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THE ORDER OF THE NORMAL APPROXIMATION FOR LINEAR  
COMBINATIONS OF ORDER STATISTICS WITH SMOOTH  
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The order of the normal approximation for linear combinations of order statistics with smooth weight functions \*)

by

R. Helmers

ABSTRACT

A Berry-Esseen bound of order  $n^{-\frac{1}{2}}$  is established for linear combinations of order statistics. The theorem requires a "smooth" weight function, and the underlying distribution function must not have "too much weight in the tails". The distribution function need not be continuous.

KEY WORDS & PHRASES: *linear combinations of order statistics, order of normal approximation.*

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## 1. INTRODUCTION AND MAIN RESULT

Let, for each  $n \geq 1$ ,  $T_n = n^{-1} \sum_{i=1}^n J\left(\frac{i}{n+1}\right) X_{i:n}$ , where  $X_{i:n}$ ,  $i = 1, \dots, n$  denotes the  $i$ -th order statistic of a random sample  $X_1, \dots, X_n$  of size  $n$  from a distribution with distribution function (df)  $F$  and  $J$  is a bounded measurable weight function on  $(0,1)$ . The inverse of a df will always be the left-continuous one. Let  $F_n^*(x) = P(T_n^* \leq x)$  for  $-\infty < x < \infty$ , where

$$(1.1) \quad T_n^* = (T_n - E(T_n)) / \sigma(T_n).$$

In theorem 2 of STIGLER (1974) it is shown that  $T_n^*$  is asymptotically  $N(0,1)$ -distributed as  $n \rightarrow \infty$ , if  $J$  is bounded and continuous a.e.  $F^{-1}$ ,  $EX_1^2 < \infty$  and  $\sigma^2(J, F) > 0$ , where

$$(1.2) \quad \sigma^2(J, F) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} J(F(x))J(F(y))(F(\min(x,y)) - F(x)F(y)) dx dy.$$

In addition these assumptions imply that  $\lim_{n \rightarrow \infty} n\sigma^2(T_n) = \sigma^2(J, F)$  (see theorem 1 of STIGLER (1974)). We also want to mention a paper of SHORACK (1972) for a related result.

The purpose of this paper is to establish a Berry-Esseen bound of order  $n^{-\frac{1}{2}}$  for the normal approximation of  $F_n^*$ . Let  $\Phi$  denote the df of the standard normal distribution. In the following theorem we state our main result.

THEOREM 1. *Suppose*

(1)  *$J$  is bounded and continuous on  $(0,1)$ . The derivative  $J'$  exists, except possibly at a finite number of points;  $J'$  satisfies a Lipschitz condition of order  $> \frac{1}{2}$  on the open intervals where it exist. The inverse  $F^{-1}$  satisfies a Lipschitz condition of order  $> \frac{1}{2}$  on neighbourhoods of the points where  $J'$  does not exist.*

(2)  *$E|X_1|^3 < \infty$  and  $\int_0^1 |J'(s)| dF^{-1}(s) < \infty$ .*

*Then  $\sigma^2(J, F) > 0$  implies that there exist a constant  $C$ , depending on  $J$  and  $F$  but not on  $n$ , such that for all  $n \geq 1$*

$$\sup_x |F_n^*(x) - \Phi(x)| \leq C n^{-\frac{1}{2}}.$$

The only other result where a Berry-Esseen bound of order  $n^{-\frac{1}{2}}$  for general linear combinations of order statistics is established is due to

BJERVE (1974). His result admits quite general weights on observations between the  $\alpha$ -th and  $\beta$ -th sample percentile ( $0 < \alpha < \beta < 1$ ), but he does not allow weights to be put on the remaining observations. In addition the df must satisfy a quite severe smoothness condition.

Theorem 1 is the first general theorem establishing a Berry-Esseen bound of order  $n^{-\frac{1}{2}}$  for linear combinations of order statistics, which allows weights to be put on all the observations. The theorem requires a "smooth" weight function, and the underlying df must not have "too much weight in the tails". The df need not be continuous.

In section 2 we shall approximate  $T_n^*$  by a random variable (rv)  $S_n^*$  such that  $T_n^* - S_n^*$  is of negligible order for our purposes. A Berry-Esseen bound of order  $n^{-\frac{1}{2}}$  for  $S_n^*$  is established in section 3 using a technique based on characteristic functions due to BICKEL (1974) (see also BJERVE (1974)).

## 2. APPROXIMATION BY $S_n^*$

Let, for each  $n \geq 1$ ,  $U_1, \dots, U_n$  be independent uniform (0,1) rv's and let  $U_{i:n}$  ( $1 \leq i \leq n$ ) denote the  $i$ -th order statistic of  $U_1, \dots, U_n$ . It is well-known (see e.g. SHORACK (1972)) that the joint distribution of  $X_1, \dots, X_n$  is the same as that of  $(F^{-1}(U_1), \dots, F^{-1}(U_n))$  for any df  $F$ . Therefore we shall identify  $X_i$  with  $F^{-1}(U_i)$  and also  $X_{i:n}$  with  $F^{-1}(U_{i:n})$ . Throughout we shall assume that all rv's are defined on the same probability space  $(\Omega, \mathcal{A}, \mathcal{P})$ . For any rv  $X$  we denote by  $X^*$  the rv  $(X - E(X))/\sigma(X)$ ; it is tacitly assumed that  $E|X| < \infty$  and  $0 < \sigma(X) < \infty$ .  $\chi_E(\cdot)$  denotes the indicator of a set  $E$ .

Define, for each  $n \geq 1$ , the rv  $S_n$  by

$$(2.1) \quad S_n = I_{1n} + I_{2n}$$

where

$$(2.2) \quad I_{1n} = -n^{-1} \sum_{i=1}^n \int_0^1 J(s) (\chi_{(0,s]}(U_i) - s) dF^{-1}(s)$$

and

$$(2.3) \quad I_{2n} = -n^{-2} \sum_{i=1}^n \sum_{j=1}^{i-1} \int_0^1 J'(s) (\chi_{(0,s]}(U_i) - s) (\chi_{(0,s]}(U_j) - s) dF^{-1}(s).$$

In this section we shall prove that under appropriate conditions  $T_n^* - S_n^*$  is of negligible order for our purposes:

$$(2.4) \quad P(|T_n^* - S_n^*| \geq n^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}}) \text{ as } n \rightarrow \infty.$$

For the purpose of our proofs we start by stating a very simple but useful lemma.

