Bounding Quantum-Classical Separations for Classes of Nonlocal Games

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Abstract

We bound separations between the entangled and classical values for several classes of nonlocal t-player games. Our motivating question is whether there is a family of t-player XOR games for which the entangled bias is 1 but for which the classical bias goes down to 0, for fixed t. Answering this question would have important consequences in the study of multi-party communication complexity, as a positive answer would imply an unbounded separation between randomized communication complexity with and without entanglement. Our contribution to answering the question is identifying several general classes of games for which the classical bias can not go to zero when the entangled bias stays above a constant threshold. This rules out the possibility of using these games to answer our motivating question. A previously studied set of XOR games, known not to give a positive answer to the question, are those for which there is a quantum strategy that attains value 1 using a so-called Schmidt state. We generalize this class to mod-m games and show that their classical value is always at least $\frac{1}{m} + \frac{m-1}{m} t^{1-t}$. Secondly, for free XOR games, in which the input distribution is of product form, we show $\beta(G) \geq \beta^*(G)^{2^t}$ where $\beta(G)$ and $\beta^*(G)$ are the classical and entangled biases of the game respectively. We also introduce so-called line games, an example of which is a slight modification of the Magic Square game, and show that they can not give a positive answer to the question either. Finally we look at two-player unique games and show that if the entangled value is $1 - \epsilon$ then the classical value is at least $1 - \mathcal{O}(\sqrt{\epsilon \log k})$ where k is the number of outputs in the game. Our proofs use semidefinite-programming techniques, the Gowers inverse theorem and hypergraph norms.

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1 Introduction

The study of multiplayer games has been extremely fruitful in theoretical computer science across diverse areas including the study of complexity classes [3], hardness of approximation [21], and communication complexity [20]. They are also a great framework in which to study Bell inequalities [2] and analyze the nonlocal properties of entanglement. A particularly simple kind of multiplayer game is an XOR game. An XOR game $G = (f, \pi)$ between t-players is defined by a function $f : X_1 \times X_2 \times \cdots \times X_t \to \{0,1\}$ and a probability distribution π over $X_1 \times \cdots \times X_t$. An input $(x_1, \ldots, x_t) \in X_1 \times \cdots \times X_t$ is chosen by a referee according to π, who then gives x_i to player i. Without communicating, player i then outputs a bit $a_i \in \{0,1\}$ with the collective goal of the players being that $a_1 \oplus \cdots \oplus a_t = f(x_1, \ldots, x_t)$. In a classical XOR game, the players' strategies are deterministic. In an XOR game with entanglement, players are allowed to share a quantum state and make measurements on this state to inform their outputs.

As players can always win an XOR game with probability at least $\frac{1}{2}$, it is common to study the *bias* of an XOR game, the probability of winning minus the probability of losing. We use $\beta(G)$ to denote the largest bias achievable by a classical protocol for the game G, and $\beta^*(G)$ to denote the best bias achievable by a protocol using shared entanglement for the game G.

Our motivating question in this paper is:

▶ **Question 1.** Is there a family of t-player XOR games $(G_n)_{n \in \mathbb{N}}$ such that $\beta^*(G_n) = 1$ and $\beta(G_n) \to 0$ as $n \to \infty$?

This question has important implications for multi-party communication complexity. For a function $f : X_1 \times \cdots X_t \to \{0,1\}$, let $R(f)$ denote the t-party randomized communication complexity of f (in the number-in-the-hand model), and let $R^*(f)$ denote the t-party randomized communication complexity of f where the parties are allowed to share entanglement. A positive answer to Question 1 gives a family of functions $(f_n)_{n \in \mathbb{N}}$ with $R^*(f_n) = O(1)$ and $R(f_n) = \omega(1)$, i.e. an unbounded separation between these two communication models.

In the reverse direction, a family of functions $(f_n)_{n \in \mathbb{N}}$ with $R^*(f_n) = O(1)$ and $R(f_n) = \omega(1)$ gives a family of games $G_n = (f_n, \pi_n)$ with $\beta^*(G_n) \geq c$ for some constant c and $\beta(G_n) \to 0$ as $n \to \infty$. Thus there is a very close connection between Question 1 and the existence of an unbounded separation between randomized communication complexity with and without entanglement.

