The Haemers bound of noncommutative graphs

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Abstract

We continue the study of the quantum channel version of Shannon's zero-error capacity problem. We generalize the celebrated Haemers bound to noncommutative graphs (obtained from quantum channels). We prove basic properties of this bound, such as additivity under the direct sum and submultiplicativity under the tensor product. The Haemers bound upper bounds the Shannon capacity of noncommutative graphs, and we show that it can outperform other known upper bounds, including noncommutative analogues of the Lovász theta function (Duan–Severini–Winter, *IEEE Trans. Inform. Theory*, 2013 and Boreland–Todorov–Winter, *arXiv*, 2019).

Keywords: Haemers bound – Noncommutative graphs – Quantum channels – Shannon capacity – Zero-error information theory

1 Introduction

The celebrated *Shannon capacity* of a graph G is defined as

$$\Theta(G) = \sup_k \sqrt[k]{\alpha(G^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^{\boxtimes k})}, \tag{1}$$

where $\alpha(G)$ denotes the independence number of G and $\boxtimes$ denotes the strong graph product [Sha56]. The logarithm of $\Theta(G)$ characterizes the amount of information that can be transmitted through a classical communication channel, with zero error, where we allow an arbitrary number of uses of the channel and we measure the average amount of information transmitted per use of the channel. (The graph G is the so-called *confusability graph* associated to the channel, see Section 2.1.) The Shannon capacity is not known to be *computable*: Even though computing the independence number is NP-complete [Kar72], there exist graphs whose Shannon capacities are not achieved by taking the strong graph product with itself *finitely* many times [GW90].

To upper bound the Shannon capacity, Lovász introduced the celebrated theta function [Lov79], which can be cast as a semidefinite program and can be used to compute, e.g., $\Theta(C_5)$. Lovász posed the question whether the Shannon capacity equals the theta function in general, which has been refuted by Haemers: He introduced another upper bound on the Shannon capacity, now known as the Haemers bound, which can be strictly smaller than the theta function on some graphs [Hae78, Hae79].

Instead of a *classical* communication channel, we could also consider a *quantum* communication channel. Doing so leads to quantum information analogues of the aforementioned questions, the study of which was systematically initiated by Duan, Severini and Winter [DSW13]. In Section 2.1 we show how the quantum setting generalizes the classical setting, which also motivates the following definitions. To (the Choi-Kraus representation of) a quantum channel $\Phi(A) = \sum_{k=1}^m E_k A E_k^\dagger$ ($\forall\ A \in$

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M_n) we associate the *noncommutative (confusability) graph* $S_\Phi = \operatorname{span}\{E_k^\dagger E_k' : \ k, k' = 1, \ldots, m\}$. The noncommutative graph S_Φ completely characterizes the number of zero-error messages one can send through the quantum channel Φ. More precisely, the independence number of $S \subseteq M_n$ is defined as the maximum number ℓ for which there exist non-zero vectors (pure quantum states) $|\psi_1\rangle, \ldots, |\psi_\ell\rangle \in \mathbb{C}^n$ satisfying that $\langle\psi_i| A |\psi_j\rangle = 0$ for all distinct $i, j \in [\ell]$ and $A \in S$. We denote this by $\alpha(S)$. The Shannon capacity of a noncommutative graph S is defined analogously as $\Theta(S) = \lim_{n\to\infty} \sqrt[n]{\alpha(S^{\otimes n})}$, where $\otimes$ is the tensor product [DSW13].

As in the classical setting, it is not known whether the Shannon capacity of noncommutative graphs is computable. We do know that computing the independence number of a noncommutative graph is QMA-hard [BS07].[1] To upper bound the Shannon capacity of noncommutative graphs, it is natural to consider lifting bounds on the Shannon capacity of classical channels to the quantum setting. Duan, Severini and Winter introduced a quantum version of the Lovász theta function on noncommutative graphs [DSW13], which "properly" generalizes the theta function to the noncommutative graph setting. Recent studies have extended many other interesting graph notions to noncommutative graphs [Sta16, Wea17, LPT18, Wea19, BTW19]. However, it remained an open question (as mentioned in [LPT18]) how to generalize the Haemers bound to noncommutative graphs.

In this paper we show how to do so. We define a Haemers bound for noncommutative graphs, which canonically generalizes the classical Haemers bound of graphs (over complex numbers). Similar to the classical case, we prove that our bound upper bounds the Shannon capacity of noncommutative graphs. Our definition is inspired by the definition of noncommutative analogue the orthogonal rank [LPT18], combined with an observation that in the classical graph setting the orthogonal rank is a positive semidefinite version of the Haemers bound [Pee96]. We also compare our bound with other existing Shannon capacity bounds of noncommutative graphs.

2 Preliminaries

Throughout, all scalars will be complex numbers. We use the Dirac notation for (unit) vectors, e.g., $|i\rangle$ stands for the i-th standard basis vector (whose dimension will be clear from context). Let $[n] = \{1, \ldots, n\}$. Let $M_{n\times m}(\mathcal{X})$ be the set of $n \times m$ complex-valued matrices whose entries are from some ring $\mathcal{X}$. When $\mathcal{X}$ is omitted, we assume $\mathcal{X} = \mathbb{C}$. Let $M_n(\mathcal{X}) = M_{n\times n}(\mathcal{X})$. For a matrix $B \in M_{n\times m}(\mathcal{X})$, we sometimes write $B = [B_{i,j}]_{i\in[n],j\in[m]}$, where $B_{i,j} \in \mathcal{X}$ denotes the (i, j)-th entry of B. A matrix $B \in M_{m\times n}$ has rank *at most* k if and only if there exist $C \in M_{k\times m}$ and $D \in M_{k\times n}$ such that $B = C^\dagger D$. We use $B \succeq 0$ to denote the positive-semidefiniteness of B and M_n^+ to denote the set of *positive semidefinite* matrices of size n-by-n. We use $\mathcal{D}_n$ to denote the set (linear space) of diagonal matrices of size n-by-n. The *trace* of a matrix $B \in M_n$ is the sum of its diagonal elements, i.e., $\operatorname{Tr}(B) = \sum_{i=1}^n B_{i,i}$. The *Hilbert-Schmidt* inner product of two matrices $A, B \in M_{n\times n'}$ is $\operatorname{Tr}(A^\dagger B)$, where $A^\dagger$ denotes the conjugate transpose of A. The tensor product of two matrices $A \in M_{n\times n'}$ and $B \in M_{m\times m'}$ is the matrix

$$A \otimes B = \begin{bmatrix} A_{1,1}B & \cdots & A_{1,n'}B \\ \vdots & & \vdots \\ A_{n,1}B & \cdots & A_{n,n'}B \end{bmatrix} \in M_{n\times n'}(M_{m\times m'}) \cong M_{nm\times n'm'}.$$

The tensor product of two linear subspaces $S \subseteq M_{n\times n'}$ and $T \subseteq M_{m\times m'}$ is $S \otimes T = \operatorname{span}\{A \otimes B : A \in S, \ B \in T\}$.

[1] In fact, computing the independence number of noncommutative graphs obtained from *entanglement-breaking* channels is already QMA-complete [BS07].

Throughout we assume that a graph is finite, simple, undirected and has no loops. Unless specified otherwise, we consider n-vertex graphs with vertices labeled by elements of the set $[n]$. For a graph $G = ([n], E)$ we use $\overline{G} = ([n], \Lambda_n \setminus E)$ to denote the complement graph of G, where $\Lambda_n = \{\{i, j\} : i, j \in [n] \text{ and } i \neq j\}$. We use $K_n = ([n], \Lambda_n)$ to denote the n-vertex complete graph and $\overline{K_n} = ([n], \emptyset)$ to denote the n-vertex empty graph.

