Group-theoretic generalisations of vertex and edge connectivities

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Abstract

Let p be an odd prime. Let P be a finite p-group of class 2 and exponent p, whose commutator quotient $P/[P,P]$ is of order p^n. We define two parameters for P related to central decompositions. The first parameter, $\kappa(P)$, is the smallest integer s for the existence of a subgroup S of P satisfying (1) $S \cap [P,P] = [S,S]$, (2) $|S/[S,S]| = p^{n-s}$, and (3) S admits a non-trivial central decomposition. The second parameter, $\lambda(P)$, is the smallest integer s for the existence of a central subgroup N of order p^s, such that P/N admits a non-trivial central decomposition.

While defined in purely group-theoretic terms, these two parameters generalise respectively the vertex and edge connectivities of graphs: For a simple undirected graph G, through the classical procedures of Baer (Trans. Am. Math. Soc., 1938), Tutte (J. Lond. Math. Soc., 1947) and Lovász (B. Braz. Math. Soc., 1989), there is a p-group P_G of class 2 and exponent p that is naturally associated with G. Our main results show that the vertex connectivity $\kappa(G)$ is equal to $\kappa(P_G)$, and the edge connectivity $\lambda(G)$ is equal to $\lambda(P_G)$. We also discuss the relation between $\kappa(P)$ and $\lambda(P)$ for a general p-group P of class 2 and exponent p, as well as the computational aspects of these parameters.

Keywords: p-groups of class 2, graph connectivity, matrix spaces, bilinear maps

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1 Introduction

The main purpose of this note is to define and explore two natural group-theoretic parameters, which are closely related to vertex and edge connectivities in graphs.

In this introduction, we first introduce the classical procedures of Baer [Bae38], Tutte [Tut47], and Lovász [Lov89] which relate graphs with p-groups of class 2 and exponent p. We then define two group-theoretic parameters. Our main result shows that the vertex and edge connectivities of a graph are equal to the two parameters we defined on the corresponding group respectively. We then compare the two parameters and discuss on their computational aspects.

Since the main goal of this note is to set up a link between graph theory and group theory, we shall include certain background information, despite that it is well-known to researchers in the respective areas.

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1.1 From graphs to groups: the Baer-Lovász-Tutte procedure

The route from graphs to groups, following Baer [Bae38], Tutte [Tut47], and Lovász [Lov89], goes via linear spaces of alternating matrices and alternating bilinear maps.

We set up some notation. For $n \in \mathbb{N}$, let $[n] := \{1, \ldots, n\}$. Let $\binom{[n]}{2}$ be the set of size-2 subsets of $[n]$. We use $\mathbb{F}$ to denote a field, and $\mathbb{F}_q$ to denote the finite field with q elements. Vectors in $\mathbb{F}^n$ are column vectors, and $\langle \cdot \rangle$ denotes the linear span over underlying field $\mathbb{F}$. Let $\Lambda(n, \mathbb{F})$ be the linear space of $n \times n$ alternating matrices over $\mathbb{F}$. Recall that an $n \times n$ matrix A over $\mathbb{F}$ is *alternating* if for any $v \in \mathbb{F}^n$, $v^t A v = 0$. That is, A represents an alternating bilinear form. Subspaces $\mathcal{A}$ of $\Lambda(n, \mathbb{F})$, denoted by $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, are called alternating matrix spaces. Fix a field $\mathbb{F}$. For $\{i, j\} \in \binom{[n]}{2}$ with $i < j$, the *elementary* alternating matrix $A_{i,j}$ over $\mathbb{F}$ is the matrix with the (i, j)th entry being 1, the (j, i)th entry being -1, and the rest entries being 0.

In this note, we only consider non-empty, simple, and undirected graphs with the vertex set being $[n]$. That is, a graph is $G = ([n], E)$ where $E \subseteq \binom{[n]}{2}$. Let $|E| = m$. Note that the non-empty condition implies that $n \geq 2$ and $m \geq 1$.

Let p be an odd prime. We use $\mathfrak{B}_{p,2}$ to denote the class of *non-abelian* p-groups of class 2 and exponent p. That is, a non-abelian group P is in $\mathfrak{B}_{p,2}$, if for any $g \in P$, $g^p = 1$, and the commutator subgroup $[P, P]$ is contained in the centre $\mathrm{Z}(P)$. For $n, m \in \mathbb{N}$, we further define $\mathfrak{B}_{p,2}(n, m) \subseteq \mathfrak{B}_{p,2}$, which consists of those $P \in \mathfrak{B}_{p,2}$ with $|P/[P, P]| = p^n$ and $|[P, P]| = p^m$. Note that the non-abelian condition implies that $n \geq 2$ and $m \geq 1$ are required for $\mathfrak{B}_{p,2}(n, m)$ to be non-empty.

We then explain the procedure from graphs to groups in $\mathfrak{B}_{p,2}$ following Baer, Tutte and Lovász.

1. Let $G = ([n], E)$ be a simple and undirected graph with m edges. Following Tutte [Tut47] and Lovász [Lov89], we construct from G an m-dimensional alternating matrix space in $\Lambda(n, \mathbb{F})$ by setting
$$\mathcal{A}_G = \langle A_{i,j} : \{i, j\} \in E \rangle. \tag{1}$$

2. Given an m-dimensional $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, let $\mathbf{A} = (A_1, \ldots, A_m) \in \Lambda(n, \mathbb{F})^m$ be an ordered basis of $\mathcal{A}$. The alternating bilinear map defined by $\mathbf{A}$, $\phi_{\mathbf{A}} : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$, is
$$\phi_{\mathbf{A}}(v, u) = (v^t A_1 u, \ldots, v^t A_m u)^t. \tag{2}$$
Since $\mathcal{A}$ is of dimension m, we have that $\phi_{\mathbf{A}}(\mathbb{F}^n, \mathbb{F}^n) = \mathbb{F}^m$.

3. Let p be an odd prime. Let $\phi : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p^m$ be an alternating bilinear map, such that $\phi(\mathbb{F}_p^n, \mathbb{F}_p^n) = \mathbb{F}_p^m$. Following Baer [Bae38], we define a p-group, $P_\phi \in \mathfrak{B}_{p,2}(n, m)$, as follows. The group elements are from $\mathbb{F}_p^n \oplus \mathbb{F}_p^m$. For $(v_i, u_i) \in \mathbb{F}_p^n \oplus \mathbb{F}_p^m$, $i = 1, 2$, the group product $\circ$ is defined as
$$(v_1, u_1) \circ (v_2, u_2) := (v_1 + v_2, u_1 + u_2 + \frac{1}{2} \cdot \phi(v_1, v_2)). \tag{3}$$
It can be verified that $P_\phi \in \mathfrak{B}_{p,2}(n, m)$, because of the condition that $\phi(\mathbb{F}_p^n, \mathbb{F}_p^n) = \mathbb{F}_p^m$.

Starting from a graph G, we follow the above three steps to obtain a p-group of class 2 and exponent p, denoted by P_G. It can be verified easily that this process preserves isomorphism types, despite that the procedure from alternating matrix sapces to alternating bilinear maps depends on choices of ordered bases; see Remark 3. That is, if the graphs G_1 and G_2 are isomorphic, then the corresponding p-groups P_{G_1} and P_{G_2} are isomorphic as well.

