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AFDELING MATHEMATISCHE BESLISKUNDE (DEPARTMENT OF OPERATIONS RESEARCH) BW 168/82 AUGUSTUS

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Preprint

kruislaan 413 1098 SJ amsterdam

BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

Printed at the Mathematical Centre, 413 Kruislaan, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O.).

1980 Mathematics subject classification: 90B99, 90C10, 90C15, 60B10

Hierarchical Vehicle Routing Problems *)

by

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ABSTRACT

Hierarchical vehicle routing problems, in which the decision to acquire a number of vehicles has to be based on imperfect (probabilistic) information about the location of future customers, allow a natural formulation as two-stage stochastic programming problems, where the objective is to minimize the sum of the acquisition cost and the length of the longest route assigned to any vehicle. For several versions of this difficult optimization problem, we show that simple heuristics have strong properties of asymptotically optimal behaviour.

KEY WORDS & PHRASES: *Multi-stage stochastic programming, hierarchical vehicle routing, travelling salesman problem, heuristic, asymptotic optimality, probabilistic analysis, convergence in expectation, convergence almost surely*

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*) This report will be submitted for publication elsewhere.

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1. INTRODUCTION

Vehicle routing problems, in which *customers* have to be served from a central *depot* by one or more *vehicles*, are usually formulated and solved under the assumption that perfect information is available about the number of customers, their demands and their locations. In actual practice, this assumption is not always justified. In particular, the medium or long term planning problem of acquiring a suitable fleet of vehicles usually has to be solved with vague and at best probabilistic information about what will ultimately be required of these vehicles.

This problem is a good example of a *hierarchical multilevel planning problem*. These problems typically involve a sequence of decisions over time, taken at an increasing level of detail and with an increasing amount of information. In the above problem, two levels can be distinguished: the *aggregate* level corresponding to the decision to acquire a certain fleet of vehicles and the *detailed* level corresponding to the actual routing of the vehicles that have been acquired. When the latter problem arises, all the required information about the customers is available; the challenge of the problem as a whole is to incorporate the initially imperfect information about the detailed level into the overall procedure, so as to arrive at a sequence of decisions that is optimal or near-optimal. Typically, the cost of acquiring extra resources at the aggregate level has to be weighed against the possible benefit of having these resources available that materializes later at the detailed level.

In an earlier paper [1], it has been argued that the natural way to formulate hierarchical problems of this type is as *multi-stage stochastic integer programming problems*, of which the various phases correspond to the levels of the hierarchical problem, and of which some parameters of later stages are usually only known in probability. The natural objective function is then to set the decision variables in each phase in such a way that the overall decision is *optimal in expectation*.

At the same time, it was pointed out in [1] that the flexibility of the stochastic programming formulation comes at a price: there is little or no hope to solve such a model to optimality within a reasonable amount of time, if only for the fact that the problem at the detailed level is usually

NP-hard [1] for any realization of the parameters. Thus *stochastic programming heuristics* are called for, and indeed any method to solve these hierarchical planning problems (e.g., so-called *hierarchical planning systems*) can rightly be viewed as such a heuristic. The question then is if any rigorous statement can be made about the quality of these heuristics beyond the familiar evaluation of necessarily arbitrary computational experiments.

In another paper [2], it was established that such a rigorous analysis is indeed possible for certain hierarchical *scheduling* problems, where the initial, aggregate level decision corresponds to the acquisition of machines, based on probabilistic information on the jobs to be processed at the detailed level. It was shown that certain simple and natural heuristics for this problem produce *asymptotically optimal* decisions in some stochastic sense, e.g. *in expectation, in probability,* or *almost surely* (a.s.). Below, we derive similar results for a formulation of the *hierarchical vehicle routing problem.*

In this formulation, we assume that the cost of the aggregate level decision is proportional to the number of vehicles acquired and that the cost of the subsequent detailed level decision is proportional to the length of the longest route assigned to any of these vehicles. This criterion is convenient since it leads to a reasonable division of labour among the vehicles. We assume initially that at the aggregate level the number of customers is known precisely, but that only probabilistic information is available about their locations: we assume in fact that the customers are uniformly distributed over a circle with the depot at its center. In Section 2, we provide a stochastic programming formulation of the problem and we propose a heuristic method to solve the first phase (aggregate) problem that is based on an estimate of the second phase detailed cost inspired by work of BEARDWOOD et al. [3] and STEELE [4].

In Section 3, we describe a heuristic for the second phase (detailed) problem that is based on the partitioning heuristic for the travelling salesman problem developed by KARP [5]. In doing so, we extend the latter heuristic to a circular shaped area and more importantly, present a much simplified analysis of its behavior.

In Section 4, we prove that the stochastic programming heuristic developed in Sections 2 and 3 satisfies the strongest possible optimality property: in the terminology of [9], it is *asymptotically* ε-*clairvoyant almost surely*, indicating that almost surely the relative loss that can be attributed to imperfect information can be made arbitrarily small.