An independent set of a graph $G = ([n], E)$ is a subset of vertices $I \subseteq [n]$ such that any two vertices in I are not adjacent, i.e., for all distinct $i, j \in I$ we have $\{i, j\} \notin E$. The independence number of G, denoted by $\alpha(G)$, is the largest cardinality among all independent sets. For two graphs $G = ([n], E)$ and $H = ([n'], F)$, the strong graph product of G and H, denoted by $G \boxtimes H$, is a graph with vertex set $[n] \times [n']$ and edge set $\{\{(i, k), (j, \ell)\} : \{i, j\} \in E \text{ or } i = j \in [n] \text{ and } \{k, \ell\} \in F \text{ or } k = \ell \in [n']\} \setminus \{\{(i, k), (i, k)\} : i \in [n], k \in [n']\}$. The independence number is supermultiplicative with respect to the strong graph product: $\alpha(G \boxtimes H) \geq \alpha(G)\alpha(H)$. The Shannon capacity of G, denoted by $\Theta(G)$, is defined as $\Theta(G) = \sup_k \sqrt[k]{\alpha(G^{\boxtimes k})} = \lim_{k\to\infty} \sqrt[k]{\alpha(G^{\boxtimes k})}$, where the limit equals the supremum due to Fekete's lemma.

We say there is a graph homomorphism from $G = ([n], E)$ to $H = ([n'], F)$, denoted by $G \to H$, if there exist a vertex map $\varphi : [n] \to [n']$ which preserves the adjacency relation, i.e., for any $\{i, j\} \in E$ we have $\{\varphi(i), \varphi(j)\} \in F$. We say that there is a graph cohomomorphism from G to H, denoted by $G \leq H$, if $\overline{G} \to \overline{H}$. The independence number can be equivalently defined in terms of graph (co)homomorphism:

$$\alpha(G) = \max\{n : K_n \to \overline{G}\} = \max\{n : \overline{K_n} \leq G\}.$$

2.1 From the Shannon capacity of graphs to the Shannon capacity of noncommutative graphs

We briefly recall the development of noncommutative graph theory, which is originally motivated by the study of the quantum channel version of Shannon's zero-error capacity problem [DSW13].

The connection between information theory and graph theory was first observed by Shannon [Sha56] in the study of the zero-error capacity problem:

How many messages can be send through a communication channel with zero error?

Here a classical communication channel N is represented by the transition probability function $N : X \to Y$ from the (finite) input alphabet X to the (finite) output alphabet Y, where $N(y|x)$ denotes the probability of getting output y conditioned on sending input x through the channel N. Two input symbols $x, x' \in X$ can be confused by the receiver if there exists an output $y \in Y$ satisfying $N(y|x) > 0$ and $N(y|x') > 0$. To transmit messages through N with zero error, these messages should be encoded into input symbols of which their outputs should not be confused by the receiver.

To estimate the zero-error capacity, Shannon associated to each channel N the *confusability* graph G_N, where the vertices of G_N are all possible input symbols in X, and two vertices $x, x' \in X$ are connected if they can be confused. A set of input symbols is said to be not confused if every pair of input symbols can not be confused. Such a set of input symbols thus forms an independent set in the confusability graph G_N. Hence, the maximum number of zero-error messages one can send via a single-use of N equals the *independence number* $\alpha(G_N)$ of G_N, which is a classical notion in graph theory. It is also not hard to see that to any graph G one can also associate a classical communication channel N such that G and G_N are the same (hence from now on we can focus on graphs instead of channels). Finally, if we are allowed to send length-k codewords through N (namely, use the channel k times nonadaptively), the confusability graph of the resulting channel is given by $G_N^{\boxtimes k}$, the k-fold strong graph product of G_N. It is then natural to see the Shannon capacity of a graph (given in Equation (1)) as the number of distinct messages per use some classical channel can communicate with no error, in the asymptotic limit.

The same zero-error capacity problem can be also studied in the context of quantum information, where classical communication channels are replaced by quantum communication channels. (We refer the readers to [NC10] for a nice introduction.) Mathematically speaking, a *quantum channel* is a *completely positive and trace preserving* linear map $\Phi : M_n \to M_{n'}$. Equivalently, $\Phi : M_n \to M_{n'}$ is a map that admits a *Choi-Kraus representation*: $\Phi(A) = \sum_{i=1}^m E_i A E_i^\dagger$ for all $A \in M_n$, where $\sum_{i=1}^m E_i^\dagger E_i = I_n$. The matrices $E_1, \ldots, E_m \in M_{n' \times n}$ are called the *Choi-Kraus operators* of Φ. The input and output symbols of a quantum channel $\Phi : M_n \to M_{n'}$ are *quantum states*, i.e. positive semidefinite matrices with trace equal to one. Two quantum states are nonconfusable (perfectly distinguishable) if and only if they are orthogonal (with respect to the Hilbert-Schmidt inner product). Note that a classical communication channel $N : X \to Y$ can be viewed as the quantum channel $\Phi_N : M_{|X|} \to M_{|Y|}$, whose Choi-Kraus representation can be chosen as

$$\Phi_N(A) = \sum_{y \in Y, x \in X} N(y|x) \left|y\right\rangle\!\left\langle x\right| A \left|x\right\rangle\!\left\langle y\right|. \tag{2}$$

To transmit (classical) zero-error messages through a quantum channel $\Phi : M_n \to M_{n'}$, one can encode the messages into quantum states and send those through the channel Φ. In order to decode the messages with zero error, the channel-outputs corresponding to different messages should be orthogonal. The maximum number of zero-error messages one can send via a single-use of the quantum channel Φ is the maximum number ℓ of distinct input states $\rho_1, \ldots, \rho_\ell$ satisfying $\mathrm{Tr}(\Phi(\rho_i)^\dagger \Phi(\rho_j)) = 0$ for all $i, j \in [\ell]$. Due to the fact that quantum channels are completely positive, we may (w.l.o.g.) assume that each ρ_i has rank 1, that is, $\rho_i = |\psi_i\rangle\langle\psi_i|$ for some unit vector $|\psi_i\rangle \in \mathbb{C}^n$. The orthogonality relations between the channel-outputs $\Phi(|\psi_i\rangle\langle\psi_i|)$ and $\Phi(|\psi_j\rangle\langle\psi_j|)$ then read as follows:

$$\mathrm{Tr}(\Phi(|\psi_i\rangle\langle\psi_i|)^\dagger \Phi(|\psi_j\rangle\langle\psi_j|)) = \sum_{k,k'=1}^m \mathrm{Tr}(E_k |\psi_i\rangle\langle\psi_i| E_k^\dagger E_{k'} |\psi_j\rangle\langle\psi_j| E_{k'}^\dagger) = \sum_{k,k'=1}^m |\langle\psi_i| E_k^\dagger E_{k'} |\psi_j\rangle|^2 = 0.$$

Observe that the maximum number of perfectly distinguishable messages, ℓ, is completely determined by the subspace $S_\Phi = \mathrm{span}\{E_k^\dagger E_{k'} : k, k' \in [m]\} \subseteq M_n$. The subspace S_Φ is named as the *noncommutative (confusability) graph* of the quantum channel Φ in [DSW13]. As we show below, the noncommutative graphs of quantum channels shall be viewed as a quantum generalization of the confusability graphs of classical channels. Note that S_Φ is self-adjoint, i.e., $A \in S_\Phi$ implies $A^\dagger \in S_\Phi$, and that $I_n \in S_\Phi$. Subspaces of M_n satisfying these two properties are known as operator systems in functional analysis. In the rest of this paper, we refer to such subspaces as noncommutative graphs. This is justified, since we can also associate to every operator system a quantum channel of which the noncommutative (confusability) graph coincides with the operator system [Dua09, CCH11].

