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A STUDY OF THE RECURSION $y_{n+1}=y_n+\tau y_n^m$

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We provide a detailed study of the recursion $y_0=1, y_{n+1}=y_n+\tau y_n^m, n=0,1,..., m>1$, which arises either as a model discretization of a nonlinear ODE or in the use of the energy method, Sharp bounds and asymptotic estimates are given for the size of the iterates y_n.

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1. Introduction

In this note we study the recursion

$$y_0=1, y_{n+1}=y_n+\tau y_n^m, \ n=0,1,2,..., \tag{1}$$

where $m>1$ is a constant and $\tau>0$ a small parameter. Clearly, for $n\tau<1/(m-1)$ the terms y_n are the approximations given by Euler's method to the value at $t=n\tau$ of the solution

$$y(t)=[1-(m-1)t]^{1}/(1-m) \tag{2}$$

of the initial value problem

$$dy/dt=y^m, y(0)=1. \tag{3}$$

Note that this solution exists only for $t<1/(m-1)$. In this connection the recurrence (1) provides a very simple example of application to a nonlinear problem of a method for the numerical integration of ordinary differential equations. The derivation of sharp bounds for the error $y_n-y(n\tau)$ may throw some light on the error propagation mechanism in nonlinear situations, which, as distinct from linear situations, is often difficult to investigate.

However, there are other instances in Numerical Analysis which lead to recursions similar to (1). Our initial motivation arose in the study of the two-dimensional system [1]

$$d\mathrm{u}/dt = (\mathrm{u}^T\mathrm{u})A\,\mathrm{u}, \ A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \tag{4}$$

which, in turn, stems from the study of the time-dependent cubic Schrödinger equation [4], [5], [6]. The skew- symmetry of A implies that, if $\mathrm{u}(t)$ is a solution of (4), then the Euclidean norm (energy) $\|\mathrm{u}(t)\|$ does not change with t. When (4) is discretized by means of Euler's rule with step-size h the norms $e_n=\|\mathrm{u}_n\|$ satisfy the recursion

$$e_{n+1}^2=e_n^2+h^2e_n^6, \ n=0,1,2,..., \tag{5}$$

which, upon defining $y_n=e_n^2/e_0^2$, $n=0,1,2,...$, $\tau = h^2e_0^4$, reduces to (1) with $m=3$. More generally, (1) or recursions similar to it, often appear in the application of the energy method in ODEs or time- dependent PDEs. From (5) we conclude that the norms e_n increase monotonically, as distinct from the situation $\|\mathrm{u}(t)\|=$constant. In Section 2 we show that the norms eventually exceed any given constant M and we obtain estimates for the smallest value of n such that $\|\mathrm{u}_n\|\geqslant M$. Section 3 is devoted to a closer study of the case $m=2$. Here we derive sharp bounds for the iterants y_n from a detailed investigation of the error $y_n-y(n\tau)$ committed by Euler's rule.

2. Asymptotic estimations

The solution $y(t)$ of (3) increases monotonically from $y(0)=1$ up to $y(1/(m-1))=\infty$. The iterants y_n also increase monotonically with n(since $y_{n+1}-y_n>0$) and for n large enough exceed any given constant $M\geqslant1$ (since $y_{n+1}-y_n=\tau y_n^m\geqslant\tau$). We shall provide asymptotic ($\tau\to0$) estimations for the smallest value of n such that $y_n\geqslant M$.

It is well known that the error $y_n-y(n\tau)$ of Euler's rule possesses an asymptotic expansion [3] (see [2] for a simple derivation)

$$y_n=y(n\tau)+\tau u(n\tau)+O(\tau^2), \ \tau\to0, \ n\tau \text{ fixed and } <1/(m-1). \tag{6}$$

Here $u(t)$ is the solution of the variational problem

$$u(0)=0, \ du/dt=my^{m-1}u-(1/2)(d^2y/dt^2). \tag{7}$$

Differentiation in (3) shows that $d^2y/dt^2= my^{2m-1}$, so that (7) can be rewritten (taking y as independent variable) in the form

$$u(0)=0, y^{-2m}(\frac{du}{dy}y^m-my^{m-1}u)=-(m/2)y^{-1}. \tag{8}$$

The problem (8) is readily integrated to yield

$$u(t) = -(m/2)y^m \ln y,\ 0 \le t < 1/(m-1), \tag{9}$$

which substituted into (6) gives

$$y_n = y(t) - \tfrac{1}{2} m\tau y(t)^m \ln y(t) + O(\tau^2),\ \tau \to 0,\ t = n\tau < 1/(m-1). \tag{10}$$

The expansion (10) is valid uniformly in t as long as t ranges in a compact interval $0 \le t \le a < 1/(m-1)$. It is clearly nonuniform for $0 \le t < 1/(m-1)$; note that for τ fixed and t near $1/(m-1)$ one has $y(t) - (m/2)\tau y(t)^m \ln y(t) << 0$.

