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NONSTATIONARY FILTRATION IN PARTIALLY SATURATED POROUS MEDIA

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Nonstationary filtration in partially saturated porous media*)

by

C.J. van Duyn & L.A. Peletier

ABSTRACT

From the mathematical formulation of a one-dimensional flow through a partially saturated porous medium, we arrive at a nonlinear free boundary problem, the boundary being between the saturated and the unsaturated regions in the medium. In particular we obtain an equation which is parabolic in the unsaturated part of the domain and elliptic in the saturated part.

Existence, uniqueness, a maximum principle and regularily properties are proved for weak solutions of a Cauchy-Dirichlet problem in the cylinder $\{(x,t): 0 \leq x \leq 1,\ t \geq 0\}$ and the nature, in particular the regularity, of the free boundary is discussed.

Finally, it is shown that solutions of a large class of Cauchy-Dirichlet problems converge towards a stationary solution as $t \to \infty$ and estimates are given for the rate of convergence.

KEY WORDS & PHRASES: *nonlinear elliptic-parabolic equation, free boundary, existence, uniqueness, maximum principle, regularity, asymptotic behaviour.*

*) This report will be submitted for publication elsewhere.

1. INTRODUCTION

During the last two decades a great deal of progress has been made on the mathematical analysis of flows through porous media [1,2,3,5,6,9,10,11, 12,14,15,18]. Much of this work, however, has been concerned with flows which were either completely saturated or completely unsaturated. In this paper we shall consider the flow of a fluid in a porous medium which is only partially saturated. This leads to a free boundary problem, the boundary being between the saturated and unsaturated regions. In the context of ground water flow this interface is called the *water table*.

Consider a homogeneous, isotropic and rigid porous medium filled with a fluid. Let q denote the macroscopic velocity of the fluid and c the volumetric moisture content. If $\bar{c}$ is the moisture content at saturation, we have $0 \le c \le \bar{c}$. Then the flow is governed by the continuity equation

$$\frac{\partial c}{\partial t} + \operatorname{div} q = 0 \tag{1.1}$$

and Darcy's law

$$q = - K(c) \operatorname{grad} \Phi, \tag{1.2}$$

where K is the hydraulic conductivity and Φ the total potential (hydraulic head) (cf. BEAR [4], p.488, RAATS and GARDNER [17]). If absorption and chemical, osmotic and thermal effects are ignored, Φ may be expressed as the sum of a hydrostatic potential ψ due to capillary suction and a gravitational potential [4]. Thus, if we choose the z-coordinate along the gravity vector, we may write

$$\Phi = \psi + z. \tag{1.3}$$

Eliminating q and Φ from (1.1)-(1.3) we obtain

$$\frac{\partial c}{\partial t} = \mathrm{div}(K(c)\ \mathrm{grad}\ \psi) + \frac{\partial}{\partial z} K(c). \tag{1.4}$$

Between the variables c and ψ there exists an empirical relationship which can be quite complicated because of hysteresis effects. However, we shall ignore these effects and assume that we may write $c = c(\psi)$, where

(i) if $\psi < 0$, $0 \le c(\psi) < \bar{c}$ and c is strictly increasing;

(ii) if $\psi > 0$, $c(\psi) = \bar{c}$.

Equation (1.4) now becomes

$$\frac{\partial}{\partial t} c(\psi) = \mathrm{div}(D(\psi)\ \mathrm{grad}\ \psi) + \frac{\partial}{\partial z} D(\psi) \tag{1.5}$$

where $D = K \circ c$. Note that in the saturated region (1.5) reduces to

$$\Delta\psi = 0.$$

Thus, equation (1.5) is of elliptic type in the saturated region and of parabolic type in the unsaturated region. Across the boundaries between these regions one would expect c and q to be continuous.

In this paper we shall restrict our attention to one-dimensional flows for which $K(c) \equiv K_0$. Then (1.5) becomes

$$(c(\psi))_t = \psi_{xx},$$

where subscripts denote differentiation, and K_0 has been set equal to unity.

Let $Q_T = \{(x,t) : 0 < x < 1,\ 0 < t \le T\}$, where T is some fixed positive constant. Then ous object is to study the Cauchy-Dirichlet problem

$$\text{(I)} \qquad \begin{cases} (c(u))_t = u_{xx} & \text{in } Q_T & (1.6) \\ u(0,t) = -1, \quad u(1,t) = +1 & \text{for } 0 < t \le T & (1.7) \\ c(u(x,0)) = v_0(x) & \text{for } 0 \le x \le 1, & (1.8) \end{cases}$$

where the function $c\colon [-a, \infty) \to [0,1]$ $(a \ge 1)$ satisfies the following hypotheses.

H1a. $c(s)$ is Lipschitz continuous and strictly increasing on $[-a,0]$. Moreover $dc(s)/ds > 0$ whenever it exists on $(-a,0)$;

H1b. $c(s) = 1$ for $s \geq 0$.

As regards the initial value v_0, we assume that there exists a function $u_0: [0,1] \to \mathbb{R}$ which satisfies the hypotheses

H2a. u_0 is Lipschitz continuous on $[0,1]$;
H2b. $u_0(0) = -1$, $u_0(1) = +1$ and $u_0(x) \geq -a$ on $[0,1]$,

such that

$$c(u_0(x)) = v_0(x) \quad \text{for } 0 \leq x \leq 1. \tag{1.9}$$

It should be observed that since $c'(s) = 0$ for $s > 0$, u_0 is not uniquely determined by v_0.

One approach to Problem I is to assume the existence of an interface $x = \zeta(t)$, to solve (1.6) on both sides of it and then to patch the solutions together at the interface, using the continuity of c and $q = -u_x$. This leads to a condition from which the function ζ then can be determined [7].

Another approach is to define a class of weak solutions on the entire cylinder $\bar{Q}_T$, to establish their existence, and to show that they have properties, which are to be expected of physical flows [13]. It is this second approach which we shall adopt in this paper.

Inspired by the class of weak solutions defined by OLEINIK, KALASHNIKOV and YUI-LIN for the porous media equation [14] we introduce the following notion of weak solution.

DEFINITION. A function $u(x,t)$, defined a.e. in $\bar{Q}_T$, will be called a weak solution of Problem I if (i) $c(u) \in C(\bar{Q}_T)$, u possibly redefined on a set of measure zero, (ii) $u - \bar{u} \in L^2(0,T;H_0^1(0,1))$, where $\bar{u}(x) = 2x-1$ and (iii) u satisfies the identity

$$\iint_{Q_T} \{\phi_x u_x - \phi_t c(u)\} dxdt = \int_0^1 \phi(x,0) v_0(x) dx \tag{1.10}$$

for all $\phi \in C^1(\bar{Q}_T)$ which vanish for $x = 0,1$ and $t = T$.

The plan of the paper is the following. In section 2 we prove the existence of a weak solution. This is done by approximating c and u_0 by sequences of smooth functions $\{c_n\}$ and $\{u_{0n}\}$ such that $c_n' \geq 1/n$ for all $n \geq 1$. For each $n \geq 1$, the equation $(c_n(u))_t = u_{xx}$ is now uniformly parabolic and has a unique smooth solution u_n which satisfies (1.7) and the initial condition $u(x,0) = u_{0n}(x)$, $0 \le x \le 1$. We then extract a subsequence u_μ, which converges weakly in $L^2(0,T;H^1(0,1))$ to an element u, which is then shown to possess all the properties of a weak solution.

In section 3 we establish the uniqueness of weak solutions and a maximum principle for the concentration $c(u)$. Then in section 4 we discuss the regularity of weak solutions.

Let $v = c(u)$, where u has been chosen so that $v \in C(\bar{Q}_T)$. Suppose there exists a point $\hat{x} \in (0,1)$ such that $v_0(x) < 1$ on $[0,\hat{x})$ and $v_0(x) = 1$ on $[\hat{x},1]$. Then we show in section 5 that there exists a function $\zeta\colon [0,T] \to (0,1)$ such that for each $t \in [0,T]$:

$$v(x,t) < 1 \quad \text{for } 0 \le x < \zeta(t)$$

$$v(x,t) = 1 \quad \text{for } \zeta(t) \le x \le 1.$$

Thus, ζ defines the interface in Problem I. Subsequently we derive a number of properties of this function ζ.