As mentioned before, the independence number of a noncommutative graph $S \subseteq M_n$ (also denoted as $\alpha(S)$) is the maximum number ℓ of unit vectors $|\psi_1\rangle, \ldots, |\psi_\ell\rangle$ for which $\langle\psi_i| A |\psi_j\rangle = 0$ for all $i \neq j$ and $A \in S_\Phi$.[2] The independence number $\alpha(S)$ is exactly the maximum number of zero-error messages one can transmit via a single-use of any quantum channel Φ whose noncommutative graph is S. Note that the use of length-k quantum codewords of a channel Ψ results in the noncommutative graph $S_\Phi^{\otimes k}$. Define the Shannon capacity of a noncommutative graph S as

$$\Theta(S) = \sup_k \sqrt[k]{\alpha(S^{\otimes k})} = \lim_{k \to \infty} \sqrt[k]{\alpha(S^{\otimes k})}. \tag{3}$$

[2] Another way to interpret this is to view matrices in S_Φ as "edges" and all (unit) vectors in $\mathbb{C}^n$ as "vertices". The equalities $\langle\psi_i| A |\psi_j\rangle = 0$ for all $A \in S$ then indicate that $|\psi_i\rangle$ and $|\psi_j\rangle$ are "nonadjacent".

The Shannon capacity of a noncommutative graph is exactly the number of distinct messages per use some quantum channel can communicate with no error, in the asymptotic limit.

Since classical channels are special cases of quantum channels. One can associate a classical channel N (viewed as the quantum channel given in Equation (2)) the noncommutative graph

$$S_N = \mathrm{span}\{|x\rangle\langle x'| : \ \exists\, y \in Y, \ \sqrt{N(y|x)N(y|x')} > 0\}.$$

Note that S_N is in one-to-one correspondence (up to graph isomorphism) with the confusability graph G_N of N: it is the matrix space whose support equals G_N (supplemented with the diagonal matrices). In other words, to every n-vertex graph $G = ([n], E)$ we associate the noncommutative graph

$$S_G := \mathrm{span}\{|i\rangle\langle j| : \ i = j \in [n] \text{ or } \{i,j\} \in E\} \subseteq M_n. \tag{4}$$

We emphasize that this correspondence is canonical for several reasons. First of all, let us call two noncommutative graphs $S, T \subseteq M_n$ *isomorphic* if there exist an $n \times n$ unitary matrix U such that $U^\dagger S U = T$. (In the classical setting this corresponds to the situation where the input symbols are permuted.) Then the graphs G and H are isomorphic if and only if S_G and S_H are isomorphic [OP15]. Second, the disjoint union of two graphs G and H is mapped to the *direct sum* of the corresponding noncommutative graph S_G and S_H and the strong graph product of two graphs G and H is mapped to the *tensor product* of the corresponding noncommutative graph S_G and S_H. Last but not least, we have $\alpha(G) = \alpha(S_G)$ [DSW13] and therefore, $\Theta(G) = \Theta(S_G)$.

2.2 Prior work on the Shannon capacities of graphs and noncommutative graphs

The graph setting. As we have mentioned before, it is not known how to compute the Shannon capacity of a graph. It is not even known whether it is computable at all. It therefore makes sense to consider bounds on the Shannon capacity. *Lower bounds* often arise from explicit constructions of stable sets in some power of the (noncommutative) graph. For instance, the state-of-the-art lower bound on $\Theta(C_7)$ comes from an independent set of size 367 in $C_7^{\boxtimes 5}$ [PS19]. A natural way to obtain *upper bounds* on the Shannon capacity is to obtain *submultiplicative* (with respect to the strong graph product) upper bounds on the independence number. Examples of such upper bounds are the fractional clique-cover number [Sha56], the Lovász theta number [Lov79], and parameters such as the (fractional) Haemers bound [Hae78, Hae79, Bla13, BC19] and the orthogonal (projective) rank [Lov79, MR16]. The last three parameters are central to this work, which is why we define them below.

Let G be a graph. The Lovász theta number of G, denoted $\vartheta(G)$, is

$$\vartheta(G) = \max\{\|I+T\| : \ T_{i,j} = 0 \text{ if } i = j \in [n] \text{ or } \{i,j\} \in E, \ I + T \succeq 0\}; \tag{5}$$

the Haemers bound of G (over $\mathbb{C}$), denoted $\mathcal{H}(G)$, is

$$\begin{aligned} \mathcal{H}(G) &= \min\{\mathrm{rank}(B) : \ B_{i,i} = 1 \ \forall\, i \in [n], \ B_{i,j} = 0 \text{ if } \{i,j\} \notin E(G)\} \\ &= \min\{\mathrm{rank}(B) : \ B_{i,i} \neq 0 \ \forall\, i \in [n], \ B_{i,j} = 0 \text{ if } \{i,j\} \notin E(G)\}; \end{aligned} \tag{6}$$

and finally the orthogonal rank[3] of G (over $\mathbb{C}$), denoted $\overline{\xi}(G)$, is

$$\overline{\xi}(G) = \min\{k : \ \exists\, |\psi_1\rangle, \dots, |\psi_n\rangle \in \mathbb{C}^k, \ \text{s.t. } \langle\psi_i|\psi_i\rangle \neq 0 \text{ and } \langle\psi_i|\psi_j\rangle = 0 \text{ if } \{i,j\} \notin E(G)\}. \tag{7}$$

[3] The name "orthogonal rank" (and its notation ξ) is normally used in the study of graph coloring, where adjacent vertices receive orthogonal vectors. In the study of independence number and Shannon capacity one usually considers $\overline{\xi}(G)$, the orthogonal rank of the complement of G, this equals the dimension of the vector space of an "orthonormal representation" (introduced by Lovász in his original paper [Lov79]), where *nonadjacent* vertices receive orthogonal vectors. In this paper we are interested in the second setting and for brevity we will refer to $\overline{\xi}(G)$ as the orthogonal rank of G.

At first glance the orthogonal rank of a graph G seems unrelated to the Haemers bound, but we point out that the orthogonal rank can be viewed as a positive-semidefinite version of the Haemers bound (which has been mentioned implicitly in [Pee96], and can also easily be obtained from [HPRS17]).

Observation 1. *For a graph $G = ([n], E)$ we have*

$$\overline{\xi}(G) = \min\{\mathrm{rank}(B) : \ B_{i,i} = 1 \ \forall \ i \in [n], \ B_{i,j} = 0 \text{ if } \{i,j\} \notin E(G), \ B \succeq 0\}.$$

Proof. Indeed, the matrix $B = \mathrm{Gram}(|\psi_1\rangle, \dots, |\psi_n\rangle) = \left[\langle\psi_i|\psi_j\rangle\right]_{i,j\in[n]}$ is feasible and has rank at most k if and only if the vectors $|\psi_1\rangle, \dots, |\psi_n\rangle$ can be taken in $\mathbb{C}^k$ and satisfy the orthogonality conditions of Equation (7). $\square$

For any graph G, we have the following inequalities:

$$\alpha(G) \le \Theta(G) \le \vartheta(G) \le \overline{\xi}(G), \quad \alpha(G) \le \Theta(G) \le \mathcal{H}(G) \le \overline{\xi}(G). \tag{8}$$

The Lovász theta function and the Haemers bound are *incomparable*: We have $\vartheta(C_5) = \sqrt{5} < 3 \le \mathcal{H}(C_5)$ and $\mathcal{H}(G) \le 7 < 9 = \vartheta(G)$ when G is taken as the complement of the *Shläfli graph* [Hae78].