For the two-player case, it is known that the answer to Question 1 is negative. It was observed by Tsirelson [30] that Grothendieck's inequality [15], a fundamental result from Banach space theory, is equivalent to the assertion that $\beta^*(G) \leq K_G \cdot \beta(G)$, where $K_G \leq 1.78 \ldots$ [24, 6] is Grothendieck's constant.

Linial and Shraibman [25] and Shi and Zhu [28] realized that the XOR bias of a game (f, π) can be used to lower bound the communication complexity of f, both in the randomized setting and the setting with entanglement. Together with Grothendieck's inequality they

used this to show that $R(f) = O(2^{2R^*(f)})$ for any partial two-party function f. Thus in the two-party case an unbounded communication separation is not possible between the randomized model with and without entanglement. Raz has given an example of a partial function f with $R(f) = 2^{\Omega(R^*(f))}$ [27], thus the upper bound of Linial-Shraibman and Shi-Zhu is essentially optimal.

In the case of three or more parties, Question 1 and the corresponding question of an unbounded separation between the entangled and non-entangled communication complexity models remain open. A striking result of Peréz-García et al. [26] shows that there is no analogue of Grothendieck's inequality in the three-player setting. In particular, they showed that there exists an infinite family of three-player XOR games $(G_n)_{n \in \mathbb{N}}$ with the property that the ratio of the entangled and classical biases of G_n goes to infinity with n. This result was later quantitatively improved by Briët and Vidick [8]. Both results rely crucially on non-constructive (probabilistic) methods, and in both separating examples the entangled bias $\beta^*(G_n)$ also goes to zero with increasing n. These works leave open the question, posed explicitly in [8], of whether there is such a family of games in which the entangled bias does not vanish with n, but instead stays above a fixed positive threshold while the classical bias decays to zero. Crucially, having a separation in XOR bias where $\beta^*(G_n)$ remains constant is what is needed to also obtain an unbounded separation between randomized communication complexity with and without entanglement.

Our contribution to answering Question 1

One approach to Question 1 is to look at different classes of games and identify which ones could possibly lead to a positive answer.

Peréz-García et al. [26] show that in any XOR game where the entangled strategy uses a GHZ state, there is a bounded gap between the classical and entangled bias: namely, the bias with a GHZ state in a t-player XOR game G is at most $K_G(2\sqrt{2})^{t-1}\beta(G)$. This bound is essentially tight as there are examples of t-player XOR games achieving a ratio between the GHZ state bias and classical bias of $\frac{\pi}{2}^t$ [32]. Briët et al. [7] later extended the Grothendieck-type inequality of Peréz-García et al. to a larger class of entangled states called Schmidt states (see Equation 1). Thus any game where there is a perfect strategy where the players share a Schmidt state cannot give a positive answer to Question 1.

Watts et al. [31] recently investigated Question 1 and found that a t-player XOR game G that is symmetric, i.e. invariant under the renaming of players, and where $\beta^*(G) = 1$ always has a perfect entangled strategy where the players share a GHZ state. Thus symmetric games also cannot give a positive answer to Question 1.

We further study games that have a perfect strategy where players share a GHZ or Schmidt state. We do this for a generalization of XOR games called mod m games. In a mod m game the players output an integer between 0 and $m-1$ and the goal is for the sum of the outputs mod m to equal a target value determined by their inputs. We show that the classical advantage over random guessing is at least $\frac{m-1}{m}t^{1-t}$ in any t player mod m game that can be won perfectly by sharing a Schmidt state (see Theorem 2).

We show this by introducing *angle games*, a class of games that can be won perfectly sharing a GHZ state and are the *hardest* of all such games. Thus a classical strategy in an angle game can be used to lower bound the winning probability of any mod m game that has a perfect Schmidt state strategy.

For small values of t we can directly analyze angle games to give bounds that are sometimes tight. One interesting consequence of our result is the following. The Mermin game G is a three-party XOR game where by sharing a GHZ state players can play perfectly,

$\beta^*(G) = 1$, while classically $\beta(G) = \frac{1}{2}$. We show that this is the maximal possible separation of any 3-party XOR game where $\beta^*(G) = 1$ via a GHZ state. In particular, this means that when one looks at the XOR repetition of the Mermin game the classical bias *does not go down at all.*

We rule out other types of games that could positively answer Question 1 as well. A t-player *free* XOR game $G = (f, \pi)$ is a game where π is a product distribution. For such games we show that $\beta(G) \geq \beta^*(G)^{2^t}$, and thus they cannot be used for a positive answer to Question 1.