In Section 5, we indicate how to extend this result to the case that the aggregate level involves a choice between vehicles of different costs and speeds. We also show how to cope with the (more realistic) cases in which each customer places an order with some fixed probability p and in which the number of customers is initially also known in probability only.

Finally, Section 6 contains some conclusions and topics for future research.

2. THE AGGREGATE LEVEL

For a precise formulation of the *hierarchical vehicle routing problem*, we assume that at the aggregate level a decision has to be made about the number k of *vehicles* that have to be acquired at cost c each, to serve n customers from a single *depot*. This decision has to be made when the exact location of these customers is not yet given: we assume that all that is known is that these will be *uniformly distributed* over a *circle* C with radius r and with the depot at its center.

Subsequently, at the detailed level, the k vehicles that have been acquired have to be routed from the depot through the n customers, a realization of whose locations is now given. If $V_i(k)$ is the *route* assigned to the i-th vehicle and $|V_i(k)|$ its *length*, then the objective at the detailed level will be to minimize the length of the *maximum route* U(k), i.e., to minimize

$$(1) \qquad |U(k)| = \max_{i=1,\dots,k} \{|V_i(k)|\}.$$

The minimal value of $|U(k)|$ and the tours in the corresponding solution will be indicated by $|U^0(k)|$ and $V_i^0(k)$ $(i = 1,\dots,k)$ respectively.

When the problem is viewed as a whole, then the solution value at the detailed level has to be interpreted as a *random variable* (to be underlined) and hence the overall objective function is a random variable as well:

$$(2) \qquad \underline{Z}(k) = ck + |\underline{U}^0(k)|.$$

As observed in Section 1, the deterministic version of the routing problem at the detailed level is already NP-hard, and a heuristic approach is required.

Our heuristic for the aggregate problem will be based on a deterministic approximation of the objective function (2) that is almost surely a lower bound. Let T^0 be an optimal *travelling salesman tour* through all the n customers, i.e. a tour whose length $|T^0|$ is minimal. Clearly,

$$(3) \qquad |U^0(k)| \geq \frac{1}{k} |T^0|$$

and we can now apply the following theorem due to STEELE [4], which extends earlier work by BEARDWOOD et al. [3].

<u>THEOREM 1</u>. *If* n *customers are distributed uniformly over a circle* c *with radius* r*, then there exists a constant* β > 0 *such that*

$$(4) \qquad \lim_{n\to\infty} \frac{|T^0|}{\sqrt{n\pi r^2}} = \beta \qquad \text{(a.s.)} \quad \square$$

It follows that, if n is large enough, the function

$$(5) \qquad Z^{LB}(k) = ck + \frac{1}{k} \beta \sqrt{n\pi r^2}$$

is almost surely a lower bound on $\underline{Z}(k)$. As a heuristic decision at the aggregate level, we now choose the number of vehicles equal to the value k^{LB} that minimizes this lower bound. Differentiation of (5) yields

$$(6) \qquad c - \frac{1}{k^2} \beta \sqrt{n\pi r^2} = 0$$

so that, if we define

$$(7) \qquad \alpha = \left(\frac{\beta\sqrt{\pi r^2}}{c}\right)^{1/2} ,$$

k^{LB} is chosen to be equal to $\lfloor \alpha n^{1/4} \rfloor$ or $\lceil \alpha n^{1/4} \rceil$, depending on which of these two integer values is the most favorable one.

3. THE DETAILED LEVEL

At the detailed level of the hierarchical vehicle routing problem, we have to route the k^{LB} vehicles that were acquired through the n customers so as to minimize the maximum route length assigned to a vehicle.

We propose to solve this problem heuristically by means of a *partitioning heuristic* that is similar in spirit to KARP's heuristic [5] for the Euclidean travelling salesman problem. In the first step of this heuristic, C is partitioned into smaller subregions, each of which contains no more than t customers for some constant t that is yet to be determined. In the second step, an optimal travelling salesman tour is constructed in each of these subregions. In the third and final step, these tours are combined in a suitable manner to form the routes $V_i^P(k^{LB})$ $(i=1,\ldots,k^{LB})$.

The partitioning of C in the first step is carried out by means of *cuts*, of which we distinguish two types. Assume that the location of each customer is represented by its polar coordinates. A *radial cut* of a region by definition splits up the region by means of the radius through the customer in the region with median angular coordinate. Similarly, a *circular cut* splits up a region by means of the circle arc (with the depot as center) through the customer in the region with median radial coordinate.

In a *round* of cutting, each subregion existing at the beginning of the round is split up exactly one. We carry out d of these rounds, with

$$(8) \qquad d = \left\lceil {}^{2}\log \frac{n-1}{t-1} \right\rceil .$$

The first d/2 rounds involve only radial cuts, thus creating $2^{d/2}$ *sectors*; the last d/2 rounds involve only circular cuts. This cutting procedure is simpler than the one proposed in [5]; it is easy to see that it results in subregions containing no more than t customers each.