The noncommutative graph setting. Previously, work has been done on constructing noncommutative analogues of the Lovász theta number [DSW13] and the orthogonal rank [Sta16, LPT18]. Our goal is to provide a noncommutative analogue of Haemers bound and therefore we will go over the (very much related) noncommutative analogue of the orthogonal rank in more detail below.

In [Sta16], the orthogonal rank of a noncommutative graph S, denoted $\overline{\xi}(S)$, is defined as

$$\overline{\xi}(S) = \min\{k : \ \exists \ \text{ quantum channel } \Phi : M_n \to M_k, \ \text{s.t. } S_\Phi \subseteq S\}. \tag{9}$$

The following proposition shows that $\overline{\xi}(S)$ is a proper noncommutative analogue of the orthogonal rank of graphs:

Proposition 2 ([Sta16, Theorem 12])**.** *Let $G = ([n], E)$ be a graph and let S_G be as in Equation (4). Then $\overline{\xi}(S_G) = \overline{\xi}(G)$.*

Proof. We first show $\overline{\xi}(S_G) \le \overline{\xi}(G)$. Let $|\psi_1\rangle, \dots, |\psi_n\rangle \in \mathbb{C}^k$ be a feasible solution of $\overline{\xi}(G)$ (as in Equation (7)) and let $\Phi : M_n \to M_k$ be the quantum channel defined as $\rho \mapsto \Phi(\rho) := \sum_{i=1}^n |\psi_i\rangle\langle i| \, \rho \, |i\rangle\langle\psi_i|$. We observe that $S_\Phi = \{|i\rangle\langle j| : \langle\psi_i|\psi_j\rangle \neq 0\}$. Since $\langle\psi_i|\psi_j\rangle = 0$ whenever $\{i,j\} \notin E$, we therefore have $S_\Phi \subseteq S_G$, which implies that $\overline{\xi}(S_G) \le \overline{\xi}(G)$.

On the other hand, let $\Phi : M_n \to M_k$ be a quantum channel, with Choi-Kraus operators $\{E_1, \dots, E_m\} \subseteq M_{k\times n}$, that is a feasible solution of $\overline{\xi}(S_G)$. Since $\sum_{j=1}^m E_j^\dagger E_j = I_n$, for each $i \in [n]$ there exist at least one $j(i) \in [m]$ such that $E_{j(i)} |i\rangle$ is nonzero. Let $|\psi_i\rangle = E_{j(i)} |i\rangle \in \mathbb{C}^k$ for $i \in [n]$. We now show that $\langle\psi_i|\psi_{i'}\rangle = 0$ for $\{i,i'\} \notin E$. Note that $S_\Phi \subseteq S_G$ implies that for $\{i,i'\} \notin E$ we have $S_\Phi \perp |i'\rangle\langle i|$. Thus $\langle\psi_i|\psi_{i'}\rangle = \langle i| E_{j(i)}^\dagger E_{j(i')} |i'\rangle = \mathrm{Tr}(E_{j(i)}^\dagger E_{j(i')} |i'\rangle\langle i|) = 0$ for all $\{i,i'\} \notin E$. It follows that the vectors $|\psi_1\rangle, \dots, |\psi_n\rangle$ are feasible for $\overline{\xi}(G)$ and therefore we have $\overline{\xi}(G) \le \overline{\xi}(S_G)$. $\square$

In [LPT18], an alternative definition of $\overline{\xi}$ for noncommutative graphs is formulated:

Proposition 3 (Proposition 14 in [LPT18])**.** *For a noncommutative graph $S \subseteq M_n$ we have*

$$\overline{\xi}(S) = \min\{\mathrm{rank}(B) : \ m \in \mathbb{N}, \ B \in M_m(S)^+, \ \sum_{i=1}^m B_{i,i} = I_n\}. \tag{10}$$

Proof. Let k and $\Phi : M_n \to M_k$ be a feasible solution of (9), where $\Phi : M_n \to M_k$ has Choi-Kraus operators $\{E_i : i \in [m]\} \subseteq M_{k\times n}$ satisfying $S_\Phi \subseteq S$. Define $B = \left[E_i^\dagger E_j\right]_{i,j\in[m]} \in M_m(S)$. Then B is a feasible solution of (10) since $\sum_{i=1}^m B_{i,i} = \sum_{i=1}^m E_i^\dagger E_i = I_n$ and B can be written as $B = C^\dagger C$, where $C = \begin{bmatrix} E_1 & \cdots & E_m \end{bmatrix} \in M_{k\times mn}$. Moreover, since $B = C^\dagger C$, we have $\mathrm{rank}(B) \leq k$.

On the other hand, let $B \in M_m(S)^+$ be a feasible solution of (10) with $\mathrm{rank}(B) = k$. Let $B = C^\dagger C$ for $C = [C_1 \cdots C_m]$, where $C_i \in M_{k\times n}$. Define $\Phi : M_n \to M_k$ by $\Phi(A) = \sum_{i=1}^m C_i A C_i^\dagger$ for any $A \in M_n$. The map $\Phi : M_n \to M_k$ is a quantum channel since $\sum_{i=1}^m C_i^\dagger C_i = \sum_{i=1}^m B_{i,i} = I_n$. Finally, we have that $S_\Phi \subseteq S$ since $C_i^\dagger C_j = B_{i,j} \in S$ for $i, j \in [m]$. Thus Φ and k form a feasible solution of (7). □

Remark: we have $\Theta(S) \leq \bar{\xi}(S)$ for all non-commutative graphs [Sta16, LPT18]. Surprisingly, in [LPT18] it was shown that there exist noncommutative graphs S for which the Duan-Severini-Winter noncommutative analogue of ϑ (see [DSW13]) is strictly larger than $\bar{\xi}$. Recently, another noncommutative analogue of ϑ has been proposed in [BTW19] from a geometric perspective, and it was shown that it always lies between $\Theta(S)$ and $\bar{\xi}(S)$.

3 Haemers bound for noncommutative graphs

3.1 Definition and consistency

By comparing the orthogonal rank (see Observation 1) to the definition of the Haemers bound for graphs (see Equation (6)), we see that both can be viewed as finding the smallest-rank matrices in a feasible region that is very similar: the only difference is that for $\bar{\xi}(G)$ the feasible matrices are additionally required to be positive semidefinite.

Motivated by Proposition 3, which gives a formulation of $\bar{\xi}$ for noncommutative graphs, we therefore define the Haemers bound for noncommutative graphs by dropping the positivity requirement on the feasible region:

Definition 4 (Haemers bound for noncommutative graphs)**.** *Let $S \subseteq M_n$ be a noncommutative graph. The Haemers bound of S (over $\mathbb{C}$) is defined as*

$$\mathcal{H}(S) = \min\{\mathrm{rank}(B) :\ m \in \mathbb{N},\ B \in M_m(S),\ \sum_{i=1}^m B_{i,i} = I_n\}. \tag{11}$$

We first show that this bound is a proper noncommutative analogue of Haemers bound.