It is convenient to define y_n for noninteger values of n by means of linear interpolation between consecutive integers. The expansion (10) is easily seen to hold even for real n, since linear interpolation within an interval of length τ has errors $O(\tau^2)$.

Let $M \ge 1$ be a fixed constant and let $n^* = n^*(\tau)$ be such that $y_{n^*} = M$. We shall compare n^* with the value n^{**} defined by the equality $y(n^{**}\tau) = M$, i.e.,

$$n^{**} = \frac{1}{(m-1)\tau} - \frac{1}{(m-1)M^{m-1}\tau}. \tag{11}$$

We note that since Euler's method is convergent, $n^*\tau$ decreases as $\tau \to 0$ towards the fixed quantity $n^{**}\tau < 1/(m-1)$, i.e., the products $n^*\tau$ vary in a region of uniformity of (10). Therefore, setting $z = y(n^*\tau)$, we can write

$$M = z - \tfrac{1}{2} m\tau z^m \ln z + O(\tau^2),$$

whence, after a straightforward calculation we conclude

$$z = M + \tfrac{1}{2} m\tau M^m \ln M + O(\tau^2).$$

Using (2) and (11) we then get $n^* = n^{**} + \frac{1}{2} m \ln M + O(\tau)$, i.e., the number of steps required in order that the computed solution reaches the fixed value M equals asymptotically the theoretical number of steps plus a constant. If we again restrict the interest to integer values of n we have the estimate

$$\left[\frac{1}{(m-1)\tau} - \frac{1}{(m-1)M^{m-1}\tau} + \frac{m}{2}\ln M\right] \tag{12}$$

for the largest integer n such that $y_n \le M$. In this paper the symbol $[x]$ denotes integer part of x The following table compares, for $m = 2$, values of the estimate (12) with true values of n

τ	m	true	estimated
0.1	10	11	11
0.1	100	13	14
0.01	10	92	92
0.01	100	103	103

As an example of application of (12) we consider Euler's method for the problem (4) with $u(0) = [1,0]^T$, $h = 0.1$, $u_0 = u(0)$. We ask for the maximum number of steps such that the norm $\|u_n\|$ does not exceed the true norm $\|u(n\tau)\| = \|u_0\| = 1$ in more than 10%, i.e., $e_n^2 \le 1.21$. As in the introduction, a change of variables brings our problem into the form (1). Then our estimate yields $n = 16$, which agrees exactly with the value found experimentally. This number of steps spans a time interval of length $nh = 1.6$. The length of the spanned interval approximately doubles when h is halved, since $n = O(\tau^{-1}) = O(h^{-2})$.

3. Bounds

In this section we restrict our attention to the case $m = 2$ in (1). We begin by computing explicitely the $O(\tau^2)$ term in the expansion (6) for the global error. Namely

$$y_n = y(n\tau) + \tau u(n\tau) + \tau^2 v(n\tau) + O(\tau^3),\ c \to 0,\ n\tau \text{ fixed and } < 1, \tag{13}$$

where, for $0 \le t < 1, y(t) = 1/(1-t), u(t) = -y^2(t)\ln y(t)$, and $v(t)$ solves the initial value problem $(' \equiv d/dt)$

$$v(0)=0,\ dv/dt=2yv-(1/6)y'''-(1/2)u''+u^2. \tag{14}$$

This problem is best treated by taking y as new independent variable as we did in (8). In this way we easily find

$$v=y^3\ln y+y^3(\ln y)^2. \tag{15}$$

We now turn to the investigation of sharp bounds for the iterants y_n using the asymptotic error relation (13).

Theorem 1. *Let $m=2$ and $n\tau<1$. The iterants y_n then satisfy*

$$\tau u(n\tau)=-\tau y^2(n\tau)\ln y(n\tau)\leqslant y_n-y(n\tau)<0. \tag{16}$$

Proof. The inequality $y_n<y(n\tau)$ is obvious in view of the geometrical interpretation of Euler's rule. We prove that

$$z_n\equiv y(n\tau)-\tau y^2(n\tau)\ln y(n\tau)\leqslant y_n,\ n=0,1,..,[\tau^{-1}]-1.$$

