

General relativistic resistive magnetohydrodynamics with robust primitive variable recovery for accretion disk simulations

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ABSTRACT

Recent advances in black hole astrophysics, particularly the first visual evidence of a supermassive black hole at the center of the galaxy M87 by the Event Horizon Telescope (EHT), and the detection of an orbiting “hot spot” nearby the event horizon of Sgr A* in the Galactic center by the Gravity Collaboration, require the development of novel numerical methods to understand the underlying plasma microphysics. Non-thermal emission related to such hot spots is conjectured to originate from plasmoids that form due to magnetic reconnection in thin current layers in the innermost accretion zone. Resistivity plays a crucial role in current sheet formation, magnetic reconnection, and plasmoid growth in black hole accretion disks and jets. We included resistivity in the three-dimensional general-relativistic magnetohydrodynamics (GRMHD) code BHAC and present the implementation of an Implicit-Explicit scheme to treat the stiff resistive source terms of the GRMHD equations. The algorithm is tested in combination with adaptive mesh refinement to resolve the resistive scales and a constrained transport method to keep the magnetic field solenoidal. Several novel methods for primitive variable recovery, a key part in relativistic magnetohydrodynamics codes, are presented and compared for accuracy, robustness, and efficiency. We propose a new inversion strategy that allows for resistive-GRMHD simulations of low gas-to-magnetic pressure ratio and highly magnetized regimes as applicable for black hole accretion disks, jets, and neutron star magnetospheres. We apply the new scheme to study the effect of resistivity on accreting black holes, accounting for dissipative effects as reconnection.

Keywords: black hole physics — accretion, accretion disks — (magnetohydrodynamics:) MHD — plasmas — relativity — methods: numerical

1. INTRODUCTION

Astrophysical phenomena typically show very distinctive time and length scales on which microscopic and macroscopic dynamics take place. Relativistic macroscopic plasma dynamics can be described by general-relativistic magnetohydrodynamics (GRMHD), coupling the fluid of charged particles to electromagnetic fields in a dynamic or static

gravitational field. This framework explains many observed astrophysical-plasma phenomena on the global scale, such as accretion onto and outflows from compact objects. Despite outstanding results achieved with GRMHD, state-of-the-art studies are affected by a lack of information on the effect of microscopic plasma physics on the macroscopic dynamics. The magnetorotational instability (MRI), for example, is a crucial mechanism of angular momentum transport in accretion disks resulting in turbulent motion (Velikhov 1959; Chandrasekhar 1960; Balbus & Hawley 1991). Magnetic reconnection and subsequent particle acceleration can occur in the turbulent disk or in the highly magnetized jet. Field-amplifying processes like the MRI, and dissipative processes like turbulence and reconnection, typically occur across a large range of scales from microscopic to macroscopic. Accurate mod-

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eling of the rarefied magnetospheres of compact objects requires knowledge of such fundamental microscopic processes affecting the macroscopic dynamics. Dynamic electric and magnetic fields determine the spatial and temporal scales on which dissipation occurs. Examples of astrophysical systems where both the electric and the magnetic fields are important include: accreting black holes in active galactic nuclei, coalescing black-hole and neutron-star binaries, neutron-star and magnetar magnetospheres, pulsar-wind nebulae, and massive stars undergoing core collapse. The magnetic field can change its topology via magnetic reconnection resulting in dissipation of the released magnetic energy that accelerates particles causing non-thermal emission. Non-thermal emission is one of the main uncertainties in the ideal-GRMHD (i.e., with infinite electrical conductivity σ) models of the Event Horizon Telescope (EHT) observations of the accretion disk of M87*, the supermassive black hole at the center of the galaxy M87 (Event Horizon Telescope Collaboration et al. 2019a; Event Horizon Telescope Collaboration et al. 2019b). Sgr A*, the black hole at the center of the Milky Way, also regularly exhibits non-thermal emission in the form of flares that have been conjectured to originate from magnetic reconnection in the accretion disk (Baganoff et al. 2001; Genzel et al. 2003; Eckart et al. 2006; Meyer et al. 2008; Neilsen et al. 2013; Dexter et al. 2014; Brinkerink et al. 2015; Gravity Collaboration et al. 2018). Dissipative (non-ideal) effects that can cause non-thermal emission are often negligible, except in regions with a strong and localized current density, e.g., in long and thin current sheets. Such effects occur on the diffusion time scale $\tau_D = L^2/\eta$, where L is the characteristic length scale of the system and η is the resistivity. Plasma is typically collisionless in astrophysical systems like Sgr A* and M87*, such that diffusion time scales are much larger than Alfvénic time scales $\tau_A = L/v_A$, with v_A the Alfvén speed. The gyroradius of electrons and ions can be considered as an effective mean free path perpendicular to the magnetic field, which is typically orders of magnitude smaller than the typical system size $r_g = GM/c^2$, the Schwarzschild radius with gravitational constant G , mass of the object M and speed of light c . The mean free path along the magnetic field is typically $\gg r_g$, such that the particles can freely travel along magnetic field lines before being deflected by Coulomb collisions (Yuan & Narayan 2014; Porth et al. 2019). Hence, $\tau_D \gg \tau_A$ and treating the plasma as an ideal magnetized fluid is a reasonable approach. However, to capture dissipation physics like magnetic reconnection, the diffusive time scale needs to be resolved in a practically near-dissipation-less system (i.e., $\tau_D \gg \tau_A$). Resolving such different time scales requires specific numerical schemes that can handle both fast and slow dynamics. Additionally, dissipative dynamics take place on a large range of length scales, which can depend both on the explicit resistivity and on the grid resolution in a

numerical simulation. Hence, a sufficiently high resolution is essential to capture small resistive length scales in GRMHD simulations.

The set of ideal-MHD (i.e., $\eta = 0$) equations cannot describe non-ideal processes due to the frozen-in condition of the magnetic field (Alfvén 1942). Ideal-GRMHD simulations have been conducted to explore magnetic reconnection in accretion disks as triggered by numerical resistivity (Ball et al. 2016; Ball et al. 2018), yet in this case there is no control on the resistivity, which purely depends on the numerical method and resolution used, rather than on an underlying physical model. The framework of general-relativistic resistive magnetohydrodynamics (GRRMHD) allows for incorporating a physical resistivity and to systematically explore dissipative effects like magnetic reconnection. In the set of non-relativistic magnetohydrodynamics equations, a resistive source term for resistivity η can be added directly to the induction equation and the electric field depends on the magnetic field \mathbf{B} , the current density $\mathbf{J} = \nabla \times \mathbf{B}$, and the fluid velocity field \mathbf{v} via a simple algebraic expression, i.e., $\mathbf{E}(\mathbf{v}, \mathbf{B}) = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J}$. In this way, the system remains hyperbolic-parabolic allowing for a standard explicit integration method. To incorporate resistivity in GRMHD however, Ampère’s law (i.e., for the evolution of the electric field) has to be solved alongside the ideal-GRMHD equations and one cannot assume $\mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J}$. Additionally, the induction equation in ideal-MHD assumes the electric field to be a purely dependent variable, i.e., $\mathbf{E}(\mathbf{v}, \mathbf{B}) = -\mathbf{v} \times \mathbf{B}$, such that it has to be replaced by Faraday’s law (i.e., for the evolution of the magnetic field) to account for a resistive electric field in relativistic MHD. For a realistically small but finite resistivity, the electric field dynamics occurs on a much shorter time scale than the GRMHD evolution. This results in a stiff source term in Ampère’s law which makes the time evolution with an explicit time integrator very inefficient.

Many methods have been developed in recent years to handle the additional complexity of the unavoidable stiff resistive source term in relativistic magnetohydrodynamics. Komisarov (2007) presented a special-relativistic resistive magnetohydrodynamics (SRRMHD) scheme based on a Strang-split method (Strang 1968), approximating the resistive source terms resulting from Ohm’s law with a semi-analytic approach. The Strang-split method requires a time-step that is proportional to the resistivity, resulting in very expensive computations for typical astrophysical plasmas with extremely low resistivity. In Dumbser & Zanotti (2009), an unstructured mesh approach was suggested to solve the SRRMHD equations. Palenzuela et al. (2009) presented the first application of an implicit-explicit (IMEX) Runge-Kutta (RK) scheme of arbitrary order to SRRMHD simulations, incorporating the full resistive Ohm’s law. There, the stiff source term is solved implicitly, while the non-stiff equations

are solved explicitly as in relativistic ideal-MHD. This overcomes the limitations otherwise imposed on the time-step due to stiffness, preventing a major slow-down compared to relativistic ideal-MHD. Takamoto & Inoue (2011) improved the semi-analytic approach of Komissarov (2007) with an implicit inversion method for the stiff source term, relaxing the restrictive time-step for SRRMHD. Bucciantini & Del Zanna (2013) and Dionysopoulou et al. (2013) applied the IMEX method to GRRMHD incorporating the full resistive Ohm’s law and Mignone et al. (2019) recently presented substantial improvements in SRRMHD. Bucciantini & Del Zanna (2013) and Palenzuela (2013) used an IMEX scheme to include Hall and dynamo effects in the GRRMHD evolution, through a generalized Ohm’s law. Numerical methods for GRRMHD have been extensively applied to neutron-star mergers (Palenzuela et al. 2013a; Palenzuela et al. 2013b; Dionysopoulou et al. 2015), the collapse of a neutron star to a black hole (Palenzuela 2013; Nathanail et al. 2017; Most et al. 2018), accretion onto black holes (Bugli et al. 2014; Qian et al. 2017; Qian et al. 2018; Vourellis et al. 2019) and in SRRMHD for relativistic reconnection (Zenitani et al. 2010; Barkov et al. 2014; Mizuno 2013; Del Zanna et al. 2016; Ripperda et al. 2019).

Numerical schemes for GRMHD require a method to recover “primitive” variables such as rest-mass density, pressure and the four-velocity from a set of “conserved” variables such as momentum and energy density. To retrieve the primitive variables it is necessary to solve one or more nonlinear equations. The solution method for the nonlinear equations is essential and is often a bottleneck for both accuracy and computational costs (Noble et al. 2006; Siegel et al. 2018). For stiff systems such as the set of GRRMHD equations, where the electric field is dynamically important, the primitive variables depend nonlinearly on the electric field and vice versa, resulting in an additional complication in the primitive variable recovery compared to ideal-GRMHD. Standard primitive-recovery methods for GRRMHD, often naively adapted from GRMHD, do not account for the nonlinear dependence of the electric field on the primitive variables (Palenzuela et al. 2009, Dionysopoulou et al. 2013, Palenzuela 2013; Qian et al. 2017), and are therefore less robust in highly magnetized plasma regions that are frequently encountered around black holes and neutron stars.

In this work, we implement the IMEX method of Bucciantini & Del Zanna 2013 combined with several novel and robust primitive-recovery methods for GRRMHD in the Black Hole Accretion Code (BHAC, Porth et al. 2017), a versatile general-relativistic magneto-fluid code based on the MPI-AMRVAC framework (van der Holst et al. 2008; Keppens et al. 2012; Porth et al. 2014; Xia et al. 2018). The designed recovery methods fully incorporate the electric field dynamics, such that highly magnetized regions around black

holes and neutron stars can be accurately resolved in the resistive regime in between the electrovacuum ($\eta \rightarrow \infty$) and the ideal-MHD limits ($\eta \rightarrow 0$). The methods are compared to the standard recovery schemes as presented by Palenzuela et al. (2009), Bucciantini & Del Zanna (2013), Dionysopoulou et al. (2013), and Mignone et al. (2019). We provide full details of the recovery procedure, such that it can be readily implemented in GRRMHD algorithms. We also propose a fall-back strategy if one or more methods fail to retrieve the primitive variables. The various methods are assessed for their accuracy, computational cost, and robustness in a survey over different parameter spaces and in several one- and multidimensional tests that are relevant for astrophysics.

In addition to having to deal with small time scales, resistive relativistic simulations have to resolve dissipative phenomena that occur across multiple spatial scales. With a uniform mesh, the computational costs of large-scale simulations with enough resolution to resolve the dissipative processes rapidly becomes prohibitive. An effective solution for problems where it is essential to simultaneously resolve microscopic and macroscopic dynamics can be found in adaptive mesh refinement (AMR) techniques. With these methods, the underlying grid on which the calculations are done is refined during the simulation. Adopting criteria that are based on the properties of the plasma dynamics, a finer grid is introduced in order to accurately resolve smaller scales in a confined area, thus dramatically reducing the computational costs (Keppens et al. 2003; Porth et al. 2017). A constrained transport (CT) method that is compatible with AMR is employed to keep the divergence of the magnetic field equal to machine precision at all times (Olivares et al. 2018; Olivares et al. 2019). The algorithm is developed to solve the GRRMHD equations in any spacetime metric in either one, two, or three spatial dimensions.

The paper is organized as follows: Sec. 2 contains the GRMHD equations and illustrates the main differences with the special relativistic and non-relativistic limits. Section 3 describes the numerical methods that are used to solve the GRRMHD equations. In Sec. 4 these methods are tested for well-known cases in special and general-relativistic magneto-hydrodynamics and the accuracy of different methods for the conserved to primitive variable transformation is explored. Our findings are summarized in Sec. 5.

2. GENERAL-RELATIVISTIC RESISTIVE MAGNETOHYDRODYNAMICS

In this section we briefly describe the covariant GRRMHD equations and introduce the notation as used in this paper. We mainly emphasize the differences between GRRMHD and the ideal-GRMHD equations solved in BHAC. More information and details on the numerical schemes and on the form of the chosen equations can be found in Porth et al. (2017).

We follow the derivation of the GRRMHD equations as in [Bucciantini & Del Zanna \(2013\)](#). For the remainder of this paper, we choose a $(-, +, +, +)$ signature for the spacetime metric. Units are adopted in which the speed of light, $c = 1$, vacuum permeability $\mu_0 = 1$, vacuum permittivity $\epsilon_0 = 1$, the gravitational constant $G = 1$, and all factors $4\pi = 1$. When considering curved spacetimes, all masses are normalized to the mass of the central object. Greek indices run over space and time, i.e., $(0,1,2,3)$, and Roman indices run over space only i.e., $(1,2,3)$.