Another class of XOR games we consider are *line games*, where the questions asked to the players are related by a geometric property. An example of a line game is a slight modification of the Magic Square game [18]. We show that line games cannot give a positive answer to Question 1 either.

Finally, we look at extensions of Question 1 beyond XOR games to more general classes of games like unique games [21], which have been deeply studied because of their application in hardness of approximation. For unique games we show that in fact if there is strategy with entanglement that can win a unique game perfectly, then there is a perfect classical strategy as well. This can be compared with the result of Cleve et al. [10] that if a two-player game with binary outputs has a perfect strategy with entanglement then it also has a perfect classical strategy. More generally, we show that if the winning probability with entanglement is $1 - \epsilon$ in a unique game with k outputs, then there is a classical strategy that wins with probability $1 - C\sqrt{\epsilon \log k}$.

In the next subsections, we discuss our results in more detail.

1.1   Perfect Schmidt strategies for MOD games

A MOD-m game is a generalization of XOR games to non-binary outputs. A nonlocal game is a MOD-m game if the players are required to answer with integers from 0 to $m - 1$, and win if and only if the sum of their answers modulo m equals the target value determined by their inputs. We denote the optimal winning probability using classical strategies by $\omega(G)$, and we write $\omega^*(G)$ for the entangled winning probability. Random play in such a game ensures that the players can always win with probability at least $\frac{1}{m}$. As with XOR games, in a MOD-m game one often considers the *bias* given by the maximum amount by which the value can exceed $\frac{1}{m}$, scaled to be in the $[0, 1]$ range. The bias is $\beta(G) = \frac{m}{m-1}(\omega(G) - \frac{1}{m})$, and similar for the entangled version. This generalizes the definition given for XOR games above.

Define a t-partite *Schmidt state* as a t-partite quantum state that can be written in the form

$$|\psi\rangle = \sum_{i=0}^{d-1} c_i |e_i^{(1)}\rangle |e_i^{(2)}\rangle \cdots |e_i^{(t)}\rangle, \tag{1}$$

where $c_i > 0$ and where the $|e_i^{(j)}\rangle$ $(i = 0, 1, ..., d-1)$ are orthogonal vectors in the j-th system. For $t = 2$ any state can be written this way, something commonly known as the Schmidt decomposition. Note that the well-known GHZ state is a Schmidt state where all the c_i are equal to $1/\sqrt{d}$. In the context of nonlocal games, define a *Schmidt strategy* as a quantum strategy that uses (only) a Schmidt state. We say a strategy is *perfect* if it achieves winning probability 1.

We consider t-player MOD-m games for which there is a perfect Schmidt strategy ("perfect Schmidt games") and for such games we give lower bounds on the classical winning probabilities. One particular set of games with this property is described by Boyer [5]. Their

entangled value is 1 but their classical value goes to 0 as the number of players goes to infinity. The authors of [31] define a closely-related class of games called *noPREF games*. This set of games is equal to the set of perfect Schmidt games when $m = 2$ and the distribution on the inputs is uniform. In [31] it is shown that checking whether a game is in this class can be done in polynomial time. Furthermore, for *symmetric t-player XOR* games they show that a game has entangled value 1 if and only if it falls in this class of perfect Schmidt games. They also provide an explicit non-symmetric XOR game with entangled value 1 that is not in this class. We introduce a *t*-player MOD-*m* game called the *uniform angle game*, denoted $\mathrm{UAG}_{t,m}$ (defined in the full version of this paper) for which there is a perfect Schmidt strategy and show a lower bound on the classical winning probability.

▶ **Theorem 2.** *Any t-player MOD-m game G with perfect Schmidt strategy satisfies $\omega(G) \geq \omega(\mathrm{UAG}_{t,m})$. Furthermore we have $\beta(\mathrm{UAG}_{t,m}) \geq t^{1-t}$.*

For $t = 3, m = 2$ (3-player XOR games) we have $\omega(\mathrm{UAG}_{3,2}) = 3/4$. In the full version we provide bounds on $\omega(\mathrm{UAG}_{t,m})$ for other values of t, m.

