Observation and Evolution of Finite-dimensional Markov Systems

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1 Markov systems

A *system* $\mathfrak{S}$ is an entity that can be in one of several *states*. Let $\mathcal{S}$ be the set of states of $\mathfrak{S}$. An *n-dimensional Markov representation* is an injective map $\rho : \mathcal{S} \to \mathcal{Q}$ onto an affine hyperplane $\mathcal{Q}$ of an n-dimensional Hilbert space $\mathcal{H}$ over $\mathbb{R}$. We denote the inner product in $\mathcal{H}$ by $\langle x|y\rangle$ and assume

$$\mathcal{Q} = \{x \in \mathcal{H} \mid \tau(x) = 1\},$$

where $\tau : \mathcal{H} \to \mathbb{R}$ is a linear functional. Given the representation ρ, we identify $\mathcal{S}$ with $\mathcal{Q}$ and speak of $\mathcal{Q}$ as the collection of *(Markov) states* of $\mathfrak{S}$.

An n-dimensional Markov system $\mathcal{S}$ admits a *standard* representation $\sigma : \mathcal{S} \to Q$ into the euclidean coordinate space $\mathbb{R}^n$ with inner product

$$\langle x|y\rangle = x^T y = \sum_{i=1}^{n} x_i y_i \quad \text{for all } x^T = (x_1, \ldots, y_n), y^T = (y_1, \ldots, y_n) \in \mathbb{R}^n.$$

and, with $\mathbf{1}^T := (1, 1, \ldots, 1)$, the affine hyperplane

$$Q = \{x \in \mathbb{R}^n \mid \tau(x) = \mathbf{1}^T x = x_1 + \ldots + x_n = 1\}.$$

However, also other representations are of interest to the mathematical modeler:

1.1 Quantum Markov systems

Motivated by the classical model of m-dimensional quantum systems, consider the (complex) Hilbert space $\mathbb{C}^{m\times m}$ of complex $(m \times m)$-matrices with inner product

$$\langle C|D\rangle = \text{tr}(D^*C),$$

where D^* is the conjugate transpose of D and $\text{tr}(A)$ denotes the trace of a matrix A. Recall that a matrix C is *self-adjoint* (or *hermitian*) if $C = C^*$ and let $\mathcal{H}$ denote the collection of all self-adjoint $(m \times m)$-matrices C. It is not difficult to see that $\mathcal{H}$ forms a real(!) Hilbert space of dimension $n = m^2$. Letting I denote the identity matrix of $\mathbb{C}^{m\times m}$, we call the members of the hyperplane

$$\mathcal{D} = \{D \in \mathcal{H} \mid \text{tr}(D) = \langle D|I\rangle = 1\}$$

Markov density matrices and refer to a system with states corresponding to Markov density matrices a *Markov quantum system.*

1.2 Quantum activity systems and quantum bits

While classical computation is based on boolean bits, quantum computation (see, *e.g.*, [8]) models activities by *quantum bits* ("*qbits*"), where one qbit has the form

$$q = \alpha|0\rangle + \beta|1\rangle \quad \text{with } \alpha, \beta \in \mathbb{C} \text{ s.t. } |\alpha|^2 + |\beta|^2 = 1.$$

The qbit q has has the interpretation that $|0\rangle$ is observed with probability $|\alpha|^2 \geq 0$ and $|1\rangle$ with probability $|\beta|^2 = 1 - |\alpha|^2 \geq 0$.

An n-dimensional quantum activity system is the n-fold tensor product $\mathcal{A} = \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_n$ of 1-dimensional quantum activity systems $\mathcal{A}_i$. An n-dimensional quantum activity state ("*n-qbit*") is therefore of the form

$$q = \sum_{k \in \{0,1\}^n} \alpha_k |k\rangle \quad \text{with } \alpha_k \in \mathbb{C} \text{ and } \textstyle\sum_k |\alpha|^2 = 1 \tag{1}$$

and corresponds to the parameter vector $v = (\alpha_k | k \in \{0,1\}^n) \in \mathbb{C}^{2^n}$ with (squared) norm

$$\|v\|^2 = v^* v = |\alpha_1|^2 + \ldots + |\alpha_n|^2 = 1.$$

Note that an n-qbit q in the form (1) cannot directly be interpreted a Markov state in standard form. The associated matrix $Q = vv^*$ is self-adjoint with trace

$$\mathrm{tr}(vv^*) = v^* v = |\alpha_1|^2 + \ldots + |\alpha_n|^2 = 1$$

and hence a Markov density (in fact, a classical quantum density).