2.1. 3 + 1 formulation of general relativity

In the context of numerically solving the GRRMHD equations, it is useful to write the equations in the 3 + 1 form based on the Arnowitt-Deser-Misner (ADM) formalism (see e.g., [Rezzolla & Zanotti 2013](#)). We introduce the foliation of space-like hypersurfaces Σ_t , defined as iso-surfaces of a scalar time function t , and a time-like unit vector that is normal to these hypersurfaces ([Porth et al. 2017](#))

$$n_\mu := -\alpha \nabla_\mu t, \quad (1)$$

where α is the lapse function. The frame of the *Eulerian observer* is defined by the four-velocity n^μ and the metric associated with each slice Σ_t can be written as

$$\gamma_{\mu\nu} := g_{\mu\nu} + n_\mu n_\nu. \quad (2)$$

The spatial projection operator is then chosen

$$\gamma^\mu{}_\nu := \delta^\mu{}_\nu + n^\mu n_\nu, \quad (3)$$

thus satisfying the constraint $\gamma^\mu{}_\nu n_\mu = 0$. This can be used to project any four-vector or tensor into its spatial and temporal component. In this formulation, any metric can be written in the form

$$g_{\mu\nu} = \begin{pmatrix} -\alpha^2 + \beta_k \beta^k & \beta_j \\ \beta_j & \gamma_{ij} \end{pmatrix}, \quad (4)$$

where β^i is the shift three-vector, and γ_{ij} is the three-metric representing the spatial part of $g_{\mu\nu}$, with determinant γ for which $(-g)^{1/2} := \alpha \gamma^{1/2}$. The corresponding inverse metric reads

$$g^{\mu\nu} = \begin{pmatrix} -1/\alpha^2 & \beta^j/\alpha^2 \\ \beta^j/\alpha^2 & \gamma^{ij} - \beta^i \beta^j/\alpha^2 \end{pmatrix}, \quad (5)$$

where γ^{ij} is the algebraic inverse of γ_{ij} , and $\beta^i = \gamma^{ij} \beta_j$. It is generally straightforward to obtain the expressions of α , β^i and γ_{ij} from the standard formulation of any general-relativistic metric (see [Porth et al. 2017](#) for commonly used metrics in BHAC). Special relativity is trivially retrieved by setting $\alpha = 1$, $\beta^i = 0$, and $\gamma^{ij} = \delta^{ij}$. In the 3 + 1 formalism, the line element is written

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij} (dx^i + \beta^i dt) (dx^j + \beta^j dt), \quad (6)$$

describing the motion of coordinate lines as seen by an Eulerian observer

$$x^i_{t+dt} = x^i - \beta^i(t, x^j) dt, \quad (7)$$

moving with four-velocity

$$n_\mu = (-\alpha, 0, 0, 0), \quad n^\mu = (1/\alpha, -\beta^i/\alpha). \quad (8)$$

A fluid element with four-velocity u^μ has a Lorentz factor $\Gamma := -u^\mu n_\mu = \alpha u^0 = (1 - v^2)^{-1/2}$ with $v^2 := v_i v^i$. This defines the fluid three-velocity

$$v^i := \frac{\gamma^i{}_\mu u^\mu}{\Gamma} = \frac{u^i}{\Gamma} + \frac{\beta^i}{\alpha}, \quad v_i := \gamma_{ij} v^j = \frac{u_i}{\Gamma}. \quad (9)$$

2.2. The fluid conservation equations

The fluid equations in general relativity are written as a set of conservation laws for mass

$$\nabla_\mu (\rho u^\mu) = 0, \quad (10)$$

and energy and momentum

$$\nabla_\mu T^{\mu\nu} = 0, \quad (11)$$

where ρ is the rest-mass density. The stress-energy tensor for a magnetized perfect fluid is written ([Dionysopoulou et al. 2013](#); [Qian et al. 2017](#))

$$T^{\mu\nu} \equiv T_{\text{fluid}}^{\mu\nu} + T_{\text{EM}}^{\mu\nu}, \quad (12)$$

where the fluid part is expressed independently of the electromagnetic fields (see e.g., [Gammie et al. 2003](#)):

$$T_{\text{fluid}}^{\mu\nu} = [\rho(1 + \epsilon) + p] u^\mu u^\nu + p g^{\mu\nu}, \quad (13)$$

with fluid pressure p and specific internal energy ϵ . The electromagnetic part is generally given by

$$T_{\text{EM}}^{\mu\nu} = F^{\mu\alpha} F^\nu{}_\alpha - \frac{1}{4} g^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}, \quad (14)$$

where $F^{\mu\nu}$ is the Maxwell tensor with Hodge dual $*F^{\mu\nu}$, the Faraday tensor.

Applying the 3 + 1 split and assuming a stationary space-time, the conservation Eqs. (10)–(11) can be written in the conservative form

$$\partial_t (\gamma^{1/2} D) + \partial_i \left[\gamma^{1/2} (-\beta^i D + \alpha v^i D) \right] = 0, \quad (15)$$

$$\begin{aligned} \partial_t (\gamma^{1/2} S_j) + \partial_i \left[\gamma^{1/2} (-\beta^i S_j + \alpha W^i{}_j) \right] = \\ \gamma^{1/2} \left(\frac{1}{2} \alpha W^{ik} \partial_j \gamma_{ik} + S_i \partial_j \beta^i - U \partial_j \alpha \right), \end{aligned} \quad (16)$$

$$\partial_t \left(\gamma^{1/2} \tau \right) + \partial_i \left[\gamma^{1/2} \left(-\beta^i \tau + \alpha \left(S^i - v^i D \right) \right) \right] = \gamma^{1/2} \left(\frac{1}{2} W^{ik} \beta^j \partial_j \gamma_{ik} + W^j_i \partial_j \beta^i - S^j \partial_j \alpha \right), \quad (17)$$

where we repeated the equations solved in (Porth et al. 2017) for ideal-GRMHD. The purely spatial variant of the stress-energy tensor W^{ij} reads

$$\begin{aligned} W^{ij} &:= \gamma^i_\mu \gamma^j_\nu T^{\mu\nu} \\ &= \rho h \Gamma^2 v^i v^j - E^i E^j - B^i B^j + \left[p + \frac{1}{2} \left(E^2 + B^2 \right) \right] \gamma^{ij}, \end{aligned} \quad (18)$$

with $h = h(\rho, p)$ the specific enthalpy of the fluid, $E^2 := E^i E_i$, $B^2 := B^i B_i$ and E^i and B^i the three-vector spatial parts of the electric and magnetic field in the Eulerian frame as defined in Eq. (27).

Equations (15)–(17) describe the evolution of *conserved quantities* as measured from an Eulerian reference frame, namely the rest-mass density

$$D := -\rho u^\mu n_\mu = \rho \Gamma, \quad (19)$$

the covariant 3-momentum density

$$S_i := \gamma^\mu_i n^\alpha T_{\alpha\mu} = \rho h \Gamma^2 v_i + \gamma^{1/2} \eta_{ijk} E^j B^k, \quad (20)$$

and the (rescaled) conserved energy density $\tau := U - D$, where

$$U := T^{\mu\nu} n_\mu n_\nu = \rho h \Gamma^2 - p + \frac{1}{2} \left(E^2 + B^2 \right). \quad (21)$$

The electric and magnetic fields are evolved through Maxwell's equations described in Sec. 2.3. In the absence of gravity, when $\alpha = 1$, $\beta^i = 0$, $\gamma^{1/2} = 1$, and $\partial_t \gamma = 0$, these reduce to the special-relativistic conservation laws. The non-relativistic (Newtonian) limit is obtained by letting $v^2 \ll 1$, $p \ll \rho$ and $E^2 \ll B^2 \ll \rho$, bearing in mind that $c = 1$.

Again, we emphasize that, unlike in ideal-GRMHD, the electric field in equations (16) and (17) cannot be substituted as $E^i = -\gamma^{-1/2} \eta^{ijk} B_j v_k$ (with η_{ijk} the spatial Levi-Civita antisymmetric symbol). In the resistive-GRMHD limit, E^i has to be obtained from Ampère's law (see next Section), resulting in a larger system of equations and therefore implying a more complex solution procedure.

2.3. The Maxwell equations

The covariant Maxwell equations in tensorial form are

$$\nabla_\nu F^{\mu\nu} = \mathcal{J}^\mu, \quad (22)$$

$$\nabla_\nu {}^* F^{\mu\nu} = 0, \quad (23)$$

where \mathcal{J}^μ is the electric 4-current. Applying the 3 + 1 split, the tensors in the Maxwell equations (22) and (23) can be

decomposed in terms of the electromagnetic fields as seen by an observer moving along the normal direction n^ν as

$$F^{\mu\nu} = n^\mu E^\nu - n^\nu E^\mu - (-g)^{-1/2} \eta^{\mu\nu\lambda\kappa} n_\lambda B_\kappa, \quad (24)$$

$${}^* F^{\mu\nu} = n^\mu B^\nu - n^\nu B^\mu + (-g)^{-1/2} \eta^{\mu\nu\lambda\kappa} n_\lambda E_\kappa. \quad (25)$$

with $\eta^{\mu\nu\lambda\kappa}$ the fully anti-symmetric symbol (see e.g., Rezzolla & Zanotti 2013) and electric and magnetic field four-vectors.

$$E^\mu := F^{\mu\nu} n_\nu, \quad B^\mu := {}^* F^{\mu\nu} n_\nu, \quad (26)$$

and their three-vector spatial parts in the Eulerian frame

$$E^i = F^{i\nu} n_\nu = \alpha F^{i0}, \quad B^i = {}^* F^{i\nu} n_\nu = \alpha {}^* F^{i0}. \quad (27)$$

Equation (23) can be written in component form, resulting in Faraday's law,

$$\partial_t \left(\gamma^{1/2} B^j \right) + \partial_i \left[\gamma^{1/2} \left(\beta^j B^i - \beta^i B^j + \gamma^{-1/2} \eta^{ijk} \alpha E_k \right) \right] = 0. \quad (28)$$

The temporal component of Eq. (23) leads to the solenoidal constraint

$$\gamma^{-1/2} \partial_i \left(\gamma^{1/2} B^i \right) = 0. \quad (29)$$

In addition to Faraday's law, Eq. (23) can be written in component form, resulting in Ampère's law for the electric field evolution in GRRMHD

$$\begin{aligned} \partial_t \left(\gamma^{1/2} E^j \right) + \partial_i \left[\gamma^{1/2} \left(\beta^j E^i - \beta^i E^j - \gamma^{-1/2} \eta^{ijk} \alpha B_k \right) \right] = \\ - \gamma^{1/2} \left(\alpha J^j - q \beta^j \right). \end{aligned} \quad (30)$$

The current density \mathcal{J}^μ is decomposed as

$$\mathcal{J}^\mu = n^\mu q + J^\mu \quad (31)$$

where $J^\mu n_\mu = 0$, $q = -\mathcal{J}^\mu n_\mu$ is the charge density, and J^μ the current density as measured by a Eulerian observer moving with four-velocity n^μ . The temporal component of Eq. (23) then provides the charge density q in equation (31)

$$\gamma^{-1/2} \partial_i \left(\gamma^{1/2} E^i \right) = q. \quad (32)$$

The spatial current density is obtained from the resistive Ohm's law in 3 + 1 split formulation (see e.g., Palenzuela et al. 2009; Bucciantini & Del Zanna 2013).

$$J^i = q v^i + \frac{\Gamma}{\eta} \left[E^i + \gamma^{-1/2} \eta^{ijk} v_j B_k - \left(v_k E^k \right) v^i \right], \quad (33)$$

with the resistivity η (not to be confused with the fully anti-symmetric symbol $\eta^{\mu\nu\lambda\kappa}$), as the reciprocal of the electrical

conductivity, i.e., $\eta = 1/\sigma$. Substituting equations (32) and (33) in (30) we obtain the final form of Ampère's law as

$$\begin{aligned} \partial_t \left(\gamma^{1/2} E^j \right) + \partial_i \left[\gamma^{1/2} \left(\beta^j E^i - \beta^i E^j - \gamma^{-1/2} \eta^{ijk} \alpha B_k \right) \right] = \\ - \gamma^{1/2} \frac{\alpha \Gamma}{\eta} \left[E^j + \gamma^{-1/2} \eta^{jik} v_i B_k - \left(v_k E^k \right) v^j \right] \\ - \left(\alpha v^j - \beta^j \right) \partial_j \left(\gamma^{1/2} E^j \right). \end{aligned} \quad (34)$$

Note that the resistivity η can depend both on space and time and that this description of the current density is valid in any metric. Hall or dynamo terms can be added by extending equation (33) to a generalized Ohm's law (e.g., [Bucciantini & Del Zanna 2013](#); [Palenzuela 2013](#)).

Finally, it is useful to introduce the fluid-frame (comoving) electric and magnetic field ([Bucciantini & Del Zanna 2013](#)),

$$e^\mu = \Gamma(E^i v_i) n^\mu + \Gamma(E^\mu + \gamma^{-1/2} \eta^{\mu\nu\lambda} v_\nu B_\lambda), \quad (35)$$

$$b^\mu = \Gamma(B^i v_i) n^\mu + \Gamma(B^\mu - \gamma^{-1/2} \eta^{\mu\nu\lambda} v_\nu E_\lambda), \quad (36)$$

allowing to rewrite the electromagnetic part $T_{EM}^{\mu\nu}$ of Eq. (12) as ([Qian et al. 2017](#))

$$\begin{aligned} T_{EM}^{\mu\nu} = (b^2 + e^2) \left(u^\mu u^\nu + \frac{1}{2} g^{\mu\nu} \right) - b^\mu b^\nu - e^\mu e^\nu \\ - u_\lambda e_\beta b_\kappa \left(u^\mu \gamma^{-1/2} \eta^{\nu\lambda\beta\kappa} + u^\nu \gamma^{-1/2} \eta^{\mu\lambda\beta\kappa} \right). \end{aligned} \quad (37)$$

The comoving electric and magnetic field strength $e^2 := e^\mu e_\mu$, $b^2 := b^\mu b_\mu$ are also employed in the definition of useful dimensionless plasma quantities, e.g., the magnetization $\sigma_{\text{mag}} := b^2/\rho$ and the gas-to-magnetic pressure ratio, or plasma- $\beta_{\text{th}} := p_{\text{gas}}/p_{\text{mag}} = 2p/b^2$.

3. NUMERICAL IMPLEMENTATION

In this Section we present the numerical approach to solve the set of GRRMHD equations in BHAC. For small resistivity, the timescales of the stiff and non-stiff parts of the system become very different and the set of equations can be regarded as a hyperbolic system with relaxation terms. These relaxation terms require special care to be captured accurately without adopting an extremely small time-step. We present and test the implementation of an IMEX RK method in BHAC, where the stiff terms are treated with an implicit step and the non-stiff parts with a standard explicit step. Our implementation differs from previous works in the use of a new primitive-recovery method (see Sec. 3.5) designed to obtain high accuracy and reliability in regimes of low resistivity and low plasma- β_{th} . We test our algorithm against several analytic and non-analytic benchmarks in Sec. 4. The methods as presented in curved spacetime are straightforwardly applicable in flat spacetime and the difficulties regarding stiff source terms are completely analogous.