Let the inputs to a game come from a set $X = X_1 \times X_2 \times ... \times X_t$ where X_i is the set of inputs for the i-th player. We say a game is *total* when all elements of X have a non-zero probability of being asked (sometimes also called having *full support*), similar to total functions in the setting of communication complexity. On the other hand, we say that a game has a *promise* on the inputs when it is not total. For the class of perfect Schmidt games we show that total games are trivial.

▶ **Lemma 3.** *When a t-player MOD-m game G with perfect Schmidt strategy is total then $\omega(G) = 1$.*

1.2 Free XOR games

In this subsection we identify two types of games, namely *free games* and *line games*, for which either the ratio of the entangled and classical biases is small, or the entangled bias itself is small. Thus these games will not be able to give a positive answer to Question 1. Free games are a general and natural class of games in which the players' questions are independently distributed. Line games appear to be less studied (see below for their definition), but turn out to be relevant in the context of parallel repetition (also see below). The main idea behind these results is that a large entangled bias implies that the games are in a sense far from random. This is quantified by the magnitude of certain norms of the game tensors. The particular norms of interest here are related to norms used in Gowers' celebrated hypergraph- and Fourier-analytic proofs of Szemerédi's Theorem. A crucial fact of these norms is that they are large if and only if there is "correlation with structure", the opposite of what one would expect from randomness. We show that this structure can be turned into good classical strategies, thus establishing a relationship between the entangled and classical biases.

▶ **Theorem 4** (Polynomial bias relation for free XOR games). *For integer $t \geq 2$ and any free t-player XOR game with entangled bias β, the classical bias is at least β^{2^t}.*

This result may be considered as an analogue of a well-known result on quantum query algorithms for total functions. It is shown in [1] that the bounded-error quantum and classical query complexities of total functions are polynomially related.

1.3 Line games

Line games are not free, but have a simple geometric structure. For a finite field $\mathbb{F}$ of characteristic at least t and positive integer n, a t-player line game is given by a map $\tau : \mathbb{F}^n \to \{0, 1\}$. In the game, the referee independently samples two uniformly random points $x, y \in \mathbb{F}^n$ and sends the point $x + (i-1)y$ to the ith player. The players win the game if and only if the XOR of their answers equals $\tau(y)$. In other words, the players' questions correspond to consecutive points (or an arithmetic progression) on a random affine line through $\mathbb{F}^n$ and the winning criterion depends only on the direction of the line. Refer to this as a line games *over* $\mathbb{F}^n$.

A small example of a line game can be obtained from a slight modification of the three-player Magic Square game, which was analyzed in [18]. The line game is played over the plane $\mathbb{F}_3^2$ and the predicate is zero only on the horizontal lines (with $y \in \{(1, 0), (2, 0)\}$. In the Magic Square game, the referee restricts only to horizontal and vertical lines.[1]

▶ **Theorem 5.** *For any $\varepsilon \in (0, 1]$, integer $t \geq 2$ and finite field $\mathbb{F}$ of characteristic at least t, there exists a $\delta(\varepsilon, t, \mathbb{F}) \in (0, 1]$ such that the following holds. For any positive integer n and any t-player line game over $\mathbb{F}^n$ with entangled bias ε, the classical bias is at least $\delta(\varepsilon, t, \mathbb{F})$.*

Note that in the above result, the value of the classical bias is independent of the dimension n of the vector space determining the players' question sets.

While it is not relevant to Question 1, the proof techniques used for Theorem 5 allow us to prove a parallel repetition theorem for a class of games that include line games. It is known that the value of free games and so-called anchored games decays exponentially under parallel repetition. Dinur et al. [13] identified a general criterion of multi-player games to behave like this, encompassing free and anchored games. They showed that it is sufficient for a certain graph that can be obtained from a game to be expanding, a well-known pseudorandom property that gives a measure of graph connectivity. Line games do not belong to this class, as their graphs are not even connected. However, we show that if a map $\tau : \mathbb{F}^n \to \{0, 1\}^n$ is pseudorandom in a different sense, then a line game defined by τ has exponential decaying value under parallel repetition. More generally, we show that this is the case for a family of XOR games over an arbitrary finite abelian group Γ. These games are given by a positive integer m, a family of affine linear maps $\psi_0, \ldots, \psi_t : \Gamma^m \to \Gamma$ such that no two are multiples of each other, and a "game map" $\rho : \Gamma \to \{0, 1\}$. In the game, the referee samples a uniform random element x from Γ^m and sends the group element $\psi_i(x)$ to the ith player. The winning criterion is given by $\rho(\psi_0(x))$. The relevant notion of pseudoranomness is quantified by the Gowers t-uniformity norm of the map $(-1)^\rho : x \mapsto (-1)^{\rho(x)}$, denoted $\|(-1)^\rho\|_{U^t}$.