LEMMA 2.1. Let  $\{X_n\}$  and  $\{Y_n\}$  be two sequences of rv's (defined on the same probability space  $(\Omega, \mathcal{A}, P)$ ), such that

- (1)  $\sigma^2(X_n - Y_n) = O(n^{-5/2})$  as  $n \rightarrow \infty$ , and  
 (2) either  $0 < \lim_{n \rightarrow \infty} n\sigma^2(X_n) < \infty$  or  $0 < \lim_{n \rightarrow \infty} n\sigma^2(Y_n) < \infty$  holds.  
 Then for any  $a > 0$   $P(|X_n^* - Y_n^*| \geq an^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}})$  as  $n \rightarrow \infty$ .

PROOF. To start with the proof we note that

$$(2.5) \quad X_n^* - Y_n^* = \frac{X_n - Y_n - E(X_n - Y_n)}{\sigma(X_n)} + (Y_n - E(Y_n)) \cdot \frac{(\sigma(Y_n) - \sigma(X_n))}{\sigma(X_n)\sigma(Y_n)}$$

and hence that

$$(2.6) \quad \sigma^2(X_n^* - Y_n^*) \leq 2\sigma^{-2}(X_n)\sigma^2(X_n - Y_n) + 2\sigma^2(Y_n) \left( \frac{\sigma(Y_n) - \sigma(X_n)}{\sigma(X_n)\sigma(Y_n)} \right)^2.$$

Obviously we may assume that  $0 < \lim_{n \rightarrow \infty} n\sigma^2(X_n) < \infty$ . Hence we know that  $\sigma^{-2}(X_n) = O(n)$  as  $n \rightarrow \infty$ . Because also  $\sigma^2(X_n - Y_n) = O(n^{-5/2})$  as  $n \rightarrow \infty$ , we have shown that the first term on the righthand side of (2.6) is  $O(n^{-3/2})$  as  $n \rightarrow \infty$ . To proceed with the second term on the righthand side of (2.6) we note that it follows from our assumptions that  $0 < \lim_{n \rightarrow \infty} n\sigma^2(Y_n) < \infty$ .  
 Now

$$\begin{aligned} 2\sigma^2(Y_n) \left( \frac{\sigma(Y_n) - \sigma(X_n)}{\sigma(X_n)\sigma(Y_n)} \right)^2 &= 2 \frac{(\sigma^2(Y_n) - \sigma^2(X_n))^2}{\sigma^2(X_n)(\sigma(X_n) + \sigma(Y_n))^2} \leq \\ &\leq 2 \frac{(2\sigma(X_n)\sigma(Y_n - X_n) + \sigma^2(Y_n - X_n))^2}{\sigma^2(X_n)(\sigma(X_n) + \sigma(Y_n))^2} \end{aligned}$$

and we can use the preceding results to find that

$$\begin{aligned} & 2 \frac{(2\sigma(X_n)\sigma(Y_n - X_n) + \sigma^2(Y_n - X_n))^2}{\sigma^2(X_n)(\sigma(X_n) + \sigma(Y_n))^2} = \\ & = O(n^2)(O(n^{-\frac{1}{2}})O(n^{-5/4}) + O(n^{-5/2}))^2 = O(n^{-3/2}) \text{ as } n \rightarrow \infty \end{aligned}$$

Hence we have shown that  $\sigma^2(X_n^* - Y_n^*) = O(n^{-3/2})$  as  $n \rightarrow \infty$ . An application of Chebychev's inequality completes the proof.  $\square$

In order to prove that (2.4) holds under appropriate conditions we need two more lemmas. In our second lemma we approximate  $T_n$  by a rv  $V_n$  given by

$$(2.7) \quad V_n = \int_0^1 J(s)F_n^{-1}(s)ds = \sum_{i=1}^n \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s)ds X_{i,n},$$

where  $F_n$  denotes the empirical df based on  $X_1, \dots, X_n$ . We shall show that  $T_n^* - V_n^*$  is of negligible order for our purposes. Let  $\|f\| = \sup_{0 < t < 1} |f(t)|$  for any function  $f$  on  $(0,1)$ . In certain cases the function  $f$  is defined on  $(0,1)$  except at a finite number of points. Then  $\|f\|$  will denote the supremum of  $|f|$  on the domain of  $f$ .

**LEMMA 2.2.** *Let  $EX_1^2 < \infty$  and suppose that condition (1) of theorem 1 is satisfied. Then  $\sigma^2(J, F) > 0$  implies that for any  $a > 0$   $P(|T_n^* - V_n^*| \geq an^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}})$  as  $n \rightarrow \infty$ . The assumption that  $J'$  satisfies a Lipschitz condition of order  $> \frac{1}{2}$  on the open intervals where it exist can be dropped from condition (1). The Lipschitz condition for  $F^{-1}$  may be of order  $\geq \frac{1}{2}$ .*

**PROOF.** It follows from  $EX_1^2 < \infty$  that  $EX_{i,n}^2 < \infty$  for any  $1 \leq i \leq n$ . Furthermore it is well-known (see ESARY, PROSCHAN and WALKUP (1967)) that for any  $x, y, i, j, n$  and  $F$  we have  $P(X_{i,n} \leq x, X_{j,n} \leq y) \geq P(X_{i,n} \leq x) P(X_{j,n} \leq y)$ . Using a representation of the covariance of two random variables given in LEHMANN (1966) this result implies directly that the covariance between  $X_{i,n}$  and  $X_{j,n}$  is finite and non-negative for all  $1 \leq i \neq j \leq n$ . Obviously this implies that

$$(2.8) \quad \sigma^2\left(\sum_{i=1}^n a_i X_{i,n}\right) \leq \sigma^2\left(\sum_{i=1}^n b_i X_{i,n}\right)$$



holds, provided  $a_i a_j \leq b_i b_j$  for all  $1 \leq i, j \leq n$ . This inequality is due to W.R. VAN ZWET and will be very useful in what follows.

Since the assumptions of this lemma imply those of theorem 1 of STIGLER (1974) (see our introduction) we know that  $\lim_{n \rightarrow \infty} n\sigma^2(T_n) = \sigma^2(J, F)$ . By assumption we have also that  $\sigma^2(J, F) > 0$ , whereas a simple application of (2.8) yields  $\sigma^2(T_n) \leq n^{-1} \|J\|^2 \sigma^2(X_1)$ . Because  $\|J\| < \infty$  and  $\sigma^2(X_1) < \infty$  by the assumptions of the lemma these results imply that  $0 < \lim_{n \rightarrow \infty} n\sigma^2(T_n) = \sigma^2(J, F) < \infty$ . Application of lemma 2.1 shows that it suffices now to prove that

$$(2.9) \quad \sigma^2(T_n - V_n) = O(n^{-5/2}) \text{ as } n \rightarrow \infty.$$

To prove (2.9) we distinguish two cases: (i)  $J$  is everywhere differentiable on  $(0,1)$ , and (ii)  $J'$  fails to exist at a finite number of points.