Finally, in section 6 we show that

$$v(x,t) \to \bar{v}(x) \equiv c(\bar{u}(x)) \quad \text{as } t \to \infty$$

where $\bar{u}(x) = 2x-1$, uniformly on $[0,1]$, and we give two estimates for the rate of convergence.

It is a pleasure to acknowledge a number of fruitful discussions with Ph. Clément and J. van Kan.

2. EXISTENCE

We begin by approximating the function c in equation (1.6) by a sequence of smooth, strictly increasing functions $\{c_n\}$.

<u>LEMMA 1</u>. *Suppose* $c\colon [-a,\infty) \to [0,1]$ *satisfies hypotheses* H1a, b. *Then there exists a sequence* $\{c_n\} \subset C^\infty(\mathbb{R})$ *and a constant* $K > 0$ *such that*

(i) $c_n(s) \to c(s)$ *as* $n \to \infty$ *uniformly on bounded subsets of* $[-a,\infty)$;

(ii) $1/n \le c_n'(s) \le K$ *on* $\mathbb{R}$ *for all* $n \ge 1$;

(iii) *if* c *is concave, then* $c_n''(s) \le 0$ *on* $\mathbb{R}$ *for all* $n \ge 1$.

<u>PROOF</u>. Let $\tilde{c}\colon \mathbb{R} \to (-\infty,1]$ be a uniformly Lipschitz continuous extension of

c to $\mathbb{R}$ such that $\tilde{c} \in C^1(-\infty,-a)$ and $\tilde{c}'(s)$ = constant and positive for $s < -a$. Let $\rho \in C_0^\infty(\mathbb{R})$ be a nonnegative function such that $\rho(s) = 0$ for $|s| \geq 1$ and $\int_{\mathbb{R}} \rho(s)ds = 1$. Then for each $n \geq 1$, we set $\rho_{1/n}(s) = n\rho(ns)$ and we define

$$c_n(s) = \frac{1}{n} s + \int_{\mathbb{R}} \rho_{1/n}(r-s)\tilde{c}(r-\frac{1}{n})\, dr. \tag{2.1}$$

Clearly $c_n \in C^\infty(\mathbb{R})$, and we assert that $\{c_n\}$ has the desired properties.

The first property follows from the Lipschitz continuity of $\tilde{c}$ and the second one from the monotonicity of $\tilde{c}$. Let $\lambda \in [0,1]$ and $s_1, s_2 \in \mathbb{R}$. Then,

$$\begin{aligned} c_n(\lambda s_1+(1-\lambda)s_2) &= \frac{1}{n}\{\lambda s_1+(1-\lambda)s_2\} + \int_{\mathbb{R}} \rho_{1/n}(r-\lambda s_1-(1-\lambda)s_2)\tilde{c}(r-\frac{1}{n})\,dr \\ &= \frac{1}{n}\{\lambda s_1+(1-\lambda)s_2\} + \int_{\mathbb{R}} \rho_{1/n}(z)\tilde{c}(\lambda(z+s_1-\frac{1}{n}) + \\ &\qquad + (1-\lambda)(z+s_2-\frac{1}{n}))\,dz \\ &\geq \frac{1}{n}\{\lambda s_1+(1-\lambda)s_2\} + \lambda\int_{\mathbb{R}} \rho_{1/n}(r-s_1)\tilde{c}(r-\frac{1}{n})\,dr \\ &\qquad + (1-\lambda)\int_{\mathbb{R}} \rho_{1/n}(r-s_2)\tilde{c}(r-\frac{1}{n})\,dr, \end{aligned}$$

where we have used the fact that if c is concave, it is possible to choose an extension $\tilde{c}$ which is also concave. Thus

$$c_n(\lambda s_1+(1-\lambda)s_2) \geq \lambda\, c_n(s_1) + (1-\lambda)c_n(s_2).$$

Since $c_n \in C^\infty(\mathbb{R})$ this establishes the third property.

Next, we approximate u_0 by a sequence of smooth functions $\{u_{0n}\}$. Set

$$\tilde{u}_0(x) = \begin{cases} -2 - u_0(-x) & -1 < x < 0 \\ u_0(x) & 0 \leq x \leq 1 \\ 2 - u_0(2-x) & 1 < x < 2 \end{cases}$$

and define for $n \geq 1$:

$$u_{0n}(x) = \int_{-1}^{2} \rho_{1/n}(x-y)\tilde{u}_0(y)\,dy \qquad 0 \leq x \leq 1. \tag{2.2}$$

LEMMA 2. *Suppose* u_0 *satisfies hypotheses* H2a,b *and* $\rho \in C_0^\infty(\mathbb{R})$ *is a non-negative function with support in* $[-1,1]$ *such that* $\int_{\mathbb{R}} \rho(x)dx = 1$, $\rho(-x) = \rho(x)$ *and* $x\rho'(x) \le 0$ *for* $x \in \mathbb{R}$. *Then the sequence* $\{u_{0n}\} \subset C^\infty([0,1])$ *defined by* (2.2) *in which* $\rho_{1/n}(x) = n\rho(nx)$ *has the following properties:*

(i) $u_{0n}(x) \to u_0(x)$ *as* $n \to \infty$ *uniformly on* $[0,1]$;

(ii) $u_{0n}(0) = -1$, $u_{0n}(1) = 1$, $u''_{0n}(0) = u''_{0n}(1) = 0$ *for all* $n \ge 1$;

(iii) *if* $u_0^{(1)}$ *and* $u_0^{(2)}$ *both satisfy* H2a,b *and* $u_0^{(1)} \ge u_0^{(2)}$ *on* $[0,1]$, *then* $u_{0n}^{(1)} \ge u_{0n}^{(2)}$ *on* $[0,1]$ *for all* $n \ge 1$.

PROOF. Parts (i) and (ii) follow from the continuity of $\tilde{u}_0$ and the symmetry properties of $\tilde{u}_0$ and ρ. (iii) Define $w = \tilde{u}_0^{(1)} - \tilde{u}_0^{(2)}$ and $w_n = u_{0n}^{(1)} - u_{0n}^{(2)}$. Then

$$w_n(x) = \int_{-1}^{2} \rho_{1/n}(x-y)w(y)dy \qquad 0 \le x \le 1.$$

It is clear that $w_n(x) \le 0$ if $1/n \le x \le 1 - 1/n$. Thus, choose $x \in (0,1/n)$. Then we can write

$$\begin{aligned} w_n(x) &= \int_{x-1/n}^{0} \rho_{1/n}(x-y)w(y)dy + \int_{0}^{x+1/n} \rho_{1/n}(x-y)w(y)dy \\ &= \int_{0}^{x+1/n} \{\rho_{1/n}(x-y) - \rho_{1/n}(x+y)\}w(y)dy \\ &\ge 0 \end{aligned}$$

in view of the asymmetry of w and the properties of ρ. Similarly, $w_n(x) \ge 0$ for $x \in (1-1/n,1]$. Hence $u_{0n}^{(1)} \ge u_{0n}^{(2)}$ on $[0,1]$.

We now consider for each $n \in \mathbb{N}$ the problem

$$I(n)\begin{cases} c_n'(u)u_t = u_{xx} & \text{in } Q_T \\ u(0,t) = -1, \quad u(1,t) = +1 & \text{for } 0 < t \le T \\ u(x,0) = u_{0n}(x) & \text{for } 0 \le x \le 1. \end{cases} \tag{2.3}$$

The properties of the functions c_n and u_{0n} guarantee that Problem I(n) has a unique solution $u_n \in C^{2+1}(\bar{Q}_T) \cap C^\infty(Q_T)$.

LEMMA 3. *Let* $m = \max\{|u_0(x)| : 0 \le x \le 1\}$ *and* $L = \text{ess sup}\{|u_0'(x)| : 0 \le x \le 1\}$ *Then*

$$|u_n(x,t)| \le M = \max\{m,1\} \quad and \quad |u_{nx}(x,t)| \le L$$

for all $n \in \mathbb{N}$ *and all* $(x,t) \in \bar{Q}_T$.