3.1. The full system of equations in BHAC

To adopt a conservative scheme we write the full system of equations treated in BHAC in the form

$$\partial_t(\gamma^{1/2} \mathbf{U}) + \partial_i(\gamma^{1/2} \mathbf{F}^i) = \gamma^{1/2} \mathbf{S}, \quad (38)$$

where \mathbf{U} represents conserved variables and \mathbf{F}^i are the fluxes,

$$\mathbf{U} = \begin{bmatrix} D \\ S_j \\ \tau \\ B^j \\ E^j \end{bmatrix}, \quad \mathbf{F}^i = \begin{bmatrix} \mathcal{V}^i D \\ \alpha W_j^i - \beta^i S_j \\ \alpha(S^i - v^i D) - \beta^i \tau \\ \beta^j B^i - \beta^i B^j + \gamma^{-1/2} \eta^{ijk} \alpha E_k \\ \beta^j E^i - \beta^i E^j - \gamma^{-1/2} \eta^{ijk} \alpha B_k \end{bmatrix}, \quad (39)$$

with the transport velocity $\mathcal{V}^i := \alpha v^i - \beta^i$. The sources read

$$\mathbf{S} = \begin{bmatrix} 0 \\ \frac{1}{2} \alpha W^{ik} \partial_j \gamma_{ik} + S_i \partial_j \beta^i - U \partial_j \alpha \\ \frac{1}{2} W^{ik} \beta^j \partial_j \gamma_{ik} + W_i^j \partial_j \beta^i - S^j \partial_j \alpha \\ 0 \\ -\alpha J^j + \beta^j q \end{bmatrix}. \quad (40)$$

The form of the GRRMHD equations as evolved in BHAC allows for a temporally and spatially dependent scalar resistivity $\eta(x^i, t)$. In our implementation of GRRMHD the resistivity can depend on any dynamic or static quantity, e.g., rest-mass density, current density or the position explicitly (see [Ripperda et al. 2019](#) for an application of non-uniform current-dependent resistivity).

3.2. Characteristic speed

The characteristic velocities are required by the Riemann solver and the Courant-Friedrichs-Lewy (CFL) condition that limits the time-step. Given the 3+1 structure of the fluxes, we obtain characteristic waves of the form

$$\lambda_\pm^i = \alpha \lambda_\pm^i - \beta^i, \quad (41)$$

with λ_\pm^i the characteristic velocity in the i -th direction in the locally flat frame $\alpha \rightarrow 1$, $\beta^j \rightarrow 0$ ([Anile 1989](#); [Del Zanna et al. 2007](#)). For simplicity we assume the characteristics to be in the limit of maximum diffusivity, i.e., the fastest waves locally travel with the speed of light, which after transforming to the Eulerian frame ([Pons et al. 1998](#); [White et al. 2016](#)) yields for each component ([Del Zanna et al. 2007](#); [Bucciantini & Del Zanna 2013](#))

$$\lambda_\pm^i = \pm \sqrt{\gamma^{ii}}. \quad (42)$$

Note that a multidimensional Riemann solver for the SR-RMHD equations was recently presented by [Mignone et al. \(2018, 2019\)](#) and [Miranda-Aranguren et al. \(2018\)](#).

3.3. Constraint equations

Our implementation of the GRRMHD equations enforces Eq. (29) to roundoff-error by means of the staggered CT scheme of Balsara & Spicer (1999), whose implementation in BHAC has been presented in detail by Olivares et al. (2019). The charge density is obtained by numerically taking the divergence of the evolved electric field as in Eq. (32).

3.4. Time stepping: IMEX method

When the resistivity of the plasma is very small yet finite, the system of Eqs. (38) becomes stiff. An explicit integration, which is commonly used in ideal-GRMHD codes, then requires time-steps that essentially scale with the resistivity, resulting in prohibitive computational costs. In Komissarov (2007), a Strang-splitting technique is applied in SRRMHD simulations such that the stiff resistive terms can be explicitly computed for the electric field evolution. However, the procedure relies on the assumption that magnetic field and the fluid velocity field remains constant during the (faster) evolution of the resistive electric field E^i . For small values of η , the solution becomes inaccurate or requires extremely small time-steps. An alternative solution is to split off the stiff part of the system of Eqs. (38) and treat it with an implicit step. With this approach no assumptions are necessary and in principle all resistivity regimes can be treated without time-step restrictions other than a standard CFL condition. This method was first proposed by Palenzuela et al. (2009) for SRRMHD and later extended by Bucciantini & Del Zanna (2013) and Dionysopoulou et al. (2013) to GRRMHD. Several improvements of the IMEX method for SRRMHD have been proposed recently by Mignone et al. (2019).

Here, we adopt the first-second order IMEX scheme as proposed by Bucciantini & Del Zanna (2013). In particular, we split the system (38) into non-stiff

$$\partial_t(\gamma^{1/2} \mathbf{X}) = \mathbf{Q}_X(\gamma^{1/2} \mathbf{X}, \gamma^{1/2} \mathbf{Y}), \quad (43)$$

and stiff equations

$$\partial_t(\gamma^{1/2} \mathbf{Y}) = \mathbf{Q}_Y(\gamma^{1/2} \mathbf{X}, \gamma^{1/2} \mathbf{Y}) + \frac{1}{\eta} \mathbf{R}_Y(\gamma^{1/2} \mathbf{X}, \gamma^{1/2} \mathbf{Y}), \quad (44)$$

with the conserved quantities \mathbf{U} split into two subsets $\{\mathbf{X}, \mathbf{Y}\}$

$$\mathbf{X} := \begin{bmatrix} D \\ S_i \\ \tau \\ B^i \end{bmatrix}, \quad \mathbf{Y} := \begin{bmatrix} E^i \end{bmatrix}, \quad (45)$$

containing the non-stiff and the stiff variables, respectively.

The time stepping involves a second-order time discretization for the non-stiff variables in \mathbf{X} , which are evolved explicitly as in Porth et al. (2017), and a first-order scheme for the

stiff variables in \mathbf{Y} , evolved implicitly. The overall solution step from time level n to $n+1$ is written

$$\begin{aligned} \tilde{\mathbf{X}}^{(1)} &= \tilde{\mathbf{X}}^n + \frac{\Delta t}{2} \mathbf{Q}_X(\tilde{\mathbf{X}}^n, \tilde{\mathbf{Y}}^n), \\ \tilde{\mathbf{Y}}^{(1)} &= \tilde{\mathbf{Y}}^n + \frac{\Delta t}{2} \mathbf{Q}_Y(\tilde{\mathbf{X}}^n, \tilde{\mathbf{Y}}^n) + \frac{\Delta t}{2\eta} \mathbf{R}_Y(\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{Y}}^{(1)}), \\ \tilde{\mathbf{X}}^{n+1} &= \tilde{\mathbf{X}}^n + \Delta t \mathbf{Q}_X(\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{Y}}^{(1)}), \\ \tilde{\mathbf{Y}}^{n+1} &= \tilde{\mathbf{Y}}^n + \Delta t \mathbf{Q}_Y(\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{Y}}^{(1)}) + \frac{\Delta t}{\eta} \mathbf{R}_Y(\tilde{\mathbf{X}}^{n+1}, \tilde{\mathbf{Y}}^{n+1}), \end{aligned} \quad (46)$$

where we have incorporated the $\gamma^{1/2}$ factors in $\tilde{\mathbf{X}} := \gamma^{1/2} \mathbf{X}$, $\tilde{\mathbf{Y}} := \gamma^{1/2} \mathbf{Y}$.

The implicit step represented by the \mathbf{R}_Y terms can be treated analytically due to the linearity (in the electric field) of the resistive Ohm's law (33). For simplicity, but without loss of generality, consider the last electric field update step in the algorithm above, from intermediate level (1) to the next time level $n+1$. Using Eq. (33), this can be written explicitly as

$$\tilde{E}^{i,n+1} = \tilde{E}^{i,*} + \frac{\alpha \Delta t \Gamma}{\eta} \left[\tilde{E}^{i,n+1} + \gamma^{-1/2} \eta^{ijk} v_j^{n+1} \tilde{B}_k^{n+1} - \left(\tilde{E}^{k,n+1} v_k^{n+1} \right) v^{i,n+1} \right], \quad (47)$$

where $\tilde{E}^i := \gamma^{1/2} E^i$ and $\tilde{B}^i := \gamma^{1/2} B^i$. Here, the explicitly updated electric field is $\tilde{E}^{i,*} = \tilde{E}^{i,n} + \Delta t \mathbf{Q}_Y(\tilde{\mathbf{X}}^{(1)}, \tilde{\mathbf{E}}^{(1)})$. Equation (47) only involves local operations (no spatial derivatives needed), hence its inversion is straightforward if the terms on the right-hand-side are known, and leads to an explicit expression for the new electric field,

$$\tilde{E}^{i,n+1} = \frac{\tilde{E}^{i,*}}{1 + \sigma_H \Gamma^{n+1}} - \frac{\Gamma^{n+1}}{1 + \sigma_H \Gamma^{n+1}} \left[\gamma^{-1/2} \eta^{ijk} v_j^{n+1} \tilde{B}_k^{n+1} - \Gamma^{n+1} \frac{\tilde{E}^{k,*} v_k^{n+1}}{\Gamma^{n+1} + \sigma_H} v^{i,n+1} \right], \quad (48)$$

where $\sigma_H := \alpha \Delta t / \eta$. In order to avoid singularities in the ideal-MHD limit $\eta \rightarrow 0$, the equation above can be recast as

$$\tilde{E}^{i,n+1} = \frac{\eta \tilde{E}^{i,*}}{\eta + \sigma_L \Gamma^{n+1}} - \frac{\sigma_L \Gamma^{n+1}}{\eta + \sigma_L \Gamma^{n+1}} \left[\gamma^{-1/2} \eta^{ijk} v_j^{n+1} \tilde{B}_k^{n+1} - \eta \Gamma^{n+1} \frac{\tilde{E}^{k,*} v_k^{n+1}}{\eta \Gamma^{n+1} + \sigma_L} v^{i,n+1} \right], \quad (49)$$

where $\sigma_L := \alpha \Delta t = \eta \sigma_H$. Note that Bucciantini & Del Zanna (2013) have a typo in their equations (33), (35), and (36) for the formulation of the implicit and explicit updates. The analytic inversion is applied in the same way at each substep of the time-stepping algorithm, hence making Eqs. (48) and (49) completely general by adjusting σ_H and σ_L with the coefficients from a Butcher tableau corresponding

to the current substep (Pareschi & Russo 2005; Palenzuela et al. 2009). Therefore, the simple first-second order IMEX algorithm (46) can be extended to arbitrary high-order accuracy while keeping the update equations for E^i unchanged. However, contrary to higher-order schemes, the first-second algorithm (46) naturally includes the ideal-MHD limit, $\eta = 0$, without suffering from numerical singularities (Bucciantini & Del Zanna 2013).

Note that the electric field update in Eqs. (48) and (49) involves the three-velocity v^i to be known at the same time level of E^i . However, v^i is a primitive quantity which depends nonlinearly on E^i . This dependence makes the update equations intrinsically implicit, and implies that the electric field update in the IMEX algorithm (46) must be carried out concurrently to a conserved-to-primitive variable inversion. The inversion recovery strategy is of key importance for GR(R)MHD simulations and high sensitivity to the physical parameters makes it a particularly challenging part of the algorithm.

3.5. Transformation of conserved to primitive variables

Throughout the solution process of the GRRMHD equations (38), a transformation of the *conserved variables* D , S_i , τ into the *primitive variables* ρ , v_i , and p is necessary. This is a local operation that requires to solve the system of nonlinear Eqs. (19), (20), and (21). The solution of such a system cannot be written in closed form, requiring a root-finding algorithm that constitutes one of the most expensive and sensitive parts of the whole solution procedure of relativistic MHD codes¹.

For most of the operations during the GRMHD evolution (i.e., as long as the electric field does not depend on primitive variables), we carry out the conserved-to-primitive inversion by solving the system of equations

$$\begin{aligned} D &:= \rho\Gamma, \\ S_i &:= \xi v_i + \gamma^{1/2} \eta_{ijk} E^j B^k, \\ \tau &:= \xi - p - D + \frac{1}{2} (E^2 + B^2), \end{aligned} \quad (50)$$

where $\xi := \rho h \Gamma^2$. Provided that E^i and B^i are known, such a system can usually be reduced to one single equation (in ξ , p , or other scalar variables, see e.g., Noble et al. 2006 or Siegel et al. 2018) and solved with a one-dimensional (1D) Newton-Raphson (NR) iteration (where 1D refers to the single scalar equation that has to be solved and not to a spatial dimension). This is the standard approach in BHAC for the solution of the ideal-GRMHD equations (van der Holst et al. 2008; Keppens et al. 2012; Porth et al. 2017).

However, the IMEX scheme presented in the previous Section for the GRRMHD equations involves an implicit update

of E^i where both the new electric field and the new three-velocity are unknown. In this case, the system of nonlinear Eqs. (50) above cannot be inverted, as E^i is not known a priori but rather an additional variable determined by Eq. (48) or (49). Therefore, during the implicit step the conserved to primitive transformation must be carried out concurrently to the implicit electric field update. The system of equations (50) is thus augmented with equation (48) or (49) for the electric field, forming again a closed set in the variables ρ , v_i , ξ , and E^i . The new system requires a robust and accurate nonlinear solution method, typically an iterative algorithm. This combined update-transformation step is a crucial operation, which heavily influences the overall performance and accuracy of the IMEX algorithm. If the iteration fails to converge, the electric field cannot be updated and the GRRMHD solution becomes inaccurate. The high failure rate in this step is an issue reported in several relativistic resistive MHD implementations, particularly in low plasma- $\beta_{\text{th}} \lesssim 0.5$ regimes (Palenzuela et al. 2009; Dionysopoulou et al. 2013; Del Zanna et al. 2016; Qian et al. 2017). A robust inversion method that is reliable in particularly demanding physical regimes (e.g., low- β_{th} , high- σ_{mag}) is essential to model accretion flows onto black holes.

Here we present a set of strategies for the inversion-update step. Based on the performance of each strategy, we design a robust approach that yields a minimal amount of failures, allowing for a wide range of simulation parameters that are unattainable with currently available methods.