This expression can be expanded as

$$\tau(y^{(1)}-y^2)+\tau^2(u^{(1)}-2uy+\tfrac{1}{2}y^{(2)})-\tau^3u^2+\sum_{k=3}^{\infty}\tau^k[\frac{y^{(k)}}{k!}+\frac{u^{(k-1)}}{(k-1)!}], \tag{17}$$

where bracketed indices denote derivates with respect to t and functions are evaluated at $t=n\tau$. The first and second terms of (17) vanish by definition of y,u (see (3), (7)), while the third is nonpositive. Therefore the proof will be finished if we show that

$$(1/k!)y^{(k)}+(1/(k-1)!)u^{(k-1)}\leqslant 0,\ \ k=3,4,.... \tag{18}$$

Differentiation of (3) leads to $y^{(k)}=k!y^{k+1}$. The derivatives of u are given by

$$u^{(k)}=-a_ky^{k+2}-(k+1)!y^{k+2}\ln y,\ k=1,2,...,$$

where $a_1=1, a_k=(k+1)a_{k-1}+k!, k=2,3,...,$ so that $a_k\geqslant k!, k=1,2,....$ Substitution of $y^{(k)}$, $u^{(k-1)}$ by their expressions in terms of y leads to (18). □

We emphasize that, after (13), for $n\tau$ fixed and $\tau\to 0$ the lower bound in (16) and the error $y_n-y(n\tau)$ differ only in $O(\tau^2)$ terms. On the other hand, for a given τ the sharpness of the lower bound decreases as n increases, since for $n\tau$ close to $1, \tau u(n\tau)>>y(n\tau)$. Later we shall show how to find sharper bounds for $n\tau$ close to 1. For the relative error we can prove

Theorem 2. *Let $c\geqslant 1, \tau\leqslant exp(-1)$. Then if $n\tau\leqslant 1-c\tau\ln\tau^{-1}$,*

$$-\frac{1}{c}\leqslant\frac{y_n-y(n\tau)}{y(n\tau)}\leqslant 0, \tag{19}$$

and

$$\tau y(n\tau)\ln y(n\tau)\leqslant 1/c. \tag{20}$$

Proof. The inequality (21) is a direct consequence of (16) and (20). The verification of (20) is straightforward. □

Remark. We conclude from (19) that the relative error is less than $1/c$ uniformly in n,τ provided that $n\tau$ is not too close to 1. Note that as τ is decreased the length of the t- interval $[0,1-c\tau\ln\tau^{-1}]$ of uniformity of the relative error tends to 1, the length of the interval of existence of $y(t)$. We shall prove later that a bound like (19) does not exist uniformly for $0\leqslant t<1$. □

We next improve the upper bound in (16):

Theorem 3. *Let $m=2$ and $n\tau \leqslant 1-\tau \ln \tau^{-1}$. The iterants y_n then satisfy*

$$y_n - y(n\tau) \leqslant \tau u(n\tau) + \tau^2 v(n\tau). \tag{21}$$

Proof. We prove that if $z_n \equiv y(n\tau) + \tau u(n\tau) + \tau^2 v(n\tau)$, then

$$z_{n+1} - z_n - \tau z_n^2 \geqslant 0, \; n\tau \leqslant 1 - \tau \ln \tau^{-1}.$$

This expression can be expanded as

$$\tau(y^{(1)} - y^2) + \tau^2(u^{(1)} - 2yu + \tfrac{1}{2}y^{(2)}) + \tau^3(v^{(1)} - 2yv - u^2 + \tfrac{1}{2}u^{(2)} + \tfrac{1}{6}y^{(3)})$$
$$-2\tau^4 uv - \tau^5 v^2 + \sum_{k=4}^{\infty} \tau^k \left(\frac{y^{(k)}}{k!} + \frac{u^{(k-1)}}{(k-1)!} + \frac{v^{(k-2)}}{(k-2)!}\right).$$

The terms in τ, τ^2, τ^3 vanish by definition of y,u,v. The series can be shown to be positive by an argument similar to that employed in the proof of Theorem 1. It remains to be proved that (σ is positive) $-2u - \tau v \geqslant 0$, or

$$2 - \tau y - \tau y \ln y \geqslant 0. \tag{22}$$

From (20) with $c=1$ we can write $\tau y \ln y \leqslant 1$, which clearly leads to (22). This concludes the proof. □

It was pointed out before that the bounds obtained so far loose their sharpness if $n\tau$ is close to 1. In the proof of our last theorem we employ a change of scale in y,t near $t=1$ which enables us to describe the behaviour of y_n when $n\tau = 1$ or is very close to 1.