Proposition 5. *Let $G = ([n], E)$ be a graph and let S_G be defined as in (4). Then $\mathcal{H}(S_G) = \mathcal{H}(G)$.*

Proof. We first show that $\mathcal{H}(S_G) \leq \mathcal{H}(G)$. Let B be a feasible solution of (6) with $\mathrm{rank}(B) = k$ and $B_{i,i} = 1$ for $i \in [n]$. Decompose B as $B = C^\dagger D$ where $C, D \in M_{k\times n}$. Denote the (normalized) columns of C and D as $\{|C_1\rangle, \ldots, |C_n\rangle\}$ and $\{|D_1\rangle, \ldots, |D_n\rangle\}$ respectively. Then $B = \left[\langle C_i|D_j\rangle\right]_{i,j\in[n]}$. Define the matrix $B' = \left[\langle C_i|D_j\rangle |i\rangle\langle j|\right]_{i,j\in[n]}$. We show that n and B' is a feasible solution of (11). First, note that $\langle C_i|D_j\rangle |i\rangle\langle j| \in S_G$ for $i, j \in [n]$, since $\langle C_i|D_j\rangle = B_{i,j} = 0$ when $\{i, j\} \notin E$. Second, we have $\sum_{i=1}^n B'_{i,i} = \sum_{i=1}^n \langle C_i|D_i\rangle |i\rangle\langle i| = \sum_{i=1}^n |i\rangle\langle i| = I_n$. To bound the rank of B', note that $B' = (C')^\dagger D'$, where $C' = \begin{bmatrix} |C_1\rangle\langle 1| & \cdots & |C_n\rangle\langle n| \end{bmatrix}$ and $D' = \begin{bmatrix} |D_1\rangle\langle 1| & \cdots & |D_n\rangle\langle n| \end{bmatrix}$. Since $C', D' \in M_{k\times n^2}$, we have $\mathrm{rank}(B') \leq k$ and therefore $\mathcal{H}(S_G) \leq \mathcal{H}(G)$.

We then show that $\mathcal{H}(G) \leq \mathcal{H}(S_G)$. Let m and B be a feasible solution of (11) with $\mathrm{rank}(B) = k$. Write $B = C^\dagger D$ for $C, D \in M_{k\times mn}$. We group the columns of C and D according to the block-structure of B: let $C = \begin{bmatrix} C_1 & \cdots & C_m \end{bmatrix}$ and $D = \begin{bmatrix} D_1 & \cdots & D_m \end{bmatrix}$ where $C_i, D_{i'} \in M_{k\times n}$ for $i, i' \in [n]$.

By the feasibility of B we have that $C_i^\dagger D_{i'} \in S_G$ for all $i, i' \in [n]$ and $\sum_{j=1}^m C_j^\dagger D_j = I_n$. By the second condition, we know that for each $i \in [n]$, we can pick a $j(i) \in [m]$ such that $\langle i| C_{j(i)}^\dagger D_{j(i)} |i\rangle \neq 0$ (if there is more than one such index, pick an arbitrary one). Let $B' = \left[\langle i| C_{j(i)}^\dagger D_{j(i')} |i'\rangle\right]_{i,i'\in[n]}$. We will show that B' is a feasible solution to (6) (the second formulation). It is easy to see that the diagonal entries of B' are nonzero, thus we only need to show that $B'_{i,i'} = \langle i| C_{j(i)}^\dagger D_{j(i')} |i'\rangle = 0$ for $\{i, i'\} \notin E$. This follows from $C_{j(i)}^\dagger D_{j(i')} \in S_G$ and $|i\rangle\langle i'| \perp S_G$ if $\{i, i'\} \notin E$. Finally, to bound the rank of B', note that B' can be written as $U^\dagger V$, where $U = \begin{bmatrix} C_{j(1)} |1\rangle & \cdots & C_{j(n)} |n\rangle \end{bmatrix} \in M_{k\times n}$ and $V = \begin{bmatrix} D_{j(1)} |1\rangle & \cdots & D_{j(n)} |n\rangle \end{bmatrix} \in M_{k\times n}$. We thus have $\mathrm{rank}(B') \leq k$ and it follows that $\mathcal{H}(G) \leq \mathcal{H}(S_G)$. □

3.2 Upper bound on the Shannon capacity of noncommutative graphs

We first show that the noncommutative analogue of the Haemers bound is submultiplicative with respect to the tensor product:

Proposition 6. *Let $S \subseteq M_n$ and $T \subseteq M_{n'}$ be noncommutative graphs, we have*

$$\mathcal{H}(S \otimes T) \leq \mathcal{H}(S)\mathcal{H}(T).$$

Proof. Let B_1 and m_1 be a feasible solution of $\mathcal{H}(S)$ and B_2 and m_2 be a feasible solution of $\mathcal{H}(T)$. We construct a feasible solution of $S \otimes T$. Let $B = B_1 \otimes B_2$ and $m = m_1 m_2$. It is easy to see that $B_1 \otimes B_2 \in M_m(S \otimes T)$. Moreover,

$$\sum_{i=1}^{m} B_{i,i} = \sum_{i_1=1}^{m_1} \sum_{i_2=1}^{m_2} (B_1)_{i_1,i_1} \otimes (B_2)_{i_2,i_2} = I_{m_1} \otimes I_{m_2} = I_m.$$

Since $\mathrm{rank}(B) = \mathrm{rank}(B_1 \otimes B_2) = \mathrm{rank}(B_1)\,\mathrm{rank}(B_2)$, we conclude that $\mathcal{H}(S\otimes T) \leq \mathcal{H}(S)\mathcal{H}(T)$. □

Remark 7. *The inequality in Proposition 6 can be strict; it was shown by Bukh and Cox that $\mathcal{H}(C_5^{\boxtimes 2}) \leq 8 < 9 \leq \mathcal{H}(C_5)^2$, see [BC19, Proposition 9].*

We next show that the Haemers bound of noncommutative graphs upper bounds the independence number of noncommutative graphs and therefore also its Shannon capacity:

Theorem 8. *Let $S \subseteq M_n$ be a noncommutative graph. We have $\alpha(S) \leq \Theta(S) \leq \mathcal{H}(S) \leq \overline{\xi}(S)$.*

Proof. The last inequality holds since $\mathcal{H}(S)$ (Equation (11)) is a relaxation of $\overline{\xi}(S)$ (Equation (10)).

To see the first two inequalities, we only need to show $\alpha(S) \leq \mathcal{H}(S)$ since by the previous proposition $\mathcal{H}(S)$ is submultiplicative.

Let $\alpha(S) = \ell$ and let $|\psi_1\rangle, \ldots, |\psi_\ell\rangle \in \mathbb{C}^n$ satisfy $\langle\psi_i| A |\psi_j\rangle = 0$ for all $i \neq j$ and $A \in S$. Let B and m be a feasible solution of $\mathcal{H}(S)$. We show that $\mathrm{rank}(B) \geq \ell$. Let $U = \begin{bmatrix} |\psi_1\rangle & \cdots & |\psi_\ell\rangle \end{bmatrix} \in M_{n\times\ell}$ and write $B = \left[B_{i,j}\right]_{i,j\in[m]}$ where $B_{i,j} \in S$ for all $i, j \in [m]$. Note that

$$\mathrm{rank}(B) \geq \mathrm{rank}((I_m \otimes U^\dagger)B(I_m \otimes U)).$$

Let $D_{i,j} := U^\dagger B_{i,j} U$ and note that $D_{i,j} = \mathrm{diag}(\langle\psi_1| B_{i,j} |\psi_1\rangle, \cdots, \langle\psi_\ell| B_{i,j} |\psi_\ell\rangle) \in M_\ell$. We set $D := \left[D_{i,j}\right]_{i,j\in[m]} = (I \otimes U^\dagger)B(I \otimes U) \in M_{m\ell}$. We claim that $\mathrm{rank}(D) \geq \ell$. To see this we use that $\sum_{j=1}^m B_{j,j} = I_n$. For any $i \in [\ell]$, there exists at least one $j(i) \in [m]$, such that $\langle\psi_i| B_{j(i),j(i)} |\psi_i\rangle \neq 0$. Then the submatrix of D consisting of the $(j(i), i)$-th row and column for all $i \in [\ell]$ is diagonal with every diagonal entry being nonzero. We conclude that $\mathrm{rank}(B) \geq \mathrm{rank}(D) \geq \ell$. □

All inequalities in the above theorem can be strict. This follows from the fact that they can be strict for graphs. Similarly, we point out that the Haemers bound of noncommutative graphs is incomparable with existing noncommutative analogues of the theta function (ϑ and $\tilde{\vartheta}$ introduced in [DSW13] and recently θ and $\hat{\theta}$ in [BTW19]). This again follows from the fact that ϑ and $\mathcal{H}$ are incomparable for graphs.

