Relative Strength of Strategy Elimination Procedures

Krzysztof R. Apt
CWI, Amsterdam, the Netherlands
and *University of Amsterdam*

Abstract

We compare here the relative strength of four widely used procedures on finite strategic games: iterated elimination of weakly/strictly dominated strategies by a pure/mixed strategy. A complication is that none of these procedures is based on a monotonic operator. To deal with this problem we use 'global' versions of these operators.

1 Introduction

In the literature four procedures of reducing finite strategic games have been widely studied: iterated elimination of weakly/strictly dominated strategies by a pure/mixed strategy. Denote the corresponding operators (the mnemonics should be clear, 'L' refers to 'local' the meaning of which will be clarified later) respectively by *LW*, *MLW*, *LS* and *MLS*. When these operators are applied to a specific game G we get the following obvious inclusions:

$$MLW(G) \subseteq LW(G) \subseteq LS(G) \text{ and } MLW(G) \subseteq MLS(G) \subseteq LS(G).$$

It is then natural to expect that these inclusions carry on to the outcomes of the iterations of these operators. It turns out that this is not completely true. Moreover, proofs of some of the apparently obvious implications are not, in our view, completely straightforward. One of the complications is that none of these operators is monotonic. To reason about them we use their 'global' versions.

More precisely, given two strategy elimination operators Φ_l and Ψ_l such that for all games G, $\Phi_l(G) \subseteq \Psi_l(G)$ we prove the inclusion $\Phi_l^\omega \subseteq \Psi_l^\omega$ between the outcomes of their iterations by means of the following generic procedure:

(i) define the corresponding 'global' versions of these operators, Φ_g and Ψ_g,

(ii) prove that $\Phi_g^\omega = \Phi_l^\omega$ and $\Psi_g^\omega = \Psi_l^\omega$,

(iii) show that for all games G, $\Phi_g(G) \subseteq \Psi_g(G)$,

(iv) show that at least one of Φ_g and Ψ_g is monotonic.

The last two steps then imply $\Phi_g^\omega \subseteq \Psi_g^\omega$ by a general lemma. The desired inclusion $\Phi_l^\omega \subseteq \Psi_l^\omega$ then follows by (ii). The main work is in proving (ii).

2 Preliminaries

2.1 Strategic games

By a ***strategic game*** (in short, a game) for n players ($n > 1$) we mean a sequence

$$(S_1, \ldots, S_n, p_1, \ldots, p_n),$$

where for each $i \in [1..n]$

- S_i is the non-empty, finite set of ***strategies*** available to player i,
- p_i is the ***payoff function*** for the player i, so $p_i : S_1 \times \ldots \times S_n \rightarrow \mathcal{R}$, where $\mathcal{R}$ is the set of real numbers.

Given a sequence of sets of strategies $S_1, \ldots, S_n$ and $s \in S_1 \times \ldots \times S_n$ we denote the ith element of s by s_i, denote $(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$ by s_{-i} and similarly with S_{-i}, and write (s'_i, s_{-i}) for $(s_1, \ldots, s_{i-1}, s'_i, s_{i+1}, \ldots, s_n)$, where we assume that $s'_i \in S_i$. We denote the strategies of player i by s_i, possibly with some superscripts.

Given a finite non-empty set A we denote by ΔA the set of probability distributions over A and call any element of ΔS_i a ***mixed strategy*** of player i. The payoff functions are extended in the standard way to mixed strategies.

We say that $G := (S_1, \ldots, S_n)$ is a ***restriction*** of a game $H := (T_1, \ldots, T_n, p_1, \ldots, p_n)$ if each S_i is a (possibly empty) subset of T_i. We identify the restriction $(T_1, \ldots, T_n)$ with H. A ***subgame*** of H is a restriction $(S_1, \ldots, S_n)$ with all S_i non-empty.

To analyze various ways of iterated elimination of strategies from an initial game H we view such procedures as operators on the set of subgames of H. A minor complication is that this set together with the componentwise inclusion on the players' strategy sets does not form a lattice. Consequently, we extend these operators to the set of all restrictions of H, which together with the componentwise set inclusion does form a lattice.