Definition 1 (The Baer-Lovász-Tutte procedure)**.** *Let $G = ([n], E)$ be an undirected simple graph with $|E| = m > 0$. The Baer-Lovász-Tutte procedure, as specified in the above three steps, takes G and a prime $p > 2$, and produces a p-group of class 2 and exponent p, $P_G \in \mathfrak{B}_{p,2}(n, m)$.*

1.2 Our results

Two group-theoretic parameters. Let H be a finite group. We use $J \leq H$ to denote that J is a subgroup of H, and $J < H$ to denote that J is a proper subgroup of H. For $S, T \subseteq H$, $ST = \{st : s \in S, t \in T\}$. If two subgroups $J, K \leq H$ satisfy that $JK = KJ$, then JK is a subgroup of H.

Recall that H is a *central product* of two subgroups J and K, if (1) every element of J commutes with every element of K, i.e. $[J, K] = 1$, and (2) H is generated by J and K, i.e. $H = JK$. See e.g. [Suz82, pp. 137]. In the following, we always assume that a central product is non-trivial, i.e., J and K are non-trivial proper subgroups of H. If such J and K exist, then we say that H admits a central decomposition.

Given $P \in \mathfrak{B}_{p,2}$, a subgroup $S \leq P$ is *regular* with respect to commutation, or simply regular for short, if $[S, S] = S \cap [P, P]$.

Definition 2 (κ and λ for p-groups of class 2 and exponent p)**.** *Let $P \in \mathfrak{B}_{p,2}(n, m)$.*

The regular-subgroup central-decomposition number *of P, denoted by $\kappa(P)$, is the smallest $s \in \mathbb{N}$ for the existence of a regular subgroup S with $|S/[S, S]| = p^{n-s}$, such that S admits a central decomposition.*

The central-quotient central-decomposition number *of P, denoted as $\lambda(P)$, is the smallest $s \in \mathbb{N}$ for the existence of a central subgroup N of order p^s, such that P/N admits a central decomposition.*

An explanation for imposing the regularity condition in the definition of $\kappa(P)$ can be found in Remark 10. In the definition of $\lambda(P)$, we can actually restrict N to be from those central subgroups contained in $[P, P]$ (cf. Observation 8 (2)).

The results. Recall that for a graph G, the *vertex connectivity* $\kappa(G)$ denotes the smallest number of vertices needed to remove to disconnect G, and the *edge connectivity* $\lambda(G)$ denotes the smallest number of edges needed to remove to disconnect G [Die17].

Given the above preparation, we can state our main result.

Theorem 1. *For an n-vertex and m-edge graph G, let $P_G \in \mathfrak{B}_{p,2}(n, m)$ be the result of applying the Baer-Lovász-Tutte procedure to G and a prime $p > 2$. Then $\kappa(G) = \kappa(P_G)$, and $\lambda(G) = \lambda(P_G)$.*

Recall that $\kappa(P_G)$ and $\lambda(P_G)$ are defined in purely group-theoretic terms, while $\kappa(G)$ and $\lambda(G)$ are classical notions in graph theory. Therefore, Theorem 1 sets up a surprising link between group theory and graph theory.

To understand these two parameters and their relation better, we consider the following question. Recall that for a graph G, it is well-known that $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ denotes the minimum degree of vertices in G (cf. e.g. [Die17, Proposition 1.4.2]). We study a question of the same type in the context of p-groups of class 2 and exponent p. For this we need the following definition.

Definition 3 (Degrees and δ for p-groups of class 2 and exponent p)**.** *For $P \in \mathfrak{B}_{p,2}(n, m)$ and $g \in P$, suppose $C_P(g) = \{h \in P : [h, g] = 1\}$ is of order p^d. Then the* degree *of g is* $\deg(g) = n + m - d$. *The* minimum degree *of P, $\delta(P)$, is the minimum degree over $g \in P \setminus [P, P]$.*

It is easy to see that for any $g \in P$, $\deg(g) \leq n - 1$ (cf. Observation 8 (3)). Therefore $\delta(P) \leq n - 1$. We then have the following.

Proposition 2. *1. For any $P \in \mathfrak{B}_{p,2}$, $\kappa(P) \leq \delta(P)$, and $\lambda(P) \leq \delta(P)$.*

2. There exists $P \in \mathfrak{B}_{p,2}$, such that $\kappa(P) > \lambda(P)$.

That is, while we can still upper bound $\kappa(P)$ and $\delta(P)$ using a certain minimum degree notion, the inequality $\kappa(\cdot) \leq \lambda(\cdot)$ does not hold in general in the p-group setting.

1.3 Related works and open ends

Related works. Alternating matrix spaces and alternating bilinear maps serve as the intermediate objects between graphs and groups in the Baer-Lovász-Tutte procedure. We elaborate more on the previous works that demonstrate their links to the two sides.

The link between graphs and alternating matrix spaces dates back to the works of Tutte and Lovász [Tut47, Lov89] in the context of perfect matchings. Let $G = ([n], E)$ be a graph, and let $\mathcal{A}_G \leq \Lambda(n, \mathbb{F})$ be the alternating matrix space associated with G as in Step 1. Tutte and Lovász realised that the matching number of G, $\mu(G)$, is equal to the maximum rank over matrices in $\mathcal{A}_G$.[1] More specifically, Tutte represented G as a symbolic matrix, that is a matrix whose entries are either variables or 0 [Tut47]. It can be interpreted as a linear space of matrices in a straightforward fashion. Lovász then more systematically studied this construction from the latter perspective [Lov89].

Recently in [BCG+19], the second author and collaborators showed that the independence number of G, $\alpha(G)$, is equal to the maximum dimension over the isotropic spaces[2] of $\mathcal{A}_G$. They also showed that the chromatic number of G, $\chi(G)$, is equal to the minimum c such that there exists a direct sum decomposition of $\mathbb{F}^n$ into c non-trivial isotropic spaces for $\mathcal{A}_G$. As the reader will see below, the proof of Theorem 1 also goes by defining appropriate parameters $\kappa(\cdot)$ and $\lambda(\cdot)$ for alternating matrix spaces, and proving that $\kappa(\mathcal{A}_G) = \kappa(G)$ and $\lambda(\mathcal{A}_G) = \lambda(G)$. This translates another two graph-theoretic parameters to the alternating matrix space setting.

The work most relevant to the current note in this direction is [LQ17] by the present authors. In that work, we adapted a combinatorial technique for the graph isomorphism problem by Babai, Erdős, and Selkow [BES80], to tackle isomorphism testing of groups from $\mathfrak{B}_{p,2}$, via alternating matrix spaces. This leads to the definition of a "cut" for alternating matrix spaces, which in turn naturally leads to the edge connectivity notion; cf. the proof of Proposition 5.

The link between alternating bilinear maps and $\mathfrak{B}_{p,2}$ dates back to the work of Baer [Bae38]. That is, from an alternating bilinear map ϕ, we can construct a group P_ϕ in $\mathfrak{B}_{p,2}$ as in Step 3. On the other hand, given $P \in \mathfrak{B}_{p,2}(n, m)$, by taking the commutator bracket we obtain an alternating bilinear map ϕ_P. A generalisation of this link to p-groups of Frattini class 2 was crucial in Higman's enumeration of p-groups [Hig60]. Alperin [Alp65], Ol'shanskii [Ol'78] and Buhler, Gupta, and Harris [BGH87] used this link to study large abelian subgroups of p-groups, a question first considered by Burnside [Bur13]. This is because abelian subgroups of P containing $[P, P]$ correspond to isotropic spaces of ϕ_P.

The works most relevant to the current note in this direction are [Wil09a, Wil09b] by James B. Wilson. He studied central decompositions of P via the link between alternating bilinear maps and $\mathfrak{B}_{p,2}$. In particular, he utilised that central decompositions of P correspond to orthogonal decompositions of ϕ_P.