PROOF. The bound on u_n is an immediate consequence of the maximum principle. To obtain the bound on u_{nx}, we first derive a uniform bound on u_{nx} at the parabolic boundary Γ_T of Q_T. Let $|u_0'| \le L$ a.e. on $(0,1)$, then it follows from (2.2) that $|u_{0n}'| \le L$ on $[0,1]$ for all $n \ge 1$. Now consider the functions $w_\pm(x) = -1 \pm Lx$. Then $w_- \le u_n \le w_+$ on Γ_T. Since w_+ and w_- both satisfy (2.3) it follows from the maximum principle that $w_- \le u_n \le w_+$ in $\bar{Q}_T$, and hence $|u_{nx}(0,t)| \le L$ for $0 \le t \le T$. In the same way we obtain that $|u_{nx}(1,t)| \le L$ for $0 \le t \le T$. Thus $|u_{nx}| \le L$ on Γ_T for all $n \ge 1$.

Next, we differentiate (2.3) with respect to x. Putting $z = u_{nx}$, this yields

$$z_t = a(x,t)z_{xx} + b(x,t)z_x$$

where $a = \{c_n'(u_n)\}^{-1}$ and $b = -c_n''(u_n)c_n'^{-2}(u_n)u_{nx}$. In view of the properties of u_n and c_n it follows that a and b are uniformly bounded in $\bar{Q}_T$, and $z \in C^{2,1}(Q_T) \cap C(\bar{Q}_T)$. Hence, by the maximum principle

$$\max_{\bar{Q}_T}|z(x,t)| \le \max_{\Gamma_T}|z(x,t)| \le L$$

which proves the second estimate.

REMARK. By a slight modification of the proof of Lemma 3 one can prove the following result: If $u_0'(x) \ge \delta > 0$ a.e. on $(0,1)$ then

$$u_{nx}(x,t) \ge \delta$$

for all $n \in \mathbb{N}$ and all $(x,t) \in \bar{Q}_T$.

Set

$$w_n = u_n - \bar{u},$$

where $\bar{u} = \bar{u}(x) = 2x - 1$ on $[0,1]$. Then it follows from Lemma 3 that $\{w_n\}$ is a bounded sequence in the space $L^2(0,T;H_0^1(0,1))$. Hence there exists a subsequence $\{w_\mu\}$ which converges weakly to an element $w \in L^2(0,T;H_0^1(0,1))$. Let

$$u = w + \bar{u} \tag{2.4}$$

then we shall show that u has all the properties required of a weak solution.

Plainly u possesses the second property. Let us next turn to property (i). For convenience we write

$$v_n(x,t) = c_n(u_n(x,t)) \quad n \in \mathbb{N} \quad (x,t) \in \bar{Q}_T.$$

LEMMA 4. *The sequence $\{v_n\}$ is uniformly Lipschitz continuous with respect to x and uniformly Hölder continuous (exponent ½) with respect to t in $\bar{Q}_T$.*

Before proving Lemma 4, we establish an auxiliary result.

PROPOSITION 1. *Let $f \in C^1([0,1])$ have the following properties: (i) $|f'| \leq A$ on $[0,1]$, (ii) $|\int_a^b f(x)dx| \leq \varepsilon$ for any $a,b \in [0,1]$. Then*

$$|f(x)| \leq \max\{2\varepsilon, \sqrt{2A\varepsilon}\} \text{ for } 0 \leq x \leq 1.$$

The proof is elementary, and we shall omit it.

Proof of lemma 4. By Lemma 3, and the properties of $\{c_n\}$:

$$|v_{nx}| = |c_n'(u_n)||u_{nx}| \leq KL \quad \text{in } \bar{Q}_T.$$

Hence, $\{v_n\}$ is uniformly Lipschitz continuous with respect to x in $\bar{Q}_T$.

Define the rectangle $R = (a,b) \times (t_1,t_2) \subset \bar{Q}_T$, and integrate (2.3) over R. This yields

$$\int_a^b \{v_n(x,t_2) - v_n(x,t_1)\}dx = \int_{t_1}^{t_2} \{u_{nx}(b,t) - u_{nx}(a,t)\}dt.$$

Hence by Lemma 3

$$\left|\int_a^b \{v_n(x,t_2) - v_n(x,t_1)\}dx\right| \leq 2L|t_2-t_1|.$$

We now apply Proposition 1 with

$$f(x) = v_n(x,t_2) - v_n(x,t_1).$$

Since $|f'| \le 2KL$, we may conclude that for $|t_2-t_1|$ small enough

$$|v_n(x,t_2) - v_n(x,t_1)| \le 2\sqrt{KL}\ |t_2-t_1|^{\frac{1}{2}}$$

for all $x \in [0,1]$.

It follows from Lemma 4 that the sequence $\{v_n\}$ has a convergent subsequence, denoted again by $\{v_\mu\}$, such that

$$v_\mu \to v \quad \text{as} \quad \mu \to \infty \quad \text{in } C^\beta(\bar{Q}_T), \tag{2.5}$$

where $\beta \in (0,1)$, and $v \in C^{0+1}(\bar{Q}_T)$.

<u>LEMMA 5</u>. *Let* u *and* v *be defined in* (2.4) *and* (2.5) *respectively. Then* $v = c(u)$ *a.e. in* $\bar{Q}_T$.

<u>PROOF</u>. Define

$$\mathcal{D} = \{(x,t) \in Q_T: v(x,t) < 1\}$$

$$\mathcal{P} = \{(x,t) \in Q_T: v(x,t) = 1\}$$

(i) Let $(x_0,t_0) \in \mathcal{D}$, and set $2\delta = 1 - v(x_0,t_0)$. Define $N_r = \{(x,t) \in Q_T: |x-x_0| + |t-t_0| < r\}$. In view of (2.5) there exists a $n_0 \ge 1$ and a $\rho > 0$ such that

$$1 - 3\delta \le v_\mu \le 1 - \delta \quad \text{in } \bar{N}_\rho$$

for all $\mu \ge n_0$. For each $n \ge 1$, we may write

$$u_n = c_n^{-1}(v_n).$$

Moreover, in view of the assumptions on c:

$$\max\{|(c_n^{-1}(s))'|: n \ge 1,\ c(-a)+\delta \le s \le 1-\delta\} = \mathcal{C}(\delta) < \infty.$$

Hence, if we set $I_\delta = [c(-a)+\delta, 1-\delta]$, then

$$|u_{\mu_1} - u_{\mu_2}| \le C(\delta)|v_{\mu_1} - v_{\mu_2}| + \max_{s \in I_\delta} |c_{\mu_1}^{-1}(s) - c_{\mu_2}^{-1}(s)|$$

$$\le C(\delta)|v_{\mu_1} - v_{\mu_2}| + C(\tfrac{1}{2}\delta) \max_{-a \le r \le 0} |c_n(r) - c_m(r)|,$$

for μ_1, μ_2 large enough. Hence $\{u_\mu\}$ is a Cauchy sequence in $C(\bar{N}_\rho)$, and therefore converges to an element $u^* \in C(\bar{N}_\rho)$. In view of the continuity of c this implies that $v = c(u^*)$ in $\bar{N}_\rho$.

Note that as $\mu \to \infty$, $w_\mu \rightharpoonup w$ in $L^2(Q_T)$ and hence $u_\mu \rightharpoonup u$ in $L^2(Q_T)$. Let $\phi \in C_0^\infty(N_\rho)$. Then

$$(\phi, u_\mu)_{L^2} \to (\phi, u)_{L^2} \quad \text{as } \mu \to \infty$$

by the weak convergence of $\{u_\mu\}$ and

$$(\phi, u_\mu)_{L^2} \to (\phi, u^*)_{L^2} \quad \text{as } \mu \to \infty$$

by the uniform convergence of u_μ in $\bar{N}_\rho$. Therefore

$$(\phi, u)_{L^2} = (\phi, u^*)_{L^2}$$

for any $\phi \in C_0^\infty(N_\rho)$ and hence $u = u^*$ a.e. in N_ρ. Thus $v = c(u)$ a.e. in N_ρ and hence, because (x_0, t_0) was an arbitrary point in $\mathcal{D}$, $v = c(u)$ a.e. in $\mathcal{D}$.