1.3 Pseudo-boolean functions and cooperative games

A real-valued set function $v : 2^N \to \mathbb{R}$ is a *pseudo-boolean function* (see [6]). Identifying the subsets $K \subseteq N$ with their associated boolean states $|k\rangle$, a pseudo-boolean function v can be viewed as a formal linear combination

$$v = \sum_{k \in \{0,1\}^n} \alpha_k |k\rangle$$

with the coefficients $\alpha_k = v(K)$.

From a game theoretic point of view, the pair $\Gamma = (N, v)$ is a *cooperative game* with *characteristic function* v. The parameter $v(K)$ is thought to reflect the "value" of the coalition $K \subseteq N$ in a given economic context. It is reasonable to assume that the game Γ is scaling-invariant. So we might equally well study the normalized game $(N, \tilde{v})$, where

$$\tilde{v} = \begin{cases} 0 & \text{if } v \equiv 0 \\ v/\|v\|^2 & \text{if } \|v\|^2 = \sum_{K \subseteq N} v(K)^2 \neq 0 \end{cases}$$

and think of a non-trivial cooperative game as a qbit with real coefficients.

Remark 1.1. *The* Hadamard transformation H *of a a* 1*-qbit is the linear transformation*

$$|k_1 \ldots k_n\rangle \mapsto H|k_1\rangle \otimes \cdots \otimes H|k_n\rangle \quad (k_1 \ldots k_n \in \{0,1\}^2). \tag{2}$$

The Hadamard coefficients $\hat{\alpha}_k$ *of* v *correspond to the* Banzhaf indices *(see [2]), well-known in social choice theory. (See,* e.g., *[7] for more applications of the Hadamard transformation to social choice problems and [5] for more on interaction indices).*

2 Observables and measurements

Returning to the general Markov state model with the n-dimensional Hilbert space $\mathcal{H}$ and $\mathcal{Q} = \{v \in \mathcal{H} \mid \tau(v) = 1\}$ relative to the system $\mathfrak{S}$, let us fix a particular basis $\mathcal{B} \subseteq \mathcal{Q}$.

Remark 2.1. *We think of $\mathcal{B}$ as the set of representatives of the "ground states" of $\mathfrak{S}$.*

We call a function $X : \mathcal{B} \to \{0,1\}$ an *information function.* So X models a "property" ground states $b \in \mathcal{B}$ may or may not have. Extending X linearly to all of $\mathcal{H}$, X corresponds to an element $x \in \mathcal{H}$ such that

$$\langle x|b\rangle = X(b) \quad \text{for all } b \in \mathcal{B}.$$

Assume that $\mathfrak{S}$ happens to be in the Markov state $q = \sum_{b\in\mathcal{B}} q_b b$ and define

$$\pi^q(r) = \sum_{b\in\mathcal{B}:X(b)=r} q_b \quad (r = 0,1).$$

We call X *(statistically) observable in the state* q if $\pi^q(r) \geq 0$ holds for $r = 0,1$.

3 Evolution of Markov systems

A *Markov (evolution) operator* relative to the Markov system $\mathfrak{S}$, represented as the hyperplane $\mathcal{Q}$ of the Hilbert space $\mathcal{H}$ is a linear transformation $\mu : \mathcal{H} \to \mathcal{H}$ such that $\mu(q) \in \mathcal{Q}$ holds for all $q \in \mathcal{Q}$.

A *(generalized) Markov chain* is a pair $(\mu, q^{(0)})$ where μ is a Markov operator and q a Markov state. The pair $(\mu, q^{(0)})$ stands short for the *Markov evolution* of states in discrete time when the Markov system $\mathfrak{S}$ is in state $q^{(0)}$ at time $t = 0$:

$$q^{(t)} = \mu(q^{(t-1)}) = \mu^t(q^{(0)}) \quad \text{for } t = 1,2,\ldots.$$

Examples of Markov chains relative to the standard representation are, of course, classical Markov chains, where μ is represented by a probability transition matrix.

Other examples arise from the Schrödinger wave evolution in quantum activity systems.

3.1 Evolution and measurement

The concept of a measurement can be naturally be put into context with evolution. We call a family $X = \{\mu_r \mid r \in \mathcal{R}\}$ of linear operators $\mu_r : \mathcal{H} \to \mathcal{H}$ a *Markov measurement* with (finite) scale $\mathcal{R}$ iff

$$\mu_X := \sum_{r\in\mathcal{R}} \mu_a \quad \text{is a Markov operator.} \tag{3}$$

In light of (3), we write (X, q) as a unifying notation for both a Markov measurement X and an associated Markov chain (μ_X, q) and refer to it as a *Markov measurement chain.* A Markov measurement chain is *invariant* if $\mu_X(q) = q$.