We first prove (2.9) in case (i). Using (2.7) and (2.8) we see that

$$(2.10) \quad \sigma^2(T_n - V_n) \leq \sigma^2\left(\sum_{i=1}^n X_{in} \left| \frac{J\left(\frac{i}{n+1}\right)}{n} - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s) ds \right|\right).$$

Applying (2.8) again and using the condition for  $J$  we find that

$$(2.11) \quad \sigma^2(T_n - V_n) \leq n^{-3} \|J'\|^2 \sigma^2(X_1).$$

Because  $\|J'\| < \infty$  and  $\sigma^2(X_1) < \infty$  by the assumptions of the lemma the proof of case (i) of the lemma is now complete.

Suppose now that we are in case (ii). Without any loss of generality we assume that  $J'$  does not exist at only one point, say  $s = s_1$ . Let  $j = [ns_1] + 1$ . Using inequality (2.8) twice we see that

$$(2.12) \quad \sigma^2(T_n - V_n) \leq 2\sigma^2\left(\sum_{\substack{i=1 \\ i \neq j}}^n X_{in} \left| \frac{J\left(\frac{i}{n+1}\right)}{n} - \int_{\frac{i-1}{n}}^{\frac{i}{n}} J(s) ds \right|\right) + \\ + 2\sigma^2\left(X_{jn} \left| \frac{J\left(\frac{j}{n+1}\right)}{n} - \int_{\frac{j-1}{n}}^{\frac{j}{n}} J(s) ds \right|\right).$$

Using condition (1) of theorem 1 and applying (2.8) once more we obtain that

$$(2.13) \quad \sigma^2(T_n - V_n) \leq 2n^{-3} \|J\|^2 \sigma^2(X_1) + 8n^{-2} \|J\|^2 \sigma^2(X_{j_n}).$$

Hence it remains to prove that  $\sigma^2(X_{j_n}) = O(n^{-\frac{1}{2}})$  as  $n \rightarrow \infty$ . Let  $g_n$  denote the beta-density of the uniform order statistic  $U_{j_n}$  (with  $j = [ns_1] + 1$ ) and let  $E_n$  be the set

$$(2.14) \quad E_n = \{u: |u - \frac{[ns_1 + 1]}{n+1}| \leq (mn^{-1} \log n)^{\frac{1}{2}}, 0 < u < 1\}$$

for some fixed  $m > 0$ . The complement of  $E_n$  in  $(0,1)$  will be denoted by  $E_n^c$ . Then we have that

$$(2.15) \quad \begin{aligned} \sigma^2(X_{j_n}) &\leq E(X_{j_n} - F^{-1}(\frac{j}{n+1}))^2 = \\ &= \int_{E_n} (F^{-1}(u) - F^{-1}(\frac{j}{n+1}))^2 g_n(u) du + \int_{E_n^c} (F^{-1}(u) - F^{-1}(\frac{j}{n+1}))^2 g_n(u) du. \end{aligned}$$

Because  $EX_1^2 < \infty$  we can use lemma 4 of STIGLER (1969) to see that the second integral on the righthand side of (2.15) is  $O(n^{-\frac{1}{2}})$  as  $n \rightarrow \infty$ , provided we choose  $m$  sufficiently large. The Lipschitz condition of  $F^{-1}$  in a neighbourhood of  $s_1$  can be used to treat the first integral on the righthand side of (2.15). Since  $\frac{j-1}{n} \leq s_1 < \frac{j}{n}$  we have for sufficiently large  $n$  and some constant  $B > 0$  that

$$(2.16) \quad \int_{E_n} (F^{-1}(u) - F^{-1}(\frac{j}{n+1}))^2 g_n(u) du \leq B \cdot E|U_{j_n} - \frac{j}{n+1}|.$$

It follows directly from this and the well-known fact that, as  $\lim_{n \rightarrow \infty} \frac{j}{n} = s_1$ , for  $0 < s_1 < 1$ ,  $E|U_{j_n} - \frac{j}{n+1}| = O(n^{-\frac{1}{2}})$ , that also the first integral on the righthand side of (2.15) is  $O(n^{-\frac{1}{2}})$  as  $n \rightarrow \infty$ . Hence we can conclude that  $\sigma^2(X_{j_n}) = O(n^{-\frac{1}{2}})$  as  $n \rightarrow \infty$ . This and (2.13) implies that  $\sigma^2(T_n - V_n) = O(n^{-5/2})$  as  $n \rightarrow \infty$ , in case (ii). This completes the proof of the lemma.  $\square$

Define for  $0 < u < 1$  the function

$$(2.17) \quad \psi(u) = \int_u^1 J(s)ds - (1-u) \int_0^1 J(s)ds$$

and let  $c = \int_0^1 J(s)ds$ . Then it is easy to check (see SHORACK (1972) for a similar approach) that

$$(2.18) \quad V_n = \int_0^1 \psi(\Gamma_n(s))dF^{-1}(s) + cn^{-1} \sum_{i=1}^n F^{-1}(U_i),$$

holds with probability 1. We use the fact that, almost surely, none of the rv's  $U_1, \dots, U_n$  take values corresponding to the discontinuities of  $F^{-1}$ . Here  $\Gamma_n$  denotes the empirical df based on  $U_1, \dots, U_n$ . This representation of  $V_n$  will be very useful.

In our third lemma we use representation (2.18) to show that  $V_n^* - S_n^*$  is of negligible order for our purposes.

LEMMA 2.3. *Let  $E|X_1|^{2+\epsilon} < \infty$  for some  $\epsilon > 0$  and suppose that condition (1) of theorem 1 is satisfied. Then  $\sigma^2(J, F) > 0$  implies that for any  $a > 0$   $P(|V_n^* - S_n^*| \geq an^{-\frac{1}{2}}) = O(n^{-\frac{1}{2}})$  as  $n \rightarrow \infty$ .*

PROOF. It follows directly from the proof of lemma 2.2 that  $0 < \lim_{n \rightarrow \infty} n\sigma^2(V_n) = \sigma^2(J, F) < \infty$ . Application of lemma 2.1 show, that it suffices now to prove that

$$(2.19) \quad \sigma^2(V_n - S_n) = O(n^{-5/2}) \text{ as } n \rightarrow \infty.$$