Given a restriction $G := (S_1, \ldots, S_n)$ of $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$ and two strategies $s_i \in T_i$ and $m_i \in \Delta T_i$ we write $m_i \succ_G s_i$ as an abbreviation for

$$\forall s_{-i} \in S_{-i}\, p_i(m_i, s_{-i}) > p_i(s_i, s_{-i})$$

and $m_i \succ_G^w s_i$ as an abbreviation for

$$\forall s_{-i} \in S_{-i}\, p_i(m_i, s_{-i}) \geq p_i(s_i, s_{-i}) \wedge \exists s_{-i} \in S_{-i}\, p_i(m_i, s_{-i}) > p_i(s_i, s_{-i}).$$

In the first case we say that s_i ***is strictly dominated on*** G ***by*** m_i and in the second one that s_i ***is weakly dominated on*** G ***by*** m_i. In particular m_i can be a pure strategy, i.e. an element of T_i.

Given an operator T on a finite lattice $(D, \subseteq)$ we denote by T^k the k-fold iteration of T, where $T^0 = D$ (so the iterations start 'at the top') and let $T^\omega := \cap_{k \geq 0} T^k$. We call T ***monotonic*** if for all G, G'

$$G \subseteq G' \text{ implies } T(G) \subseteq T(G').$$

When comparing two ways of eliminating strategies from a strategic game, represented by the operators T and U on the lattice of all restrictions of H, we would like to deduce $T^\omega \subseteq U^\omega$ from the fact that for all G, $T(G) \subseteq U(G)$. Unfortunately, in general this implication does not hold; a revealing example is provided in Section 4. What does hold is the following simple lemma that relates to steps (iii) and (iv) of the generic procedure from the Introduction and reveals the importance of the monotonicity.

Lemma 1 *Consider two operators T and U on a finite lattice $(D, \subseteq)$, such that*

- *for all G, $T(G) \subseteq U(G)$,*
- *at least one of T and U is monotonic.*

Then $T^\omega \subseteq U^\omega$.

Proof. We prove by induction that for all $k \geq 0$ we have $T^k \subseteq U^k$. The claim holds for $k = 0$. Suppose it holds for some k. Then by the assumptions and the induction hypothesis we have the following string of inclusions and equalities:

- if T is monotonic: $T^{k+1} = T(T^k) \subseteq T(U^k) \subseteq U(U^k) = U^{k+1}$,
- if U is monotonic: $T^{k+1} = T(T^k) \subseteq U(T^k) \subseteq U(U^k) = U^{k+1}$. □

3 Strict dominance

From now we fix an initial game $H = (T_1, \ldots, T_n, p_1, \ldots, p_n)$. Given a restriction $G := (S_1, \ldots, S_n)$ of H we denote S_i by G_i. In particular we denote T_i by H_i.

First we focus on two operators on the restrictions of H:

$$LS(G) := G',$$

where for all $i \in [1..n]$

$$G'_i := \{s_i \in G_i \mid \neg \exists s'_i \in G_i \; s'_i \succ_G s_i\},$$

and *MLS* defined analogously but with $G'_i := \{s_i \in G_i \mid \neg \exists m_i \in \Delta G_i \; m_i \succ_G s_i\}$.

Starting with Luce and Raiffa [1957], the iterated elimination of strictly dominated strategies is customarily defined as the outcome of the iteration of the *MLS* operator starting with the initial game H.

To reason about the above two operators we introduce two related operators defined by ('G' stands for 'global'):

$$GS(G) := G'$$

where for all $i \in [1..n]$

$$G'_i := \{s_i \in G_i \mid \neg \exists s'_i \in H_i \; s'_i \succ_G s_i\}.$$

and *MGS* defined analogously but with $G'_i := \{s_i \in G_i \mid \neg \exists m_i \in \Delta H_i \; m_i \succ_G s_i\}$.

So in the *LS* and *MLS* operators we limit our attention to strict dominance by a pure/mixed strategy in the *current* game, G, while in the other two operators we consider strict dominance by a pure/mixed strategy in the *initial* game, H.

The difference is crucial because the operators *LS* and *MLS* are not monotonic, while *GS* and *MGS* are. To see the former just take the following game H:

	X
A	$1, 0$
B	$0, 0$

Note that $LS(H) = MLS(H) = (\{A\}, \{X\})$ and $LS(\{B\}, \{X\}) = MLS(\{B\}, \{X\}) = (\{B\}, \{X\})$. So $(\{B\}, \{X\}) \subseteq H$, while neither $LS(\{B\}, \{X\}) \subseteq LS(H)$ nor $MLS(\{B\}, \{X\}) \subseteq MLS(H)$.