Now consider concatenating measurements ($w := r_1 ... r_n$)

$$\mu_w(q) := \mu_{r_n}(...(\mu_{r_1}(q))...)$$

and observe that, by multinomial expansion, $\mu_X^t = \sum_{w \in \mathcal{R}^t} \mu_w$. We call a Markov measurement chain (X, q) *(statistically) observable* iff

$$\tau(\mu_w(q)) \geq 0 \quad \text{for all} \quad w \in \mathcal{R}^*.$$

3.2 Equivalence and minimality of Markov measurements

We call two Markov measurement chains

$$X_1 = (\{\mu_r : \mathcal{H}_1 \to \mathcal{H}_1 \mid r \in \mathcal{R}\}, q_1) \quad \text{and} \quad X_2 = (\{\rho_r : \mathcal{H}_2 \to \mathcal{H}_2 \mid r \in \mathcal{R}\}, q_2)$$

where, possibly, $\dim \mathcal{H}_1 \neq \dim \mathcal{H}_2$, *equivalent* iff

$$\tau_1(\mu_{\bar{r}}(q_1)) = \tau_2(\mu_{\bar{r}}(q_2)) \quad \text{for all} \quad \bar{r} \in \mathcal{R}^* = \sum_{t \geq 0} \mathcal{R}^t.$$

We write

$$(X_1, q_1) \sim (X_2, q_2)$$

in that case.

We call a Markov measurement chain (X, q) on $\mathcal{H}$ *minimal* iff $\dim \mathcal{H}$ is minimal among all Markov measurement chains that are equivalent to (X, q). (See also [4] for details on how to perform equivalence tests efficiently.)

3.3 Decomposition of Markov measurements

We present the following new theorem:

Theorem 3.1 (Decomposition of invariant Markov measurement chains)**.** *Let* $X = (\{\mu_r : \mathcal{H} \to \mathcal{H} \mid r \in \mathcal{R}\}, q)$ *be a minimal, observable, invariant Markov measurement chain. Let* $d := \dim(\mathrm{Eig}_{\mu_X}(1)$*. Then there are minimal, observable, invariant Markov measurement chains*

$$X_i := (\{\mu_r^{(i)} : \mathcal{H}_i \to \mathcal{H}_i \mid r \in \mathcal{R}\}, q_i) \quad i = 1, ..., d$$

such that

(i) $q = q_1 + ... + q_d$

(ii) $(X, q_i) \sim (X_i, q_i)$

(iii) $\dim(\mathrm{Eig}_{\mu_{X_i}}(1)) = 1$.

(iv) $\mathcal{H} \cong \mathcal{H}_1 \otimes ... \otimes \mathcal{H}_d$.

Remark 3.2. $\dim \mathrm{Eig}_{\mu_X}(1) \geq 1$*, see [3].*

One may perceive this theorem as a building block for a unifying theory of classification for, for example, hidden Markov processes, quantum random walks and action-based cooperation systems emerging from game theory [10].

References

[1] J.-P. Aubin: *Cooperative fuzzy games.* Math. of Operations Research 6 (1981), 1-13.

[2] J.F. Banzhaf: *Weighted voting does not work: A mathematical analysis.* Rutgers Law Review 19 (1965), 317-347.

[3] U. Faigle and A. Schönhuth: *Asymptotic mean stationarity of sources with finite evolution dimension*, IEEE Trans. Information Theory 53 (2007), 2342-2348.

[4] U. Faigle and A. Schönhuth: *Efficient tests for equivalence of hidden Markov processes and quantum random walks*, IEEE Trans. Information Theory, 57 (2011), 1746-1753.

[5] M. Grabisch, J.-L. Marichal, R. Mesiar and E. Pap, *Aggregation Functions.* Encyclopedia of Mathematics and its Applications vol. 127, Cambridge, 2009.

[6] P.L. Hammer and S. Rudeanu, *Boolean Methods in Operations Research and Related Areas.* Springer-Verlag, 1968.

[7] G. Kalai: *A Fourier-theoretic perspective on the Condorcet paradox and Arrow's theorem.* Advances in Applied Mathematics 29 (2002), 412-426.

[8] A. Yu. Kitaev, A.H. Shen and M.N. Vyalyi, *Classical and Quantum Computation*, Graduate Studies in Mathematics vol. 47, American Mathematical Society, 2002.

[9] A. Schönhuth, *Discrete-valued stochastic vector spaces.* Doctoral dissertation (in German), Universität zu Köln, 2006.

[10] J. Voss, *A System-theoretic Approach to Multi-Agent Models.* Doctoral dissertation (in German), Universität zu Köln, 2012.