(ii) Let $\varepsilon > 0$ and let

$$\mathcal{P}_\varepsilon = \{(x,t) \in Q_T: \ v(x,t) > 1 - \varepsilon\}.$$

By (2.5) there exists an $n_1 \in \mathbb{N}$ such that if $\mu \ge n_1$

$$v_\mu > 1 - 2\varepsilon \quad \text{in } \mathcal{P}_\varepsilon.$$

Hence, if $\mu \ge n_1$,

$$u_\mu = c_\mu^{-1}(v_\mu) \ge c_\mu^{-1}(1 - 2\varepsilon). \tag{2.6}$$

Let $\phi \in C_0^\infty(\mathcal{P}_\varepsilon)$ be nonnegative. Then (2.6) and the weak convergence of $\{u_\mu\}$ imply that

$$0 \le \lim(\phi, u_\mu - c_\mu^{-1}(1-2\varepsilon))_{L^2}$$

$$= (\phi, u-c^{-1}(1-2\varepsilon))_{L^2}.$$

Thus

$$u \ge c^{-1}(1-2\varepsilon) \quad \text{a.e. in } \mathcal{P}_\varepsilon. \tag{2.7}$$

Because $\mathcal{P} \subset \mathcal{P}_\varepsilon$ for any $\varepsilon > 0$, it follows that (2.7) holds in $\mathcal{P}$ for any $\varepsilon > 0$ and hence, letting ε tend to zero we obtain

$$u \ge 0 \quad \text{a.e. in } \mathcal{P}.$$

This implies that $c(u) = 1$ a.e. in $\mathcal{P}$. Since $v = 1$ in $\mathcal{P}$ we have proved that $v = c(u)$ a.e. in $\mathcal{P}$. This completes the proof of the lemma.

Thus by Lemma 5 and (2.5) u has property (i), and it remains to verify that u has property (iii).

Let $\phi \in C^1(\bar{Q}_T)$ be a test function, i.e. it vanishes at $x = 0,1$ and at $t = T$. Then, because u_n is a classical solution of Problem I(n) we have

$$\iint_{Q_T} \{\phi_x u_{nx} - \phi_t c_n(u_n)\} dxdt = \int_0^1 \phi(x,0) c_n(u_{0n}(x)) dx. \tag{2.8}$$

If we pass in (2.8) to the limit through the subsequence $\{u_\mu\}$ and use (2.5) we obtain

$$\iint_{Q_T} \{\phi_x u_x - \phi_t v\} dxdt = \int_0^1 \phi(x,0) v_0(x) dx.$$

Since, by Lemma 5, $v = c(u)$ a.e. in Q_T, it follows that u satisfies (1.10) and hence possesses property (iii).

Thus the function u defined by (2.4) is indeed a weak solution of Problem I. This completes the proof of the existence theorem:

<u>THEOREM 1</u>. *Suppose that the function* c *satisfies hypotheses* H1a,b *and that* v_0 *is such that there exists a function* u_0 *which satisfies* H2a,b. *Then there exists a weak solution of Problem* I.

3. UNIQUENESS AND A MAXIMUM PRINCIPLE

In this section we shall establish uniqueness of the weak solution u defined in the previous section, and a maximum principle for the concentration.

THEOREM 2. *Let the function* c *satisfy hypotheses* H1a,b. *Then Problem* I *has at most one weak solution.*

PROOF. Let u_1 and u_2 be two weak solutions of Problem I. Then if we substitute them into (1.10) and substract, we obtain the identity.

$$\iint_{Q_T} [\phi_x(u_1-u_2)_x - \phi_t\{c(u_1)-c(u_2)\}]dxdt = 0 \tag{3.1}$$

for all $\phi \in C^1(\bar{Q}_T)$ which vanish at $x = 0$, $x = 1$ and $t = T$. It follows from a completion argument that (3.1) continuous to hold if ϕ is taken from the set $H^1(Q_T) \cap C(\bar{Q}_T)$ and vanishes at $x = 0$, $x = 1$ and $t = T$. Hence, for any $t_1 \in (0,T]$, (3.1) holds if we substitute

$$\phi(x,t) = \begin{cases} \int_t^{t_1} \{u_1(x,s)-u_2(x,s)\}ds & 0 \le x \le 1, \quad 0 \le t < t_1 \\ 0 & 0 \le x \le 1, \quad t_1 \le t \le T \end{cases}$$

and we obtain

$$\tfrac{1}{2}\int_0^1 \{\phi_x(x,0)\}^2dx + \iint_{Q_{t_1}} (u_1-u_2)\{c(u_1)-c(u_2)\}dxdt = 0.$$

Since both integrals are nonnegative, it follows, that both must vanish, and hence that

$$\tfrac{1}{2}\int_0^{t_1} \{u_{1x}(x,s)-u_{2x}(x,s)\}ds = 0 \quad \text{a.e. on } [0,1]. \tag{3.2}$$

Let us write $w = u_1 - u_2$, and let $\chi(a,b)$ denote the characteristic function of the interval $(a,b) \subset \mathbb{R}$. Then utilizing the fact that t_1 is an arbitrary point in $(0,T]$, it follows from (3.2) that

$$\iint_{Q_T} \chi(x_1,x_2)\chi(t_1,t_2)w_x dxdt = 0$$

for arbitrary intervals $(x_1,x_2) \subset (0,1)$ and $(t_1,t_2) \subset (0,T)$. Hence $w_x = 0$ a.e. in Q_T. Because, by property (ii) of weak solutions $w \in L^2(0,T;H_0^1(0,1))$, this implies that $w = 0$ a.e. in Q_T, and hence that $u_1 = u_2$ a.e. in Q_T.

COROLLARY. *By Lemma 5, the uniqueness of* u *implies the uniqueness of the concentration* $v = c(u)$ *in* $\bar{Q}_T$.

THEOREM 3. *Suppose the function* c *satisfies hypotheses* H1a,b *and* v_{01} *and* v_{02} *are such that there exist functions* u_{01} *and* u_{02} *which satisfy* H2a,b. *Let* u_1 *and* u_2 *be the weak solutions of Problem* I, *corresponding to, respectively* v_{01} *and* v_{02}. *Then, if* $v_{01} \geq v_{02}$ *on* $[0,1]$, *it follows that* $c(u_1) \geq c(u_2)$ a.e. *in* $\bar{Q}_T$.

PROOF. By (1.9)

$$c(u_{01}(x)) \geq c(u_{02}(x)) \quad \text{on } [0,1]. \tag{3.3}$$

We shall show that u_{01} and u_{02} can be chosen so that

$$u_{01}(x) \geq u_{02}(x) \qquad \text{on } [0,1]. \tag{3.4}$$

Suppose at a point $x_0 \in (0,1)$,

$$u_{01}(x_0) < u_{02}(x_0). \tag{3.5}$$

Then it follows from the monotonicity of c that

$$c(u_{01}(x_0)) \leq c(u_{02}(x_0)).$$

In view of (3.3) only equality can apply, and this is only compatible with (3.5) if $u_{01}(x_0) \geq 0$ and hence $u_{02}(x_0) > 0$.

Thus (3.4) can only be violated at points in (0,1) where both u_{01} and u_{02} are nonnegative. However at these points we may modify u_{01} and u_{02}, provided they remain nonnegative and Lipschitz continuous. Thus, at points where $u_{02} \geq 0$, we redefine u_{02}, so that $u_{02} = 0$, except for a sufficiently small interval $(1-\delta,1]$. Then, in view of the Lipschitz continuity of u_{01}, we can achieve inequality (3.4).