For the purpose of this proof we define, for each  $n \geq 1$ , the rv.  $W_n$  given by

$$(2.20) \quad W_n = \int_0^1 (\psi(s) + (\Gamma_n(s) - s)\psi'(s) + \frac{(\Gamma_n(s) - s)^2}{2} \psi''(s))dF^{-1}(s) + cn^{-1} \sum_{i=1}^n F^{-1}(U_i).$$

Note that the assumptions of the lemma guarantee that  $W_n$  is well-defined. It will be convenient to prove

$$(2.21) \quad \sigma^2(V_n - W_n) = O(n^{-5/2}) \text{ as } n \rightarrow \infty$$

and

$$(2.22) \quad \sigma^2(W_n - S_n) = O(n^{-5/2}) \text{ as } n \rightarrow \infty$$

rather than (2.19). We first prove (2.22). Using (2.17) we find that

$$(2.23) \quad W_n = \int_0^1 \psi(s) dF^{-1}(s) - \int_0^1 J(s)(\Gamma_n(s) - s) dF^{-1}(s) - \\ - \int_0^1 J'(s) \frac{(\Gamma_n(s) - s)^2}{2} dF^{-1}(s) + \\ + c \int_0^1 (\Gamma_n(s) - s) dF^{-1}(s) + cn^{-1} \sum_{i=1}^n F^{-1}(U_i).$$

Because  $\Gamma_n(s) = n^{-1} \sum_{i=1}^n \chi_{(0,s]}(U_i)$  for all  $0 < s < 1$  and  $n \geq 1$  we have

$$(2.24) \quad \int_0^1 (\Gamma_n(s) - s) dF^{-1}(s) = n^{-1} \sum_{i=1}^n \left( \int_{(0, U_i)} (-s) dF^{-1}(s) + \right. \\ \left. + \int_{[U_i, 1)} (1-s) dF^{-1}(s) \right).$$

Now integration by parts, the finiteness of  $E|X_1|$  and the fact that, almost surely, none of the rv's  $U_1, \dots, U_n$  take values corresponding to the discontinuities of  $F^{-1}$  shows that

$$(2.25) \quad \int_0^1 (\Gamma_n(s) - s) dF^{-1}(s) = -n^{-1} \sum_{i=1}^n F^{-1}(U_i) + \int_0^1 F^{-1}(s) ds$$

holds with probability 1.

Thus

$$(2.26) \quad W_n - E(W_n) \stackrel{\text{a.s.}}{=} -n^{-1} \sum_{i=1}^n \int_0^1 J(s) (\chi_{(0,s]}(U_i) - s) dF^{-1}(s) - \\ - 2^{-1} n^{-2} \sum_{i=1}^n \sum_{j=1}^n \int_0^1 J'(s) (\chi_{(0,s]}(U_i) - s) (\chi_{(0,s]}(U_j) - s) \\ dF^{-1}(s) +$$

$$+ 2^{-1} n^{-1} \int_0^1 J'(s) s(1-s) dF^{-1}(s).$$

Combining (2.26) with (2.1), (2.2) and (2.3) and using the assumptions of the lemma together with Fubini's theorem to verify that  $ES_n = 0$  we find that

$$(2.27) \quad W_n - S_n - E(W_n - S_n) \stackrel{\text{a.s.}}{=} -2^{-1} n^{-2} \sum_{i=1}^n \int_0^1 J'(s) ((\chi_{(0,s]}(U_i) - s)^2 - s(1-s)) dF^{-1}(s),$$

and hence that

$$(2.28) \quad \sigma^2(W_n - S_n) = 2^{-2} n^{-3} \sigma^2 \left( \int_0^1 J'(s) (\chi_{(0,s]}(U_1) - s)^2 dF^{-1}(s) \right).$$

To see that the variance on the righthand side of (2.28) is finite note that

$$(2.29) \quad \begin{aligned} & \sigma^2 \left( \int_0^1 J'(s) (\chi_{(0,s]}(U_1) - s)^2 dF^{-1}(s) \right) \leq \\ & \leq E \left( \int_0^1 J'(s) (\chi_{(0,s]}(U_1) - s)^2 dF^{-1}(s) \right)^2 = \\ & = E \int_0^1 \int_0^1 J'(s) J'(v) (\chi_{(0,s]}(U_1) - s)^2 (\chi_{(0,v]}(U_1) - v)^2 dF^{-1}(s) dF^{-1}(v) = \\ & = \int_0^1 \int_0^1 J'(s) J'(v) E(\chi_{(0,s]}(U_1) - s)^2 (\chi_{(0,v]}(U_1) - v)^2 dF^{-1}(s) dF^{-1}(v) \leq \\ & \leq \int_0^1 \int_0^1 |J'(s) J'(v)| (E(\chi_{(0,s]}(U_1) - s)^4 E(\chi_{(0,v]}(U_1) - v)^4)^{\frac{1}{2}} dF^{-1}(s) dF^{-1}(v) \leq \\ & \leq 2 \left( \int_0^1 |J'(s)| (s(1-s))^{\frac{1}{2}} dF^{-1}(s) \right)^2, \end{aligned}$$

where the interchange of the expectation and the integrals is a consequence of Fubini's theorem. The validity of this application of Fubini's theorem can be inferred from the moment condition of the lemma, the boundedness of

$J'$  on its domain and the continuity of  $F^{-1}$  at the points where  $J'$  is undefined. These conditions also imply that  $\int_0^1 |J'(s)|(s(1-s))^{\frac{1}{2}} dF^{-1}(s)$  is finite.

Thus we have shown that  $\sigma^2(W_n - S_n) = O(n^{-3})$  as  $n \rightarrow \infty$ . This completes the proof of (2.22).

Next we prove (2.21). As in the second part of the proof of lemma 2.2 we distinguish two cases. First we assume (case (i)) that  $J$  is everywhere differentiable on  $(0,1)$ . Using (2.18), (2.20) and Taylor's theorem, together with the Lipschitz condition for  $J'$  on  $(0,1)$ , we see that for all  $n \geq 1$  and some constant  $A > 0$   $|V_n - W_n| \leq A \int_0^1 |\Gamma_n(s) - s|^{5/2} dF^{-1}(s)$  and hence that

$$(2.30) \quad \sigma^2(V_n - W_n) \leq E(V_n - W_n)^2 \leq A^2 E\left(\int_0^1 |\Gamma_n(s) - s|^{5/2} dF^{-1}(s)\right)^2.$$

Applying Fubini's theorem, the Cauchy-Schwarz inequality, and making some simple moment calculations it follows that for some constant  $B > 0$

$$(2.31) \quad \sigma^2(V_n - W_n) \leq Bn^{-5/2} \left(\int_0^1 (s(1-s))^{\frac{1}{2}} dF^{-1}(s)\right)^2.$$

The moment assumption of the lemma ensures that the integral on the right-hand side of (2.31) is finite. This completes the proof of (2.21) for case (i).