3.5.1. “1D” fixed-point strategy

The most commonly used approach in GRRMHD consists of reducing the system of nonlinear equations to one (Palenzuela et al. 2009; Dionysopoulou et al. 2013), or two (Bucciantini & Del Zanna 2013; Qian et al. 2017) scalar equation(s). A usual choice is to solve the energy equation

$$\xi = p + D + \tau - \frac{1}{2} (E^2(v_i) + B^2) \quad (51)$$

for the scalar variable ξ , hence the “1D” fixed-point notation, or alternatively “2D” fixed-point for two scalar variables (see e.g., Noble et al. 2006 and Qian et al. 2017 for an iteration on Γ and $W := (\rho + p\hat{\gamma}/(\hat{\gamma} - 1))\Gamma^2$). The pressure $p(\rho, \xi)$ is determined by eliminating the dependence on ρ by substitution with $\rho = D/\Gamma$, and reduced to a function of ξ only via the relation

$$\Gamma = \sqrt{1 + \frac{S'^2}{\xi^2 - S'^2}}, \quad (52)$$

which follows from Eq. (50) above. Here, $S'^2 := S'^i S'_i$, with $S'_i := S_i - \gamma^{1/2} \eta_{ijk} E^j B^k$. The dependence of E^i on v_i [Eqs. (48) or (49)], however, is nonlinear and cannot be recast into an explicit relation $E^i(\xi)$. As a consequence, the usual solution approach consists of a hybrid NR iteration where

¹ Note that B^i is both a conserved and a primitive variable, hence an inversion step is not needed for the magnetic field.

the dependence of E^i on ξ is not taken into account, and the electric field is obtained with a fixed-point iteration. Starting from an initial guess for ξ and E^i , each nonlinear iteration is composed of the following steps:

1. Compute the velocity as

$$v_i = \frac{S_i - \gamma^{1/2} \eta_{ijk} E^j B^k}{\xi^{(m)}}. \quad (53)$$

2. Compute Γ and p , and the electric field from equation (48) or (49).
3. Compute the residual,

$$f(\xi) = \xi - p - D - \tau + \frac{1}{2}(E^2 + B^2), \quad (54)$$

and its derivative neglecting the dependence of E^i on ξ ,

$$\frac{df}{d\xi} = 1 - \frac{dp}{d\xi}. \quad (55)$$

4. Update the value of ξ at the m -th iteration with a NR step,

$$\xi^{(m+1)} = \xi^{(m)} - f(\xi^{(m)}) \left(\frac{df(\xi^{(m)})}{d\xi} \right)^{-1}. \quad (56)$$

5. Track the absolute change in the iteration variables, $|\xi^{(m+1)} - \xi^{(m)}|$ and $|E^{i,(m+1)} - E^{i,(m)}|$. The iteration is stopped if this difference falls below a prescribed tolerance, which we normally take to be 10^{-14} .

Step 3 above is where a crucial assumption is introduced. Computing the electric field with a fixed-point strategy of this type and neglecting the dependence $E^i(\xi)$ is equivalent to assuming that the electric field only varies slightly between successive Newton steps. This is not always true, especially when the system is very stiff. The stiffness of the nonlinear equations can originate from a parameter choice (e.g., for low values of the resistivity η), or from the physical regime described by the conserved quantities (e.g., large electromagnetic energy density compared to the rest-mass density or pressure resulting in low β_{th} and high σ_{mag}). In such cases, the electric field becomes dynamically important, and its variation with respect to other quantities cannot be neglected.

Most implementations of IMEX schemes employing the 1D (or 2D) fixed-point scheme above report numerical issues related to combinations of low- η , high- σ_{mag} , and low- β_{th} regimes (Palenzuela et al. 2009; Qian et al. 2017). Failures in the inversion-update step can sometimes be mitigated by reducing the time step (Palenzuela et al. 2009), which effectively reduces the stiffness in the nonlinear system of equations to invert. However, this is an undesirable constraint especially

for production runs of accretion flows, where large regions of low- β_{th} (i.e. the ambient surrounding the accretion disk) or high- σ_{mag} (i.e. the jet) can rapidly determine a degradation of computational performance.

3.5.2. “3D/4D” fully-consistent strategies

A more robust approach to the inversion-update problem is to eliminate any assumption on the importance of the dynamics of E^i , and treat the whole system of nonlinear equations simultaneously and consistently. The system of equations (50) for the fluid variables, together with the relation $\Gamma = (1 - v^2)^{-1/2}$, involves in principle 6 independent unknowns; the augmented system including equation (48) or (49) for the electric field, increases the number of total unknowns to 9. It is essential to reduce the problem to a smaller set of equations, in order to improve the robustness of the iterative solution procedure. A larger number of unknowns implies a higher computational cost and involves a larger solution space, thus decreasing the likelihood of convergence. In our analysis, we find that the problem can be reduced to a minimal system of three or four scalar equations, depending on the quantities chosen as iteration variables.

In general, the system of equations is described by a set of nonlinear residuals $\mathbf{f}(\mathbf{x})$ in the unknowns \mathbf{x} , which contains either three or four components (hence 3D/4D). Starting from an initial guess, we adopt an iterative strategy that progressively decreases the residuals until $\mathbf{f}(\mathbf{x}) \simeq \mathbf{0}$. In BHAC, the iteration is typically carried out with a multi-dimensional NR scheme (using a hardcoded analytic Jacobian $\mathbf{H}(\mathbf{x}) = \partial \mathbf{f}(\mathbf{x}) / \partial \mathbf{x}$). For robustness and flexibility, we have the option of selecting a Jacobian-free Newton-Krylov (NK) scheme that does not require the full Jacobian but only directional derivatives, thus allowing for new strategies to be easily implemented (see e.g., Kelley 1995 for a reference implementation of NK schemes).

In our analysis of the inversion-update equations, we find that the iterative scheme yields the lowest failure rates when applied to the following reduced systems:

- *3D system on u_i* : since the new electric field is an explicit function of v_i [Eq. (48) or (49)], we find that it is possible to reduce the iteration to the three components of the fluid velocity, recast as the normalized 3-momentum $u_i = \Gamma v_i$, with $\Gamma = \sqrt{1 + \gamma^{ij} u_i u_j}$. The iteration variables are then $\mathbf{x} = \mathbf{u} = (u_1, u_2, u_3)$ and the residuals take the form

$$\mathbf{f}(\mathbf{u}) = u_i - \frac{S_i - \gamma^{1/2} \eta_{ijk} E^j B^k}{Dh}, \quad (57)$$

where E^j is determined by Eq. (48) or (49) as a function of \mathbf{u} , and $h(\rho, p)$ is computed as a function of \mathbf{u} only, by substituting $\rho = D/\Gamma$ and by recasting $p = p(\mathbf{u})$. The latter approach may not be possible for an arbitrary or

tabulated equation of state (EoS); however, this operation is straightforward for the EOS choices available in BHAC and typically employed for simulations of accretion flows onto black holes. For instance, the closure equation for perfect fluids with polytropic index $\hat{\gamma}$ can be written as $p = \rho(\hat{\gamma} - 1)\epsilon$, where

$$\epsilon = \Gamma \frac{\tau'}{D} - z \frac{\sqrt{S'^2}}{D} + \frac{z^2}{1 + \Gamma} \quad (58)$$

with $\tau' := \tau - (E^2 + B^2)/2$ and $z^2 := \Gamma^2 - 1$. With all quantities written as explicit functions of the iteration variables \mathbf{u} , the Newton algorithm can be applied to minimize the residuals (57). We note that a similar 3D approach has been presented by [Bucciantini & Del Zanna \(2013\)](#) and [Mignone et al. \(2019\)](#).

- *4D system on (ξ, u_i)* : as an alternative to the 3D system above, we can choose to retain ξ as an additional unknown related to the energy of the system. The iteration variables in this case are $\mathbf{x} = (\xi, \mathbf{u})$ and the residuals read

$$\mathbf{f}(\xi, \mathbf{u}) = \begin{bmatrix} \xi - p - D - \tau + (E^2 + B^2)/2 \\ u_i - \Gamma \left(S_i - \gamma^{1/2} \eta_{ijk} E^j B^k \right) / \xi \end{bmatrix}. \quad (59)$$

Here, the electric field is still computed from \mathbf{u} , and the pressure is determined from ξ and Γ as in the 1D approach of Sec. 3.5.1. This strategy involves a larger system of equations to handle compared to the 3D case above, but requires less operations at each nonlinear iteration.

- *4D system on (z, E^i)* : as a final alternative we recast the system of equations to a formulation involving z and the electric field \mathbf{E} as iteration variables, $\mathbf{x} = (z, \mathbf{E})$. The residuals for this case are written

$$\mathbf{f}(z, \mathbf{E}) = \begin{bmatrix} z - \sqrt{S'^2}/(Dh) \\ E^i - f_E^i(v_i) \end{bmatrix}, \quad (60)$$

where $f_E^i(v_i)$ are the right-hand sides of equation (48) or (49). The velocity is explicitly computed from the iteration variables as $v_i = S'_i/(Dh\Gamma)$, and the specific enthalpy is retrieved from the pressure $p(z, \mathbf{E})$ similarly to the 3D strategy above.

By considering several possible strategies for the inversion-update step, we are allowed to explore a wide range of important properties such as computational cost and convergence rate in the parameter space of interest. As a general approach, in the schemes above we prefer to rely on variables that are not constrained between specific limiting values due to physical consistency, e.g., we make use of $u_i \in (-\infty, \infty)$ rather than $v_i \in (-1, 1)$, or $z \in (-\infty, \infty)$, which is preferable to $\Gamma \in [1, \infty)$

([Galeazzi et al. 2013](#)). In this way, the iterative scheme is less likely to fail due to the variables assuming out-of-range values. Additionally, the availability of several schemes allows for designing a robust backup strategy: in the case of failure of a primary inversion scheme, a second one can be employed that relies on different variables. The choice of iteration variables affects the convergence properties of each scheme, and we show in Sec. 4.1 how a backup strategy can be designed such that the smallest number of failures is achieved. As initial guess for the iterative schemes, we typically employ the value of the unknown quantities at the previous time-step (we find that different initial guesses produce little variations in the overall performance). When convergence is reached, we check the final values for all primitive quantities for physical consistency.

3.5.3. Entropy inversion

In highly magnetized regions of accretion flows (e.g., the relativistic jet in accretion flow simulations), the evolution equation for the conserved energy τ can sometimes become too numerically inaccurate, resulting in unphysical solutions to the conserved to primitive inversion problem. In such situations, BHAC relies on an additional backup strategy for the conserved to primitive inversion, based on the entropy κ . For most of the code operations that involve a standard inversion step (i.e. when E^i is known at the current time-step), this entropy “switch” consists of finding the root of a single nonlinear equation, typically in the unknown Γ . The energy τ is discarded in the process, and replaced with a value consistent with the newly recovered primitives. For details on this standard procedure we refer the reader to the corresponding Section in [Porth et al. \(2017\)](#).

When the entropy-switch is needed during the more complicated inversion-update step in the implicit part of our IMEX scheme, the system cannot be reduced to one single equation. We approach the problem by applying the same 3D/4D strategies presented above, with slight modifications. For the 3D scheme in u_i and the 4D scheme in (z, E^i) , we replace any closure relation with an ideal-gas law for the enthalpy ([Rezzolla & Zanotti 2013](#))

$$h = 1 + \frac{\hat{\gamma}}{\hat{\gamma} - 1} \frac{p}{\rho}, \quad (61)$$

augmented with the polytropic (isentropic) EOS $p = \kappa \rho^{\hat{\gamma}}$. The pressure can still be computed explicitly from the iteration variables, therefore leaving the scheme essentially unchanged. For the 4D scheme in (ξ, \mathbf{u}) , we replace the equation for ξ such that the residuals read

$$\mathbf{f}(\xi, \mathbf{u}) = \begin{bmatrix} \xi - \rho h \Gamma^2 \\ u_i - \Gamma \left(S_i - \gamma^{1/2} \eta_{ijk} E^j B^k \right) / \xi \end{bmatrix}, \quad (62)$$

where the enthalpy is still given by the polytropic EOS as above.

If the entropy-switch strategy is activated, it is applied by default whenever the primary inversion fails. Particularly for accretion flow simulations, the entropy-switch is also applied upon successful primary inversion in regions where a low $\beta_{\text{th}} < 10^{-2}$ is detected. Switching between different systems of nonlinear equations, which is required for the entropy-based inversion, is easily handled in BHAC with the NK subroutines, which do not require the full Jacobian for each different inversion strategy. In case of failure of the entropy-based inversion, the last-resort solutions include the replacement of the primitive variables in the faulty cells with averages from nearby converged zones. Alternatively, floor values in pressure, velocity, and rest-mass density can be set if convergence in the inversion step is not reached. Note that the electric field is not floored to an arbitrary value, but rather recalculated with equation (48) or (49) by using the floor value of v_i .

4. NUMERICAL TESTS

In this Section we present a number of validation tests for our implementation of the GRRMHD algorithm. We include comparisons on the reliability of the inversion-update strategies presented in Sec. 3.5, based on which we design a robust strategy that is employed as our default choice in BHAC. We also analyse the performance of the entropy-based inversion scheme with dedicated tests. We present one- and two-dimensional case studies and compare the results to reference high-resolution runs and analytic solutions. Finally, we show an example application of our algorithm in the astrophysically relevant case of a torus accreting onto a black hole.

4.1. Exploration of parameter space for primitive-variable recovery

In order to test the inversion-update strategies presented in Sec. 3.5, we explore the performance of each algorithm in a large parameter space. We choose to study vastly different GRMHD regimes with $\sigma_{\text{mag}} \in [10^{-2}, 10^2]$ and $\beta_{\text{th}} \in [10^{-10}, 10^5]$, by selecting appropriate sets of primitive variables. By picking primitive quantities $\Gamma - 1 \in [10^{-2}, 10^3]$ and $\eta \in [10^{-14}, 10^6]$, we can construct complete sets of corresponding conserved variables. These are then run through the inversion-update algorithm, whose performance can be evaluated by comparing the retrieved primitives with the manufactured initial sets. Our calculations are performed in flat spacetime, hence without considering the effect of spacetime curvature onto the inversion-update process.

The 3D- u_i , 4D- (ξ, u_i) , and 4D- (z, E^i) inversion schemes are applied with and without the entropy-switch strategy discussed in Sec. 3.5.3. These results are compared to the performance of the 1D- ξ scheme with fixed-point calculation of E^i . Our comparison includes the results of a “backup”

strategy designed to yield the maximum rate of successful inversions. This is constructed by applying one of the three multi-D strategies (typically the 3D- u_i) and then switching to another scheme upon failure of the inversion process. As a final backup measure, the entropy-switch strategy in u_i is called in case the three primary strategies fail in retrieving a valid set of primitives (for these tests, we do not apply the β_{th} threshold described in Sec. 3.5.3). This backup combined strategy shows dramatic improvements over the standard 1D- ξ scheme, and is the default choice in BHAC for production runs. In the unit tests below, the order of the strategies used in the “backup” approach is 3D- u_i , 4D- (ξ, u_i) , 4D- (z, E^i) , and finally the entropy-switch. In all cases, we use a NR approach with hardcoded Jacobian (we find no significant differences in the convergence rates when applying a NK scheme instead).