Let u_{in} be the solution of Problem I(n) with initial value u_{0in} $(i = 1,2)$. Then by (3.4) and Lemma 2, $u_{01n} \geq u_{02n}$ on $[0,1]$ and hence, by the maximum principle

$$u_{1n} \geq u_{2n} \qquad \text{in } \bar{Q}_T$$

for every $n \geq 1$. This implies, by the monotonicity of c_n that

$$c_n(u_{1n}) \geq c_n(u_{2n}) \qquad \text{in } \bar{Q}_T$$

for every $n \geq 1$ and hence, by (2.5) and Lemma 5,

$$c(u_1) \geq c(u_2) \qquad \text{a.e. in } \bar{Q}_T.$$

Because $c(u_1)$ and $c(u_2)$ are uniquely determined by v_{01} and v_{02} the theorem is proved.

4. REGULARITY

In this section we derive three results about the regularity of weak solutions of Problem I.

<u>THEOREM 4</u>. *Let* u *be the weak solution of Problem* I, *in which* c *and* v_0 *satisfy the hypotheses imposed in Theorem* 1. *Then* $u \in L^2(0,T;H^2(0,1))$.

<u>PROOF</u>. Let $\varepsilon \in (0,\frac{1}{2})$ and $Q_T^\varepsilon = (\varepsilon,1-\varepsilon) \times (\varepsilon,T]$. Let u_n be the solution of Problem I(n). Then $u_n \in C^\infty(\bar{Q}_T^\varepsilon)$ and hence, if we multiply (2.3) by u_{nt} and integrate over Q_T^ε, we obtain

$$\int_\varepsilon^T \int_\varepsilon^{1-\varepsilon} c_n'(u_n)u_{nt}^2 dxdt = \int_\varepsilon^T [u_{nx}u_{nt}]_\varepsilon^{1-\varepsilon} dt + \tfrac{1}{2}\int_\varepsilon^{1-\varepsilon} u_{nx}^2(x,\varepsilon)dx$$

$$- \tfrac{1}{2}\int_\varepsilon^{1-\varepsilon} u_{nx}^2(x,T)dx.$$

At this point we let $\varepsilon \to 0^+$. Because $u_n \in C^{2+1}(\bar{Q}_T)$ and $u_{nt}(0,t) = u_{nt}(1,t) = 0$ for $t \in [0,T]$, we obtain in the limit

$$\int_0^T \int_0^1 c_n'(u_n)u_{nt}^2 dxdt \leq \tfrac{1}{2}\int_0^1 u_{0n}'^2 dx$$

or because $|c_n'(s)| \le K$ for all $s \in \mathbb{R}$ and $n \ge 1$,

$$\iint_{Q_T} u_{nxx}^2 dxdt \le \tfrac{1}{2} K \int_0^1 u_{0n}'^2 dx.$$

Remembering that u_n and u_{nx} are uniformly bounded with respect to n in $\bar{Q}_T$, it follows that the sequence $\{u_n\}$ is bounded in $L^2(0,T;H^2(0,1))$. This implies that there exists a subsequence of $\{u_\mu\}$ which converges weakly to an element $\tilde{u} \in L^2(0,T;H^2(0,1))$. Plainly $\tilde{u} = u$.

In the region $\mathcal{D}$, where $v < 1$, the equation is parabolic and u can be shown to be a classical solution of Problem I. This is the content of the following theorem. We shall impose an additional condition on the function c.

H1c. The restriction of c to $[-a,0]$ belongs to $C^2([-a,0])$ and $c'' \le 0$.

THEOREM 5. *Let* u *be a weak solution of Problem* I, *in which* c *and* v_0 *satisfy the hypotheses imposed in Theorem* I *and* H1c. *Then* u *is a classical solution of equation* (1.6) *in the region*

$$\mathcal{D} = \{(x,t) \in Q_T: v(x,t) < 1\}.$$

PROOF. Let $(x_0,t_0) \in \mathcal{D}$. Then, because $v \in C(\bar{Q}_T)$, there exists a neighbourhood $N_0 \subset \mathcal{D}$ of (x_0,t_0) and a $\delta_1 > 0$ such that $v < 1 - 3\delta_1$ in $\bar{N}_0$. Since $v_\mu \to v$ as $\mu \to \infty$ in $C(\bar{Q}_T)$, there exists a $\mu_0 \ge 1$ such that if $\mu \ge \mu_0$

$$v_\mu = c_\mu(u_\mu) > 1 - 2\delta_1 \quad \text{in } \bar{N}_0. \tag{4.1}$$

By Lemma 1 (i) there exists a $\mu_1 \ge 1$ such that if $\mu \ge \mu_1$

$$|c_\mu(s)-c(s)| < \delta_1 \quad \text{for } s \in [-M,M]. \tag{4.2}$$

Thus, if $\mu \ge \mu^* = \max\{\mu_0,\mu_1\}$, it follows from Lemma 3 that

$$c(u_\mu) > 1 - \delta_1 \quad \text{in } \bar{N}_0. \tag{4.3}$$

In view of the fact that $c \in C^2([-a,0])$ and $c' > 0$ on $[-a,0)$ (4.3) implies the existence of a constant $\delta_2 > 0$ such that if $\mu \ge \mu^*$

$$-M \le u_\mu \le -\delta_2 \quad \text{in } \bar{N}_0. \tag{4.4}$$

We now consider equation (2.3), writing it as

$$u_t = a_\mu(x,t)u_{xx}, \tag{4.5}$$

where

$$a_\mu(x,t) = \{c'_\mu(u_\mu(x,t))\}^{-1}.$$

Since $c' > 0$ on $[-a,0)$, there exists a $\delta_3 > 0$ such that

$$c'(s) \geq \delta_3 \quad \text{for } s \in [-M,-\delta_2]. \tag{4.6}$$

Let $\tilde{c}$ be a concave extension of c to $\mathbb{R}$, as in Lemma 1. Clearly we may assume that $\tilde{c} \in C^2((-\infty,0])$. Then for $s \in [-a,0]$

$$c'_n(s)-c'(s) = \frac{1}{n} + \int_{-1/n}^{1/n} \rho_{1/n}(z)\{\tilde{c}'(s+z-\tfrac{1}{n})-\tilde{c}'(s)\}dz. \tag{4.7}$$

Because $\tilde{c}$ is concave, the integral in (4.7) is nonnegative and hence

$$c'_n(s) - c'(s) \geq \frac{1}{n} \quad \text{for } s \in [-a,0]. \tag{4.8}$$

Thus, by (4.4) and (4.6) we have that if $\mu \geq \mu^*$

$$\delta_3 < c'_\mu(u_\mu) \leq K \quad \text{in } \bar{N}_0 \tag{4.9}$$

and hence

$$1/K \leq a_\mu < 1/\delta_3 \quad \text{in } \bar{N}_0.$$

Therefore (4.5) is uniformly parabolic and the coefficient a_μ is uniformly bounded away from zero in $\bar{N}_0$ if μ is large enough.

By Lemma 4, $v_n \in C^{0+1}(\bar{N}_0)$ and hence, in view of (4.9) $u_\mu \in C^{0+1}(\bar{N}_0)$ as well. Therefore $a_\mu \in C^{0+1}(\bar{N}_0)$ and it can easily be verified that the norm of a_μ in $C^{0+1}(\bar{N}_0)$ is uniformly bounded with respect to μ. By standard regularity theory ([8], p.64) this implies that there exists a neighbourhood $N_1 \subset N_0$ of (x_0,t_0) such that the solution u_μ of (4.5) belongs to $C^{2+1}(\bar{N}_1)$, the norm being uniformly bounded with respect to $\mu \geq \mu^*$. Hence there exists a subsequency of $\{u_\mu\}$ which converges in $C^{2+\alpha}(\bar{N}_1)$ $(0 < \alpha < 1)$ to an element $u^* \in C^{2+1}(\bar{N}_1)$. It is clear then that $u^* = u$ a.e. in N_1 and satisfies (1.6).

Whereas the definition of a weak solution and Theorem 4 supply information about the dependence of u on x, we have as yet very little information about the dependence of u on t outside the region $\mathcal{D}$. In the following theorem we partially fill this gap, however only at the expense of an additional condition on u_0.