[1] This is straightforward to see if the underlying field $\mathbb{F}$ is large enough. If $\mathbb{F}$ is small, it follows e.g. as a consequence of the linear matroid parity theorem; cf. the discussion after [Lov89, Theorem 4].

[2] A subspace $U \leq \mathbb{F}^n$ is an isotropic space of $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, if for any $u, u' \in U$, and any $A \in \mathcal{A}$, $u^t A u' = 0$.

Finally, we recently learnt of the work [RV19] of Rossmann and Voll, who study those p-groups of class 2 and exponent p obtained from graphs through the Baer-Lovász-Tutte procedure in the context of zeta functions of groups.

Open ends. The most interesting questions to us are the computational aspects of these parameters. That is, given the linear basis of an alternating matrix space $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, compute $\kappa(\mathcal{A})$ and $\lambda(\mathcal{A})$ (see Definition 4). When $\mathbb{F} = \mathbb{F}_q$ with q odd, there is a randomised polynomial-time algorithm to decide whether $\kappa(\mathcal{A}) = \lambda(\mathcal{A}) = 0$ by Wilson [Wil09b]. When $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, by utilising certain machineries from [IQ19], Wilson's algorithm can be adapted to yield a deterministic polynomial-time algorithm to decide whether $\kappa(\mathcal{A}) = \lambda(\mathcal{A}) = 0$. However, to directly use Wilson's algorithm to compute $\kappa(\mathcal{A})$ or $\lambda(\mathcal{A})$ seems difficult, as when $\kappa(\mathcal{A}) = \lambda(\mathcal{A}) = 0$, a non-trivial orthogonal decomposition can be nicely translated to a certain idempotent in an involutive algebra associated with any linear basis of $\mathcal{A}$; for details, see [Wil09b].

2 Proofs

2.1 Preparations

Some notation has been introduced at the beginning of sections 1.1 and 1.2. We add some more here. For a field $\mathbb{F}$ and $d, e \in \mathbb{N}$, we use $\mathrm{M}(d \times e, \mathbb{F})$ to denote the linear space of $d \times e$ matrices over $\mathbb{F}$, and $\mathrm{M}(d, \mathbb{F}) := \mathrm{M}(d \times d, \mathbb{F})$. The ith standard basis vector of $\mathbb{F}^n$ is denoted by e_i.

Some notions for alternating matrix spaces. We introduce some basic concepts, and then define κ and λ, for alternating matrix spaces.

Let $\mathcal{A}, \mathcal{B} \leq \Lambda(n, \mathbb{F})$. We say that $\mathcal{A}$ and $\mathcal{B}$ are *isometric*, if there exists $T \in \mathrm{GL}(n, \mathbb{F})$, such that $\mathcal{A} = T^t\mathcal{B}T := \{T^tBT : B \in \mathcal{B}\}$.

For a d-dimensional $W \leq \mathbb{F}^n$, let T be an $n \times d$ matrix whose columns span W. Then the restriction of $\mathcal{A}$ to W via T is $\mathcal{A}|_{W,T} := \{T^tAT : A \in \mathcal{A}\} \leq \Lambda(d, \mathbb{F})$. For a different $n \times d$ matrix T' whose columns also span W, $\mathcal{A}|_{W,T'}$ is isometric to $\mathcal{A}|_{W,T}$. So we can write $\mathcal{A}|_W$ to indicate a restriction of $\mathcal{A}$ to W via some such T.

Let $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ be of dimension m. An *orthogonal decomposition* of $\mathcal{A}$ is a direct sum decomposition of $\mathbb{F}^n$ into $U \oplus V$, such that for any $u \in U$, $v \in V$, and $A \in \mathcal{A}$, $u^tAv = 0$. An orthogonal decomposition is non-trivial, if neither U nor V is the trivial space. In the following, we always assume an orthogonal decomposition to be non-trivial unless otherwise stated.

In the degenerate case when $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ is the zero space, we define it to have an orthogonal decomposition for any $n \in \mathbb{N}$. When $\mathcal{A} = \langle A \rangle \leq \Lambda(n, \mathbb{F})$ is of dimension 1 and $n > 2$, $\mathcal{A}$ always admits an orthogonal decomposition. This can be seen easily from the canonical form for alternating matrices [Lan02, Chap. XV, Sec. 8].

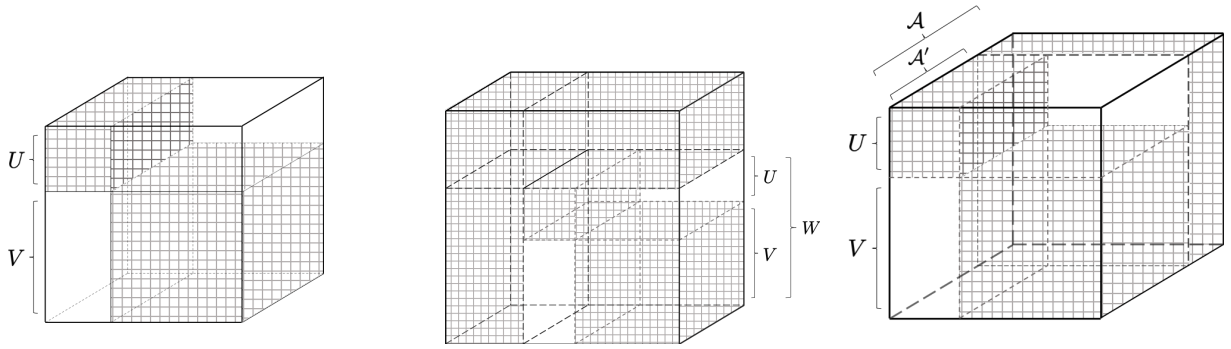
Definition 4 (κ and λ for alternating matrix spaces)**.** *Let $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ be of dimension m.*

We define the restriction-orthogonal number *of $\mathcal{A}$, $\kappa(\mathcal{A})$, as the minimum $c \in \mathbb{N}$ for the existence of a dimension-$(n-c)$ subspace $W \leq \mathbb{F}^n$, such that $\mathcal{A}|_W$ admits an orthogonal decomposition.*

We define the subspace-orthogonal number *of $\mathcal{A}$, $\lambda(\mathcal{A})$, as the minimum $c \in \mathbb{N}$ for the existence of a dimension-$(m-c)$ subspace $\mathcal{A}' \leq \mathcal{A}$, such that $\mathcal{A}'$ admits an orthogonal decomposition.*

Clearly, $\mathcal{A}$ itself admits an orthogonal decomposition if and only if $\kappa(\mathcal{A}) = \lambda(\mathcal{A}) = 0$. Since we defined the zero alternating matrix space to have an orthogonal decomposition, $\kappa(\mathcal{A}) \leq n-1$ and $\lambda(\mathcal{A}) \leq m$.

Suppose we are given a dimension-m $\mathcal{A} = \langle A_1, \ldots, A_m \rangle \leq \Lambda(n, \mathbb{F})$. We form a 3-tensor $\mathtt{A} \in \mathbb{F}^{n \times n \times m}$ such that $\mathtt{A}(i,j,k) = A_k(i,j)$. We illustrate the existence of an orthogonal decomposition for $\mathcal{A}$, the existence of W such that $\mathcal{A}|_W$ has an orthogonal decomposition, and the existence of $\mathcal{A}' \leq \mathcal{A}$ with an orthogonal decomposition, up to appropriate basis changes, in Figure 1.