As a first test, we explore the $(\eta, \sigma_{\text{mag}})$ parameter space. Considering a wide range of values for the resistivity allows for investigating the importance of the dynamics of E^i compared to the ideal-GRMHD limit for both high and low magnetization. The corresponding sets of primitives are constructed by choosing $B^2 = 1$, $\Gamma = 2$, and $\beta_{\text{th}} = 0.1$ as fixed parameters, as applicable for a relativistic magnetized plasma. For the electric field update, we choose an electric field strength $(E^*)^2 = 0.1$ (that is normally obtained from an explicit update) and a time-step $\Delta t = 0.01$. The polytropic index is fixed to $\hat{\gamma} = 2$.

Figure 1 shows the results of 10^6 conserved-to-primitive inversions expressed in terms of the number of iterations needed to reach a solution of the nonlinear system with an absolute accuracy of 10^{-14} in the computed primitives. The maximum amount of iterations allowed is 100, after which the algorithm is stopped and the inversion is considered as having failed (denoted by dark red dots in the plots). Note that, in production runs, additional checks are applied on the iteration error when the maximum iteration number (we typically allow for 100 iterations) is reached. In this case, if the final error is only slightly larger than the prescribed tolerance, the solution can still be considered valid via a larger, user-defined acceptance tolerance. The two 4D strategies (middle columns) show rather complementary regions of failure, with the (ξ, u_i) scheme being more reliable for high- σ_{mag} zones and the (z, E^i) scheme converging more easily for low- σ_{mag} zones. The 3D scheme in u_i (leftmost column) shows the highest rate of successful inversions, with no specific regions of failed recovery. The entropy-switch applied to the three strategies as backup options (bottom row) shows slightly larger success rate, but does not change the convergence regions qualitatively. All strategies show superior performance compared to the standard 1D strategy in ξ with fixed-point calculation of E^i (top-right panel), both in terms of convergence rate and number of required iterations. The combined “backup” strategy (our default choice for calculations in BHAC) there-

fore provides a dramatic improvement over the often applied fixed-point strategies, with zero failures in the explored parameter space, compared to a $\sim 13\%$ failure rate for the 1D scheme in ξ , although the set of equations are admittedly slightly different if the entropy-switch is applied.

As a second test, we explore the $(\eta, \beta_{\text{th}})$ parameter space. Considering the variation of β_{th} relates the resistive dynamics of E^i (defined by η) with the case of magnetically dominated (i.e., low- β_{th}) or thermally dominated energy (i.e., high- β_{th}) plasma. The primitive sets are constructed upon fixing $B^2 = 1$, $\Gamma = 2$, $\sigma_{\text{mag}} = 10$, $(E^*)^2 = 0.1$, and $\Delta t = 0.01$. The results are shown in Fig. 2, which illustrates how the performance of the three new schemes is similar to the previous case, with almost complementary convergence zones for the 4D schemes, and seemingly scattered failures for the 3D scheme in u_i . For this case, the entropy-switch greatly increases the success rate of the inversion procedure. Overall, the new backup-strategy combined with entropy-switch (bottom-right panel) yields a $\sim 0.005\%$ failure rate, a major improvement over the standard 1D scheme in ξ (top-right panel). The latter shows a large region of no convergence, with an overall $\sim 76\%$ failure rate. Failures in the fixed-point strategy seem to be mostly driven by simultaneous conditions of low resistivity and low- β_{th} , which could preclude modelling large zones of accretion flows which are magnetically-dominated and nearly-ideal ($\eta \rightarrow 0$).

As a final test, we consider the $(\Gamma, \sigma_{\text{mag}})$ parameter space. High- σ_{mag} , high- Γ regions are common in accretion flow simulations, e.g., in the highly magnetized jet emerging from compact objects, where the fluid can be accelerated to high Lorentz factors. Here we manufacture the primitive sets by fixing $B^2 = 1$, $\eta = 0.1$, $\beta_{\text{th}} = 0.1$, $(E^*)^2 = 0.1$, and $\Delta t = 0.01$. The results in Fig. 3 show a generally large region of failure at high- Γ for all strategies. The entropy-switch significantly improves the convergence rate for all the strategies. Compared to the standard 1D strategy in ξ (showing a $\sim 48\%$ failure rate), our combined backup strategy shows failures only in a restricted $\sim 0.007\%$ of the considered parameter space.

In summary: our tests clearly indicate that the new inversion-update strategies presented in Sec. 3.5 are necessary to properly handle a wide range of regimes typically encountered in GR(R)MHD simulations of accretion flows and of compact-binary mergers. We observe dramatic improvements over the standard approach of a 1D inversion scheme on a scalar with fixed-point calculation of E^i , with recorded failures only in an extremely limited number of cases (e.g., 49 failures over a total of 10^6 points considered in the $(\eta, \beta_{\text{th}})$ space, or zero failures in the considered $(\eta, \sigma_{\text{mag}})$ space). However, one should be careful when considering the entropy-switch as a reliable backup strategy: using the entropy results in a different physical system, where a lower temperature is assumed as applicable in the isentropic limit

$d(p/\rho^\gamma)/dt = 0$. In all cases, we detect a final error on the computed primitives of the same order of the iteration error (hence below 10^{-14} in case of convergence).

4.2. Shock-tube tests

As a second test actually solving the set of GRRMHD equations, we have considered a one-dimensional shock-tube in flat spacetime. Such tests are very restrictive for code validation and show strong nonlinear behavior and steep discontinuities. The ability of the code to handle a range of resistivities for such a problem is essential for astrophysical applications where shocks are ubiquitous. We use the shock-tube setup as proposed by Brio & Wu (1988) to test the code performance, and compare to the results in the ideal-GRMHD limit from Porth et al. (2017). For nonzero resistivity we compare the efficiency and performance of the different inversion methods, including the benchmark Strang-split scheme of Komissarov (2007). Considering the lack of exact solutions for shock-tubes with nonzero resistivity, we postpone testing convergence properties to Sections 4.3, 4.4, and 4.5.

The initial conditions are given by:

$$(\rho, p, v^x, v^y, v^z, B^x, B^y, B^z) = \quad (63)$$

$$\begin{cases} (1.0, & 1.0, 0.0, 0.0, 0.0, 0.5, & 1.0, 0.0) & x < 0 \\ (0.125, & 0.1, 0.0, 0.0, 0.0, 0.5, & -1.0, 0.0) & x > 0 \end{cases}$$

with an adiabatic index of $\hat{\gamma} = 2$. These settings result in $\beta_{\text{th}} = 1.6$ for $x < 0$ and $\beta_{\text{th}} = 0.16$ for $x > 0$. We use a uniform grid with 1024 points spanning $x \in [-1/2, 1/2]$. We adopt a second-order TVD limiter (Koren 1993) for spatial reconstruction with CFL number of 0.4.

All tests have been performed with both the Strang-split scheme of Komissarov (2007) and with all inversion-update methods for the IMEX scheme. Note that for these tests we do not activate the entropy fix, nor do we replace faulty cells or apply the floor models since for the multi-D inversion strategies no failures are encountered.

In the right panel of Fig. 4, the results for the B^y -component of the magnetic field are shown for all resistivities $\eta \in [0, 10^4]$ considered, at $t = 0.2$. The results obtained with the IMEX and Strang schemes cannot be distinguished visually for $\eta \geq 10^{-5}$ cases. For $\eta \geq 10$ the results correspond to the zero conductivity case and for $\eta \leq 10^{-5}$ no visual differences are observed between resistive and ideal-MHD.

In the left panel we compare the runtime for the different primitive-recovery methods, normalized to the runtime of ideal-GRMHD. The Strang split method of Komissarov (2007) performs best for high resistivity $\eta \geq 10^{-3}$, since no additional iterations on the electric field are included in the conserved to primitive transformation. However, for $\eta < 10^{-3}$ the proportionality of the time-step to the resistivity rapidly decreases the performance to prohibitively long runtimes.

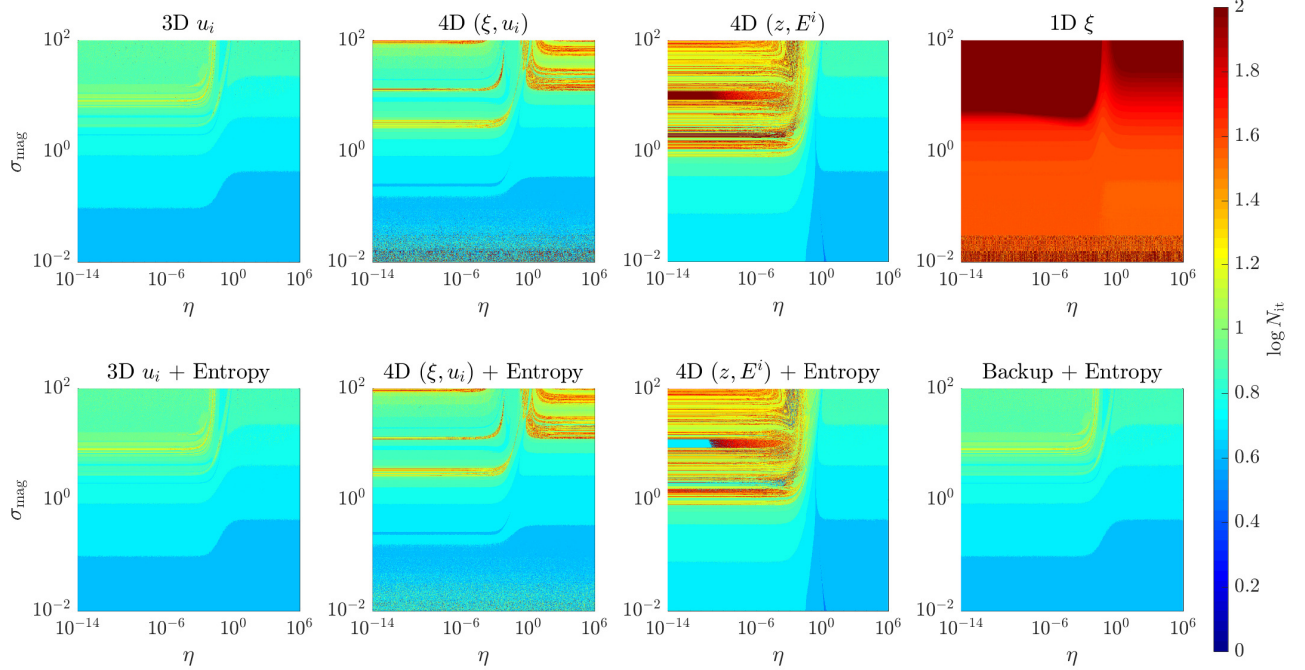


Figure 1. Convergence plots for the inversion-update strategies applied to the $(\eta, \sigma_{\text{mag}})$ parameter space, in terms of number of iterations needed (limited to 100). The manufactured sets of primitive variables are constructed by choosing $B^2 = 1$, $\Gamma = 2$, $\beta_{\text{th}} = 0.1$, $(E^*)^2 = 0.1$, and $\Delta t = 0.01$. Each panel represents for 10^6 conserved-to-primitive inversions. The inversion schemes are applied without (top row) and with (bottom row) entropy-switch as a backup strategy. The new combined “backup” scheme (bottom right) shows zero total failures, dramatically surpassing the performance of the 1D scheme (top right).

For the IMEX scheme with a 1D ξ inversion method we observe a similar trend, and without reducing the CFL condition no convergence is reached for $\eta < 10^{-3}$. For example, the 1D ξ inversion scheme needs a CFL condition of 0.06 for $\eta = 10^{-4}$, a CFL condition of 0.006 for $\eta = 10^{-5}$, and 0.0006 for $\eta = 10^{-6}$ (these results are not shown in Fig. 4, where we keep the CFL fixed). Palenzuela et al. (2009) reached a similar conclusion, stating that the 1D primitive variable recovery for more demanding Riemann problems (such as this shock-tube case) lacks robustness for ratios of $\beta_{\text{th}} \lesssim 0.4$. Note that the shock-tube as tested in Dumbser & Zanotti (2009), Bucciantini & Del Zanna (2013), and Qian et al. (2017) is less restrictive for the inversion scheme, due to the smaller discontinuity in plasma- $\beta_{\text{th}} \in [0.45, 0.4]$ between the left and right state in the initial conditions. Our tests confirm that this less demanding setup can be correctly modeled for any η without reducing the time-step with all IMEX inversion schemes (including the 1D ξ method) considered in this work.

The 3D and 4D inversion schemes presented in Sec. 3.5 perform similarly and always produce the correct solution, with a runtime that is comparable to the 1D method. The runtime always remains within a factor ~ 2 of the runtime of the ideal-GRMHD solver in BHAC. If using a hardcoded Jacobian for the primitive recovery, such that a NR iterative method can be applied, these schemes become even faster and always produce a correct solution for the shock-tube within

~ 1.5 times the ideal-GRMHD runtime, for any value of the resistivity.