We number the 2^d subregions by starting with an arbitrary sector, numbering the subregions according to increasing distance from the depot, and continuing on the adjacent sector in, say, clockwise direction until all subregions have been numbered. The j-th region will be denoted by $Y_j (j=1,\dots,2^d)$ (Figure 1). In each region Y_j, an optimal travelling salesman tour $T^0(Y_j)$ through its customers (including those on the boundary) is now formed by means of some suitable optimization method. The union of these tours defines a *spanning walk* W, i.e. a connected network in which each node has even degree, on the set of all customers. It is well known that there exists a route through W of length $|W| = \Sigma_j |T^0(Y_j)|$ in which each edge is visited exactly once. It is also easy to see (cf.[5]) that such a route can be transformed into an ordinary travelling salesman tour of no greater length.

We proceed to assign each customer to a specific vehicle, in such a way that the routelength for each vehicle is approximately equal to $|W|/k^{LB}$. We do so in the obvious manner, by considering $Y_1, Y_2, Y_3, \dots$ until we find the greatest ℓ such that

(9) $$\delta = |W|/\, k^{LB} - \textstyle\sum_{j=1}^{\ell} |T^0(Y_j)| \geq 0 .$$

If $\delta > 0$, we divide $T^0(y_{\ell+1})$ into two parts. The customers on the first part, which has length δ, together with the customers on tours $T^0(y_1),\dots,T^0(y_\ell)$ are assigned to the first vehicle. The customers on the second part are the first ones to be assigned to the second vehicle. We continue this procedure until each vehicle has a set of tours (including at most two partial tours) assigned to it whose lengths sum exactly to $|W|/k^{LB}$.

The union of these tours does not necessarily define a spanning walk. It will generally have the form depicted in Figure 2 in heavy lines. As indicated in the figure, at most eight additional dotted segments may be necessary to create a spanning walk. Two additional segments undicated by +++ are needed to connect the depot to the customer that is closest to it. It is easy to see that the total length of these additional segments is bounded by a constant γ depending only on r. The resulting spanning walk is transformed into a tour in the standard manner [5]. The longest of the resulting routes $V_i^P(k^{LB})$ has length $|U^P(k^{LB})|$; this is the value produced by the heuristic.

It is not difficult to see that, subject to the usual assumption that each elementary operation on real numbers requires one time unit, the above heuristic can be implemented to require a running time that is polynomial in the number of customers.

In the first step,all customers have to be sorted with respect to their angular as well as to their radial coordinates. In addition, each round of cutting takes linear time. Altogether, this step requires $O(n \log n + nd) = O(n \log n)$ time.

The second step, calculation of the optimal travelling salesman tours in each subregion, can be carried out in $O(\theta^t)$ time per region, for some constant $\theta > 2$ and hence in $O(n\theta^t/t)$ time overall.

In the third step, the assignment of each customer to a vehicle takes time that is linear in the number of subregions and in the number of vehicles. This includes the time needed to create the extra segments, which is proportional to t^2k^{LB}.

It follows that the overall running time is $O(n \log n + n\theta^t/t + t^2n^{1/4})$ which is polynomial in n for any fixed choice of t.

4. ANALYSIS OF THE HEURISTIC

The value produced by the stochastic programming heuristic described in Sections 2 and 3 is the random variable

$$Z^H = ck^{LB} + |U^P(k^{LB})|. \tag{10}$$

Our analysis of this value starts from an error analysis of the detailed level partitioning heuristic.

Consider a subregion Y_j, and let $T^0 \cap Y_j$ denote the intersection of the optimal tour through all n customers with Y_j. $T^0 \cap Y_j$ may consist of various segments; their total length is denoted by $|T^0 \cap Y_j|$. Let $per(Y_j)$ be the length of the parameter of Y_j. The following lemma is proved in [5].

LEMMA 1.

$$|T^0(Y_j)| - |T^0 \cap Y_j| \le \frac{3}{2} \; per(Y_j) \quad \Box \tag{11}$$

It follows that

$$\sum_{j=1}^{2^d} |T^0(Y_j)| \le \sum_{j=1}^{2^d} |T^0 \cap Y_j| + \frac{3}{2} \sum_{j=1}^{2^d} per(Y_j), \tag{12}$$

i.e.

(13) $$|W| \le |T^0| + \frac{3}{2} \sum_{j=1}^{2^d} \text{per}(Y_j).$$

Our cutting procedure, which is different from the one in [5], yields an upper bound on $\Sigma_j \text{ per}(Y_j)$.

LEMMA 2.