▶ **Lemma 6.** *Let m, t be positive integers and let Γ be a finite abelian group. Let $\psi_0, \ldots, \psi_t : \Gamma^m \to \Gamma$ be affine linear maps such that no two are multiples of each other and let $\rho : \Gamma \to \{0, 1\}$. Let G be the t-player XOR game given by the system $\{\psi_0, \ldots, \psi_t, \rho\}$. Then, for every positive integer k,*

$$\omega(G^k) \leq \Big(\frac{1 + \|(-1)^\rho\|_{U^t}}{2} \Big)^k.$$

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[1] Though this is not the typical description of the game, it is easily seen to be equivalent.

1.4 Unique games

We know that the answer to Question 1 is negative in the two-player case, but we can generalize the question by dropping the XOR restriction. The set of XOR games is part of a larger class of games called unique games for which we investigate the relation between classical and entangled values. A two-player nonlocal game is a unique game if for every pair of questions, for every possible answer of the first player there is exactly one answer of the second player that lets them win, and vice versa. Stated differently, for every question there is a matching between the answers of the two players such that only the matching pairs of answers let the player win.

The Unique Games Conjecture (UGC) of Khot [21] states that for any $\epsilon, \delta > 0$, for any $k > k(\epsilon, \delta)$, it is NP-hard to distinguish instances of unique games with winning probability at least $1 - \epsilon$ from those with winning probability at most δ, where k is the number of possible answers. This conjecture has important consequences because it implies several hardness of approximation results. For example, for the Max-Cut problem, Khot et al. [22] showed that the UGC implies that obtaining an approximation ratio better than ≈ 0.878 is NP-hard. Other results include inapproximability for Vertex Cover [23] and graph coloring problems [14].

Our results relate the quantum and classical winning probabilities in the regime of near-perfect play and are based on a result in [9].

▶ **Theorem 7.** *Let $\epsilon \geq 0$. There is an efficient algorithm that, given any two-player unique game with entangled value $1 - \epsilon$, outputs a classical strategy with winning probability at least $1 - C\sqrt{\epsilon \log k}$, where C is a constant independent of the game.*

Note that for $\epsilon = 0$ this means a perfect quantum strategy implies a perfect classical strategy. Furthermore, the above result only beats a trivial strategy when $\epsilon = \mathcal{O}(1/\log k)$.

Work in a similar direction includes [19]. They show that entangled version of the UGC is false, by providing an efficient algorithm that gives an explicit quantum strategy with winning probability at least $1 - 6\epsilon$ when the true entangled value is $1 - \epsilon$. In the classical case, [9] gives an algorithm that outputs a classical strategy with winning probability $1 - \mathcal{O}(\sqrt{\epsilon \log k})$ when the true *classical value* is $1 - \epsilon$. We extend this result by showing that this classical strategy also does the job when, not the classical, but the entangled value is $1 - \epsilon$.

2 Techniques

This section provides an overview of the proof techniques that we employed. We give sketches of the main ideas which are worked out in full detail in later sections.