4.3. Self-similar current sheet

The third test case is the evolution of a thin current sheet first considered by Komissarov (2007). Once the layer has expanded over several times its initial width, a self-similar evolution ensues. The analytic solution at time t is described by

$$B^y(x, t) = \text{erf}\left(\frac{x}{2\sqrt{\eta t}}\right), \quad (64)$$

for the magnetic field, while the electric field evolves as

$$E^z(x, t) = \sqrt{\frac{\eta}{\pi t}} \exp\left(-\frac{x^2}{4\eta t}\right), \quad (65)$$

and we set $t = 1$ as initial condition, to start with a resolved state in the self-similar phase. Rest-mass density and pressure are homogeneous and set to $\rho = 1$ and $p = 5000$, while all remaining GRRMHD variables are set to zero. The dynamics takes place in the x -direction which is resolved between $x \in [-1.5, 1.5]$ by 256 grid-points. Here we fix the resistivity to $\eta = 0.01$. The test is reproduced with all 3D and 4D inversion methods. Note that for these tests we do not activate the entropy fix, nor do we replace faulty cells or apply the floor models. In all cases we apply an HLL reconstruction scheme

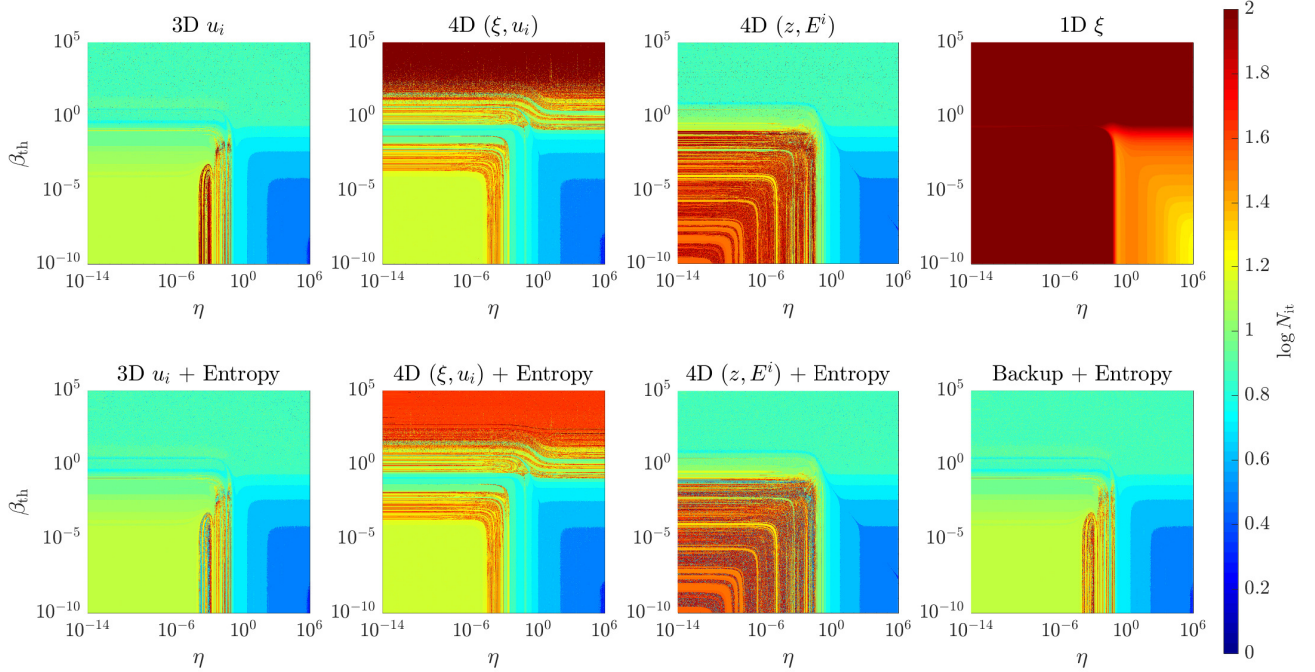


Figure 2. Convergence plots for the inversion-update strategies applied to the $(\eta, \beta_{\text{th}})$ parameter space, in terms of number of iterations needed (limited to 100). The manufactured sets of primitive variables are constructed by choosing $B^2 = 1$, $\Gamma = 2$, $\sigma_{\text{mag}} = 10$, $(E^*)^2 = 0.1$, and $\Delta t = 0.01$. Each panel represents for 10^6 conserved-to-primitive inversions. The inversion schemes are applied without (top row) and with (bottom row) entropy-switch as a backup strategy. The new combined “backup” scheme (bottom right) shows a $\sim 0.005\%$ failure rate, a major improvement over the $\sim 76\%$ failure rate of the 1D scheme (top right).

with a Koren-type limiter (Koren 1993) and we keep a CFL ratio of 0.5.

The analytic solution for B^y and E^z is shown in Fig. 5 at time $t = 10$ (black line). The numerical results (red line) cannot be distinguished visually. In order to assess the accuracy of the evolution, we study the order of convergence of the numerical solution. We measure the L_1 and L_∞ norm of the error in the numerical solution by progressively increasing the number of grid-points and comparing to a high-resolution run with 8192 grid-points. We choose not to compare the numerical results with the analytic solution above, which is only valid in the limit of infinite pressure (as pointed out by Bucciantini & Del Zanna 2013). The error trend thus obtained is reported in Fig. 6. We observe that, for low-resolution runs, the accuracy of the scheme is above second order (as expected by the use of a Koren limiter for a smooth solution), a sign that spatial errors dominate over temporal inaccuracies. For high resolutions, where spatial errors become progressively less important, the scheme tends to first-order accuracy, as is expected from the application of the first-second order IMEX scheme from Bucciantini & Del Zanna (2013).

Finally, in order to test the implementation of the fluxes in 3+1 split formulation, we run the setup under different gauges. These are summarized in table 1. The results including gauge effects are indistinguishable from Fig. 5 once the coordinate-transformations have been accounted for.

Table 1. Coordinates for the gauge effect tests.

Case	α	β^i	γ_{11}	γ_{22}	γ_{33}
A	1	(0,0,0)	1	1	1
B	2	(0,0,0)	1	1	1
C	1	(0.4,0,0)	1	1	1
D	1	(0,0,0)	4	1	1
E	1	(0,0,0)	1	4	1
F	2	(0.4,0,0)	4	9	1

4.4. Charged vortex

Mignone et al. (2019) has recently proposed the first exact two-dimensional equilibrium solution of the SRRMHD equations, which describes a rotating flow with a uniform rest-mass density in a vertical magnetic field and a radial electric field. Adopting a set of cylindrical coordinates (r, ϕ, z) , the solution

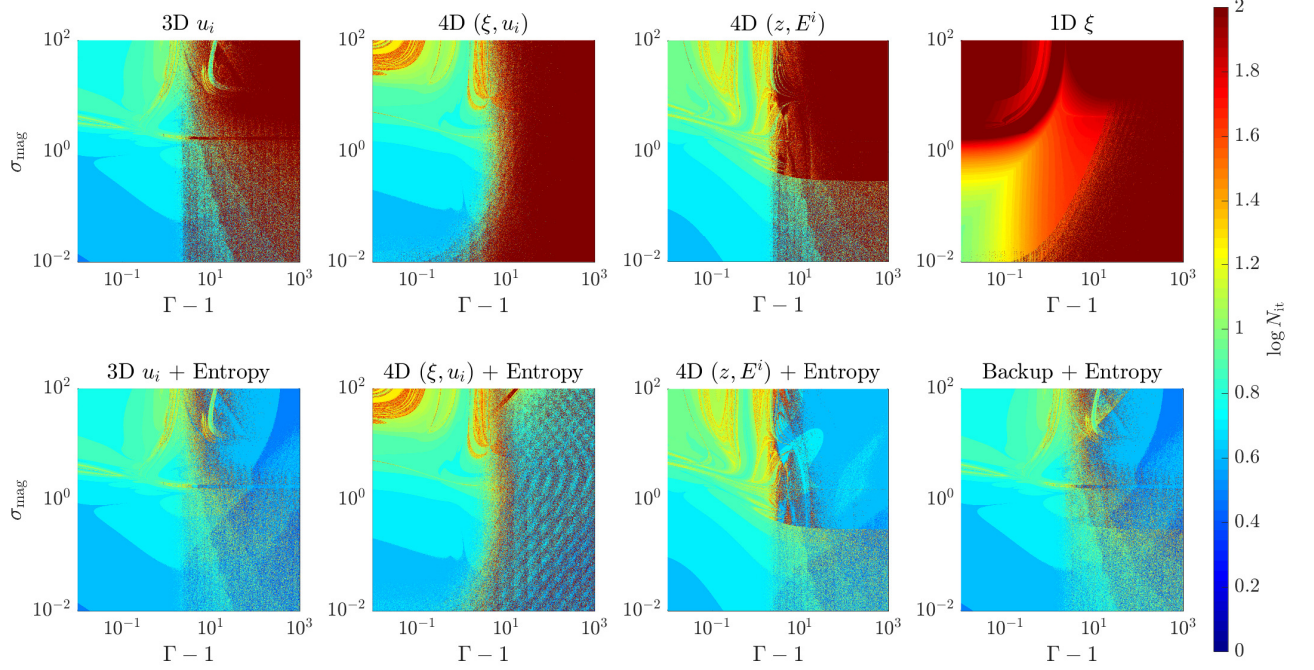


Figure 3. Convergence plots for the inversion-update strategies applied to the $(\Gamma, \sigma_{\text{mag}})$ parameter space, in terms of number of iterations needed (limited to 100). The manufactured sets of primitive variables are constructed by choosing $B^2 = 1$, $\eta = 0.1$, $\beta_{\text{th}} = 0.1$, $(E^*)^2 = 0.1$, and $\Delta t = 0.01$. Each panel represents for 10^6 conserved-to-primitive inversions. The inversion schemes are applied without (top row) and with (bottom row) entropy-switch as a backup strategy. The new combined “backup” scheme (bottom right) shows a $\sim 0.007\%$ failure rate, with a major improvement over the $\sim 48\%$ failure rate of the 1D scheme (top right).

is given by

$$\begin{aligned}
 E_r &= \frac{q_0}{2} \frac{r}{r^2 + 1}, \\
 B_z &= \frac{\sqrt{(r^2 + 1)^2 - q_0^2/4}}{r^2 + 1}, \\
 v_\phi &= -\frac{q_0}{2} \frac{r}{\sqrt{(r^2 + 1)^2 - q_0^2/4}}, \\
 p &= -\frac{\rho(\hat{\gamma} - 1)}{\hat{\gamma}} + \left(p_0 + \frac{\rho(\hat{\gamma} - 1)}{\hat{\gamma}} \right) \left(\frac{4r^2 + 4 - q_0^2}{(r^2 + 1)(4 - q_0^2)} \right)^{\frac{\hat{\gamma}}{2(\hat{\gamma}-1)}}, \tag{66}
 \end{aligned}$$

with radial coordinate $r := x^2 + y^2$ and on the axis $r = 0$, we choose the charge density $q_0 = 0.7$, pressure $p_0 = 0.1$ and uniform rest-mass density $\rho = 1$ in the whole domain in accordance with Mignone et al. (2019). The adiabatic index is set as $\hat{\gamma} = 4/3$ and the resistivity as $\eta = 10^{-3}$. Note that Mignone et al. (2019) evolve the charge density with a separate evolution equation and set it initially as $q = q_0/(r^2 + 1)^2$, whereas in BHAC it is obtained as the divergence of the evolved electric field [cf., Eq. (32)].

The simulation is carried out on a two-dimensional Cartesian grid with $x, y \in [-10, 10]$ with a uniform resolution of $[N_x \times N_y]$ and $(N_x = N_y = 32, 64, 128, 256, 512)$ cells until time $t = 5$. We apply continuous extrapolation of all quanti-

ties at the boundaries. We use the 3D- u_i primitive-recovery method from Sec. 3.5 (similar to the inversion method used by Mignone et al. 2019) and we do not activate the entropy fix, nor do we replace faulty cells or apply the floor models since no inversion failures are encountered.

In Fig. 7 we show a horizontal cut at $y = 0$ of the charge density q at $t = 0$ and $t = 2$ (left panel), and the second-order convergence of the L_1 and L_∞ norms on the difference in pressure p , between the initial and final time, as a function of resolution (right panel). We point out that our solution is in accordance with the results obtained by the application of the CT method to control the divergence of both the electric and magnetic field variables of Mignone et al. (2019). The large-amplitude oscillations observed by Mignone et al. (2019) in q and E_y and attributed to a general Lagrange-multiplier method for the divergence cleaning are absent in our evolution, where we only control the magnetic field divergence by a CT method instead of both the electric and magnetic field divergences. Second-order convergence is obtained by maintaining the equilibrium solution, showing that spatial errors dominate over temporal errors.

4.5. Magnetized spherical accretion

As a first test in general relativity, we consider the problem of spherical accretion onto a Schwarzschild black hole with a strong radial magnetic field (Gammie et al. 2003; Vil-

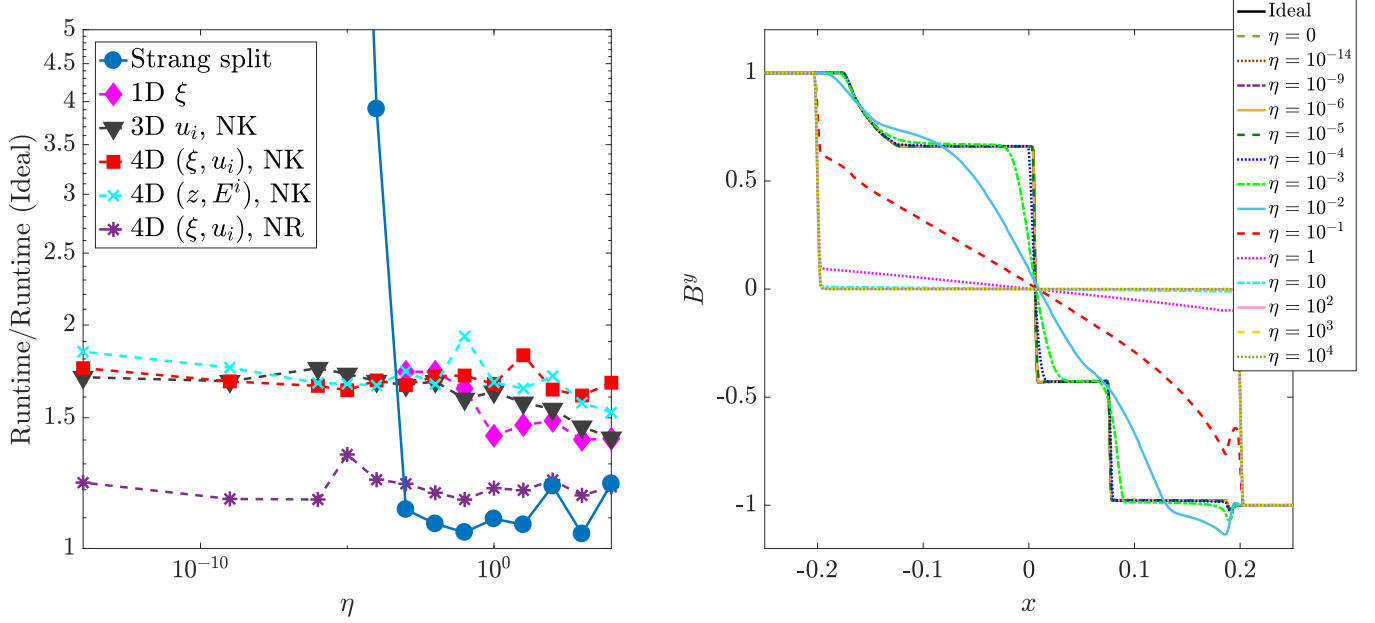


Figure 4. Shock-Tube test as in [Brio & Wu \(1988\)](#). Left panel: runtime normalized by the runtime of ideal-GRMHD in BHAC for all resistivities and a selected number of inversion schemes. Right panel: B^y component of the magnetic field for a range of resistivities $\eta \in [0, 10^4]$. Also the ideal-GRMHD ($\eta = 0$) result of [Porth et al. \(2017\)](#) is shown. The results for the different methods cannot be distinguished visually. Cases with $\eta \leq 10^{-6}$ have not been reproduced with the Strang-split method of [Komissarov \(2007\)](#) due to the extremely small time-step needed. For $\eta \leq 10^{-6}$ there is no visual difference between ideal-GRMHD and GRRMHD solutions.