(14) $$\sum_{j=1}^{2^d} \text{per}(Y_j) = O(\sqrt{n/t}).$$

PROOF. After d/2 radial cuts, the sum of the parameters of the sectors is clearly equal to

(15) $$2^{d/2}(2r) + 2\pi r.$$

In the first round of circular cuts, all sectors are split by circle arcs, the sum of which is certainly smaller than $2\pi r$, so that (15) is increased by no more than $4\pi r$. In the second round, the increase is bounded in a similar manner by $8\pi r$. Hence, the overall increase is bounded by

(16) $$(2^{d/2}-1)\, 4\pi r.$$

Since $d = \lceil \log^2((n-1)/(t-1)) \rceil$, (15) and (16) together imply (14). □

It follows from (13) and (14) that the route length for each vehicle, and hence $|U^P(K^{LB})|$, is bounded from above by

(17) $$\frac{|W|}{k^{LB}} + \gamma \le \frac{|T^0|}{k^{LB}} + \frac{O(\sqrt{n/t})}{k^{LB}}$$

and hence

(18) $$\underline{Z}^H \le ck^{LB} + \frac{1}{k^{LB}} |\underline{T}^0| + \frac{O(\sqrt{n/t})}{k^{LB}}.$$

We note that a comparable detailed level heuristic, in which the customers are first divided among the vehicles and routes are formed only afterwards, would be much harder to analyse: either the shape of a subregion or its number of customers would be random, and Theorem 1 could not be ap-

plied to each individual route.
We wish to compare the upper bound (18) with a lower bound $\underline{Z}^D$, the value of the *distribution problem,* which is found by defining the random variable $\underline{k}^D$ as the number of vehicles minimizing (2) as a function of each realization of customer locations and setting

$$(19) \qquad \underline{Z}^D = \underline{Z}(\underline{k}^D).$$

This random variable represents the minimum cost achievable with perfect foresight into the customer locations, that in our formulation of the problem of course become known only after acquisition of the vehicles. Note, however, that since

$$(20) \qquad \underline{Z}(k) \geq Z^{LB}(k) = ck + \frac{1}{k}\beta\sqrt{n\pi r^2} \qquad \text{(a.s.)}$$

(cf.(5)), we find that

$$(21) \qquad \underline{Z}^D \geq \min_k Z^{LB}(k) = Z^{LB}(k^{LB}) \qquad \text{(a.s.)}$$

Hence, combining (18), (21) and Theorem 1, we obtain the following result.

THEOREM 2.

$$(22) \qquad \frac{\underline{Z}^H}{\underline{Z}^D} \leq 1 + O(\frac{1}{\sqrt{t}}) \qquad \text{(a.s.)} \qquad \square$$

In the terminology of [9], the stochastic programming heuristic is *asymptotically ε-clairvoyant almost surely:* the error that can be ascribed to the lack of perfect information at the aggregate level and to the use of a heuristic (suboptimal) method at the detailed level can almost surely be made arbitrarily small through an appropriate choice of t. This is the strongest possible asymptotic optimality result that can be found for such heuristics. In particular, it implies [6] that if k^0 is defined to be the value that minimizes the standard stochastic programming objective function $E\underline{Z}(k) = ck + E|\underline{U}^0(k)|$, then

$$(23) \qquad \lim_{n \to \infty} E\left(\frac{\underline{Z}^H}{\underline{Z}(k^0)}\right) = 1 + O(\sqrt{\frac{1}{t}}).$$

It is not difficult to extend this so as to prove that

$$(24)\qquad \lim_{n\to\infty} \frac{EZ^H}{E\underline{Z}(k^0)} = 1 + O(\sqrt{\tfrac{1}{t}})$$

and thus the heuristic is also *asymptotically ε-optimal in expectation.*

Not surprisingly, the lower bound function almost surely provides a good asymptotic description of $\underline{Z}(k)$.

THEOREM 3. *For every* k

$$(25)\qquad \lim_{n\to\infty} \frac{\underline{Z}(k)}{ck+\frac{1}{k}\beta\sqrt{n\pi r^2}} = 1 \qquad \text{(a.s.)}$$

PROOF. Clearly, as in (17) and (18), the heuristic implies that

$$(26)\qquad 1 \le \frac{\underline{Z}(k)}{ck+\frac{1}{k}\beta\sqrt{n\pi r^2}} \le \frac{ck+\frac{1}{k}|\underline{T}^0| + O(\frac{1}{k}\sqrt{\frac{n}{t}})}{ck+\frac{1}{k}\beta\sqrt{n\pi r^2}} \qquad \text{(a.s.)}$$

Noting that this inequality holds for any choice of t, the theorem is now an immediate consequence of Theorem 1. □

Theorem 3 provides an almost surely asymptotically exact deterministic approximation of the objective function $\underline{Z}(k)$.

Next, it is also easy to prove that not only the value of the heuristic solution but also the solution at the aggregate level itself almost surely converges to the optimal one.