Suppose now that  $J'$  fails to exist at a finite number of points (case (ii)). To prove (2.21) in this case is somewhat more delicate. It seems convenient to introduce at this point the well-known Kolmogorov-Smirnov statistic  $D_n = n^{\frac{1}{2}} \sup_{0 < s < 1} |\Gamma_n(s) - s|$ . It was shown by DVORETZKY, KIEFER and WOLFOWITZ (1956) that  $P(D_n \geq \lambda_n) \leq c \exp(-2\lambda_n^2)$ , for all  $n \geq 1$ ,  $\lambda_n \geq 0$  and a positive constant  $c$  independent of  $n$  and  $\lambda_n$ . Obviously this implies that  $P(D_n \geq (2^{-1} m \log n)^{\frac{1}{2}}) = O(n^{-m})$  as  $n \rightarrow \infty$ , for any fixed  $m > 0$ . Let us denote by  $\chi_n$  the indicator of the set  $\{D_n \geq (2^{-1} m \log n)^{\frac{1}{2}}\}$ . Without loss of generality we assume that  $J'$  does not exist at only one point  $s_1 \in (0,1)$ .

We first show that  $E(V_n - W_n - E(V_n - W_n))^2 \chi_n = O(n^{-5/2})$  as  $n \rightarrow \infty$  holds for an appropriate value of  $m$ . Since  $\sigma^2(W_n - S_n) = O(n^{-3})$  as  $n \rightarrow \infty$ , and hence that  $E(W_n - S_n - E(W_n - S_n))^2 \chi_n = O(n^{-3})$  as  $n \rightarrow \infty$  for any  $m > 0$ , was obtained

earlier in this proof, it suffices to show that

$$(2.32) \quad E(V_n - S_n - E(V_n - S_n))^2 \chi_n = O(n^{-5/2}) \text{ as } n \rightarrow \infty.$$

To prove (2.32) we apply Hölder's inequality to obtain for any  $0 < \eta < 1$

$$E(V_n - S_n - E(V_n - S_n))^2 \chi_n \leq (E(V_n - S_n - E(V_n - S_n))^{2+2\eta})^{1/(1+\eta)} (E\chi_n^{1+1/\eta})^{\eta/(1+\eta)},$$

and hence, using the  $c_r$ -inequality (see e.g. LOEVE (1955), page 155), that

$$(2.33) \quad E(V_n - S_n - E(V_n - S_n))^2 \chi_n \leq 16(E|V_n|^{2+2\eta} + E|S_n|^{2+2\eta})^{1/(1+\eta)} (P(\chi_n=1))^{\eta/(1+\eta)}.$$

Since  $P(\chi_n=1) = O(n^{-m})$  as  $n \rightarrow \infty$ , it follows that  $(P(\chi_n=1))^{\eta/(1+\eta)} = O(n^{-5/2})$  as  $n \rightarrow \infty$ , provided we choose  $m > 5/\eta$ . Now using (2.1), (2.2), (2.3) and (2.7) and applying integration by part we see that

$$|V_n| \leq n^{-1} \|J\| \sum_{i=1}^n |F^{-1}(U_i)|$$

and

$$|S_n| \leq n^{-1} (\|J\| + \|J'\|) \sum_{i=1}^n (|F^{-1}(U_i)| + \int_0^1 |F^{-1}(s)| ds)$$

holds for all  $n \geq 1$  with probability one. Combining this result with the finiteness of  $E|X_1|^{2+2\eta}$  for any  $0 < \eta < \varepsilon/2$  and some  $\varepsilon > 0$  satisfying the moment condition of the lemma and applying the  $c_r$ -inequality we find that the expectations in (2.32) are uniformly bounded in  $n$  for any  $\eta \in (0, \varepsilon/2)$ . Hence we have shown that (2.32) holds for any fixed  $m > (10/\varepsilon)$ .

To complete the proof of (2.21) in case (ii) it remains to show that  $E(V_n - W_n - E(V_n - W_n))^2 \chi_n^c = O(n^{-5/2})$  as  $n \rightarrow \infty$  for some fixed  $m > (10/\varepsilon)$ . It follows from (2.18) and (2.20) that  $V_n - W_n = \int_0^1 g_n(s) dF^{-1}(s)$  where

$$(2.34) \quad g_n(s) = \psi(\Gamma_n(s)) - \psi(s) - (\Gamma_n(s) - s)\psi'(s) - \frac{(\Gamma_n(s) - s)^2}{2} \psi''(s),$$

for all  $0 < s < 1$ , except  $s = s_1$ , and any  $n \geq 1$ . Note that the fact that  $g_n$  remains undefined in  $s = s_1$  causes no problem because  $F^{-1}$  puts no mass

at  $s_1$ .

Taking the set  $E_n$  as in (2.14) we write

$$(2.35) \quad V_n - W_n = \int_{E_n} g_n(s) dF^{-1}(s) + \int_{E_n^c} g_n(s) dF^{-1}(s),$$

and hence that

$$(2.36) \quad E(V_n - W_n)^2 \chi_n^c \leq 2E\left(\int_{E_n} g_n(s) dF^{-1}(s) \cdot \chi_n^c\right)^2 + 2E\left(\int_{E_n^c} g_n(s) dF^{-1}(s) \cdot \chi_n^c\right)^2.$$

On the set where  $\chi_n^c = 1$ , we have that  $|g_n(s)| = O((\Gamma_n(s) - s)^2) = O(n^{-1} \log n)$  as  $n \rightarrow \infty$ , uniformly for all  $0 < s < 1$  except  $s = s_1$ . Using the Lipschitz condition for  $F^{-1}$  we find that the first righthand term of (2.36) is of order  $O(n^{-5/2})$  as  $n \rightarrow \infty$ . On the set  $E_n^c$  we can argue that, because  $\chi_n^c = 1$ , the closed interval  $[s, \Gamma_n(s)]$  does not contain the point  $s_1$  where  $J'$  does not exist for all sufficiently large  $n$ . Together with (2.34) and the Lipschitz condition for  $J'$  this implies for  $s \in E_n^c$  and some constant  $A > 0$  that  $|g_n(s)| \leq A |\Gamma_n(s) - s|^{5/2}$  for all sufficiently large  $n$ . Now this and the moment assumption of the lemma ensures that the second righthand term of (2.36) is  $O(n^{-5/2})$  as  $n \rightarrow \infty$ . Hence we can conclude that  $E(V_n - W_n)^2 \chi_n^c = O(n^{-5/2})$  as  $n \rightarrow \infty$ . From this we find easily that also  $E(V_n - W_n - E(V_n - W_n))^2 \chi_n^c = O(n^{-5/2})$  as  $n \rightarrow \infty$ . This completes the proof of the lemma.  $\square$