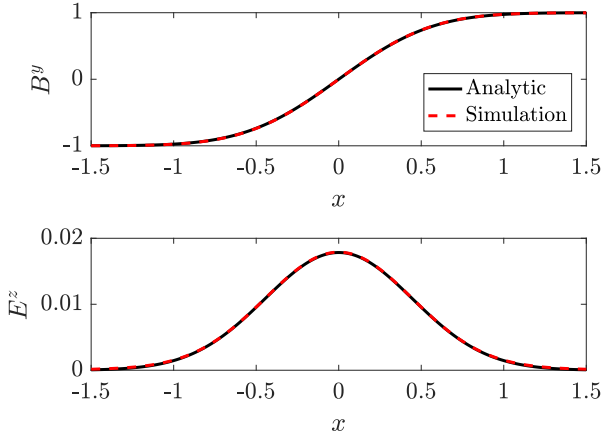


Figure 5. Self-similar current sheet solution at $t = 10$ as in [Komissarov \(2007\)](#) on 256 grid-points.

[liers & Hawley 2003](#)) and compare to the steady-state solution ([Bondi 1952](#); [Michel 1972](#)). This test is a particular challenge for the primitive-recovery method due to the low β_{th} regions close to the event horizon. We set the mass of the black hole $M = 1$ and dimensionless spin $a = 0$, such that distances and times are measured in terms of M . We follow the initial conditions given by [Hawley et al. \(1984\)](#) and we parametrize the field strength through the magnetization $\sigma_{\text{mag}} = 10^3$ at

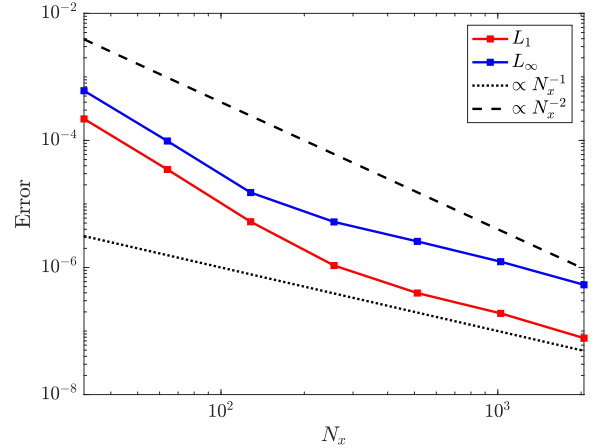


Figure 6. Convergence study for the current sheet evolution at increasing number of grid-points N_x . The L_1 and L_∞ norms of the difference between a high-resolution run and the numerical results indicate first-order convergence for high-resolution runs (where temporal discretization errors dominate over spatial errors), as expected from the properties of the first-second IMEX scheme by [Bucciantini & Del Zanna \(2013\)](#).

the inner edge of the domain at $r = 1.9M$. To test the inversion method, the resistivity is set to $\eta = 0$, while the full set of GRRMHD equations is solved, allowed by the first-second IMEX scheme. We employ two-dimensional modified spheri-

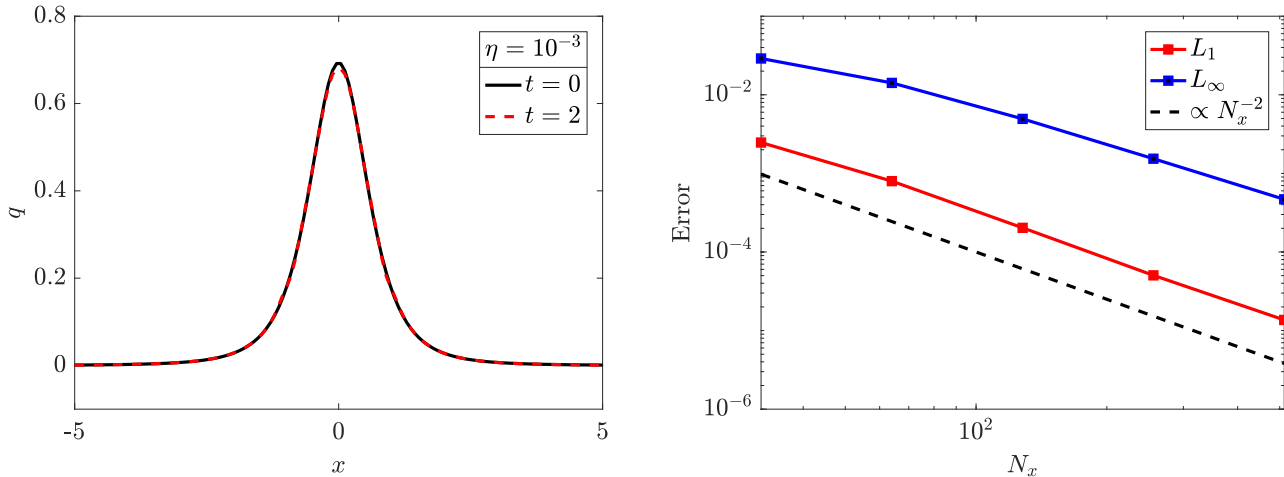


Figure 7. Charged vortex as in Mignone et al. (2019), with in the left panel a horizontal ($y = 0$) cut of the charge density at $t = 0$ (black line) and $t = 2$ (red dashed line) for 256^2 cells, and in the right panel the L_1 (red line) and L_∞ (blue line) norm of the pressure difference between initial and final state versus resolution, showing second-order scaling.

cal Kerr-Schild (MKS) coordinates as described in Porth et al. (2017) with $r \in [1.9M, 10M]$; $\theta \in [0, \pi]$ and a uniform radial resolution $N_r = 200$ and angular resolution $N_\theta = 100$. The steady-state effectively reduces to a one-dimensional problem due to the purely radial dependence of the equilibrium solution. The analytic solution is fixed at the radial boundaries. The test has been reproduced with all 3D and 4D inversion methods. The entropy-switch is activated if $\beta_{\text{th}} \leq 10^{-2}$ (for $r \lesssim 8M$, see Fig. 8) or if the primary inversion procedure fails.

Figure 8 shows the radial profiles of rest-mass density ρ , radial three-velocity v^r , β_{th} , and σ_{mag} as found with primitive recovery with entropy switch (green dashed line) and without (red dashed line) compared to the analytic solution (black solid line). For the radial three-velocity (top right panel) it is clear that the entropy-switch improves the solution close to the event horizon $r \lesssim 6M$. The improvement provided by the entropy-switch is also visible in the L_1 and L_∞ norm of the rest-mass density (Fig. 9), increasing the numerical accuracy by approximately a factor four. Second-order convergence is obtained both with and without the entropy-switch.

4.6. Resistive accreting torus

Finally, we simulate accretion from a magnetized test-fluid torus (Fishbone & Moncrief 1976) around a Kerr black hole. We again set the mass of the black hole $M = 1$ and the dimensionless spin $a = 0.9375$. We employ MKS coordinates on a two-dimensional domain where $r \in [1.29, 2500]$ and $\theta \in [0, \pi]$ with a uniform resolution of $N_r \times N_\theta = 512 \times 256$ cells. At the initial state, the inner edge of the torus is located at $r = 6$ and the maximum rest-mass density is localized at $r = 12$. To simulate the vacuum region outside the torus we

set the rest-mass density and the pressure in the atmosphere as $\rho_{\text{atm}} = \rho_{\text{min}} r^{-3/2}$ and $p_{\text{atm}} = p_{\text{min}} r^{-5/2}$. The rest-mass density and the pressure are reset whenever they fall below these floor values. The normalization of the power-law floor model is set to $\rho_{\text{min}} = 10^{-4}$, $p_{\text{min}} = (1/3) \times 10^{-6}$.

The initial magnetic field configuration consists of a weak single loop given by the vector potential

$$A_\phi \propto \max(\rho/\rho_{\text{max}} - 0.2, 0), \quad (67)$$

where ρ_{max} is the global maximum rest-mass density in the torus. The field strength is determined such that $2p_{\text{max}}/b_{\text{max}}^2 = 100$, where the spatial locations where p_{max} and b_{max}^2 are found do not necessarily coincide. Note that this configuration does not result in an exact MHD equilibrium.

At the polar axis, we impose symmetric boundary conditions for all scalar variables, the radial and poloidal vector components for all scalar variables, the radial and poloidal vector components v^r , B^r , v^ϕ , B^ϕ , and the azimuthal component E^θ ; antisymmetric boundary conditions are imposed for the azimuthal vector components v^θ and B^θ , and for the radial and poloidal components E^r , E^ϕ (see Porth et al. 2017 and Porth et al. 2019 for a discussion on the boundary conditions in GRMHD simulations of magnetized accretion flows). At the inner and outer radial boundaries we impose zero-inflow boundary conditions. We choose an ideal-gas EOS with $\hat{\gamma} = 4/3$.

The initial equilibrium configuration is perturbed with random, low-amplitude pressure oscillations. This triggers the MRI during the accretion of gas from the torus onto the central object. The instability amplifies the initial magnetic field and drives the disruption of the equilibrium towards a quasi-steady state around $t \sim 500M$. During the process, the resistivity determines the development of diffusive processes. Here, we

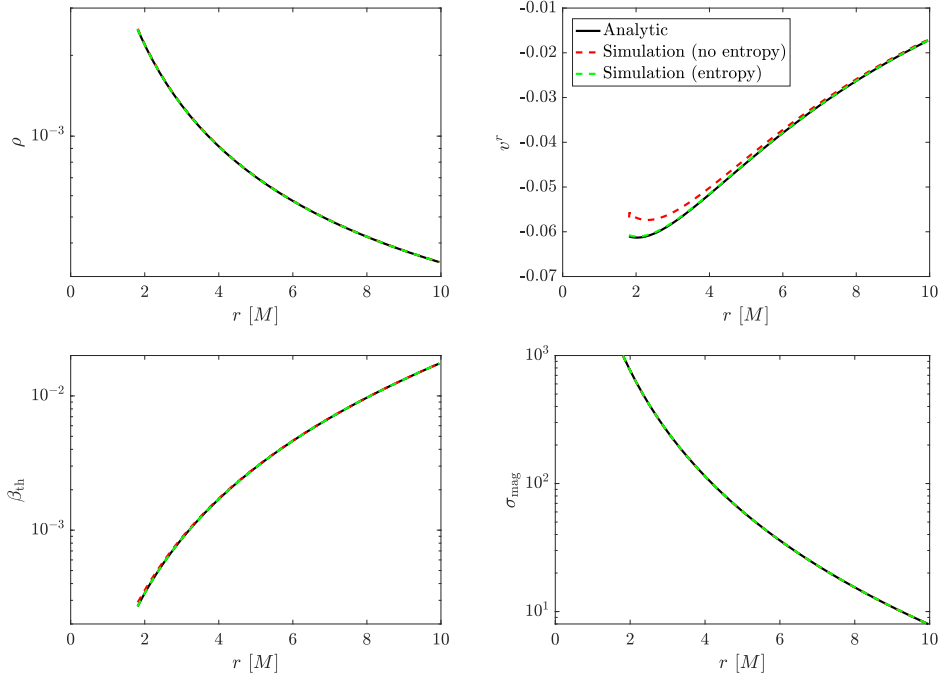


Figure 8. Radial profiles of ρ , v^r , β_{th} , and σ_{mag} for magnetized spherical accretion at $t = 100M$ in MKS coordinates, with $\sigma_{\text{mag}} = 10^3$ at the inner edge of the domain and a uniform resolution $N_r = 200$. The black solid line indicates the initial exact solution, the dashed red line shows the standard primitive-recovery method, and the dashed green line shows the standard treatment supplemented with the entropy-switch. The entropy-switch is activated for $\beta_{\text{th}} \leq 10^{-2}$, i.e., for $r \lesssim 8M$. The error in the radial three-velocity v^r shows the clearest advantage of the entropy-switch.

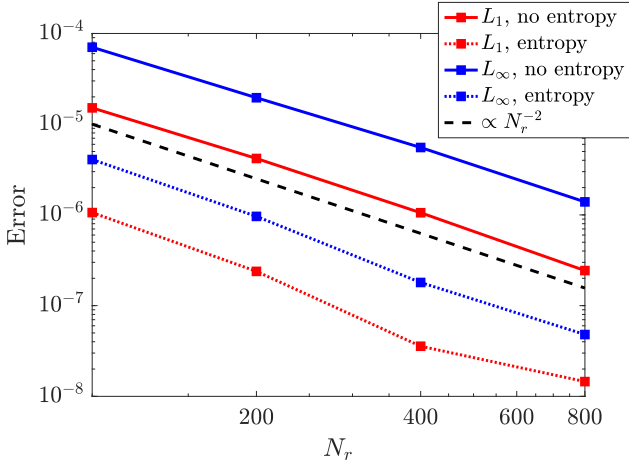


Figure 9. Convergence study for the Bondi accretion test at increasing number of grid-points N_r . The test is run with (dotted lines) and without (solid lines) the entropy-switch as a backup strategy for the conserved to primitive inversion. The L_1 and L_∞ norms of the difference between the analytic solution at $t = 0$ and the numerical result at $t = 100$ show second-order convergence in all cases, proving that spatial errors dominate over temporal inaccuracies.

choose a range of values for $\eta \in [10^{-14}, 10^{-2}]$ and we simulate the development of the MRI until $t = 2000M$. We also study the exact ideal-MHD limit $\eta = 0$ (allowed by the first-second IMEX scheme). Additionally, we perform the same simu-

lation with the ideal-GRMHD version of BHAC. The latter differs from the $\eta = 0$ case simulated with our resistive algorithm in many aspects, most notably by solving the induction equation for the magnetic field by assuming that the electric field is a purely dependent quantity $E^i = -\gamma^{-1/2} \eta^{ijk} v_j B_k$ (as presented in Porth et al. 2017). Comparing the results of the resistive scheme in the ideal-MHD limit with the purely ideal-MHD implementation is therefore a strict and important benchmark. The resistive runs were all completed within a runtime of a factor $\sim 2-3$ longer compared to the ideal-MHD case.