H2c. $u_0 \in C^{2+\gamma}([0,1])$ $(0 < \gamma \le 1)$, $u_0''(0) = u_0''(1) = 0$ and

$$u_0''(x) \ge -\kappa\, c'(u_0(x)) \quad \text{on } [0,1]$$

for some $\kappa > 0$, whenever $u_0(x) \ne 0$.

<u>THEOREM 6</u>. *Let* u *be the weak solution of Problem* I *in which* c *and* v_0 *are such that hypotheses* H1a-c *and* H2a-c *are satisfied. Then there exists a function* $u^*: \bar{Q}_T \to \mathbb{R}$ *such that* $u^* = u$ a.e. *in* Q_T *and*

$$u^*(x,t_1) - u^*(x,t_2) \ge -\kappa(t_1-t_2)$$

for all (x,t_1), $(x,t_2) \in \bar{Q}_T$ *such that* $0 \le t_2 \le t_1 \le T$.

<u>PROOF</u>. Let u_n be the solution of Problem I(n) in which the initial function u_{0n} has been replaced by u_0. Then $u_n \in C^{2+\gamma}(\bar{Q}_T)$ and hence

$$c_n'(u_0(x))u_{nt}(x,0) = u_0''(x) \quad \text{on } [0,1].$$

In view of H2c and (4.8) this means that

$$u_{nt}(x,0) > -\kappa \quad \text{on } [0,1].$$

At the lateral boundaries we have $u_{nt}(0,t) = u_{nt}(1,t) = 0$, hence $u_{nt} > -\kappa$ on the parabolic boundary Γ_T of Q_T for every $n \ge 1$.

Since $u_n \in C^\infty(Q_T)$ we may differentiate (2.3) with respect to t. Writing $u_{nt} = q$, this yields the equation

$$q_t = \frac{1}{c_n'(u_n)}\, q_{xx} - \frac{c_n''(u_n)}{c_n'(u_n)}\, q^2$$

$$\ge \frac{1}{c_n'(u_n)}\, q_{xx},$$

because of the concavity of c_n. Since $q \in C(\bar{Q}_T)$ and $q > -\kappa$ along Γ_T it follows from the maximum principle that $q > -\kappa$ in $\bar{Q}_T$, i.e.

$$u_{nt}(x,t) > -\kappa \quad \text{for } (x,t) \in \bar{Q}_T, \quad n \geq 1. \tag{4.10}$$

For any $n \geq 1$ and $(x,t) \in \bar{Q}_T$ we define

$$w_n(x,t) = \int_0^t u_n(x,s)ds \qquad w(x,t) = \int_0^t u(x,s)ds,$$

where u is the weak solution of Problem I. Since $\{u_\mu\}$ converges weakly to u in $L^2(0,T;H^1(0,1))$ it follows that w_μ converges weakly to w in $H^1(0,T;H^1(0,1))$. But $H^1(0,T;H^1(0,1))$ is compactly imbedded in $C(\bar{Q}_T)$. Hence w_μ converges strongly to w in $C(\bar{Q}_T)$.

Next we define for $n \geq 1$

$$z_n(x,t) = w_n(x,t) + \tfrac{1}{2}\kappa t^2$$

$$z(x,t) = w(x,t) + \tfrac{1}{2}\kappa t^2.$$

Then for each $x \in [0,1]$, it follows from (4.10) that

$$z_{ntt} = u_{nt} + \kappa > 0.$$

Hence the function $z_n(x,\cdot): [0,T] \to \mathbb{R}$ is convex for any $n \geq 1$ and any $x \in [0,1]$. Since $z_n \to z$ in $C(\bar{Q}_T)$ this implies that $z(x,\cdot): [0,T] \to \mathbb{R}$ is also convex for any $x \in [0,1]$. Thus, the right derivative $\partial^+ z(x,t)/\partial t$ exists for all $t \in [0,T)$ and is nondecreasing with respect to t.

Now we define for $(x,t) \in [0,1] \times [0,T)$

$$u^*(x,t) = \frac{\partial^+}{\partial t} z(x,t) - \kappa t.$$

Clearly $u^* = u$ a.e. in Q_T. Moreover if $0 \leq t_2 \leq t_1 < T$

$$u^*(x,t_1) + \kappa t_1 \geq u^*(x,t_2) + \kappa t_2.$$

To complete the proof we define

$$u^*(x,T) = \frac{\partial^-}{\partial t} z(x,T) - \kappa T \qquad 0 \leq x \leq 1$$

and we obtain, in view of the convexity of $z(x,\cdot)$ for

$$u^*(x,T) + \kappa T = \frac{\partial^-}{\partial t} z(x,T) \geq \frac{\partial^+}{\partial t} z(x,t_2) = u^*(x,t_2) + \kappa t_2,$$

whenever $t_2 \in [0,T)$, and $x \in [0,1]$.

REMARK. In what follows we shall often refer to u^* as *the* weak solution of Problem I, in the cases that hypothesis H1a-c and H2a-c are satisfied.

REMARK. The convexity condition H2c imposed on u_0 is reminiscent of the convexity condition introduced by ARONSON [3] and, more recently KNERR [12], to derive an equation for the interface in the Cauchy problem for the porous media equation

$$u_t = (u^m)_{xx} \qquad x \in \mathbb{R},\ t > 0$$

$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}.$$

In this problem the condition is: $(u_0^{m-1}(x))'' > -\kappa$ at points $x \in \mathbb{R}$, where $u_0(x) > 0$.

5. THE INTERFACE

Let u be a weak solution of Problem I and $v = c(u)$ the associated concentration profile. As in the proof of Lemma 5 we set

$$\mathcal{D} = \{(x,t) \in Q_T;\ v(x,t) < 1\}$$

$$\mathcal{P} = \{(x,t) \in Q_T;\ v(x,t) = 1\}.$$

It is immediately clear from Lemma 3 that

$$(0,L^{-1})\times(0,T] \subset \mathcal{D} \qquad \text{and} \qquad [1-L^{-1},1)\times(0,T] \subset \mathcal{P}. \tag{5.1}$$

For each $t \in [0,T]$ we define

$$\zeta^-(t) = \sup\{x \in (0,1) : (\xi,t) \in \overline{\mathcal{D}} \quad \text{for all } \xi \in [0,x)\}$$

$$\zeta^+(t) = \inf\{x \in (0,1) : (\xi,t) \in \overline{\mathcal{P}} \quad \text{for all } \xi \in (x,1]\}.$$

Then, in view of (5.1),

$$L^{-1} \leq \zeta^-(t) \leq \zeta^+(t) \leq 1-L^{-1} \qquad \text{for } 0 \leq t \leq T.$$

In this section we shall show that if $\zeta^-(0) = \zeta^+(0)$, then $\zeta^-(t) = \zeta^+(t)$ for all $t \in [0,T]$, and hence, that there exists a function $\zeta\colon [0,T] \to (0,1)$ such that

$$\mathcal{D} = \{(x,t)\colon 0 < x < \zeta(t), \quad 0 < t \leq T\} \tag{5.2}$$

$$\mathcal{P} = \{(x,t)\colon \zeta(t) \leq x < 1 \quad 0 < t \leq T\}. \tag{5.3}$$

This function ζ will be called the *interface*. Having proved its existence, we shall derive a few properties.

<u>THEOREM 7</u>. *Let the hypotheses* H1a-c *and* H2a,b *be satisfied. Suppose* $\zeta^-(0) = \zeta^+(0)$. *Then there exists a function* $\zeta\colon [0,T] \to (0,1)$*, such that* $\mathcal{D}$ *and* $\mathcal{P}$ *are given by* (5.2) *and* (5.3).

<u>PROOF</u>. Clearly it is enough to prove that $\zeta^-(t) \geq \zeta^+(t)$ for $t \in [0,T]$. Thus, suppose to the contrary that for some $\tau \in (0,T]$, $\zeta^-(\tau) < \zeta^+(\tau)$. Then, since $v \in C(\bar{Q}_T)$, there exist numbers $x_1, x_2 \in [\zeta^-(\tau), \zeta^+(\tau)]$ such that $x_1 < x_2$, $v(x_1,\tau) = v(x_2,\tau) = 1$ and $v(x,\tau) < 1$ on (x_1,x_2).