(a) $\kappa(\mathcal{A}) = \lambda(\mathcal{A}) = 0$ (b) $\kappa(\mathcal{A}) \leq n - \dim(W)$ (c) $\lambda(\mathcal{A}) \leq m - \dim(\mathcal{A}')$

Figure 1: Pictorial descriptions of the alternating matrix space parameters. The white regions indicate that the entries there are all zero. For example, in (a), suppose $U \oplus V$ is an orthogonal decomposition for $\mathcal{A}$. Then up to a change of basis, the upper-right and the lower-left corners of $\mathtt{A}$ have all-zero entries. (b) and (c) also indicate the situations with appropriate changes of bases.

Some notions for alternating bilinear maps. We introduce basic concepts, and then define κ and λ, for alternating bilinear maps.

Let $\phi, \psi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ be two alternating bilinear maps. Following [Wil09a], we say that ϕ and ψ are *pseudo-isometric*, if they are the same under the natural action of $\mathrm{GL}(n, \mathbb{F}) \times \mathrm{GL}(m, \mathbb{F})$.

For $U \leq \mathbb{F}^n$, ϕ naturally restricts to U to give $\phi|_U : U \times U \to \mathbb{F}^m$. For $X \leq \mathbb{F}^m$, ϕ naturally induces $\phi/_X : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m/X$ by composing ϕ with the projection from $\mathbb{F}^m$ to $\mathbb{F}^m/X$.

Let $\phi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ be an alternating bilinear map. An orthogonal decomposition of ϕ is a direct sum decomposition of $\mathbb{F}^n = U \oplus V$, such that for any $u \in U$, $v \in V$, we have $\phi(u, v) = 0$. In the following, unless stated otherwise, we always assume an orthogonal decomposition of ϕ to be non-trivial, i.e., neither U nor V is the trivial space.

Definition 5 (κ and λ for alternating bilinear maps)**.** *Let $\phi : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ be an alternating bilinear map.*

The restriction-orthogonal number *of ϕ, $\kappa(\phi)$, is the minimum $c \in \mathbb{N}$ for the existence of a dimension-$(n-c)$ subspace $U \leq \mathbb{F}^n$, such that $\phi|_U$ admits an orthogonal decomposition.*

The quotient-orthogonal number *of ϕ, $\lambda(\phi)$, is the minimum $c \in \mathbb{N}$ for the existence of a dimension-c $X \leq \mathbb{F}^m$, such that $\phi/_X$ admits an orthogonal decomposition.*

Remark 3 (From alternating matrix spaces to bilinear maps)**.** This connection is simple but may deserve some discussion. Recall that, given an m-dimensional alternating matrix space $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, we can fix an ordered basis of $\mathcal{A}$ as $\mathbf{A} = (A_1, \ldots, A_m) \in \Lambda(n, \mathbb{F})^m$, and construct an alternating

bilinear map $\phi_{\mathbf{A}} : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ as in Equation 2. Furthermore, $\phi_{\mathbf{A}}(\mathbb{F}^n, \mathbb{F}^n) = \mathbb{F}^m$ because $\mathcal{A}$ is of dimension m.

In the above transformation, we shall need $\mathbf{A} \in \Lambda(n, \mathbb{F})^m$ as an intermediate object. For a different ordered bases $\mathbf{A}'$, $\phi_{\mathbf{A}'}$ is pseudo-isometric to $\phi_{\mathbf{A}}$. Because of this, we shall write $\phi_{\mathcal{A}}$ to indicate $\phi_{\mathbf{A}}$ with some ordered basis $\mathbf{A}$ of $\mathcal{A}$.

Furthermore, if $\mathcal{A}$ and $\mathcal{B}$ are isometric and $\mathbf{A}$ (resp. $\mathbf{B}$) is an ordered basis for $\mathcal{A}$ (resp. $\mathcal{B}$), then $\phi_{\mathcal{A}}$ and $\phi_{\mathcal{B}}$ are pseudo-isometric as well.

2.2 Proof of Theorem 1

The proof of Theorem 1 goes by showing that the parameters $\kappa(\cdot)$ and $\lambda(\cdot)$ defined for graphs, alternating matrix spaces, alternating bilinear maps, and groups from $\mathfrak{B}_{p,2}$, are preserved in the three steps of the Baer-Lovász-Tutte procedure. The first step, from graphs to alternating matrix spaces, is the tricky one, at least for $\lambda(\cdot)$. The other two steps are rather straightforward.

From graphs to alternating matrix spaces.

Proposition 4. *Let $G = ([n], E)$ be a graph, and let $\mathcal{A}_G \leq \Lambda(n, \mathbb{F})$ be defined in Step 1. Then $\kappa(G) = \kappa(\mathcal{A}_G)$.*

Proof. We first show $\kappa(\mathcal{A}_G) \leq \kappa(G)$. Let $I \subseteq [n]$ be a subset of vertices of size $d = n - \kappa(G)$, such that the induced subgraph of G on I is disconnected. Let $W = \langle e_i : i \in I \rangle$, and T be the $n \times d$ matrix over $\mathbb{F}$ whose columns are $e_i \in \mathbb{F}^n$, $i \in I$. It is straightforward to verify that $\mathcal{A}_G|_{W,T}$ admits an orthogonal decomposition.

We then show $\kappa(\mathcal{A}_G) \geq \kappa(G)$. Let $W \leq \mathbb{F}^n$ be a subspace of dimension $d = n - \kappa(\mathcal{A}_G)$, such that $\mathcal{A}|_W$ admits an orthogonal decomposition. That is, there exists $W = U \oplus V$ such that

$$\forall u \in U, v \in V, \forall A \in \mathcal{A}, u^t A v = 0. \tag{4}$$

Suppose $\dim(U) = b$ and $\dim(V) = c$, so $d = b + c$. Construct an $n \times d$ matrix $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ where T_1 (resp. T_2) is of size $n \times b$ (resp. $n \times c$) and its columns form a basis of U (resp. V). Let the ith row of T_1 be u_i^t where $u_i \in \mathbb{F}^b$, and let the jth row of T_2 be v_j^t where $v_j \in \mathbb{F}^c$, for $i, j \in [n]$. Then by Equation 4, for any $\{i, j\} \in E$,

$$T_1^t (e_i e_j^t - e_j e_i^t) T_2 = u_i v_j^t - u_j v_i^t \tag{5}$$

is the all-zero matrix of size $b \times c$.

Because T is of rank d, there exists a $d \times d$ submatrix R of T of rank d. Let $I \subseteq [n]$ be the set of row indices of this submatrix R. We claim that the induced subgraph of G on I, $G[I]$, is disconnected. To show this, we exhibit a partition of $I = I_1 \uplus I_2$ such that no edges in $G[I]$ go across I_1 and I_2.

Recall that R is of rank d. As an easy consequence of the Laplace expansion, there exists a partition of I, $I = I_1 \uplus I_2$ with $|I_1| = b$, $|I_2| = d - b = c$, such that the following holds. Let R_1 be the $b \times b$ submatrix of R with row indices from I_1 and column indices from $[b]$, and R_2 the $c \times c$ submatrix of R with row indices from I_2 and column indices from $[d] \setminus [b]$. Then R_1 and R_2 are both full-rank. Note that $\{u_i^t : i \in I_1\}$ is the set of rows of R_1 and $\{v_j^t : j \in I_2\}$ is the set of rows of R_2.