Monotonicity of GS and MGS follows directly from their definitions. The MGS operator is studied in Brandenburger, Friedenberg and Keisler [2006b] (it is their operator Φ) and in Apt [2007b]. For the case of strict dominance by a pure strategy it was introduced for arbitrary games in Milgrom and Roberts [1990, pages 1264-1265], studied for compact games with continuous payoffs in Ritzberger [2001, Section 5.1] and considered for arbitrary games in the presence of transfinite iterations in Chen, Long and Luo [2005] and Apt [2007b].

The fact that the MGS operator is monotonic has some mathematical advantages. For example, by virtue of a general result established in Apt [2007b], it is automatically order independent. Moreover, thanks to monotonicity, as argued in Apt [2007a], it can be used in the epistemic framework of game theory based on possibility correspondences as 'a stand alone' concept of rationality.

The following result, the second part of which was proved (as Proposition 2.2 (ii)) in Brandenburger, Friedenberg and Keisler [2006b], relates the original operators to their global versions and corresponds to step (ii) of the generic procedure from the Introduction.

Lemma 2 *$GS^\omega = LS^\omega$ and $MGS^\omega = MLS^\omega$.*

Proof. See the appendix. $\square$

This allows us to establish the expected inclusion.

Theorem 1 *$MLS^\omega \subseteq LS^\omega$.*

Proof. This is a consequence of the mentioned generic procedure, since step (iii) holds: for all restrictions G, $MGS(G) \subseteq GS(G)$, and step (iv) holds: both MGS and GS are monotonic. $\square$

4 Weak dominance

Next we compare the LS and MLS operators with their weak dominance counterparts, LW and MLW, defined in the same way, but using the $\succ_G^w$ relation instead of $\succ_G$. In the literature, starting with Luce and Raiffa [1957], the iterated elimination of weakly dominated strategies is customarily defined as the outcome of the iteration of the MLW operator starting with the initial game H.

To reason about these two operators we use the corresponding 'global' versions, GW and MGW, defined in the same way as for strict dominance. We have following counterpart of Lemma 2, the second part of which was proved (as Lemma F.1) in Brandenburger, Friedenberg and Keisler [2006a].

Lemma 3 $GW^{\omega} = LW^{\omega}$ *and* $MGW^{\omega} = MLW^{\omega}$.

Proof. See the appendix. □

This allows us to establish the following result.

Theorem 2 $LW^{\omega} \subseteq LS^{\omega}$ *and* $MLW^{\omega} \subseteq MLS^{\omega}$.

Proof. Again, this is a consequence of the generic procedure from the Introduction. Indeed, step (iii) holds: for all restrictions G, $GW(G) \subseteq GS(G)$ and $MGW(G) \subseteq MGS(G)$, and step (iv) holds: both GS and MGS are monotonic. □

However, surprisingly, the inclusion $MLW^{\omega} \subseteq LW^{\omega}$ does not hold.

Example 1 Consider the following game H:

	X	Y	Z
A	$2,1$	$0,1$	$1,0$
B	$0,1$	$2,1$	$1,0$
C	$1,1$	$1,0$	$0,0$
D	$1,0$	$0,1$	$0,0$

Applying to it the MLW operator we get

	X	Y
A	$2,1$	$0,1$
B	$0,1$	$2,1$

Another application of MLW yields no change. In contrast, after three iterations of the LW operator to the initial game we reach

	X
A	$2,1$

□

Since the conclusion of Lemma 1 does not apply here, while Lemma 3 holds, we conclude that none of the operators LW, MLW, GW and MGW is monotonic.

Acknowledgement

We acknowledge helpful email discussions with Adam Brandenburger.

References

K. R. Apt

[2007a] Epistemic analysis of strategic games with arbitrary strategy sets, in: *Proc. 11th Conference on Theoretical Aspects of Reasoning about Knowledge (TARK07)*, ACM Press. To appear.

[2007b] The many faces of rationalizability, *The B.E. Journal of Theoretical Economics, 7(1).* (Topics), Article 18, 39 pages. Available from `http://arxiv.org/abs/cs.GT/0608011`.

A. Brandenburger, A. Friedenberg, and H. Keisler

[2006a] Admissibility in games. Working paper. Available from `http://pages.stern.nyu.edu/~abranden`.

[2006b] Fixed points for strong and weak dominance. Working paper. Available from `http://pages.stern.nyu.edu/~abranden/`.