To conclude this section we remark that to show that  $P(|T_n^* - S_n^*| \geq n^{-1/2}) = O(n^{-1/2})$  as  $n \rightarrow \infty$ , first use lemma 2.2 to see that  $P(|T_n^* - V_n^*| \geq 2^{-1} n^{-1/2}) = O(n^{-1/2})$  as  $n \rightarrow \infty$ . Next apply lemma 2.3 to find that  $P(|V_n^* - S_n^*| \geq 2^{-1} n^{-1/2}) = O(n^{-1/2})$  as  $n \rightarrow \infty$ ; Hence, since the conditions of lemma 2.3 imply those of lemma 2.2,  $P(|T_n^* - S_n^*| \geq n^{-1/2}) = O(n^{-1/2})$  as  $n \rightarrow \infty$ , is shown to hold under the conditions of lemma 2.3.

### 3. THE ORDER OF NORMAL APPROXIMATION FOR $S_n^*$

In this section we shall show that the conditions of theorem 1 ensure that the normal approximation for  $S_n^*$  is of order  $n^{-1/2}$ . As we have already shown in section 2 that, under the conditions of lemma 2.3, we may approximate  $T_n^*$  by  $S_n^*$ , the proof of theorem 1 will then be completed.



The rv  $S_n^*$  is given by  $S_n^* = J_{1n} + J_{2n}$ , where  $J_{mn} = I_{mn}/\sigma(S_n)$  for  $m = 1, 2$  and all  $n \geq 1$ . For convenience we shall write  $\sigma_n = \sigma(S_n)$ . Since our proof will depend on characteristic functions (c.f.) let us denote by  $\rho_n^*$  and  $\rho_{n1}$  the c.f. of  $S_n^*$  and  $J_{1n}$ . The c.f. of a summand of  $n\sigma_n J_{1n}$ , that is of

$$(3.1) \quad - \int_0^1 J(s) (\chi_{(0,s]}(U_1) - s) dF^{-1}(s)$$

will be denoted by  $\rho$ . Clearly we have  $\rho_{n1}(t) = \rho^n(t/n\sigma_n)$  for all  $t$  and  $n \geq 1$ .

Following BICKEL (1974) we shall first show that there exist  $\varepsilon_1 > 0$ ,  $D_1$  and a natural number  $n_1$ , depending on  $J$  and  $F$  but not on  $n$ , such that for all  $n \geq n_1$

$$(3.2) \quad \int_{|t| < \varepsilon_1 n^{\frac{1}{2}}} |\rho_{n1}(t) - e^{-t^2/2}| \cdot |t|^{-1} dt \leq D_1 n^{-\frac{1}{2}}.$$

Secondly we show that there exist  $\varepsilon_2 > 0$ ,  $D_2$  and a natural number  $n_2$ , depending on  $J$  and  $F$  but not on  $n$ , such that for all  $n \geq n_2$

$$(3.3) \quad \int_{|t| < \varepsilon_2 n^{\frac{1}{2}}} |\rho_n^*(t) - \rho_{n1}(t)| \cdot |t|^{-1} dt \leq D_2 n^{-\frac{1}{2}}.$$

The Berry-Esseen bound of order  $n^{-\frac{1}{2}}$  for  $S_n^*$  then follows directly from (3.2), (3.3) and the usual argument based on Esseen's smoothing lemma (see e.g. FELLER (1966)).

We first prove (3.2)

LEMMA 3.1. *Let  $E|X_1|^3 < \infty$  and suppose that condition (1) of theorem 1 is satisfied. Then  $\sigma^2(J, F) > 0$  implies (3.2).*

PROOF. To start with the proof we note that the conditions of lemma 2.3 are satisfied. Since it was already shown in the proof of lemma 2.3 that  $0 < \lim_{n \rightarrow \infty} n\sigma^2(V_n) = \sigma^2(J, F) < \infty$  and that  $\sigma^2(V_n - S_n) = O(n^{-5/2})$  as  $n \rightarrow \infty$  it follows that  $0 < \lim_{n \rightarrow \infty} n\sigma_n^2 = \sigma^2(J, F) < \infty$ . However, to prove the lemma we shall need the more precise result that

$$(3.4) \quad \frac{\sigma^2(J, F)}{n\sigma_n^2} = 1 + O(n^{-1}) \quad \text{as } n \rightarrow \infty.$$

To see that (3.4) holds, note first that, using the boundedness of  $J$  and  $J'$  (on its domain), the continuity of  $F^{-1}$  at the points where  $J'$  does not exist, the finiteness of  $E|X_1|^{2+\varepsilon}$  for some  $\varepsilon > 0$ , and applying Fubini's theorem, we find that  $E I_{1n} = E I_{2n} = 0$  and  $E I_{1n} I_{2n} = 0$ . Hence the covariance between  $I_{1n}$  and  $I_{2n}$  is zero. This implies that

$$(3.5) \quad \sigma_n^2 = \sigma^2(I_{1n}) + \sigma^2(I_{2n}).$$

Note also that  $\sigma^2(I_{1n}) = n^{-1} \sigma^2(J, F)$  and  $\sigma^2(I_{2n}) = O(n^{-2})$  as  $n \rightarrow \infty$ . Combining this with (3.5) we have proved (3.4). Hence

$$(3.6) \quad J_{1n} = \tau_n I_{1n}^*$$

where  $I_{1n}^* = I_{1n} / \sigma(I_{1n})$  and  $\tau_n = 1 + O(n^{-1})$  as  $n \rightarrow \infty$ . Remark that  $I_{1n}^*$  is a properly standardized sum of independent, identically distributed, random variables.

Secondly we will show that the summands of  $n \sigma_n \tau_n I_{1n}^*$  (that is of (3.1)) have finite absolute third moment. Note that

$$(3.7) \quad \left| \int_0^1 J(s) (\chi_{(0,s]}(U_1) - s) dF^{-1}(s) \right| \leq \\ \leq \|J\| \left( \int_{(0, U_1)} s dF^{-1}(s) + \int_{[U_1, 1)} (1-s) dF^{-1}(s) \right).$$

Using integration by parts, the finiteness of  $E|X_1|^3 < \infty$ , and applying the  $c_r$ -inequality (see LOEVE (1955)), we find that

$$(3.8) \quad E \left( \int_{(0, U_1)} s dF^{-1}(s) \right)^3 = E \left| U_1 F^{-1}(U_1) - \int_0^{U_1} F^{-1}(s) ds \right|^3 \leq \\ \leq 4(E|U_1 F^{-1}(U_1)|^3 + E \left( \int_0^{U_1} |F^{-1}(s)| ds \right)^3) \leq 4(E|X_1|^3 + (E|X_1|)^3) < \infty.$$

Now (3.7), (3.8) and a symmetry argument ensure that the summands of  $n \sigma_n \tau_n I_{1n}^*$  have finite absolute third moment.