We then claim that no edges in $G[I]$ go across I_1 and I_2. By contradiction, suppose there is an edge $\{i,j\}$, $i \in I_1$ and $j \in I_2$, in $G[I]$. Then the same edge $\{i,j\}$ is also in G. By Equation 5, we have $u_i v_j^t - u_j v_i^t$ is the all-zero matrix. Since R_1 and R_2 are full-rank, we have u_i and v_j are nonzero vectors. This implies that $u_j = \alpha u_i$ and $v_i = (1/\alpha) v_j$ for some nonzero $\alpha \in \mathbb{F}$. But this implies that $\begin{bmatrix} u_j^t & v_j^t \end{bmatrix} = \alpha \begin{bmatrix} u_i^t & v_i^t \end{bmatrix}$, that is, the ith and jth rows of T are linearly dependent. Noting that these rows are in R which is full-rank, we arrive at the desired contradiction. This concludes the proof. □

Proposition 5. *Let $G = ([n], E)$ be a graph, and let $\mathcal{A}_G \leq \Lambda(n, \mathbb{F})$ be defined in Step 1. Then $\lambda(G) = \lambda(\mathcal{A}_G)$.*

Proof. We first show $\lambda(\mathcal{A}_G) \leq \lambda(G)$. Let D be a size-$\lambda(G)$ subset of E such that $G' = ([n], E \setminus D)$ is disconnected. Let $\mathcal{A}_{G'} = \langle A_{i,j} : \{i,j\} \in E \setminus D \rangle \leq \mathcal{A}_G$. It is straightforward to verify that $\mathcal{A}_{G'}$ admits an orthogonal decomposition.

We then show $\lambda(\mathcal{A}_G) \geq \lambda(G)$. For this, it is convenient to introduce an equivalent formulation of $\lambda(\cdot)$ for alternating matrix spaces, which is originated from [LQ17].

Given a direct sum decomposition $\mathbb{F}^n = U \oplus V$ with $\dim(U) = b$ and $\dim(V) = c = n - b$, let T_1 (resp. T_2) be a $n \times b$ (resp. $n \times c$) matrix whose columns form a basis of U (resp. V). Given an m-dimensional $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, let $\mathcal{C}_{U,V,T_1,T_2}(\mathcal{A}) = \{T_1^t A T_2 : A \in \mathcal{A}\} \leq \mathrm{M}(b \times c, \mathbb{F})$. Note that different choices of T_1 and T_2 result in a subspace of $\mathrm{M}(b \times c, \mathbb{F})$ which can be transformed to $\mathcal{C}_{U,V,T_1,T_2}(\mathcal{A})$ by left-multiplying some $X \in \mathrm{GL}(b, \mathbb{F})$ and right-multiplying some $Y \in \mathrm{GL}(c, \mathbb{F})$. So we can write $\mathcal{C}_{U,V}$ to indicate $\mathcal{C}_{U,V,T_1,T_2}$ via some such T_1 and T_2. We claim that

$$\lambda(\mathcal{A}) = \min\{\dim(\mathcal{C}_{U,V}(\mathcal{A})) : \forall \text{ non-trivial } \mathbb{F}^n = U \oplus V\}. \tag{6}$$

To see this, let $\mathcal{A}' \leq \mathcal{A}$ be of dimension $m - \lambda(\mathcal{A})$ which admits an orthogonal decomposition $\mathbb{F}^n = U \oplus V$. It is easy to verify that $\dim(\mathcal{C}_{U,V}(\mathcal{A})) \leq m - (m - \lambda(\mathcal{A})) = \lambda(\mathcal{A})$. On the other hand, let $\mathbb{F}^n = U \oplus V$ be a direct sum decomposition such that $\dim(\mathcal{C}_{U,V}(\mathcal{A}))$ is minimal. Let T_1 (resp. T_2) be a matrix whose columns form a basis of U (resp. V). Let $\mathcal{A}' = \{A \in \mathcal{A} : T_1^t A T_2 = 0\}$. We then have $\dim(\mathcal{A}') = m - \dim(\mathcal{C}_{U,V}(\mathcal{A}))$, and clearly $\mathcal{A}'$ admits an orthogonal decomposition. This gives $\lambda(\mathcal{A}) \leq m - \dim(\mathcal{A}') = \dim(\mathcal{C}_{U,V}(\mathcal{A}))$.

After introducing this formulation, let $\mathbb{F}^n = U \oplus V$ be a direct sum decomposition with $\dim(U) = b$ and $\dim(V) = c = n - b$, such that $\dim(\mathcal{C}_{U,V}(\mathcal{A}_G)) = \lambda(\mathcal{A}_G) = d$. Construct an $n \times n$ full-rank matrix $T = \begin{bmatrix} T_1 & T_2 \end{bmatrix}$ where T_1 (resp. T_2) is a $n \times b$ (resp. $n \times c$) matrix whose columns form a basis of U (resp. V). Let the ith row of T_1 be u_i^t where $u_i \in \mathbb{F}^b$, and let the jth row of T_2 be v_j^t where $v_j \in \mathbb{F}^c$. We distinguish two cases:

1. Suppose for any $i \in [n]$, $u_i \neq 0$ if and only if $v_i = 0$. Then there exists $[n] = I_1 \uplus I_2$ with $|I_1| = b$ and $|I_2| = c$, such that $i \in I_1$ if and only if $u_i \neq 0$, and $j \in I_2$ if and only if $v_j \neq 0$. Furthermore, vectors in $\{u_i : i \in I_1\}$ are linearly independent, and vectors in $\{v_j : j \in I_2\}$ are linearly independent. We claim that there is no more than d edges of G crossing I_1 and I_2. Suppose not, then there exists $\{\{i_1, j_1\}, \ldots, \{i_{d+1}, j_{d+1}\}\} \subseteq E$, such that $i_k \in I_1$, and $j_k \in I_2$ for $k \in [d+1]$. Note that

$$T_1^t (e_{i_k} e_{j_k}^t - e_{j_k} e_{i_k}^t) T_2 = u_{i_k} v_{j_k}^t - u_{j_k} v_{i_k}^t = u_{i_k} v_{j_k}^t \in \mathcal{C}_{U,V}(\mathcal{A}_G) \tag{7}$$

   for all $k \in [d+1]$. It is straightforward to verify that $u_{i_k} v_{j_k}^t$, $k \in [d+1]$, are linearly independent, which contradicts that $\mathcal{C}_{U,V}(\mathcal{A}_G)$ is of dimension d.

2. Suppose there exists $i \in [n]$, such that both u_i and v_i are nonzero. Suppose by contradiction that $\lambda(G) > d$. It follows that the vertex i is of degree at least $d+1$. Suppose i is adjacent to $j_1, \ldots, j_{d+1} \in [n]$. Then $u_i v_{j_k}^t - u_{j_k} v_i^t \in \mathcal{C}_{U,V}(\mathcal{A}_G)$ for $k \in [d+1]$ by Equation 7. Since $\dim(\mathcal{C}_{U,V}(\mathcal{A}_G)) = d$, the matrices $u_i v_{j_k}^t - u_{j_k} v_i^t$, $k \in [d+1]$, are linearly dependent. It follows that there exist $\alpha_k \in \mathbb{F}$ for $k \in [d+1]$, at least one of which is nonzero, such that

$$\sum_{k=1}^{d+1} \alpha_k (u_i v_{j_k}^t - u_{j_k} v_i^t) = 0.$$

This implies that

$$u_i (\sum_{k=1}^{d+1} \alpha_k v_{j_k}^t) = (\sum_{k=1}^{d+1} \alpha_k u_{j_k}) v_i^t$$

as two rank-1 matrices. From the above, and by the assumption that u_i and v_i are nonzero, we have that $\beta u_i = \sum_{k=1}^{d+1} \alpha_k u_{j_k}$ and $\beta v_i = \sum_{k=1}^{d+1} \alpha_k v_{j_k}$ for some nonzero $\beta \in \mathbb{F}$. Since at least one of α_k's is nonzero, this means that the rows in T with indices $\{i, j_1, \ldots, j_{d+1}\}$ are linearly dependent, which contradicts that T is full-rank.