Y.-C. Chen, N. V. Long, and X. Luo

[2005] Iterated strict dominance in general games. Available from `http://www.sinica.edu.tw/~xluo/pa10.pdf`.

R. D. Luce and H. Raiffa

[1957] *Games and Decisions*, John Wiley and Sons, New York.

P. Milgrom and J. Roberts

[1990] Rationalizability, learning, and equilibrium in games with strategic complementarities, *Econometrica*, 58, pp. 1255–1278.

K. Ritzberger

[2001] *Foundations of Non-cooperative Game Theory*, Oxford University Press, Oxford.

Appendix

Proof of Lemma 2.

This result is a corollary to Theorem 5 of Apt [2007b]. We provide here a direct proof. Note first that for all restrictions G we have $GS(G) \subseteq LS(G)$ so by Lemma 1 we have $GS^\omega \subseteq LS^\omega$.

To establish the first equality we prove by induction that for all $k \geq 0$

$$LS^k \subseteq GS^k.$$

We only need to establish the induction step. Take $s_i \in LS_i^{k+1}$. By definition $s_i \in LS_i^k$ and $\neg\exists s'_i \in LS_i^k \; s'_i \succ_{LS^k} s_i$. By the induction hypothesis $s_i \in GS_i^k$. Let

$$A := \{s'_i \in H_i \mid s'_i \succ_{LS^k} s_i\}.$$

Suppose by contradiction that $A \neq \emptyset$. Choose a maximal element $s_i^* \in A$ w.r.t. the $\succ_{LS^k}$ ordering on A. Then for all $l \in [0..k]$, $s_i^* \in LS_i^l$, so in particular $s_i^* \in LS_i^k$, which contradicts the assumption about s_i. So $A = \emptyset$, which implies by the induction hypothesis that $\neg\exists s'_i \in H_i \; s'_i \succ_{GS^k} s_i$. So $s_i \in GS_i^{k+1}$. □

Proof of Lemma 3.

We prove by induction that for all $k \geq 0$

$$LW^k = GW^k.$$

Again, we only need to establish the induction step. We prove first that $LW^{k+1} \subseteq GW^{k+1}$. Take $s_i \in LW_i^{k+1}$. By the induction hypothesis $LW^k = GW^k$, so $s_i \in GW_i^k$. Let

$$A := \{s'_i \in H_i \mid s'_i \succ_{LW^k} s_i\}.$$

Suppose by contradiction that $A \neq \emptyset$. Define the function $index : A \to [0..k]$ by

$$index(s'_i) := \begin{cases} k & \text{if } s'_i \in LW_i^k \\ j & \text{if } s'_i \in LW_i^j \setminus LW_i^{j+1} \end{cases}$$

Let B be the set of elements of A with the maximal index, i.e.,

$$B = \arg\max_{s'_i \in A} index(s'_i).$$

Let now j_0 be the *index* of the elements in B and s_i^* a maximal element of B w.r.t. the $\succ^w_{LW^{j_0}}$ ordering on B.

Note that $j_0 = k$. Indeed, otherwise $s_i^* \notin LW_i^{j_0+1}$. Then for some $s''_i \in LW_i^{j_0}$ we have $s''_i \succ^w_{LW^{j_0}} s_i^*$. But $s_i^* \succ_{LW^k} s_i$ and $LW^k \subseteq LW^{j_0}$, so $s''_i \succ^w_{LW^k} s_i$. Hence $s''_i \in B$, which contradicts the choice of s_i^*.

So $j_0 = k$, which means that $s_i^* \in LW_i^k$ and $s_i^* \succ^w_{LW^k} s_i$. But this contradicts the fact that $s_i \in LW_i^{k+1}$. So $A = \emptyset$, which implies by the induction hypothesis that $\neg\exists s'_i \in H_i \; s'_i \succ^w_{GW^k} s_i$. So $s_i \in GW_i^{k+1}$.

Next we prove that $GW^{k+1} \subseteq LW^{k+1}$. Take $s_i \in GW_i^{k+1}$. Then $s_i \in GW_i^k = LW_i^k$. Also $\neg\exists s'_i \in H_i \; s'_i \succ^w_{GW^k} s_i$, so $\neg\exists s'_i \in LW_i^k \; s'_i \succ^w_{GW^k} s_i$, so by the induction hypothesis $\neg\exists s'_i \in LW_i^k \; s'_i \succ^w_{LW^k} s_i$, which means that $s_i \in LW_i^{k+1}$.

□