We are now in a position to prove (3.2). Remark first that using (3.6) and applying a change of variables we get

$$\begin{aligned}
 (3.9) \quad & \int_{|t| < \epsilon_1 n^{\frac{1}{2}}} |\rho_{n1}(t) - e^{-t^2/2}| |t|^{-1} dt \leq \\
 & \leq \int_{|t| < \epsilon_1 n^{\frac{1}{2}} \tau_n} |E e^{it I_{1n}^*} - e^{-t^2/2}| |t|^{-1} dt + \\
 & + \int_{|t| < \epsilon_1 n^{\frac{1}{2}} \tau_n} |e^{-t^2/2} - e^{-t^2/2\tau_n^2}| |t|^{-1} dt.
 \end{aligned}$$

Since  $I_{1n}^*$  is a properly standardized sum of independent, identically distributed, random variables with finite absolute third moment and  $\tau_n = O(1)$  as  $n \rightarrow \infty$ , we can simply follow the argument leading to the Berry-Esseen theorem (FELLER (1966)) to see that the first integral on the right-hand side of (3.9) is  $O(n^{-\frac{1}{2}})$ , as  $n \rightarrow \infty$ .

To treat the second integral on the right-hand side of (3.9) we note that because  $\tau_n = 1 + O(n^{-1})$  as  $n \rightarrow \infty$  we have from an application of the mean value theorem that for all sufficiently large  $n$

$$\begin{aligned}
 & \int_{|t| < \epsilon_1 n^{\frac{1}{2}} \tau_n} |e^{-t^2/2} - e^{-t^2/2\tau_n^2}| |t|^{-1} dt \leq \\
 & \leq A n^{-1} \int_{-\infty}^{\infty} |t| e^{-t^2/4} dt
 \end{aligned}$$

holds for some constant  $A > 0$ . This completes the proof of the lemma.  $\square$

Next we shall be concerned with the problem of showing that (3.3) holds under appropriate conditions. To estimate  $|\rho_n^*(t) - \rho_{n1}(t)|$  is a rather delicate matter. We start with the very simple remark that since  $|\rho_n^*(t) - \rho_{n1}(t)| = |E e^{it J_{1n}} (e^{it J_{2n-1}})|$  we have (see BICKEL (1974)) for all  $t$  and any  $m$  and  $n \geq 1$

$$(3.10) \quad |\rho_n^*(t) - \rho_{n1}(t)| \leq \left| \sum_{l=1}^{2m-1} \frac{(it)^l}{l!} E e^{it J_{1n}} (J_{2n})^l \right| + \frac{t^{2m}}{(2m)!} E (J_{2n})^{2m}.$$

Estimates for  $|\mathbb{E}e^{itJ_{1n}(J_{2n})^1}|$  and  $|E(J_{2n})^{2m}|$  which are adequate for our purposes will be given in the following lemma. The basic idea of this lemma is similar to that of lemma 6.2 and 6.3 of BICKEL (1974) (see also BJERVE (1974) where the same idea is exploited).

**LEMMA 3.2.** *Suppose the conditions (1) and (2) of theorem 1 are satisfied. Then  $\sigma^2(J,F) > 0$  implies that there exist a constant  $A > 0$ , depending on  $J$  and  $F$  but not on  $l, m$  and  $n$ , such that for all  $t$  and any  $n \geq 1$ .*

- (i)  $|\mathbb{E}e^{itJ_{1n}J_{2n}}| \leq A t^2 n^{-\frac{1}{2}} |\rho(t/n\sigma_n)|^{n-2}$
- (ii)  $|\mathbb{E}e^{itJ_{1n}(J_{2n})^1}| \leq A^1 n^{1/2} |\rho(t/n\sigma_n)|^{n-2l} \quad \text{for } 1 \leq 2l \leq n,$
- (iii)  $E(J_{2n})^{2m} \leq A^{2m} n^{-m} m^{2m} \quad \text{for } 1 \leq m \leq n.$

**PROOF.** For convenience we shall write

$$(3.11) \quad g(U_i) = - \int_0^1 J(s) (\chi_{(0,s]}(U_i) - s) dF^{-1}(s) \quad \text{for } 1 \leq i \leq n$$

and

$$(3.12) \quad h(U_i, U_j) = - \int_0^1 J'(s) (\chi_{(0,s]}(U_i) - s) (\chi_{(0,s]}(U_j) - s) dF^{-1}(s) \\ \text{for } 1 \leq j < i \leq n.$$

It follows from this, (2.1) - (2.3) and the definitions of  $J_{1n}$  and  $J_{2n}$  given earlier in this section that

$$(3.13) \quad J_{1n} = (n\sigma_n)^{-1} \sum_{i=1}^n g(U_i), \quad J_{2n} = (n^2\sigma_n)^{-1} \sum_{i=1}^n \sum_{j=1}^{i-1} h(U_i, U_j).$$

To prove statement (i) we follow Bickel's idea (see BICKEL (1974)) and remark that

$$(3.14) \quad |\mathbb{E}e^{itJ_{1n}J_{2n}}| = |(n^2\sigma_n)^{-1} \sum_{i=1}^n \sum_{j=1}^{i-1} \mathbb{E}e^{itJ_{1n}h(U_i, U_j)}| \leq \\ \leq \sigma_n^{-1} \cdot |\rho(t/n\sigma_n)|^{n-2} \cdot \int_0^1 |J'(s)| \cdot |\mathbb{E}e^{\frac{it}{n\sigma_n} g(U_1)} (\chi_{(0,s]}(U_1) - s)|^2 dF^{-1}(s),$$

where the interchange of expectation and integral follows from an application of Fubini's theorem. The validity of this application follows from the

finiteness of  $E|X_1|^{2+\varepsilon} < \infty$  for some  $\varepsilon > 0$  (as implied by condition (2)), the boundedness of  $J'$  on its domain and the continuity of  $F^{-1}$  at the points where  $J'$  is undefined. Thus we have for  $0 < s < 1$  and  $n \geq 1$

$$\begin{aligned}
 (3.15) \quad & \left| E e^{\frac{it}{n\sigma_n} g(U_1)} (\chi_{(0,s]}(U_1) - s) \right|^2 \leq \\
 & \leq t^2 (n\sigma_n)^{-2} (E|g(U_1)| \cdot |\chi_{(0,s]}(U_1) - s|)^2 \leq \\
 & \leq t^2 (n\sigma_n)^{-2} E g^2(U_1) \cdot s(1-s).
 \end{aligned}$$