In Fig. 10 we show the spatial distribution of the characteristic quantities β_{th} and σ_{mag} for progressively decreasing η , and for the simulation run with the ideal-GRMHD version of BHAC. The results are averaged in time between $t = 500M$ and $t = 1000M$ to account for statistical fluctuations in the quasi-steady state accretion. The $\eta = 10^{-2}$ case most evidently shows the diffusive effects of resistivity, quenching the MRI-induced turbulence. Turbulent features become progressively more apparent as η decreases, until the point where numerical resistivity dominates over the explicit physical resistivity. For the resolution considered here, this threshold can be identified around $\eta \sim 10^{-4}$, at which point the results become visually indistinguishable from simulations at lower resistivity values (including the $\eta = 0$ limit). Minor visual differences between the ideal-GRMHD result of BHAC and the $\eta \leq 10^{-4}$ cases are

attributed to the differences in the numerical scheme in the two cases (e.g., the different characteristic speed employed, see Sec. 3.2).

To remove the smoothing introduced by the time averaging, Fig. 11 shows a close-up view of the accretion region at time $t = 1600M$ during the evolution of the system. The $\eta = 10^{-2}$ run (left) shows no sign of turbulence, which is almost completely suppressed by the diffusive processes introduced by the high resistivity. The $\eta = 10^{-14}$ case (right), on the contrary, clearly shows the formation of characteristically turbulent structures, with steep gradients both in β_{th} and σ_{mag} .

Finally, for a more quantitative comparison, we monitor the accretion rate \dot{M} and magnetic flux through the black hole event horizon Φ_B for all runs, defined as

$$\dot{M} := \int_0^{2\pi} \int_0^\pi \rho u^r \sqrt{-g} d\theta d\phi, \quad (68)$$

$$\Phi_B := \frac{1}{2} \int_0^{2\pi} \int_0^\pi |B^r| \sqrt{-g} d\theta d\phi. \quad (69)$$

In Fig. 12 the evolution in time of both quantities is shown for the $\eta = 10^{-4}, 10^{-3}, 10^{-2}$ runs, together with the results from the $\eta = 0$ run and the ideal-MHD run. The plots show how large resistivity, that is above the numerical resistivity threshold $\eta > 10^{-4}$, can affect the evolution of the system, delaying the instability in time and decreasing the final semi-steady state values. These results are consistent with the findings by Qian et al. (2017), establishing that a high resistivity significantly quenches the MRI. Additionally, due to the robust conserved to primitive strategies presented in Sec. 3.5, we find no difficulty in simulating the demanding cases where $\eta \rightarrow 0$. As shown in Fig. 12, the development time of the MRI and the final steady-state values for both \dot{M} and Φ_B are in good agreement between the $\eta = 0$ run and the ideal-GRMHD simulation. These are also consistent with the $\eta = 10^{-4}$ results, confirming that the numerical resistivity is of the order of $\eta < 10^{-3}$ for the resolution considered here. Identifying this threshold is of major importance, since dissipative length scales that need to be resolved in resistive simulations are proportional to the resistivity. Hence, the necessary resolution depends on the resistivity as $N \propto \eta^{-1}$. If the resolution is lower than the necessary threshold to capture the resistive dynamics, the numerical resistivity is prevailing. With explicit resistivity we can explore new physical regimes that are unattainable in ideal-GRMHD, and explore the effect of dissipative length scales on the development of the MRI. Explicit treatment of viscosity in GRMHD (e.g., Fragile et al. 2018; Fujibayashi et al. 2018) in combination with resistivity will soon allow for the investigation of turbulent black hole accretion without relying on numerical dissipation.

5. CONCLUSIONS

We presented the implementation of a resistive module in the general relativistic magneto-fluid code BHAC. The new GRRMHD algorithm is tested and used in this work in combination with AMR and a recently implemented staggered CT method to ensure solenoidal magnetic fields (Olivares et al. 2018; Olivares et al. 2019).

The GRRMHD equations are solved with the first-second order IMEX scheme from Bucciantini & Del Zanna (2013) and the performance is compared to the Strang split scheme of Komissarov (2007). The IMEX scheme uses a first-order iterative implicit step to solve the resistive, stiff term and solves the non-stiff terms with a second-order explicit scheme as in the ideal-GRMHD module in BHAC. We find that the time-step in the IMEX scheme does not depend on the resistivity. This results in a speedup compared to the Strang-split scheme that is of the order of $1/\eta$. Particularly for cases with $\eta \lesssim 10^{-4}$, this results in a major speedup, since for this regime the time-step in the Strang-split scheme is dominated by the resistive stiff terms. The implemented IMEX scheme can be straightforwardly extended to higher order. Based on the current implementation, it is also straightforward to incorporate additional physics like Hall and dynamo dynamics (see e.g., Bucciantini & Del Zanna 2013; Palenzuela 2013; Bugli et al. 2014). The system of GRRMHD equations can also be extended to evolve an extra equation for radiation dynamics, where the IMEX scheme is applied to the stiff terms due to the optically thick plasma (see e.g., Zanotti et al. 2011; Roedig et al. 2012; Sadowski et al. 2013; Sadowski et al. 2014; McKinney et al. 2014, and Melon Fuksman & Mignone 2019 in SRRMHD).

Well-established GRRMHD methods struggle with regimes where both η and plasma- β_{th} are small (Palenzuela et al. 2009; Qian et al. 2017), e.g., in highly magnetized accretion flows and jets in the surroundings of black holes and neutron stars. Here, the dynamic electric field makes the recovery of primitive variables, a key part of all GRMHD codes, particularly demanding. We designed and presented several novel primitive-recovery methods taking the nonlinear dependence of the dynamic (resistive) electric field on the primitive variables fully into account. Compared to existing primitive-recovery methods for GRRMHD presented by Dionysopoulou et al. (2013), and Palenzuela (2013), our methods are very robust in a large parameter space and can accurately handle nonzero and non-uniform resistivity ranging from the ideal-MHD limit $\eta \rightarrow 0$ to the electrovacuum limit $\eta \rightarrow \infty$ in highly magnetized regions of high σ_{mag} and low β_{th} . The exact ideal-MHD limit $\eta = 0$ is recovered for several analytic tests and for a realistic accreting torus simulation, due to the nature of the first-second order IMEX scheme of Bucciantini & Del Zanna (2013). We note that the 3D primitive-recovery method by Bucciantini & Del Zanna

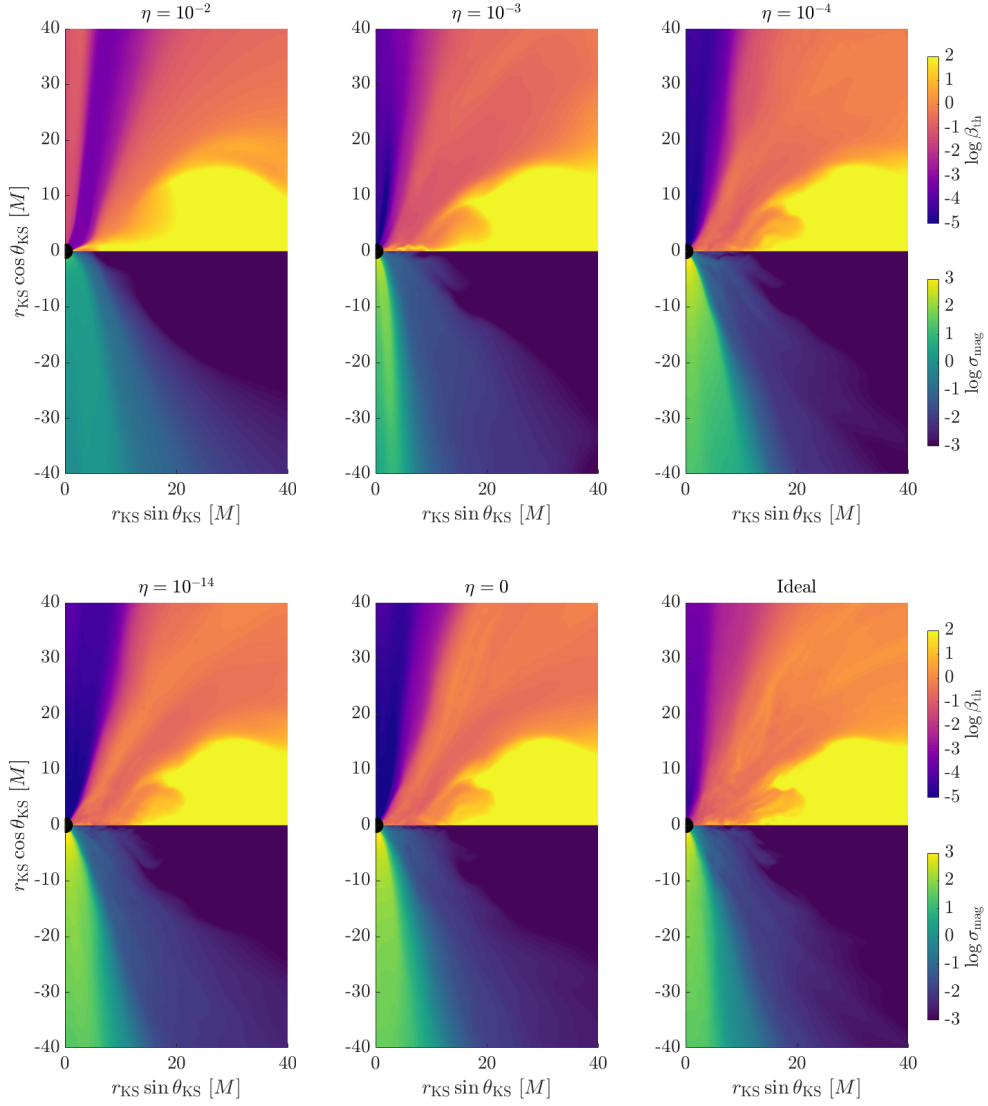


Figure 10. Resistive accreting-torus simulations with $\eta = 10^{-2}$, 10^{-3} , 10^{-4} (top row, from left to right) and $\eta = 10^{-14}$, 0 (bottom row, from left to right) compared to the ideal-GRMHD run (bottom right) showing the logarithmic $\beta_{\text{th}} = 2p/b^2$ (upper half) and magnetization $\sigma_{\text{mag}} = b^2/\rho$ (lower half) averaged over $t \in [500M, 1000M]$. The higher resistivity runs show significant diffusion and suppression of turbulent structures in the accretion flow. The results for lower resistivity $\eta \leq 10^{-4}$ are statistically similar, confirming that the numerical resistivity is of the order $\eta \sim 10^{-4}$ for the considered resolution, hence playing little to no role in the evolution of the system.

(2013) and Mignone et al. (2019) performs similarly well in the tests presented in this work.

Additionally, we proposed a backup system of recovery methods, combined with an entropy-switch. This combined method turned out to be essential to accurately resolve highly magnetized regions in black hole accretion simulations. We explored a parameter space of $\sigma_{\text{mag}} \in [10^{-2}, 10^2]$, $\beta_{\text{th}} \in [10^{-10}, 10^5]$, $\Gamma - 1 \in [10^{-2}, 10^3]$ and $\eta \in [10^{-14}, 10^6]$, which should be representative for all regions normally encountered in simulating high-energy astrophysical phenomena. Combined with the AMR strategy in BHAC, the new GRMHD algorithm allows for resolving both the global accre-

tion features governed by the MRI-induced turbulence, and the dissipative reconnection physics that are conjectured to be responsible for non-thermal radiation. These non-thermal processes can be subsequently modeled in BHAC with first-principle approaches, e.g., the newly implemented general-relativistic (charged) particle module (Ripperda et al. 2018; Bacchini et al. 2018; Bacchini et al. 2019).

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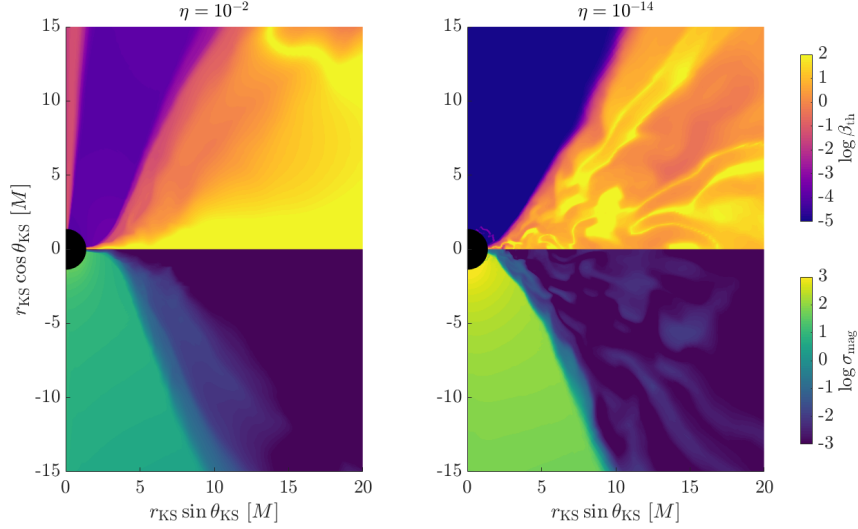


Figure 11. Close-up view of the accretion region in the resistive torus simulations with $\eta = 10^{-2}$ (left) and $\eta = 10^{-14}$ (right), showing the logarithmic $\beta_{\text{th}} = 2p/b^2$ (upper half) and magnetization $\sigma_{\text{mag}} = b^2/\rho$ (lower half) at time $t = 1600M$. The high-resistivity run shows almost no sign of turbulent structures, which are instead clearly visible in the low-resistivity case.

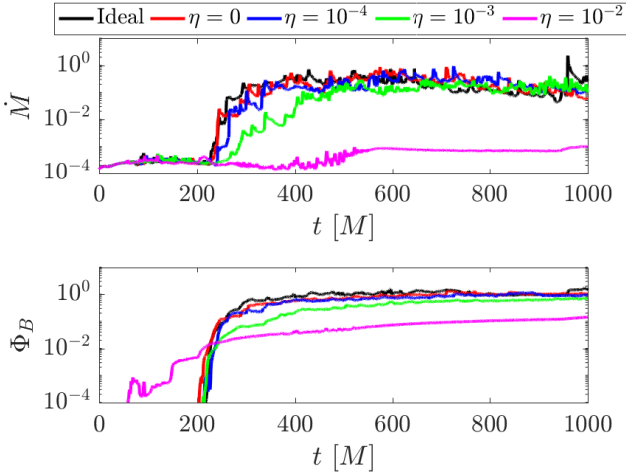


Figure 12. Evolution in time of the mass accretion rate \dot{M} (top) and magnetic flux through the horizon Φ_B (bottom) for the resistive Fishbone-Moncrief torus with $\eta = 10^{-2}$ (magenta lines), 10^{-3} (green lines), and 10^{-4} (blue lines). These are compared to the $\eta = 0$ case (red lines, ideal-MHD limit) and the purely ideal-GRMHD run with BHAC (black lines). All cases with $\eta \leq 10^{-4}$ show excellent agreement in the MRI development time and steady-state values with the ideal-MHD run.

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Software: BHAC (Porth et al. 2017; Olivares et al. 2019)

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