THEOREM 4.

$$(27)\qquad \lim_{n\to\infty} \frac{\underline{k}^D}{k^{LB}} = 1 \qquad \text{(a.s.)}$$

PROOF. Suppose that there exists an $\varepsilon > 0$ such that for each n_0 there is an $n \ge n_0$ with

$$(28)\qquad \underline{k}^D > (1+\varepsilon)\, k^{LB} \qquad \text{(a.s.)}$$

Since $Z^{LB}(k)$ is a unimodal function of k, this would imply that

(29) $$\underline{Z}^D > c\ (1+\varepsilon)k^{LB} + \frac{1}{(1+\varepsilon)k^{LB}}\ \beta\sqrt{n\pi r^2} \qquad \text{(a.s.)}$$

i.e., for n sufficiently large,

(30) $$\underline{Z}^D > (1+\varepsilon+\frac{1}{1+\varepsilon})(\beta c\sqrt{n\pi r^2})^{1/2} \qquad \text{(a.s.)}$$

and hence from Theorem 3, for some $\varepsilon' > 0$,

(31) $$\underline{Z}^D > (1+\varepsilon')\ \underline{Z}(k^{LB}) \qquad \text{(a.s.)}$$

which contradicts the definition of $\underline{Z}^D$. Thus, there is an n_0 such that for all $n > n_0$

(32) $$\underline{k}^D \leq (1+\varepsilon)k^{LB} \qquad \text{(a.s.)}$$

We prove similarly that

(33) $$\underline{k}^D \geq (1-\varepsilon)k^{LB} \qquad \text{(a.s.)}$$

which establishes the desired result. □

We finally obtain the following analogue of Theorem 1.

<u>THEOREM 5</u>.

(34) $$\lim_{n\to\infty} \frac{\underline{Z}^D}{2(\beta c\sqrt{n\pi r^2})^{1/2}} = 1 \qquad \text{(a.s.)}$$

<u>PROOF</u>. Immediate from Theorem 3 and 4. □

5. EXTENSIONS OF THE MODEL

In this section, we consider three natural extensions of the hierarchical vehicle routing model introduced in Section 2.

We first consider the case that, at the aggregate level the problem is to select a subset K of vehicles from a set $\mathcal{K}$, where the i-th vehicle now has a specific *cost* c_i and a *speed* s_i. At the detailed level, the problem is to form routes $V_i(K)$ ($i \in K$) for the i-th vehicle, so as to minimize the maximum time to traverse any route

(35) $$|U(K)| = \max_{i \in K} \{|V_i(K)|/s_i\}.$$

If $|U^0(K)|$ denotes the minimum value of (35), the overall objective function is given by

(36) $$\underline{W}(K) = \sum_{i \in K} c_i + |\underline{U}^0(K)|.$$

If we assume that there exist constants c^L, c^U, s^L and s^U, such that $c^L \le c_i \le c^u$ and $s^L \le s_i \le s^U$ for all $i \in K$, it turns out that this extension can be analyzed in the same fashion as the extension from *identical* to *uniform* machines in the case of hierarchical machine scheduling models [2], although the final result here is stronger.

Proceeding as we did in Section 2, we first define $c(K) = \Sigma_{i \in K} c_i$ and $s(K) = \Sigma_{i \in K} s_i$ and observe that the following function is an obvious lower bound (a.s.) on $\underline{W}(K)$:

(37) $$W^{LB}(K) = c(K) + \frac{1}{s(K)} \beta\sqrt{n\pi r^2}.$$

As in [2], it is easy to see that finding a subset K^{LB} that minimizes (37) over all choices $K \subset \mathcal{K}$ is an NP-hard problem. Hence, a *greedy heuristic* is applied to solve the problem at the aggregate level: vehicles are selected in order of nondecreasing c_i/s_i ratio until the lower bound function starts to increase. The same arguments as in [2] will yield that the subset K^G selected in this way satisfies

(38) $$W^{LB}(K^G) \le W^{LB}(K^{LB}) + c^U$$

and hence the absolute error caused by using the greedy heuristic is bounded by a constant.

In the second phase, at the detailed level, we first apply the same heuristic as described in Section 3 to construct a spanning walk W through the customers.

Rather than cutting the walk into pieces of equal length, we allocate a part of length $s_i |W| / s(K^G)$ to the i-th vehicle and transform this part into a route in the manner described in Section 3 as well.

It is easily verified that the value of this stochastic programming heuristic $\underline{W}^H$ is related to the value of the corresponding distribution problem $\underline{W}^D$ by

$$(39) \qquad \frac{\underline{W}^H}{\underline{W}^D} \le 1 + \frac{1}{c(K^G) + \beta\sqrt{n\pi r^2}/s(K^G) - c^U} O(\sqrt{n/t}) \qquad \text{(a.s.)}$$

Analogously to Lemma 6 in [2], we can prove that $c(K^G) = \Theta(n^{1/4})$ and $s(K^G) = \Theta(n^{1/4})$, to conclude that

$$(40) \qquad \frac{\underline{W}^H}{\underline{W}^D} \le 1 + O(\frac{1}{\sqrt{t}}) \qquad \text{(a.s.)}$$

so that the extended heuristic is also asymptotically ε-clairvoyant almost surely.

The two other extensions of the original model that are dealt with in this section allow for additional uncertainty about the detailed level, when the decision to acquire vehicles has to be taken. We first consider the situation in which it is no longer certain that each of the n customers has to be visited; rather, each customer places an order with some fixed probability p. Subsequently, we consider the more difficult case in which the number of customers is itself a random variable.

If each customer orders with fixed probability p, the number of customers to be visited is a random variable $\underline{m}$, distributed according to a *binomial* distribution with parameters n and p.