3.3 Properties of the Haemers bound and examples

We first give an upper bound on the size of the block-matrix needed in the definition of the Haemers bound of noncommutative graphs. The result is similar to Proposition IV.7 from [LPT18], where they show that for $\overline{\xi}(S)$ we may restrict to $m \leq 2n^3$ in Equation (10).

Proposition 9. *Let $S \subseteq M_n$, the optimal solution of $\mathcal{H}(S)$ can be achieved with $m \leq n^4$.*

Proof. First, let us note that $\mathcal{H}(S) \leq \overline{\xi}(S) \leq n$ since $B = I_n \in M_1(S)$ is a feasible solution of rank n.

Let $m \in \mathbb{N}$ and $B \in M_m(S)$ be feasible for $\mathcal{H}(S)$ with $\mathrm{rank}(B) = k \leq n$. Let us write $B = C^\dagger D$ with $C, D \in M_{k\times mn}$. Say $C = \begin{bmatrix} C_1 & \cdots & C_m \end{bmatrix}$ and $D = \begin{bmatrix} D_1 & \cdots & D_m \end{bmatrix}$ where $C_i, D_i \in M_{k\times n}$. Then, feasibility of B implies that $\sum_{i=1}^m C_i^\dagger D_i = I_n$. The crucial observation is now that there can be at most kn linearly independent matrices C_i (likewise for the matrices D_i). It follows that $I_n = \sum_{i=1}^m C_i^\dagger D_i = \sum_{i\in I, j\in J} \alpha_{i,j} C_i^\dagger D_j$ for some index sets $I, J \subseteq [m]$ with $|I|, |J| \leq kn$ and coefficients $\alpha_{i,j} \in \mathbb{C}$ (for $i \in I, j \in J$). We will now construct a matrix $B' \in M_{(kn)^2}(S)$ which is feasible for $\mathcal{H}(S)$ and has rank at most k. For $i \in I$ set

$$\bar{C}_i := \big[\underbrace{C_i \ \cdots \ C_i}_{|J| \text{ times}}\big] \in M_{k\times|J|n}, \qquad \bar{D}_i \quad := \big[(\alpha_{i,j_1} D_{j_1}) \ (\alpha_{i,j_2} D_{j_2}) \ \cdots \ (\alpha_{i,j_{|J|}} D_{j_{|J|}})\big] \in M_{k\times|J|n},$$

where $j_1, \ldots, j_{|J|}$ are the elements of J (for later use, let similarly $I = \{i_1, \ldots, i_{|I|}\}$). Next define

$$C' = \begin{bmatrix} \bar{C}_{i_1} \ \bar{C}_{i_2} & \cdots & \bar{C}_{i_{|I|}} \end{bmatrix} \in M_{k\times|I||J|n}$$
$$D' = \begin{bmatrix} \bar{D}_{i_1} \ \bar{D}_{i_2} & \cdots & \bar{D}_{i_{|I|}} \end{bmatrix} \in M_{k\times|I||J|n}.$$

Finally, set $B' = (C')^\dagger D'$ and observe that $B' \in M_{|I||J|}(S)$, that the sum of the diagonal blocks of B' is $\sum_{i\in I, j\in J} \alpha_{i,j} C_i^\dagger D_j = I_n$, and that $\mathrm{rank}(B') \leq k$.

To conclude the proof it suffices to note that $|I|, |J| \leq kn$ and therefore we may restrict our attention to $m \leq (kn)^2 \leq n^4$ in the definition of $\mathcal{H}(S)$ (see Equation (11)). $\square$

The above proposition implies that the Haemers bound is computable.

Corollary 10. *The Haemers bound of noncommutative graphs is computable.*

Proof. Let $S \subseteq M_n$ be a noncommutative graph. Proposition 9 tells us that in Definition 4 we may restrict our attention to $m \leq n^4$, i.e., matrices B of size polynomial in n. In the proof of the previous proposition we have seen that $\mathcal{H}(S) \leq n$, therefore it is an integer between 1 and n and we may compute it by solving several feasibility problems. Each feasibility problem asks whether there exists a matrix $B \in M_{n^4}(S)$ whose diagonal blocks sum up to the identity matrix has rank at most k for some $k \in [n]$. Such a problem can be viewed as asking whether or not a system of polynomial equations has a common root (in $\mathbb{C}$): The condition that $\mathrm{rank}(B) \leq k$ is equivalent to the condition that all the $(k+1) \times (k+1)$-minors of B are equal to zero.

Hilbert's Nullstellensatz implies that a system of polynomials has a common root if and only if 1 does not belong to the ideal generated by those polynomials. The latter can be tested using *Gröbner bases*. We refer to, for instance, [CLO15] for more details. $\square$

We now give a formulation of $\mathcal{H}(S)$ that is similar to Stahlke's definition of $\bar{\xi}(S)$ (see Equation (9)). For this, recall that a general trace-preserving linear map $\Psi : M_n \to M_{n'}$ can be written as $\Psi(X) = \sum_{i=1}^m E_i X F_i^\dagger$ for all $X \in M_n$, where $E_1, \ldots, E_m, F_1, \ldots, F_m \in M_{n' \times n}$ satisfy $\sum_{i=1}^m F_i^\dagger E_i = I_n$, see [Wat18, Theorem 2.26]. Let $T_\Psi = \operatorname{span}\{F_i^\dagger E_j : \ i, j \in [m]\}$. We have the following:

Proposition 11. *Let $S \subseteq M_n$, we have*

$$\mathcal{H}(S) = \min\{k : \ \exists \ \textit{trace-preserving linear map } \Psi : M_n \to M_k \textit{ s.t. } T_\Psi \subseteq S\}. \tag{12}$$

Proof. To show "$\geq$", let $m \in \mathbb{N}$ and $B \in M_m(S)$ be feasible for $\mathcal{H}(S)$ in Equation (11) with $\operatorname{rank}(B) = k$. Let us write $B = C^\dagger D$ with $C, D \in M_{k \times mn}$. Say $C = \begin{bmatrix} C_1 & \cdots & C_m \end{bmatrix}$ and $D = \begin{bmatrix} D_1 \cdots D_m \end{bmatrix}$ where $C_i, D_i \in M_{k \times n}$. Then, feasibility of B implies that $\sum_{i=1}^m C_i^\dagger D_i = I_n$. Let $\Psi : M_n \to M_k$ be defined as $\Psi(A) = \sum_{i=1}^m D_i A C_i^\dagger$ for any $A \in M_n$. Then Ψ is trace-preserving and $T_\Psi = \operatorname{span}\{C_i^\dagger D_j : \ i, j \in [m]\} \subseteq S$.

Conversely, to show "$\leq$", let $\Psi : M_n \to M_k$ be a feasible solution of the right-hand side of Equation (12), and let $E_1, \ldots, E_m, F_1, \ldots, F_m \in M_{k \times n}$ be such that $\Psi(A) = \sum_{i=1}^m E_i A F_i^\dagger$ for all $A \in M_n$ and $\sum_{i=1}^m F_i^\dagger E_i = I_n$. Define the matrix $B = \begin{bmatrix} F_i^\dagger E_j \end{bmatrix}_{i,j \in [m]}$. Then B and m is a feasible solution of $\mathcal{H}(S)$ with $\operatorname{rank}(B) \leq k$. □

With this characterization we show that the Haemers bound is monotone with respect to noncommutative graph *cohomomorphism*, a notion that was introduced in [Sta16] (see also [LZ18]). Let $S \subseteq M_n$ and $T \subseteq M_{n'}$ be two noncommutative graphs. We say there is a *cohomomorphism* from S to T, denoted as $S \leq T$, if there exists a quantum channel $\Phi : M_n \to M_{n'}$ with Choi-Kraus operators $E_1, \ldots, E_m \in M_{n \times n'}$, such that for every $B \in T$ and $i, j \in [m]$ we have $E_i^\dagger B E_j \in S$. It is called cohomomorphism because of its interpretation for graphs: for graphs G and H we have $S_G \leq S_H$ if and only if there is a homomorphism from $\overline{G}$ to $\overline{H}$ [Sta16, LZ18].