For $\varepsilon > 0$, let I_ε denote a subinterval of (x_1,x_2) in which $v < 1-\varepsilon$, and let G_ε be the component of the set $\{(x,t) \in Q_T\colon v(x,t) < 1-\varepsilon\}$ which is connected with I_ε. Finally let Γ_ε denote the part of the boundary of G_ε for which $t < \tau$. It follows from the continuity of v that I_ε, G_ε and Γ_ε are non-empty for ε small enough.

Let Γ_T denote the parabolic boundary of Q_T. Then, for ε sufficiently small, we distinguish the following two cases:

(i) $\Gamma_\varepsilon \cap \Gamma_T = \emptyset$. Since $v \in C(\bar{Q}_T)$, this implies that $v|_{\Gamma_\varepsilon} = 1-\varepsilon$. Moreover, because $G_\varepsilon \subset \mathcal{D}$, it follows from Theorem 5 that in G_ε, v satisfies the equation

$$v_t = (c^{-1}(v))_{xx}$$

in a classical sense. Here, c^{-1} denotes the inverse of c, which is well-defined for the values taken on by v in $\bar{G}_\varepsilon$.

Let $\min_{(x,t)\in\bar{G}_\varepsilon} v(x,t) = v(x_0,t_0) < 1-\varepsilon$. Then because $v|_{\Gamma_\varepsilon} = 1-\varepsilon$, (x_0,t_0)

is either an interior point of G_ε or it lies on a segment of the line $t = \tau$, which is part of the boundary of G_ε. In both cases we can apply the maximum principle {[16], p.169, Theorem 2} to show that $v(x,t_0) = v(x_0,t_0) < 1-\varepsilon$ for all x on the line segment $t = t_0$ which lies in G_ε and contains the point (x_0,t_0). Plainly this contradicts the fact that $v|_{\Gamma_\varepsilon} = 1-\varepsilon$. Therefore $v|_{\bar{G}_\varepsilon} \geq 1-\varepsilon$ and in particular $v|_{I_\varepsilon} \geq 1-\varepsilon$. Thus we arrive at a contradiction.

(ii) $\Gamma_\varepsilon \cap \Gamma_T \neq \emptyset$. Since $v|_{\Gamma_\varepsilon \cap Q_T} = 1-\varepsilon$ there exists a point $(\tilde{x},\tilde{t}) \in \Gamma_T \cap \Gamma_\varepsilon$ such that $v(\tilde{x},\tilde{t}) = 1-\varepsilon$. Because $v(0,t) = c(-1)$ and $v(1,t) = 1$, $t \in [0,T]$, it follows that for ε sufficiently small, $\tilde{t} = 0$ and $\tilde{x} \in (0,\zeta(0))$. By assumption $v_0 < 0$ on $[0,\zeta(0))$. Hence there exists a $\delta > 0$ such that $v_0(x) < 1-\delta$ on $[0,\tilde{x}]$.

Let ℓ_ε denote a curve in G_ε which connects $(\tilde{x},0)$ with an arbitrary point (x_3,τ) of I_ε. Plainly $v|_{\ell_\varepsilon} < 1-\delta$. Then, we consider the domain D enclosed by the arcs

$$\ell_1 = \{(x,t): \quad t = 0, \quad 0 \leq x \leq \tilde{x}\}$$

$$\ell_2 = \{(x,t): \quad x = 0, \quad 0 \leq t \leq T\}$$

$$\ell_3 = \{(x,t): \quad t = T, \quad 0 \leq x \leq x_3\}$$

and ℓ_ε. Let $\{v_\mu\}$ be the sequence which converges to v in $C^\beta(\bar{Q}_T)$ (cf. (2.5)). Then there exists a $\mu^* \geq 1$ such that if $\mu \geq \mu^*$, $v_\mu < 1-\delta/2$ on $\ell_1 \cup \ell_2 \cup \ell_\varepsilon$. It now follows from the maximum principle that $v_\mu < 1-\delta/2$ in $\bar{D}$ and in particular in the point (x_1,τ). Hence $v(x_1,\tau) < 1-\delta/2 < 1$, and we have again obtained a contradiction.

In what follows, we shall always assume that $\zeta^-(0) = \zeta^+(0)$, and that $\mathcal{D}$ and $\mathcal{P}$ are given by (5.2) and (5.3).

In $\mathcal{D}$ the weak solution u is a classical solution of equation (1.6) by Theorem 5. If u_0 satisfies the convexity condition H2c, we can say in addition that u_x is continuous up to $\zeta(t)$.

<u>THEOREM 8</u>. *Let* u *be the weak solution of Problem* I, *in which* c *and* v_0 *are such that hypotheses* H1a-c *and* H2a-c *are satisfied. Then for each* $t_0 \in (0,T]$

$$\lim_{x \uparrow \zeta(t_0)} u_x(x,t_0) \quad \textit{exists.}$$

<u>PROOF</u>. Consider the sequence $\{w_n\}$ defined by

$$w_n(x,t) = u_n(x,t) + \tfrac{1}{2}\kappa K\{x-\zeta(t)\}^2,$$

where u_n is the solution of Problem I(n) (with u_{0n} replaced by u_0). Then, by the proof of Theorem 6

$$w_{nxx} = u_{nxx} + \kappa K > \kappa\{-c_n'(u_n)+K\} \geq 0.$$

Let $(x_0,t_0) \in \mathcal{D}$. Then, by the proof of Theorem 5, $w_{\mu xx}(x_0,t_0) \to w_{xx}(x_0,t_0)$ as $\mu \to \infty$ and hence

$$w_{xx}(x_0,t_0) \geq 0 \qquad 0 < x_0 < \zeta(t_0).$$

By Lemma 3 it follows in a similar manner that

$$w_x(x_0,t_0) \leq L + \kappa K \qquad 0 \leq x_0 < \zeta(t_0).$$

Therefore

$$\lim_{x_0 \uparrow \zeta(t_0)} u_x(x_0,t_0) = \lim_{x_0 \uparrow \zeta(t_0)} w_x(x_0,t_0) \qquad \text{exists.}$$

In the next theorem we discuss the nature of the weak solution u in the saturated region $\mathcal{P}$.

<u>THEOREM 9</u>. *Let* u *be the weak solution of Problem* I *in which* c *and* v_0 *satisfy the hypotheses of Theorem* 1. *Suppose that* $\zeta(t)$ *is continuous on an interval* (a,b) ⊂ (0,T]. *Then*

$$u(x,t) = \frac{x-\zeta(t)}{1-\zeta(t)} \qquad \zeta(t) \leq x \leq 1 \tag{5.4}$$

almost everywhere on (a,b). *If in addition hypotheses* H1c *and* H2c *are satisfied,* (5.4) *holds for all* $t \in (a,b)$.

<u>PROOF</u>. Define the set $A = \{(x,t) \in Q_T\colon \zeta(t) < x < 1,\ a < t < b\}$. This set is open because $\zeta \in C(a,b)$.

Let $\phi \in C_0^\infty(A)$. Then, because $c(u) = 1$ in A, the integral identity (1.10) yields

$$0 = \iint_A \{\phi_x u_x - \phi_t\}dxdt = \iint_A \phi_x u_x dxdt.$$

By Theorem 4, $u \in L^2(0,T;H^2(0,1))$. Hence

$$\iint_A \phi u_{xx} dxdt = 0.$$

Thus $u_{xx} = 0$ a.e. in A from which (5.4) follows almost everywhere on (a,b).

Next, we assume that H1c and H2c are satisfied. By the first part of the theorem there exists a set E, which is dense in (a,b), where (5.4) holds. Let $t_0 \in (a,b)\backslash E$. We shall show that (5.4) also holds for $t = t_0$.