These conclude the proof that $\lambda(\mathcal{A}_G) \geq \lambda(G)$. □

Remark 6 (Cuts in alternating matrix spaces)**.** The alternative formulation of $\lambda(\cdot)$ as in Equation 6 rests on a natural generalisation of the notion of cuts in graphs. Proposition 5 then indicates that for an alternating matrix space $\mathcal{A}_G$ constructed from a graph G, the minimum cut sizes of $\mathcal{A}_G$ and G are equal.

From alternating matrix spaces to alternating bilinear maps. We now relate the parameters $\kappa(\cdot)$ and $\lambda(\cdot)$ for alternating matrix spaces and alternating bilinear maps in the following easy proposition. Note that we use the notation $\phi_{\mathcal{A}}$ due to the discussions in Remark 3.

Proposition 7. *For an m-dimensional $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, let an alternating bilinear map $\phi_{\mathcal{A}} : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^m$ be defined in Step 2. Then we have $\kappa(\mathcal{A}) = \kappa(\phi_{\mathcal{A}})$, and $\lambda(\mathcal{A}) = \lambda(\phi_{\mathcal{A}})$.*

Proof. The equality $\kappa(\mathcal{A}) = \kappa(\phi_{\mathcal{A}})$ is straightforward to verify.

To show that $\lambda(\mathcal{A}) \geq \lambda(\phi_{\mathcal{A}})$, let $\mathcal{A}' \leq \mathcal{A}$ be a dimension-$(n - \lambda(\mathcal{A}))$ subspace of $\mathcal{A}$ admitting an orthogonal decomposition. Let $c = \lambda(\mathcal{A})$. We fix an ordered basis of $\mathcal{A}$, $\mathbf{A} = (A_1, \ldots, A_m)$, such that $\{A_1, \ldots, A_{m-c}\}$ spans $\mathcal{A}'$. Let $X \leq \mathbb{F}^m$ be the linear span of the last c standard basis vectors. We claim that $\phi_{\mathbf{A}}/_X$ admits an orthogonal decomposition. Indeed, let $U \oplus V$ be an orthogonal decomposition of $\mathcal{A}'$. Then for any $u \in U, v \in V$, we have $\phi_{\mathbf{A}}(u, v) \in X$, which means that $U \oplus V$ is also an orthogonal decomposition for $\phi_{\mathbf{A}}/_X$.

To show that $\lambda(\mathcal{A}) \leq \lambda(\phi_{\mathcal{A}})$, let $\mathbf{A} = (A_1, \ldots, A_m)$ be an ordered basis of $\mathcal{A}$, and let $c = \lambda(\phi_{\mathcal{A}})$. Let X be a dimension-c subspace of $\mathbb{F}^m$, such that $\phi_{\mathbf{A}}/_X$ admits an orthogonal decomposition $\mathbb{F}^n = U \oplus V$. That is, for any $u \in U$ and $v \in V$, $\phi_{\mathbf{A}}(u, v) \in X$. Form an ordered basis of $\mathbb{F}^m$, $(w_1, \ldots, w_m)$, where $w_i = \begin{bmatrix} w_{i,1} \\ \vdots \\ w_{i,m} \end{bmatrix} \in \mathbb{F}^m$, such that the last c vectors form a basis of X. Let $A'_i = \sum_{j \in [m]} w_{i,j} A_j$ be another ordered basis of $\mathcal{A}$, and $\mathbf{A}' = (A'_1, \ldots, A'_m)$. Then for any $u \in U$ and $v \in V$, since $\phi_{\mathbf{A}}(u, v) \in X$, the first $m - c$ entries of $\phi_{\mathbf{A}'}(u, v)$ are zero. In particular, this implies that $\mathbb{F}^n = U \oplus V$ is an orthogonal decomposition for $\mathcal{A}' = \langle A'_1, \ldots, A'_{m-c} \rangle$, where $\mathcal{A}'$ is of dimension $m - c$. □

From alternating bilinear maps to groups from $\mathfrak{B}_{p,2}$. To start with, we observe the following basic properties of κ, λ, and δ for groups from $\mathfrak{B}_{p,2}(n,m)$.

Observation 8. *Let $P \in \mathfrak{B}_{p,2}(n,m)$. Then we have the following.*

1. *Suppose $P = JK$ is a central decomposition. Let $J' = J[P,P]$, and $K' = K[P,P]$. Then J' and K' form a central decomposition of P, and both of them properly contain $[P,P]$.*
2. *If for a central subgroup N, P/N admits a central decomposition, then $P/(N \cap [P,P])$ admits a central decomposition.*
3. *For any $g \in P$, $\deg(g) \leq n-1$.*

Proof. (1): To show that J' and K' form a central decomposition of P, we only need to verify that J' and K' are proper. For the sake of contradiction, suppose $P = J' = J[P,P]$. Since $[P,P]$ is the Frattini subgroup of P, it follows that $J = P$, contradicting that J is proper.

To show that J' properly contains $[P,P]$, again for the sake of contradiction suppose $J' \leq [P,P]$. Then $P = J'K' \leq [P,P]K' = K'$, a contradiction to K' being a proper subgroup of P.

(2): If $N \leq [P,P]$, the conclusion holds trivially. Suppose otherwise. Let J/N and K/N be a central product of P/N for $J, K \leq P$. That is, for any $j \in J$ and $k \in K$, $jkj^{-1}k^{-1} \in N$, so in fact $jkj^{-1}k^{-1} = [j,k] \in N \cap [P,P]$. It then follows easily that $J/(N \cap [P,P])$ and $K/(N \cap [P,P])$ form a central product of $P/(N \cap [P,P])$.

(3): If $g \in \mathrm{Z}(P)$, $\deg(g) = 0$. If $g \notin \mathrm{Z}(P)$, then $C_P(g)$ contains the subgroup generated by g and $[P,P]$, which is of order $\geq p^{m+1}$. □

Recall that in Step 3, we start from bilinear map $\phi : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p^m$ satisfying $\phi(\mathbb{F}_p^n, \mathbb{F}_p^n) = \mathbb{F}_p^m$, and construct P_ϕ, a p-group of class 2 and exponent p. Then $[P_\phi, P_\phi] \cong \mathbb{Z}_p^m$, and $P_\phi/[P_\phi, P_\phi] \cong \mathbb{Z}_p^n$.

It is easily checked that, by Equation 3, subspaces of $\mathbb{F}_p^m$ correspond to subgroups of $[P_\phi, P_\phi]$, and subspaces of $\mathbb{F}_p^n$ correspond to subgroups of $P_\phi/[P_\phi, P_\phi]$. We then set up the following notation. For $U \leq \mathbb{F}_p^n$, let Q_U be the subgroup of $P_\phi/[P_\phi, P_\phi]$ corresponding to U, and let S_U be the *smallest* subgroup of P_ϕ satisfying $S_U[P,P]/[P,P] = Q_U$. Note that S_U is regular with respect to commutation, that is, $S_U \cap [P,P] = [S_U, S_U]$. For $X \leq \mathbb{F}_p^m$, let N_X be the subgroup of $[P_\phi, P_\phi]$ corresponding to X.