To bound $\underline{m}$ from above and below, we apply Chernoff's *inequalities* [7] according to which, for all $\varepsilon > 0$,

$$(41) \qquad \sum_{i=0}^{(1-\varepsilon)np} \binom{n}{i} p^i (1-p)^{n-i} < \exp(-\varepsilon^2 np/2),$$

$$(42) \qquad \sum_{i=(1+\varepsilon)np}^{n} \binom{n}{i} p^i (1-p)^{n-i} < \exp(-\varepsilon^2 np/3).$$

Applying the Borel-Cantelli lemma [6], we obtain that

$$(1-\varepsilon)np \le \underline{m} \le (1+\varepsilon)np \qquad (a.s.) \tag{43}$$

As suggested by (43), we obtain a stochastic programming heuristic for this model by choosing the number of vehicles at the aggregate level equal to the most favorable integer approximation $k^{H(p)}$ of

$$\left(\frac{\beta\sqrt{p\pi r^2}}{c} \right)^{1/2} n^{1/4}. \tag{44}$$

Note that - not surprisingly - $k^{H(p)} \to k^{LB}$ as $p \to 1$.
The detailed level heuristic remains the same one as described in Section 3.

We now use the lower and upper bounds implied by (43) in conjunction with Theorem 1 to analyze the quality of this heuristic H(p):

$$\underline{Z}^{H(p)} \le ck^{H(p)} + \frac{1}{k^{H(p)}} \beta\sqrt{(1+\varepsilon)np\pi r^2} + \frac{1}{k^{H(p)}} O\left(\sqrt{\frac{(1+\varepsilon)np}{t}}\right) \qquad (a.s.) \tag{45}$$

In a similar fashion, we derive for the value $\underline{Z}^{D(p)}$ of the distribution problem that

$$\underline{Z}^{D(p)} \ge 2\sqrt{1-\varepsilon}\, (c\beta \sqrt{np\pi r^2})^{1/2} \qquad (a.s.) \tag{46}$$

We conclude that

$$\frac{\underline{Z}^{H(p)}}{\underline{Z}^{D(p)}} \le \frac{1+\sqrt{1+\varepsilon}}{2\sqrt{1-\varepsilon}} + O\left(\frac{1}{\sqrt{t}}\right) \qquad (a.s.) \tag{47}$$

By appropriate choices of ε and t, the right hand side of (47) can be made arbitrarily close to 1, and once again, the heuristic is asymptotically ε-clairvoyant almost surely.

The final extension, in which the number of customers is a random variable $\underline{n}$, is more complicated. We assume that $\underline{n}$ has mean μ and variance σ^2. We shall prove that for μ sufficiently large and σ^2 fixed, we can obtain a heuristic H' that is asymptotically optimal in expectation. This heuristic is based on selecting the number $k^{H'}$ of vehicles at the aggregate level to be equal to the most favorable integer approximation of

$$(48)\qquad \left(\frac{\beta\, E(\sqrt{\underline{n}})\sqrt{\pi r^2}}{c}\right)^{1/2},$$

the natural generalization of k^{LB}. The analysis of this heuristic is based on appropriate lower and upper bounds on $E(|\underline{T}^0|)$ for this model.

We first observe that for every $\varepsilon > 0$, we can find $n(\varepsilon)$ such that for $n > n(\varepsilon)$ the conditional expectation of the optimal tour length satisfies

$$(49)\qquad \left|E(|\underline{T}^0| \mid \underline{n}=n) - \beta\sqrt{n\pi r^2}\right| \le \varepsilon \qquad \text{(a.s.)}$$

Hence,

$$\begin{aligned}
E(|\underline{T}^0|) &= \sum_{n=1}^{\infty} E(|\underline{T}^0| \mid \underline{n}=n)\Pr\{\underline{n}=n\} \\
&= \sum_{n=1}^{n(\varepsilon)} E(|\underline{T}^0| \mid \underline{n}=n)\Pr\{\underline{n}=n\} + \\
&\quad + \sum_{n=n(\varepsilon)+1}^{\infty} E(|\underline{T}^0| \mid \underline{n}=n)\Pr\{\underline{n}=n\} \\
(50)\qquad &\ge \sum_{n=1}^{n(\varepsilon)} E(|\underline{T}^0| \mid \underline{n}=n)\Pr\{\underline{n}=n\} + E(\sqrt{\underline{n}})\beta\sqrt{\pi r^2} \\
&\quad - \sum_{n=1}^{n(\varepsilon)} \beta\sqrt{n\pi r^2}\,\Pr\{\underline{n}=n\} - \sum_{n=n(\varepsilon)}^{\infty} \Pr\{\underline{n}=n\} \\
&\ge \sum_{n=1}^{n(\varepsilon)} \left[E(|\underline{T}^0| \mid \underline{n}=n) - \beta\sqrt{n\pi r^2}\right]\Pr\{\underline{n}=n\} + \\
&\quad + E(\sqrt{\underline{n}})\beta\sqrt{\pi r^2} - \varepsilon .
\end{aligned}$$