Because as in the proof of lemma 3.1 the conditions of lemma 2.3 are satisfied we can repeat the argument given in the first part of that proof to find that  $0 < \lim_{n \rightarrow \infty} n\sigma_n^2 = \sigma^2(J, F) < \infty$ ,  $\sigma^2(I_{1n}) = n^{-1} \sigma^2(J, F)$  and hence that  $\sigma^2(g(U_1)) = \sigma^2(J, F)$ . We can conclude that for some constant  $A > 0$  the left-hand side of (3.15) is bounded by  $At^2 n^{-1} s(1-s)$  for  $0 < s < 1$ , all  $t$  and  $n \geq 1$ . In view of (3.14) we have obtained statement (i).

To prove statement (ii) we note that for  $l \geq 1$

$$(J_{2n})^l = (n^2 \sigma_n^2)^{-l} \sum_{\substack{(i_\nu, j_\nu) \\ \nu=1, \dots, l}} \prod_{\nu=1}^l h(U_{i_\nu}, U_{j_\nu}),$$

where the summation is over all pairs  $(i_\nu, j_\nu)$ ,  $1 \leq j_\nu < i_\nu \leq n, \nu = 1, \dots, l$ . Following again Bickel's idea (see BICKEL (1974)) we note that this implies

$$\begin{aligned}
 (3.16) \quad & \left| E e^{itJ_{2n}} (J_{2n})^l \right| \leq (n^2 \sigma_n^2)^{-l} |\rho(t/n\sigma_n)|^{n-2l} \\
 & \cdot E \left( \sum_{i=1}^n \sum_{j=1}^{i-1} |h(U_i, U_j)| \right)^l.
 \end{aligned}$$

Applying the  $c_r$ -inequality (see LOËVE (1955)) and using (3.12) we find

$$(3.17) \quad E \left( \sum_{i=1}^n \sum_{j=1}^{i-1} |h(U_i, U_j)| \right)^l \leq n^{2l} E |h(U_1, U_2)|^l.$$

Finally note that it follows from (3.12) that

$$(3.18) \quad E |h(U_1, U_2)|^l \leq \left( \int_0^1 |J'(s)| dF^{-1}(s) \right)^l.$$

Combining this with condition (2) of theorem 1 and using (3.16) and (3.17) we have proved statement (ii).

The proof of statement (iii) is essentially that of lemma 6.2 of BICKEL (1974). We use (3.18) and condition (2) of theorem 1 to guarantee the existence of some constant  $B > 0$  such that  $E|h(U_1, U_2)|^{2m} \leq B^{2m}$  (because in BICKEL (1974)  $h$  is bounded at the outset, BICKEL does not encounter this problem). This completes the proof of the lemma.  $\square$

We are now in a position to prove (3.3).

LEMMA 3.3. *Suppose the conditions (1) and (2) of theorem 1 are satisfied. Then  $\sigma^2(J, F) > 0$  implies (3.3).*

PROOF. The proof is essentially Bickel's proof. See BICKEL (1974) p.17 and 18. Remark first that it follows directly from lemma 3.2 and the conditions of this lemma that the statements (i), (ii) and (iii) of lemma 3.2 hold.

It follows from statement (iii) of lemma 3.2 that for  $|t| < \varepsilon_2 n^{\frac{1}{2}}$

$$\frac{t^{2m}}{(2m)!} E(J_{2n})^{2m} \leq \varepsilon_2^{2m} n^m (2m)^{-2m} e^{2m} A^{2m} n^{-m} m^{2m} \leq (2\varepsilon_2 A)^{2m}.$$

Following BICKEL (1974) we take  $\varepsilon_2 = p/(2A)$  for some  $0 < p < 1$  and  $m = \frac{[\log n]+1}{2|\log p|} \wedge n$  to obtain that

$$(3.22) \quad \frac{t^{2m}}{(2m)!} E(J_{2n})^{2m} \leq p^{2m} < n^{-1} \quad \text{for } |t| < \varepsilon_2 n^{\frac{1}{2}}.$$

Because  $\rho$  is the c.f. of a rv. with expectation zero and variance  $0 < \sigma^2(J, F) < \infty$  and  $\lim_{n \rightarrow \infty} n\sigma_n^2 = \sigma^2(J, F)$  (see the proof of lemma 3.1) there exist, for  $p$  sufficiently small, a  $\tau > 0$  such that for  $|t| < \varepsilon_2 n^{\frac{1}{2}}$

$$(3.23) \quad \log|\rho(t/n\sigma_n)| \leq -\frac{\tau t^2}{n}$$

From (3.10) with  $m = 1$ , and lemma 3.2 (i) and (iii) we have for all  $t$  and  $n \geq 1$

$$\begin{aligned}
|\rho_n^*(t) - \rho_{n1}(t)| &\leq |t| \cdot |E e^{itJ_{1n} J_{2n}}| + \frac{t^2}{2} E(J_{2n})^2 \leq \\
&\leq A|t|^{3n^{-\frac{1}{2}}} |\rho(t/n\sigma_n)|^{n-2} + A^2 t^2 n^{-1}.
\end{aligned}$$

Combining this with (3.23) we find that

$$(3.24) \quad \int_{|t| < n^{\frac{1}{4}}} |\rho_n^*(t) - \rho_{n1}(t)| \cdot |t|^{-1} dt = O(n^{-\frac{1}{2}}) \quad \text{as } n \rightarrow \infty.$$

We also have, using (3.23) and statement (ii) of lemma 3.2, that for  $n^{\frac{1}{4}} \leq |t| < \varepsilon_2 n^{\frac{1}{2}}$  and  $1 < 2m$

$$|E e^{itJ_{1n}(J_{2n})^1}| \leq A^1 n^{1/2} \exp(-\tau n^{\frac{1}{2}}(1-4m/n)).$$

But then we obtain for  $n^{\frac{1}{4}} \leq |t| < \varepsilon_2 n^{\frac{1}{2}}$

$$(3.25) \quad \left| \sum_{l=1}^{2m-1} \frac{(it)^l}{l!} E e^{itJ_{2n}(J_{1n})^l} \right| = O(n^{-1}) \quad \text{as } n \rightarrow \infty.$$

Now combine (3.22), (3.24) and (3.25) with (3.10). This completes the proof of the lemma.  $\square$

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