Proposition 9. *Let $\phi : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p^m$ and $P_\phi \in \mathfrak{B}_{p,2}(n,m)$ be as above. Then $\kappa(\phi) = \kappa(P_\phi)$, and $\lambda(\phi) = \lambda(P_\phi)$.*

Proof. To show that $\kappa(\phi) \geq \kappa(P_\phi)$, suppose there exists a $(n - \kappa(\phi))$-dimensional $U \leq \mathbb{F}_p^n$ such that $\phi|_U$ admits an orthogonal decomposition. It can be verified easily that this induces a central decomposition for the regular subgroup $S_U \leq P_\phi$. Furthermore, by the second isomorphism theorem, $S_U/[S_U, S_U] = S_U/(S_U \cap [P_\phi, P_\phi]) \cong S_U[P_\phi, P_\phi]/[P_\phi, P_\phi] = Q_U$, which is of order $p^{n-\kappa(\phi)}$.

To show that $\kappa(\phi) \leq \kappa(P_\phi)$, suppose that a regular $S \leq P_\phi$ satisfying $|S/[S,S]| = p^{n-\kappa(P_\phi)}$ admits a central decomposition $S = JK$. Appyling Observation 8 (1) to S, we can assume that J and K both properly contain $[S,S]$. Let U_S (resp. U_J, U_K) be the subspace of $\mathbb{F}_p^n$ corresponding to $S[P_\phi, P_\phi]/[P_\phi, P_\phi]$ (resp. $J[P_\phi, P_\phi]/[P_\phi, P_\phi]$, $K[P_\phi, P_\phi]/[P_\phi, P_\phi]$). Then it can be verified, using Equation 3, that U_J and U_K form an orthogonal decomposition for $\phi|_{U_S}$. Furthermore, by the second isomorphism theorem, $S[P_\phi, P_\phi]/[P_\phi, P_\phi] \cong S/[S,S]$, which holds with S replaced by J or K as well. In particular we have $\dim(U_S) = n - \kappa(P_\phi)$.

To show that $\lambda(\phi) \geq \lambda(P_\phi)$, we translate a subspace $X \leq \mathbb{F}_p^m$ such that $\phi/_X$ admits an orthogonal decomposition, to a subgroup $N_X \leq [P_\phi, P_\phi]$. Then it can be verified easily that the orthogonal decomposition of $\phi/_X$ yields a central decomposition of P_ϕ/N_X.

To show that $\lambda(\phi) \leq \lambda(P_\phi)$, suppose $N \leq P_\phi$ is a central subgroup of order $p^{\lambda(P_\phi)}$ such that P_ϕ/N admits a central decomposition. By Observation 8 (2), we can assume that $N \leq [P_\phi, P_\phi]$. Let X be the subspace of $\mathbb{F}_p^m$ corresponding to N. Let $J/N, K/N \leq P_\phi/N$ be a central decomposition of P_ϕ/N for $J, K \leq P_\phi$. Applying Observation 8 (1) to P_ϕ/N, we can assume that J/N and K/N both properly contain $[P_\phi/N, P_\phi/N] = [P_\phi, P_\phi]/N$. In particular, J and K properly contain $[P_\phi, P_\phi]$, so $J/[P_\phi, P_\phi]$ (resp. $K/[P_\phi, P_\phi]$) corresponds to a non-trivial proper subspace $U_J \leq \mathbb{F}_p^n$ (resp. $U_K \leq \mathbb{F}_p^n$). Then it can be verified that U_J and U_K span $\mathbb{F}_p^n$, and for any $u \in U_J$ and $u' \in U_K$, we have $\phi(u, u') \in X$. Therefore U_J and U_K form an orthogonal decomposition for $\phi/_X$. □

Remark 10 (On the regular condition)**.** The reason for imposing the regular condition is to rule out the following central decompositions, which is not well-behaved regarding the correspondence between ϕ and P_ϕ. Suppose that $S \leq P_\phi$ satisfies $[S, S] < [P_\phi, P_\phi]$. Then S and $[P_\phi, P_\phi]$ form a central decomposition of $S[P_\phi, P_\phi]$. Translating back to ϕ, $[S, S] < [P_\phi, P_\phi]$ just says that $\phi(U_S, U_S)$ is a proper subspace of $\mathbb{F}^m$, which is not related to whether $\phi|_{U_S}$ admits an orthogonal decomposition.

2.3 Proof of Proposition 2

We shall work in the setting of alternating matrix spaces. So we state the correspondence of Definition 3 in this setting, which was proposed in [Qia19] and has been used in [BCG+19].

Definition 6 (Degrees and δ for alternating matrix spaces)**.** *Let $\mathcal{A} \leq \Lambda(n, \mathbb{F})$. For $v \in \mathbb{F}^n$, the* degree *of v in $\mathcal{A}$ is the dimension of $\mathcal{A}v := \{Av : A \in \mathcal{A}\}$. The* minimum degree *of $\mathcal{A}$, denoted as $\delta(\mathcal{A})$, is the minimum degree over all $0 \neq v \in \mathbb{F}^n$.*

To translate from groups in $\mathfrak{B}_{p,2}(n, m)$ to alternating matrix spaces, we recall the following procedure which consists of inverses of the last two steps of the Baer-Lovász-Tutte procedure.

For any $P \in \mathfrak{B}_{p,2}(n, m)$, let $V = P/[P, P] \cong \mathbb{Z}_p^n$ and $U = [P, P] \cong \mathbb{Z}_p^m$. The commutator map $\phi_P : V \times V \to U$ is alternating and bilinear. After fixing bases of V and U as $\mathbb{F}_p$-vector spaces, we can represent $\phi_P : \mathbb{F}_p^n \times \mathbb{F}_p^n \to \mathbb{F}_p^m$ as $(A_1, \ldots, A_m) \in \Lambda(n, \mathbb{F}_p)^m$, which spans an m-dimensional $\mathcal{A}_P \leq \Lambda(n, \mathbb{F}_p)$. It is easy to check that isomorphic groups yield isometric alternating matrix spaces. Furthermore, this procedure preserves κ and λ, by essentially the same proof for Proposition 7 and 9, and δ, by a straightforward calculation.

The following proposition then implies Proposition 2 (1).

Proposition 11. *Given $\mathcal{A} \leq \Lambda(n, \mathbb{F})$, we have $\kappa(\mathcal{A}) \leq \delta(\mathcal{A})$ and $\lambda(\mathcal{A}) \leq \delta(\mathcal{A})$.*

Proof. We first show that $\kappa(\mathcal{A}) \leq \delta(\mathcal{A})$. Take some $v \in \mathbb{F}^n$ such that $\deg(v) = \delta(\mathcal{A})$. If $\delta(\mathcal{A}) = n-1$, then the inequality holds trivially. Otherwise, let $U = \{u \in \mathbb{F}^n : \forall\ A \in \mathcal{A}, u^t Av = 0\}$. Note that $\dim(U) = n - \deg(v) \geq 2$, and $v \in U$. Let V be any complement space of $\langle v \rangle$ in U. Then $\langle v \rangle \oplus V$ is an orthogonal decomposition of $\mathcal{A}|_U$. It follows that $\kappa(\mathcal{A}) \leq n - \dim(U) = \deg(v) = \delta(\mathcal{A})$.