Theorem 4. *Assume that $\tau = 1/N$, N an integer. Then the value y_N corresponding to the last grid point $t=1$ in $[0,1]$ behaves like $N/\ln N$. More precisely,*

$$\lim_{N\to\infty} \frac{y_n \ln N}{N} = 1. \tag{23}$$

Proof. Let $c \geqslant 1$ be a fixed constant and set

$$n^* = [\frac{1}{\tau} - c \ln \frac{1}{\tau}] \geqslant \frac{1}{\tau} - c \ln \frac{1}{\tau} - 1.$$

Here τ is assumed to be small enough in order to guarantee that $n^* \geqslant 0$. From (19),

$$y_{n^*} \geqslant (1 - \frac{1}{c}) y(n^*\tau) \geqslant (1 - \frac{1}{c}) y(1 - c\tau \ln \frac{1}{\tau} - \tau) = \frac{F(c)}{\tau \ln \tau^{-1} + \tau c^{-1}},$$

where we have set $F(c) = (c-1)/c^2$. In order to describe the behaviour of the iterates y_n for $n \geqslant n^*$ we introduce the scaled iterates

$$Y_k = y_{n^*+k} \{(\tau \ln \frac{1}{\tau} + \frac{\tau}{c})/F(c)\}, \; k = 0,1,..,N-n^*.$$

These satisfy

$$Y_0 \geqslant 1, \tag{24}$$
$$Y_{k+1} = Y_k + \frac{F(c)}{\ln \tau^{-1} + c^{-1}} Y_k^2, \; k = 0,1,...,N-n^*-1,$$

i.e., a recursion of the form (1). If we set $n^{**} = N - n^*$, we have $n^{**} = \tau^{-1} - [\tau^{-1} - c \ln \tau^{-1}]$, so that the lower bound in (16) applied to the new recursion (24) yields

$$Y_{n^{**}} \geqslant \psi(\tau,c)(1 - \frac{F(c)}{\ln \tau^{-1} + c^{-1}} \psi(\tau,c) \ln \psi(\tau,c)), \tag{25}$$

where

$$1/\psi(\tau,c) = 1 - (\frac{1}{\tau} - [\frac{1}{\tau} - c \ln \frac{1}{\tau}]) \frac{F(c)}{\ln \tau^{-1} + c^{-1}}.$$

Note that it is allowed to have $Y_0>1$ in (24) as all iterates increase with Y_0. Returning to the original iterates y_n, (25) can be written to

$$(\tau \ln \frac{1}{\tau}+\frac{\tau}{c})y_N \geqslant F(c)\psi(\tau,c)(1-\frac{F(c)}{\ln \tau^{-1}+c^{-1}}\psi(\tau,c)\ln \psi(\tau,c)).$$

Letting now $\tau \rightarrow 0$ we find

$$\lim\inf(\tau \ln \frac{1}{\tau}+\frac{\tau}{c})y_N \geqslant F(c)/(1-cF(c)) = \frac{c^2-c}{c^2},$$

whence

$$\liminf(\tau \ln \frac{1}{\tau})y_N \geqslant \frac{c^2-c}{c^2}.$$

Since this holds for every $c \geqslant 1$ we conclude that

$$\liminf(\tau \ln \frac{1}{\tau})y_N \geqslant 1.$$

It remains to be proved that

$$\limsup(\tau \ln \frac{1}{\tau})y_N \leqslant 1.$$

This can be done in a similar way using now the bounds of Theorem 3 rather than (16). □

Corollary. *Assume that $\tau=1/N, N$ an integer. Then the relative error for the value y_{N-1} corresponding to the last grid point $t=1-\tau$ in* [0,1) *satisfies*

$$\frac{y_{N-1}-y(1-\tau)}{y(1-\tau)} \sim \frac{1}{\ln N}-1,\ \tau \rightarrow 0, \tag{26}$$

and thus approaches -1 as $\tau \rightarrow 0$.

Proof. From the recursion formula we have $y_{N-1}=\frac{1}{2}N(\sqrt{1+4y_N/N}-1)$, so that, using (23), $y_{N-1} \sim N/\ln N$ as $N \rightarrow \infty$. Relation (26) is trivial now, since $y(1-\tau)=N$. □

Finally we provide some numerical illustrations of the theorems above. When $\tau=0.05$, $n=10$ the theoretical solution $y(n\tau)$ takes the value 2. The approximation is $y_n=1.8844$ with error $-.1156$. The lower bound (16) yields -.1387 and the upper bound (21) gives -.1151. The table illustrates the estimates of Theorem 4 and its corollary.

τ	y_N	$N^{-1}y_N \ln N$	$N^{-1}y_{N-1}-1$	$1/\ln N-1$
1.E-1	6.12E0	1.41	-.571	-.566
1.E-2	3.03E1	1.39	-.756	-.783
1.E-3	1.93E2	1.33	-.834	-.855
1.E-4	1.39E3	1.28	-.876	-.891
1.E-5	1.08E4	1.24	-.901	-.913
1.E-6	8.81E4	1.21	-.918	-.927
1.E-7	7.41E5	1.19	-.931	-.938
1.E-8	6.39E6	1.17	-.940	-.946
1.E-9	5.61E7	1.16	-.948	-.952

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