Proposition 12. *For noncommutative graphs $S \subseteq M_n$ and $T \subseteq M_{n'}$, $S \leq T$ implies $\mathcal{H}(S) \leq \mathcal{H}(T)$.*

Proof. Let $\Psi : M_{n'} \to M_k$ be a trace-preserving linear map acting as $\Psi(A) = \sum_{i=1}^m E_i A F_i^\dagger$ for any $A \in M_{n'}$, where $E_1, \ldots, E_m, F_1, \ldots, F_m \in M_{k \times n'}$. And let Ψ be feasible to $\mathcal{H}(T)$ as in Equation (12). Let $\Phi : M_n \to M_{n'}$ be a quantum channel with Choi-Kraus operators $D_1, \ldots, D_{m'} \in M_{n' \times n}$, such that for any $A \in T$ and $i, j \in [m']$, $D_i^\dagger A D_j \in S$. We claim that for the linear map $\Psi' : M_n \to M_k$ acting as $\Psi'(A) = \Psi(\Phi(A))$ for any $A \in M_n$, Ψ' is trace preserving and $T_{\Psi'} = \operatorname{span}\{D_j^\dagger F_i^\dagger E_{i'} D_{j'} : \ i, i' \in [m], \ j, j' \in [m']\} \subseteq S$. Ψ' being trace preserving is easy to see since Φ and Ψ are trace-preserving. Since for every $i, i' \in [m]$, $F_i^\dagger E_{i'} \in T_\Psi \subseteq T$. We have $D_j^\dagger F_i^\dagger E_{i'} D_{j'} \in S$ for any $j, j' \in [m']$ and $i, i' \in [m]$. Thus Ψ' is a feasible solution of $\mathcal{H}(S)$ as in Equation (12), which implies $\mathcal{H}(S) \leq \mathcal{H}(T)$. □

The above monotonicity result allows us to give an alternative proof of Proposition 8: Note that $\alpha(S) = \max\{\ell : \ \mathcal{D}_\ell \leq S\}$ [Sta16] (See also [LZ18, Lemma 14]). Letting $\alpha(S) = \ell$, we obtain $\mathcal{H}(S) \geq \mathcal{H}(\mathcal{D}_\ell) = \ell$.

The following proposition lists some other basic properties of the bound $\mathcal{H}(S)$.

Proposition 13. *Let $S \subseteq M_n$ and $T \subseteq M_{n'}$ be noncommutative graphs. The following holds:*

(1) For any $n \times n$ unitary matrix U we have $\mathcal{H}(S) = \mathcal{H}(U^\dagger S U)$.

(2) $\mathcal{H}(S \oplus T) = \mathcal{H}(S) + \mathcal{H}(T)$.

Proof. (1) It suffices to show $\mathcal{H}(U^\dagger SU) \le \mathcal{H}(S)$. Let m, B be a feasible solution of $\mathcal{H}(S)$, then m, $B' = (I_m \otimes U^\dagger)B(I_m \otimes U)$ is a feasible solution of $\mathcal{H}(U^\dagger SU)$ and $\mathrm{rank}(B') = \mathrm{rank}(B)$, therefore $\mathcal{H}(U^\dagger SU) \le \mathcal{H}(S)$.

(2) We first show that $\mathcal{H}(S \oplus T) \le \mathcal{H}(S) + \mathcal{H}(T)$. Let B_1 and m_1 be a feasible solution of $\mathcal{H}(S)$ and B_2 and m_2 be a feasible solution of $\mathcal{H}(T)$. Without loss of generality we assume $m_1 = m_2 = m$. Let $B_1 = \left[X_{i,j}\right]_{i,j\in[m]}$ and $B_2 = \left[Y_{i,j}\right]_{i,j\in[m]}$ with $X_{i,j} \in S$ and $Y_{i,j} \in T$. The matrix

$$B = \left[\begin{bmatrix} X_{i,j} & 0 \\ 0 & Y_{i,j} \end{bmatrix}\right]_{i,j\in[m]}$$

and m is a feasible solution of $\mathcal{H}(S \oplus T)$ with $\mathrm{rank}(B) = \mathrm{rank}(B_1) + \mathrm{rank}(B_2)$.

We now show that $\mathcal{H}(S \oplus T) \ge \mathcal{H}(S) + \mathcal{H}(T)$. Let B and m be a feasible solution of $\mathcal{H}(S \oplus T)$. We have $M_m(S \oplus T) \simeq M_m(S) \oplus M_m(T)$ where the isomorphism is given by a permutation, let us denote the permutation with π. Then $\pi B \pi^\dagger = B_1 \oplus B_2$ where $B_1 \in M_m(S)$ and $B_2 \in M_m(T)$ are feasible for $\mathcal{H}(S)$ and $\mathcal{H}(T)$ respectively. It remains to observe that

$$\mathrm{rank}(B) = \mathrm{rank}(B_1 \oplus B_2) = \mathrm{rank}(B_1) + \mathrm{rank}(B_2) \ge \mathcal{H}(S) + \mathcal{H}(T). \qquad \square$$

Let us now compute the value of $\mathcal{H}(S)$ for some basic noncommutative graphs S.

Example 14. *Let $S = \mathbb{C}I_n$ be the subspace containing only scalar multiples of the $n \times n$ identity matrix, then $\mathcal{H}(S) = n$.*

Indeed, any feasible solution B of $\mathcal{H}(\mathbb{C}I_n)$ has at least one non-zero diagonal block $B_{i,i} \in S = \mathbb{C}I_n$. This diagonal block provides a submatrix of B with rank n. Therefore $\mathcal{H}(\mathbb{C}I_n) \ge n$. It is also easy to see that $B = I_n \in M_1(\mathbb{C}I_n)$ provides a feasible solution with rank exactly n.

Example 15. *Let $S = \mathcal{D}_n$ be the subspace of M_n containing all diagonal matrices, then, by part (2) of Proposition 13, we have $\mathcal{H}(\mathcal{D}_n) = n$.*

Lemma 16. *Let $S \subseteq M_n$ be a noncommutative graph. Then $\mathcal{H}(S) = 1$ if and only if $S = M_n$.*

Proof. Suppose $S = M_n$ and set $u = \oplus_{i=1}^n |i\rangle$. Then $B = uu^\dagger$ is feasible for $\mathcal{H}(M_n)$ and has rank equal to 1. Therefore $\mathcal{H}(M_n) = 1$.