The first term of (50) can be bounded from below by

$$\begin{aligned}
(51)\qquad &\beta\sqrt{n(\varepsilon)\pi r^2}\,\Pr\{\underline{n} = n(\varepsilon)\} - \varepsilon - \sum_{n=1}^{n(\varepsilon)} \beta\sqrt{n\pi r^2}\,\Pr\{\underline{n} = n\} \\
&\ge -\beta\sqrt{n(\varepsilon)\pi r^2}\,\Pr\{\underline{n} < n(\varepsilon)\} - \varepsilon .
\end{aligned}$$

We can choose $\mu(\varepsilon)$ in such a way that for $\mu > \mu(\varepsilon)$

$$(52)\qquad \Pr\{n < n(\varepsilon)\} < \frac{\varepsilon}{\beta\sqrt{n(\varepsilon)\pi r^2}} .$$

More specifically, if we choose

$$(53)\qquad \mu(\varepsilon) = n(\varepsilon) + \left(\frac{\beta}{\varepsilon}\sqrt{n(\varepsilon)\pi r^2}\right)^{1/2}\sigma$$

then Chebyshev's inequality implies that (52) is satisfied.

Together, (50), (51) and (52) imply that for μ sufficiently large

$$(54)\qquad E(|\underline{T}^0|) \geq E(\sqrt{\underline{n}})\beta\sqrt{\pi r^2} - 3\varepsilon.$$

In a similar fashion we can also prove that

$$(55)\qquad E(|\underline{T}^0|) \leq E(\sqrt{\underline{n}})\beta\sqrt{\pi r^2} + 3\varepsilon.$$

Based on (54) and (55), it is once again easy to prove that the error produced by the stochastic programming heuristic H' can be made arbitrarily small in expectation by suitable choices for ε and t.

6. CONCLUDING REMARKS

In the previous sections we have seen that simple and natural heuristics have very strong asymptotic optimality properties in solving various difficult hierarchical vehicle routing problems.

It is interesting to observe that the routes produced by the heuristic at the detailed level are very similar in structure to those produced by the well known *sweep heuristic* [8]. If it is important to divide the workload evenly among the vehicles, as in the case of a *minmax* objective, such a heuristic is attractive. If the objective would be to minimize the sum of all distances, this would not necessarily be the case.

In the model as formulated in Section 2, it would not be fruitful to consider this modified second phase objective function: there is no incentive to use more than one vehicle. Such a *minsum* objective is only relevant if *capacity constraints* are imposed on every single vehicle. These constraints may refer to the maximum distance or time to be allowed for any vehicle, or to the maximum number of customers or more generally, the maximum delivery load that a vehicle can be assigned to.

We have been able to analyze the latter model in more detail; it turns out to be closely related asymptotically to certain familiar plant location problems. These results will be the subject of future publications.

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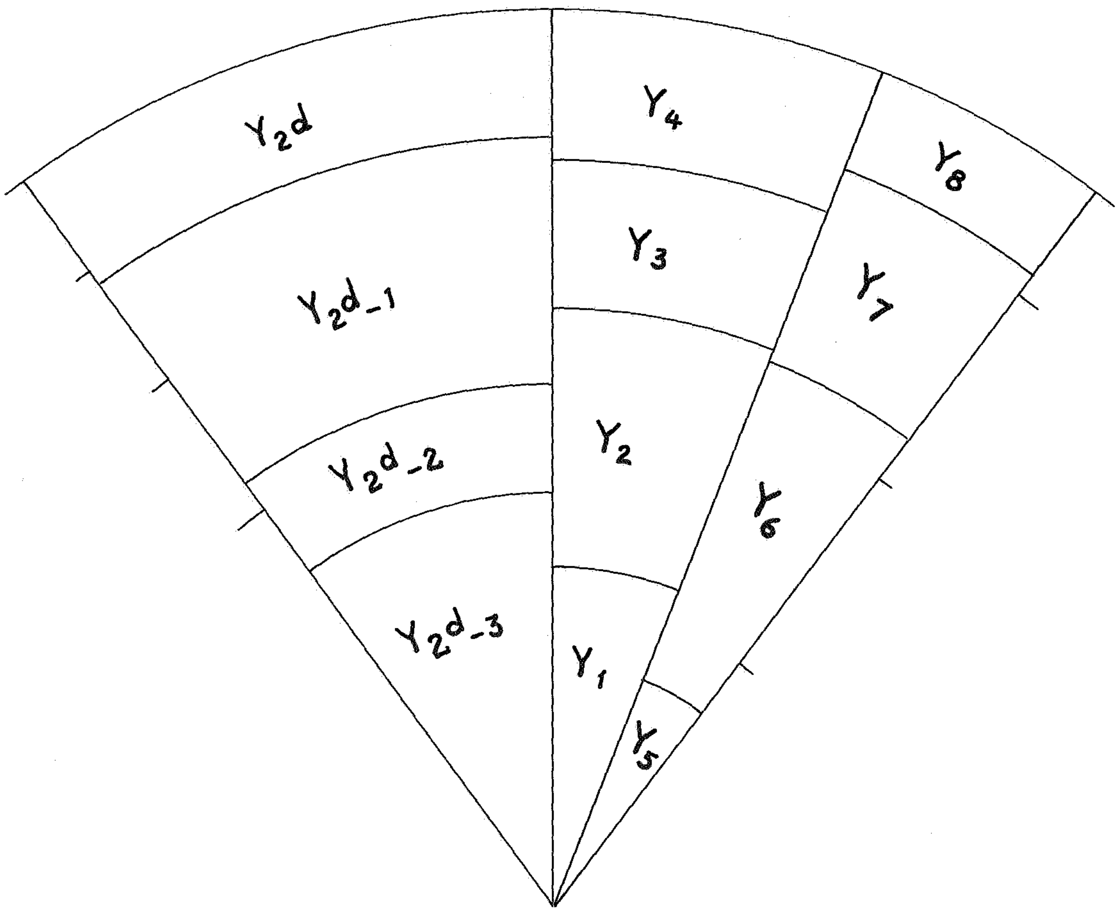

