, Carole picks x_t randomly.

Analysis. We will show the following.

Theorem 3.3. *The majority rule strategy achieves $\mathbb{E}[V(n,T)] = O(\sqrt{n}\ln n)$. More precisely, the probability that any chip i has position $\geq k\sqrt{n}$ at time T is $ne^{-\Omega(k)}$.*

Proof. Consider some time t, and a chip i that is at a non-zero position at the end of $t-1$. We claim that chip i basically does a random walk with drift towards zero.

Look at the other non-zero coordinates (other than i), and suppose there are ℓ of them. We consider two cases depending on whether ℓ is even or odd.

1. ℓ is even. Consider the random directions of the ℓ chips other than i, as given by v_t. If these directions are evenly split which occurs with probability $\varepsilon \sim K\ell^{-1/2} \geq Kn^{-1/2}$, then the majority direction is determined by $v_t(i)$ and so chip i goes towards the origin.

   Else if the ℓ directions are not split evenly, then at least $\ell/2+1$ chips of these ℓ chips have one direction (and at most $\ell/2-1$ the other). So $v_t(i)$ has no effect on the outcome of the majority rule, and as $v_t(i)$ is random and independent of the other ℓ directions, chip i moves randomly.

2. ℓ is odd. If strictly more than $(\ell+1)/2$ of the ℓ chips have one direction, then the sign of i does not affect the majority outcome. So as above, the chip i moves randomly.

   Else, exactly $(\ell+1)/2$ chips have one direction (say $+$) and $(\ell-1)/2$ have $(-)$. As the directions are random this happens with probability $\varepsilon \geq Kn^{-1/2}$. Conditioned on this event, with probability $1/2$, the direction of i chip is also $+$, in which case there is a strict majority for $+$, and chip i goes towards the origin. Else i picks the direction $-$ with probability $1/2$, resulting in an overall tie, in which case Carole (and hence chip i) moves randomly.

So in either case, each chip does a random walk on non-negative integers with a reflection at 0 and with drift at least $\epsilon/2$ towards the origin. That is, from 0 it goes to 1, and from $y \neq 0$ it goes to $y-1$ with probability at least $\frac{1}{2}(1+\frac{\varepsilon}{2})$, and else to $y+1$. So the stationary distribution at positions $y > 0$ for this chip, is dominated by the stationary distribution for an (imaginary) chip that goes to $y-1$ with probability $\frac{1}{2}(1+\frac{\varepsilon}{2})$ and to $y+1$ otherwise. This stationary distribution u_y satisfies

$$u_y = \frac{1-(\varepsilon/2)}{2}u_{y+1} + \frac{1+(\varepsilon/2)}{2}u_{y-1}. \tag{3.28}$$

This has the solution

$$u_y = K_\epsilon \left(\frac{1 - (\varepsilon/2)}{1 + (\varepsilon/2)} \right)^y \tag{3.29}$$

and in particular,

$$\Pr[y \geq \alpha\varepsilon^{-1}] = \Theta(e^{-\alpha}) \tag{3.30}$$

Taking $\alpha = (1 + \delta) \ln n$, the probability of any particular chip being at $\alpha\varepsilon^{-1}$ or higher is $o(n^{-1})$ so with probability $1 - o(1)$ all the chips are $\leq \alpha\varepsilon^{-1}$. So the value $V = V(n, T) = O(\alpha\varepsilon^{-1}) = O(\sqrt{n} \ln n)$ with high probability. $\qquad\square$

3.3 A strategy with $O(n^{1/2})$ bound

We now describe a strategy that achieves $V(n, T) = O(\sqrt{n})$ with high probability. It will be based on combining the ideas from the strategy for $V(n, n)$ from section 2 (call this rule 1) and the majority rule from section 3.2 (call this rule 2).

The strategy.   It is convenient to view the process as the chip game defined in section 1.2. Now, chips will also be colored either *green* or *red*. Initially, all the chips begin at 0 and are colored *green*. Starting at $t = 1$, we do the following.

1. At (odd) time steps t, choose the sign x_t by applying rule 1 on the green chips.

2. At (even) time steps t, choose x_t by applying rule 2 on all the chips (one could do even better by applying rule 2 on the red chips, but it is not necessary).

The color of the chips evolves as follows. When the potential Φ (given by (2.5)) for rule 1 exceeds H, *all* the chips become red. When a red chip reaches 0, it becomes green.

Analysis.   We will show the following.

Theorem 3.4. *The strategy above achieves* $V(n, T) = O(n^{1/2})$ *with high probability.*

Proof. The result will follow from some simple observations combined with properties of rule 1 and rule 2 that we proved earlier.

First, when x_t is determined by applying rule 1 on the green chips, the red chips move randomly. This follows as for any red chip i, $v_t(i)$ is independent of x_t.

Second, if we do a good strategy on a chip at alternate time steps, and on other time steps we pick x_t randomly, then this is still a good strategy. In particular, for the good strategy rule 2, this halves the negative drift which makes no qualitative difference. For rule 1, this halves the negative drift due to the L term (while Q does not change). But the ratio between $\mathbb{E}[|L|]$ and Q can be made to be an arbitrarily large constant by modifying the algorithm parameters, without any qualitative difference in the results.

Third, when we calculate Φ to apply rule 1 on the green chips, we will assume (for the purposes of calculation of Φ only) that the red chips are at position 0, and they do not move (that is $v_t(i) = 0$ for them) until they become green. Lemma 2.2 and hence Theorem 2.1 remain true in this setting, as Q can only decrease if some $v_t(i) = 0$, and in the bound for $|L|$ is not affected as we did not the contribution of class 0 in Lemma 2.7.

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We now give the argument. Let us divide the time into phases, where a new phase begins when the potential Φ of rule 1 on green chips reaches H and all the chips become red. Note that a chip can only turn red when a phase begins and it is at position $O(n^{1/2})$ when this happens. As red chips have an expected drift $cn^{-1/2}$ toward zero under rule 2 (and move randomly otherwise), in time (say) n^4 all red chips will become green with probability exponentially close to 1. Next, by the third observation above, the length of the phase is at least as long as the time until Φ reaches H in Theorem 2.1, which is $\exp(n^\gamma)$ for some $\gamma > 0$. Together, we get that for any time t, the probability that there is any red chip is exponentially small, and the result follows. $\square$

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