For the other direction, let S be a noncommutative graph for which $\mathcal{H}(S) = 1$. Let $B \in \mathrm{M}_m(S)$ be a feasible solution to $\mathcal{H}(S)$ with rank equal to 1. Say $B = uv^\dagger$ where $u, v \in \mathbb{C}^{mn}$. Decompose u as $u = \oplus_{i=1}^m |\psi_i\rangle$ where $|\psi_i\rangle \in \mathbb{C}^n$ for each $i \in [m]$. Similarly, let $v = \oplus_{i=1}^m |\phi_i\rangle$. From the feasibility of B it follows that $\sum_{i=1}^m |\psi_i\rangle\langle\phi_i| = I_n$. In particular this implies that

$$\mathrm{span}\{|\psi_1\rangle, \ldots, |\psi_m\rangle\} = \mathbb{C}^n = \mathrm{span}\{|\phi_1\rangle, \ldots, |\phi_m\rangle\}.$$

This in turn implies that $\mathrm{span}\{|\psi_i\rangle\langle\phi_j| : i, j \in [m]\} = M_n$. At the same time, since $B \in \mathrm{M}_m(S)$, we have that $|\psi_i\rangle\langle\phi_j| \in S$ for each $i, j \in [m]$. Therefore $S = M_n$. $\square$

Example 17. *Let $S_n = \mathrm{span}\{I_n, |i\rangle\langle j| : \ i \neq j \in [n]\} \subseteq M_n$. It follows from the above Lemma 16 that $\mathcal{H}(S_n) \ge 2$ whenever $n \ge 2$. When $n = 2$, i.e., for $S_2 = \left\{\begin{pmatrix} a & b \\ c & a \end{pmatrix} : a, b, c \in \mathbb{C}\right\}$, this lower bound is tight. Indeed, $B = I_2 \in M_1(S_2)$ is feasible for $\mathcal{H}(S_2)$ and has rank exactly equal to 2.*

For $k \ge 2$, we have $\mathcal{H}(S_k \otimes S_{k^2}) \le \overline{\xi}(S_k \otimes S_{k^2}) \le k^2 < k^3 \le \vartheta(S_k \otimes S_{k^2}) \le \tilde{\vartheta}(S_k \otimes S_{k^2})$ where all but the first inequality were shown in [LPT18]. Thus, the ratio $\mathcal{H}(S)/\vartheta(S)$ (and $\mathcal{H}(S)/\tilde{\vartheta}(S)$) can be arbitrarily small.

Example 18. *In [WD18], they presented a family of noncommutative graphs S_γ (with parameter γ) where the Duan-Severini-Winter noncommutative analogue of ϑ, $\tilde{\vartheta}(S_\gamma)$ can be strictly larger than the entanglement-assisted Shannon capacity. Explicitly,*

$$S_\gamma = \operatorname{span}\{|1\rangle\langle 3|\,, |3\rangle\langle 1|\,, \sin^2\gamma\,|2\rangle\langle 2| + |3\rangle\langle 3|\,, \cos^2\gamma\,|2\rangle\langle 2| + |1\rangle\langle 1|\} \subseteq M_3,$$

and $\tilde{\vartheta}(S_\gamma) = 2 + \cos^2\gamma + \cos^{-2}\gamma \geq 4$ when $\gamma \in [0, \pi/2)$. On the other hand, note that $\mathcal{H}(S_\gamma) \leq 3$ for every γ, which is strictly smaller than $\tilde{\vartheta}(S_\gamma)$.

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References

[BC19] B. Bukh and C. Cox. On a Fractional Version of Haemers' Bound. *IEEE Transactions on Information Theory*, 65(6):3340–3348, June 2019.

[Bla13] Anna Blasiak. *A Graph-theoretic Approach to Network Coding*. PhD thesis, Cornell University, 2013.

[BS07] Salman Beigi and Peter W. Shor. On the Complexity of Computing Zero-error and Holevo Capacity of Quantum Channels. arXiv:0709.2090, 2007.

[BTW19] Gareth Boreland, Ivan G. Todorov, and Andreas Winter. Sandwich Theorems and Capacity Bounds for Non-commutative Graphs. arXiv:1907.11504, 2019.

[CCH11] T. S. Cubitt, J. Chen, and A. W. Harrow. Superactivation of the Asymptotic Zero-Error Classical Capacity of a Quantum Channel. *IEEE Transactions on Information Theory*, 57(12):8114–8126, Dec 2011.

[CLO15] D.A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*. Springer Publishing Company, Incorporated, 4th edition, 2015.

[DSW13] Runyao Duan, Simone Severini, and Andreas Winter. Zero-Error Communication via Quantum Channels, Noncommutative Graphs, and a Quantum Lovász Number. *IEEE Transactions on Information Theory*, 59(2):1164–1174, Feb 2013.

[Dua09] Runyao Duan. Super-activation of Zero-error Capacity of Noisy Quantum Channels. arXiv: 0906.2527, 2009.

[GW90] F. Guo and Y. Watanabe. On Graphs in which the Shannon Capacity is Unachievable by Finite Product. *IEEE Transactions on Information Theory*, 36(3):622–623, May 1990.

[Hae78] Willem Haemers. An Upper Bound for the Shannon Capacity of a Graph. *Colloquia Mathematica Societatis János Bolyai*, 25:267–272, 1978.

[Hae79] Willem Haemers. On Some Problems of Lovász Concerning the Shannon Capacity of a Graph. *IEEE Transactions on Information Theory*, 25(2):231–232, 1979.

[HPRS17] Leslie Hogben, Kevin F. Palmowski, David E. Roberson, and Simone Severini. Orthogonal Representations, Projective Rank, and Fractional Minimum Positive Semidefinite Rank: Connections and New Directions. *Electronic Journal of Linear Algebra*, 32:98–115, 2017.

[Kar72] Richard M. Karp. Reducibility among Combinatorial Problems. In *Complexity of computer computations (Proc. Sympos., IBM Thomas J. Watson Res. Center, Yorktown Heights, N.Y., 1972)*, pages 85–103. Plenum, New York, 1972.

[Lov79] László Lovász. On the Shannon Capacity of a Graph. *IEEE Transactions on Information Theory*, 25(1):1–7, 1979.

[LPT18] R. H. Levene, V. I. Paulsen, and I. G. Todorov. Complexity and Capacity Bounds for Quantum Channels. *IEEE Transactions on Information Theory*, 64(10):6917–6928, Oct 2018.

[LZ18] Yinan Li and Jeroen Zuiddam. Quantum Asymptotic Spectra of Graphs and Non-commutative Graphs, and Quantum Shannon Capacities. arXiv:1810.00744, 2018.

[MR16] Laura Mančinska and David E. Roberson. Quantum Homomorphisms. *J. Combin. Theory Ser. B*, 118:228–267, 2016.

[NC10] Michael A. Nielsen and Isaac L. Chuang. *Quantum Computation and Quantum Information*. Cambridge university press, 2010.

[OP15] Carlos M. Ortiz and Vern I. Paulsen. Lovász Theta Type Norms and Operator Systems. *Linear Algebra and its Applications*, 477:128–147, 2015.

[Pee96] René Peeters. Orthogonal representations over finite fields and the chromatic number of graphs. *Combinatorica*, 16(3):417–431, 1996.

[PS19] Sven Polak and Alexander Schrijver. New lower bound on the Shannon capacity of C7 from circular graphs. *Information Processing Letters*, 143:37–40, 2019.

[Sha56] Claude E. Shannon. The Zero Error Capacity of a Noisy Channel. *Institute of Radio Engineers, Transactions on Information Theory*, IT-2:8–19, 1956.

[Sta16] Dan Stahlke. Quantum Zero-Error Source-Channel Coding and Non-Commutative Graph Theory. *IEEE Transactions on Information Theory*, 62(1):554–577, Jan 2016.

[Wat18] John Watrous. *The Theory of Quantum Information*. Cambridge University Press, 2018.

[WD18] X. Wang and R. Duan. Separation Between Quantum Lovász Number and Entanglement-Assisted Zero-Error Classical Capacity. *IEEE Transactions on Information Theory*, 64(3):1454–1460, March 2018.

[Wea17] Nik Weaver. A "Quantum" Ramsey Theorem for Operator Systems. *Proceedings of the American Mathematical Society*, 145(11):4595–4605, 2017.

[Wea19] Nik Weaver. The "Quantum" Turan Problem for Operator Systems. *Pacific Journal of Mathematics*, 301(1):335–349, 2019.