2.1 Reduction to angle games

To prove Theorem 2 we introduce a new set of t-player MOD-m games that we call *angle games*. We define a particular angle game called the *uniform angle game*, denoted by $\mathrm{UAG}_{t,m}$ and show that it is the hardest of these games. In an angle game, players receive complex phases $e^{i\phi}$ (angles) satisfying a promise, and the winning answer depends only on the product of the inputs $e^{i\phi_1} \cdot e^{i\phi_2} \cdots e^{i\phi_t}$. We prove the theorem by extracting from any perfect Schmidt strategy a set of complex phases that satisfy such a promise, and thereby reducing any such game to the $\mathrm{UAG}_{t,m}$ game. Let us sketch how this is accomplished. Assume that a perfect Schmidt strategy exists, and let $\{P_1^{(j,x_j)}, ..., P_m^{(j,x_j)}\}$ be the projective measurement

done by player j on input x_j so that $P_i^{(j,x_j)}$ corresponds to output i. Now define unitaries $U^{(j,x_j)} = \sum_i \omega_m^i P_i^{(j,x_j)}$, where $\omega_m = e^{2\pi i/m}$ is an m-th root of unity. Since the strategy is perfect we have for every input $(x_1, ..., x_t)$ that

$$\omega_m^{M(x_1,...,x_t)} = \langle \psi | U^{(1,x_1)} \otimes U^{(2,x_2)} \otimes ... \otimes U^{(t,x_t)} | \psi \rangle.$$

Using the definition of a Schmidt state, we show that this equality implies that these unitaries must be of a simple form and their entries satisfy the promise of an angle game. We prove Theorem 2 and Lemma 3 in the full version, where we also provide classical strategies for the uniform angle game and show that these are tight in the case of 3-player XOR games.

2.2   Norming hypergraphs and quasirandomness

Our main tool for proving Theorem 4 is a relation between the entangled and classical biases and a norm on the set of game tensors. For t-tensors, this norm is given in terms of a certain t-partite t-uniform hypergraph H. Recall that such a hypergraph consists of t finite and pairwise disjoint vertex sets $V_1, \ldots, V_t$ and a collection of t-tuples $E(H) \subseteq V_1 \times \cdots \times V_t$, referred to as the edge set of H. For a t-tensor $T \in \mathbb{R}^{n_1 \times \cdots \times n_t}$, the norm has the following form:

$$\|T\|_H = \left(\mathbf{E}_{\phi_i : V_i \to [n_i]} \Big[\prod_{(v_1, \ldots, v_t) \in E(H)} T\big(\phi_1(v_1), \ldots, \phi_t(v_t)\big) \Big] \right)^{\frac{1}{|E(H)|}}, \tag{2}$$

where the expectation taken with respect to the uniform distribution over all t-tuples of mappings ϕ_i from V_i to $[n_i]$. Expressions such as (2) play an important role in the context of graph homomorphisms [4]. If T is the adjacency matrix of a bipartite graph with left and right node sets $[n_1]$ and $[n_2]$ respectively, then each product in (2) is 1 if and only if the maps ϕ_1 and ϕ_2 preserve edges.

Criteria for H under which (2) defines a norm or a semi-norm were determined by Hatami [17, 16] and Conlon and Lee [12]. Famous examples of graph norms include the Schatten-p norms for even $p \geq 4$ (in which case H is a p-cycle) and a well-known family of hypergraph norms are the Gowers octahedral norms. The latter were introduced for the purpose of quantifying a notion of quasirandomness of hypergraphs as an important part of Gowers' graph-theoretic proof of Szemerédi's theorem on arithmetic progressions. Having large Gowers norm turns out to imply *correlation with structure*, as opposed to quasirandomness. This is true also for the norm relevant for our setting. In particular, it turns out that the structure with which a game tensor correlates can be turned into a classical strategy for the game. As such, a large norm of the game tensor implies a large classical bias of the game itself. At the same time, we show that the entangled bias is bounded from above by the norm of the game tensor, provided the game is free. Putting these observations together gives the proof of Theorem 4, which we give in the full version of this paper.

The particular hypergraph norm relevant in our setting was introduced in [11] and can be obtained recursively as follows. Starting with a t-partite t-uniform hypergraph H with vertex set $V_1 \cup \cdots \cup V_t$, write $\mathrm{db}_i(H)$ for the t-partite t-uniform hypergraph obtained by making two vertex-disjoint copies of H and gluing them together so that the vertices in the two copies of V_i are identified. We obtain our hypergraph by starting with a single edge $e = (v_1, \ldots, v_t)$ (and vertex sets of size 1), and applying this operation to all parts, forming the hypergraph $\mathrm{db}_1(\mathrm{db}_2(\ldots \mathrm{db}_t(e)))$ with vertex sets of size 2^{t-1} and 2^t edges. The fact that this hypergraph defines a norm via (2) was proved in [12].