fig. 1

The subregions

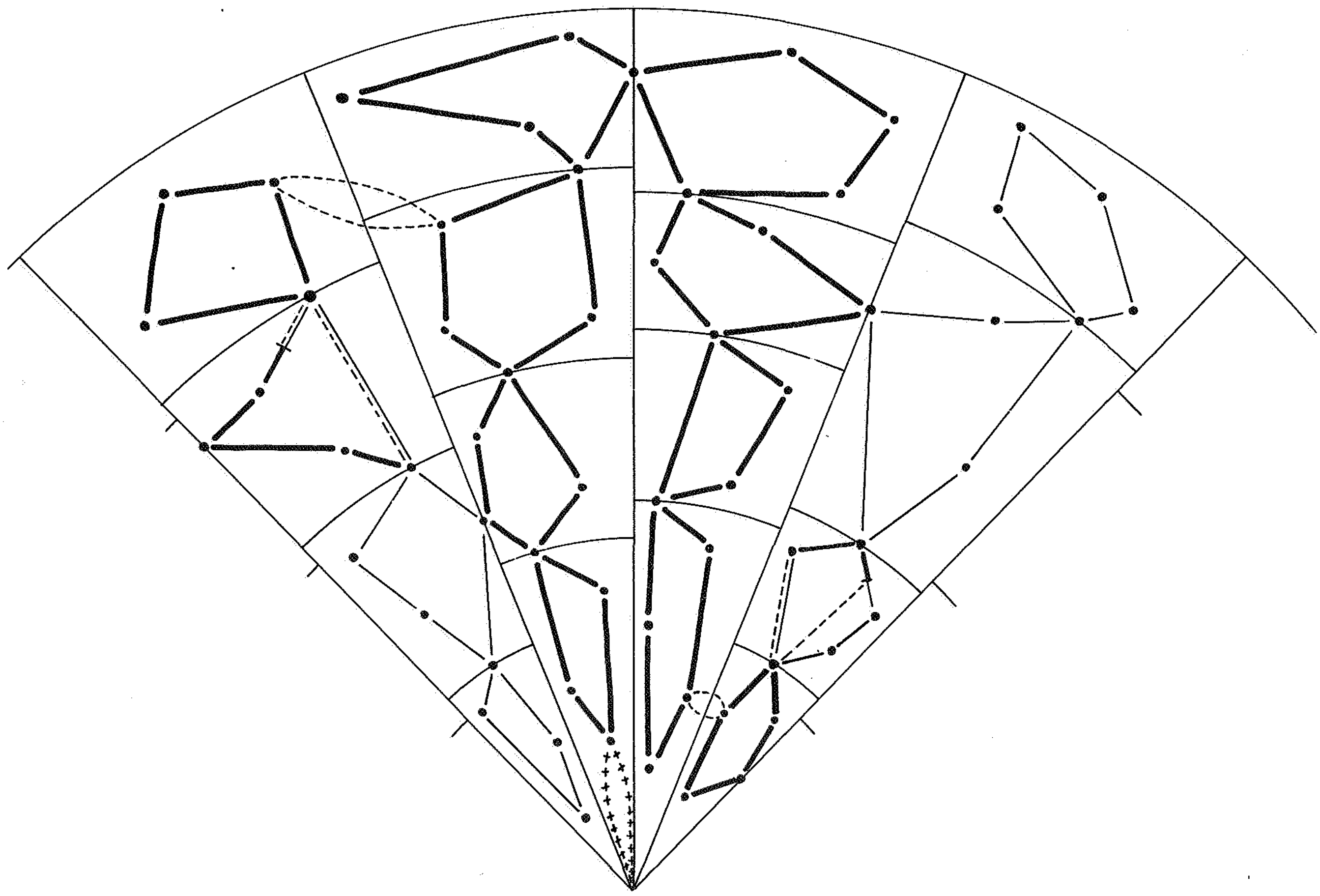

fig. 2

From a set of tours to a spanning walk

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