We then show that $\lambda(\mathcal{A}) \leq \delta(\mathcal{A})$. Take some $v \in \mathbb{F}^n$ such that $\deg(v) = \delta(\mathcal{A})$. Let W be any complement subspace of $\langle v \rangle$ in $\mathbb{F}^n$, and let T_W be an $n \times (n-1)$ matrix whose columns form a basis of W. The space $v^t \mathcal{A} T_W = \{v^t A T_W : A \in \mathcal{A}\} \leq \mathrm{M}(1 \times (n-1), \mathbb{F})$ is of dimension $\deg(v)$. By Equation 6, we then have $\lambda(\mathcal{A}) \leq \dim(v^t \mathcal{A} T_W) = \deg(v) = \delta(\mathcal{A})$. □

In contrast to the graph setting, we show that it is possible that $\kappa(\mathcal{A}) > \lambda(\mathcal{A})$ over $\mathbb{Q}$ and $\mathbb{F}_q$, therefore proving Proposition 2 (2). For this we need the following definition.

Definition 7. *We say that $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ is* fully connected*, if for any linearly independent $u, v \in \mathbb{F}^n$, there exists $A \in \mathcal{A}$, such that $u^t A v \neq 0$.*

An observation on fully connected $\mathcal{A}$ follows from the definition easily.

Observation 12. *Suppose that $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ is fully connected. Then $\kappa(\mathcal{A}) = n - 1$.*

We shall construct a fully connected $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ with $\lambda(\mathcal{A}) < n - 1 = \kappa(\mathcal{A})$. To do this we need the fully connected notion in the (not necessarily alternating) matrix space setting. That is, $\mathcal{B} \leq \mathrm{M}(s \times t, \mathbb{F})$ is fully connected, if for any nonzero $u \in \mathbb{F}^s$ and nonzero $v \in \mathbb{F}^t$, there exists $B \in \mathcal{B}$, such that $u^t B v \neq 0$. The following fact is well-known.

Fact 13. *Let $\mathbb{F}$ be a finite field or $\mathbb{Q}$. Then over $\mathbb{F}$, there exists a fully connected matrix space in $\mathrm{M}(s, \mathbb{F})$ of dimension s.*

Proof. Let $\mathbb{K}$ be a degree-s field extension of $\mathbb{F}$. The regular representation of $\mathbb{K}$ on $\mathbb{F}^s$ gives an s-dimensional $\mathcal{C} \leq \mathrm{M}(s, \mathbb{F})$, such that each nonzero $C \in \mathcal{C}$ is of full rank. Let $(C_1, \ldots, C_s)$ be an ordered basis of $\mathcal{B}$. Let $B_i \in \mathrm{M}(s, \mathbb{F})$, $i \in [s]$, be defined by $B_i = \begin{bmatrix} C_1 e_i & C_2 e_i & \ldots & C_s e_i \end{bmatrix}$. That is, the jth column of B_i is the ith column of C_j. Then $\mathcal{B} = \langle B_1, \ldots, B_s \rangle \leq \mathrm{M}(s, \mathbb{F})$ is of dimension s and fully connected. Indeed, if $\mathcal{B}$ is not fully connected, then there exist nonzero $v \in \mathbb{F}^s$ and nonzero $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_s \end{bmatrix} \in \mathbb{F}^s$ such that $v^t B_i u = 0$ for any $i \in [s]$. But this just means that v is in the left kernel of $C' = u_1 C_1 + \cdots + u_s C_s$, contradicting that C' is of full rank. □

Let $s, t \in \mathbb{N}$ and $n = s + t$. Let $\mathcal{B} \leq \mathrm{M}(s \times t, \mathbb{F})$ be a fully connected matrix space of dimension $d < n - 1$. We shall use $\mathcal{B}$ to construct a fully connected $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ such that $\lambda(\mathcal{A}) \leq d < n - 1 = \kappa(\mathcal{A})$.

Suppose $\mathcal{B}$ is spanned by $B_1, \ldots, B_d \in \mathrm{M}(s \times t, \mathbb{F})$. Let $A_i = \begin{bmatrix} 0 & B_i \\ -B_i^t & 0 \end{bmatrix}$ for $i \in [d]$. For $1 \leq i < j \leq s$, let $C_{i,j} = \begin{bmatrix} E_{i,j} & 0 \\ 0 & 0 \end{bmatrix} \in \Lambda(n, \mathbb{F})$, where $E_{i,j} = e_i e_j^t - e_j e_i^t \in \Lambda(s, \mathbb{F})$ is an elementary alternating matrix. For $1 \leq i < j \leq t$, let $D_{i,j} = \begin{bmatrix} 0 & 0 \\ 0 & F_{i,j} \end{bmatrix} \in \Lambda(n, \mathbb{F})$, where $F_{i,j} = e_i e_j^t - e_j e_i^t \in \Lambda(t, \mathbb{F})$ is an elementary alternating matrix. Let $\mathcal{A}$ be spanned by $\{A_i : i \in [d]\} \cup \{C_{i,j} : 1 \leq i < j \leq s\} \cup \{D_{i,j} : 1 \leq i < j \leq t\}$.

Proposition 14. *Let $\mathcal{A} \leq \Lambda(n, \mathbb{F})$ be as above. Then $\mathcal{A}$ is fully connected.*

Proof. Assume there exist linearly independent $u, v \in \mathbb{F}^n$ such that for any $A \in \mathcal{A}$, $u^t A v = 0$. Take $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ and $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, where $u_1, v_1 \in \mathbb{F}^s$ and $u_2, v_2 \in \mathbb{F}^t$. Note that for any $1 \leq i < j \leq s$,

$$\begin{bmatrix} u_1^t & u_2^t \end{bmatrix} \begin{bmatrix} E_{i,j} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = u_1^t E_{i,j} v_1 = 0.$$

Similarly, we have $u_2^t F_{i,j} v_2 = 0$ for all $1 \leq i < j \leq t$.

We then distinguish among the following cases.

1. v_1 and v_2 are both nonzero. In this case we have $u_1 = \lambda v_1$ and $u_2 = \mu v_2$ for some $\lambda \neq \mu \in \mathbb{F}$. Therefore, we have

$$\begin{bmatrix} u_1^t & u_2^t \end{bmatrix} \begin{bmatrix} 0 & B_i \\ -B_i^t & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -u_2^t B_i^t v_1 + u_1^t B_i v_2 = -\mu v_2^t B_i^t v_1 + \lambda v_1^t B_i v_2 = (\lambda - \mu) v_1^t B_i v_2.$$

   Since $\mathcal{B}$ is fully connected, this implies that $v_1 = 0$ or $v_2 = 0$, a contradiction to the assumption of this case.

2. v_1 is zero and v_2 is nonzero. Then $u_2 = \lambda v_2$, and u_1 cannot be zero. Therefore, we have

$$\begin{bmatrix} u_1^t & u_2^t \end{bmatrix} \begin{bmatrix} 0 & B_i \\ -B_i^t & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -u_2^t B_i^t v_1 + u_1^t B_i v_2 = u_1^t B_i v_2 = 0,$$

   which is a contradiction to the full connectivity of $\mathcal{B}$.

3. v_1 is nonzero and v_2 is zero. This case is in complete analogy with the previous case.

This concludes the proof that $\mathcal{A}$ is fully connected. □

We then have $\kappa(\mathcal{A}) = n - 1$ by Observation 12. Now observe that the subspace of $\mathcal{A}$ spanned by $C_{i,j}$ and $D_{i,j}$ admits a central decomposition. This gives that $\lambda(\mathcal{A}) \leq d < n - 1 = \kappa(\mathcal{A})$. Over $\mathbb{F}_q$ and $\mathbb{Q}$, such $\mathcal{B}$ exists for $s > 1$ by Fact 13. This concludes the proof of Proposition 2 (2).

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