2.3 Line games and Gowers uniformity norms

The proof of Theorem 5 is based on two fundamental results from additive combinatorics: the generalized von Neumann inequality and the Gowers Inverse Theorem. The former easily shows that the classical bias of a line game is bounded from above by the Gowers t-uniformity norm of the game map. We show that in fact the same upper bound holds for the entangled bias as well. A large entangled bias thus implies a large uniformity norm for the game map. Analogous to the above-mentioned octahedral norms for tensors, uniformity norms were introduced to quantify a notion of pseudorandomness for bounded maps over abelian groups as an important step in Gowers' other proof of Szemerédi's Theorem, based on higher-order Fourier analysis. The highly non-trivial Gowers Inverse Theorem of Tao and Ziegler [29] establishes that high uniformity norm again implies correlation with structure. Although structure in this context means something quite different than for tensors, we show that it still implies a lower bound on the classical bias. The above observations together prove Theorem 5, details of which can be found in the full version.

2.4 Semidefinite programming relaxation

The proof of Theorem 7 is a small modification of a proof in [9]. They consider a semidefinite programming (SDP) relaxation of the optimization problem for the classical value and then give two algorithms for rounding the result of the SDP to a classical strategy. In the SDP relaxation the objective is to optimize $\mathbb{E}_{x,y} \sum_{i=1}^{k} \langle u_i^{(x)} \mid v_{\pi_{xy}(i)}^{(y)} \rangle$ where $u_i^{(x)}, v_j^{(y)} \in \mathbb{R}^d$ are vectors corresponding to questions x, y and answers i, j. Furthermore, π_{xy} is the matching of correct answers on questions x, y. A classical strategy would correspond to the case where the vectors are integers instead, such that for each x exactly one $u_i^{(x)}$ is equal to 1 and all other $u_i^{(x)}$ are equal to zero and similar for the $v_j^{(y)}$. A quantum strategy also gives rise to a set of vectors, but satisfying different constraints [19]. One of the constraints of the SDP considered in [9] is $0 \leq \langle u_i \mid v_{\pi_{xy}(i)} \rangle \leq |u_i|^2$ which is valid for classical strategies, but in general not for quantum strategies. For our proof, we consider the same SDP but with this constraint dropped. In that case it is also a relaxation for the entangled case and with a few changes one of the rounding algorithms in [9] is also valid when the constraint is dropped. Note that the result only beats a trivial strategy when $\epsilon = \mathcal{O}(1/\log k)$ whereas the other rounding algorithm in [9] is non-trivial for any ϵ. However this other algorithm is more dependent on the extra constraint and it is not clear if it can be dropped there as well.

To get some intuition for the rounding algorithm, we sketch a solution for $\epsilon = 0$ here. In this case one can show that for each question pair x, y the set of vectors $|u_i^{(x)}\rangle$ $(i = 1, ..., k)$ known by the first player is the same set of vectors as the set $|v_i^{(y)}\rangle$ $(i = 1, ..., k)$ known to the second player. In particular, the vector $|u_i^{(x)}\rangle$ is the same as the matching vector $|v_{\pi_{xy}(i)}^{(y)}\rangle$ of the other player. Using shared randomness they can sample a random vector $|g\rangle$ and compute the overlaps $\xi_i^{(x)} = \langle g | u_i^{(x)} \rangle$ and $\xi_i^{(y)} = \langle g | v_i^{(y)} \rangle$ respectively. As they have the same vectors, the players will have the same values for answers in the matching: $\xi_i^{(x)} = \xi_{\pi_{xy}(i)}^{(y)}$. Now both players simply output the answer i for which $|\xi_i^{(x)}|$ (and $|\xi_i^{(y)}|$ for the other player) has the largest value. With probability one this will yield correct answers. For $\epsilon > 0$ the sets of vectors will not be exactly equal and therefore the values $\xi_i^{(x)}, \xi_{\pi_{xy}(i)}^{(y)}$ will be close but not exactly equal. The discrepancy in these values will be bigger for vectors $|u_i^{(x)}\rangle$ with a small norm. In the full version we provide the rounding algorithm in full detail and show how this issue is solved.

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