Let $\{t_n\} \subset E \cap (a,t_0)$ be such that $t_n \to t_0$ as $n \to \infty$. Let $x \in (\zeta(t_0),1)$. Then by the continuity of ζ, $x \in (\zeta(t_n),1)$ for n large enough and hence, by Theorem 6 and (5.4):

$$u(x,t_0) \geq \frac{x-\zeta(t_n)}{1-\zeta(t_n)} - \kappa(t_0,t_n).$$

If we now let n tend to infinity we obtain

$$u(x,t_0) \geq \frac{x-\zeta(t_0)}{1-\zeta(t_0)} \,. \tag{5.5}$$

Next, suppose there exists a point $x_1 \in (\zeta(t_0),1)$ at which we have strict inequality in (5.5), i.e.

$$u(x_1,t_0) = \frac{x_1-\zeta(t_0)}{1-\zeta(t_0)} + \varepsilon$$

where $\varepsilon > 0$. Let $\{t_m\} \subset (t_0,b) \cap E$ be such that $t_m \to t_0$ as $m \to \infty$. Then by Theorem 6 and (5.4) for m large enough

$$\frac{x_1-\zeta(t_m)}{1-\zeta(t_m)} \geq \frac{x_1-\zeta(t_0)}{1-\zeta(t_0)} + \varepsilon - \kappa(t_m-t_0).$$

Letting m tend to infinity, we obtain a contradiction.

To complete this section we derive a few regularity properties of the interface $\zeta(t)$. We begin with an auxiliary lemma.

<u>LEMMA 6</u>. *Let hypotheses* H1a-c *and* H2a,b *be satisfied. Let* $(x_0,t_0) \in \mathcal{P}$ *and suppose that there exists a positive constant* α *such that*

$$\ell_\alpha(x_0,t_0) = \{(x,t)\in Q_T\colon\ t=t_0-\alpha(x_0-x),\ 0 < x < x_0\} \subset \mathcal{D}.$$

Then there exist positive constants β *and* m *such that*

$$u(x,t_0) \le \beta\{1-e^{\alpha m(x_0-x)}\} \qquad 0 \le x \le x_0$$

and $x_0 = \zeta(t_0)$.

<u>PROOF</u>. Let D denote the open triangle enclosed by $x = 0$, ℓ_α and $t = t_0$. Then, because $u < 0$ on ℓ_α, it follows from Theorem 7 that $u < 0$ in D, whence, by Theorem 6 u is a classical solution of the equation

$$u_{xx} - c'(u)u_t = 0 \qquad \text{in D.}$$

Define the function $u(x,t) = \beta[1-e^{m\{t-t_0+\alpha(x_0-x)\}}]$, where β and m are positive contants, which we shall select in due course. We have

$$\begin{aligned} w_{xx}-c'(u)w_t &= \{-(\alpha m)^2+c'(u)m\}e^{m\{t-t_0+\alpha(x_0-x)\}} \\ &\le \{-(\alpha m)^2+Km\}e^{m\{t-t_0+\alpha(x_0-x)\}} \\ &< 0 \end{aligned}$$

if we choose $m > K/\alpha^2$.

Along ℓ_α we have $w = 0$ and along $\{x=0\}$

$$\begin{aligned} w(0,t) &= \beta[1-e^{m(t-t_0+\alpha x_0)}] \\ &\ge \beta[1-e^{m\alpha x_0}] \\ &> -1 \end{aligned}$$

for β sufficiently small.

Set $z = w-u$. Then

$$z_{xx} - c'(u)z_t < 0 \qquad \text{in D}$$

and $z \ge 0$ along the parabolic boundary of D. Let $D_\varepsilon \subset D$ be the triangle enclosed by $x = 0$, ℓ_α and $t = t_0-\varepsilon$, where $\varepsilon \in (0,\alpha x_0)$. Then, by the maximum principle, $z \ge 0$ in $\bar{D}_\varepsilon$ and in particular

$$u(x,t_0-\varepsilon) \le w(x,t_0-\varepsilon) \qquad \text{for } 0 \le x \le x_0-\varepsilon/\alpha.$$

or

$$v(x,t_0-\varepsilon) \le c(w(x,t_0-\varepsilon)) \quad \text{for } 0 \le x \le x_0-\varepsilon/\alpha.$$

If we now let ε tend to zero and use the fact that $v \in C(\bar{\Omega}_T)$, we obtain

$$v(x,t_0) \le c(w(x,t_0)) \qquad \text{for } 0 \le x \le x_0$$

from which the result follows.

We are now in a position to prove the first regularity result.

<u>THEOREM 10</u>. *Let the hypotheses* H1a-c *and* H2a-c *be satisfied and suppose* ζ *is continuous on* $(0,T)\backslash\{t_1,\ldots,t_N\}$ $(t_k \in (0,T),\ k = 1,2,\ldots,N)$, *such that*

$$\zeta(t_k^-) = \lim_{t\uparrow t_k} \zeta(t) \quad \text{and} \quad \zeta(t_k^+) = \lim_{t\downarrow t_k} \zeta(t) \qquad k = 1,2,\ldots,N$$

exist. Then ζ *is continuous on* $(0,T)$. *If* $\zeta(T^-)$ *exists, then* $\zeta \in C((0,T])$.

<u>PROOF</u>. Let $t_i \in \{t_k\}$. Since $v \in C(\bar{\Omega}_T)$ it follows that $(\zeta(t_i^{\pm}),t_i) \in \mathcal{P}$, and hence

$$\zeta(t_i) \le \min\{\zeta(t_1^-), \zeta(t_i^+)\} \tag{5.6}$$

(i) Suppose $\zeta(t_i^+) < \zeta(t_i^-)$. Then, by (5.6)

$$\zeta(t_i) < \zeta(t_i^-). \tag{5.7}$$

Since ζ is continuous on (t_{i-1},t_i) (or $(0,t_1)$ if $i = 1$), there exists a constant $\alpha > 0$ such that $\ell_\alpha(\zeta(t_i^-),t_i) \subset \mathcal{D}$. Hence by Lemma 6, $\zeta(t_i^-) = \zeta(t_i)$ which contradicts (5.7).

(ii) Suppose

$$\zeta(t_i^-) < \zeta(t_i^+). \tag{5.8}$$

Let $\{\tau_n\} \subset (t_{i-1},t_i)$, such that $\tau_n \to t_i$ as $n \to \infty$. Then, because $\zeta \in C(t_{i-1},t_i)$ we obtain, using Theorems 6 and 9:

$$u(x,t) \ge \frac{x-\zeta(\tau_n)}{1-\zeta(\tau_n)} - \kappa(t-\tau_n) \tag{5.9}$$

if $x \in [\zeta(\tau_n),1]$ and $t \geq \tau_n$. Let $\{\tau_m\} \subset (t_i,T)$ such that $\tau_m \to t_i$ as $m \to \infty$. Then (5.9) implies that

$$0 \geq \frac{\zeta(\tau_m)-\zeta(\tau_n)}{1-\zeta(\tau_n)} - \kappa(\tau_m-\tau_n)$$

if $\zeta(\tau_m) \geq \zeta(\tau_n)$. In view of (5.8) this will be the case of m and n are large enough. Hence if we let m and n tend to infinity we obtain

$$0 \geq \frac{\zeta(t_i^+)-\zeta(t_i^-)}{1-\zeta(t_i)},$$

which contradicts (5.8).

Thus, $\zeta(t_i^+) = \zeta(t_i^-)$ whence ζ is continuous at $t = t_i$, it follows that $\zeta \in C(0,T)$.

Finally, if $\zeta(T^-)$ exists, it is clear that $\zeta(T) \leq \zeta(T^-)$ and it follows from the argument given in case (i) that in fact $\zeta(T) = \zeta(T^-)$.

In Theorem 10, we assume a certain degree of regularity of the interface ζ, and we proved on the basis of this a stronger regularity result. In the following theorem we shall make no initial regularity assumptions about ζ. Instead we impose a monotonicity assumption on u_0.

<u>THEOREM 11</u>. *Let hypotheses* H1a-c *and* H2a-b *be satisfied, and let* v_0 *be such that* u_0 *can be chosen to satisfy the condition* $u_0'(x) \geq \delta > 0$ *a.e. on* (0,1). *Then* $\zeta \in C([0,T])$.

<u>PROOF</u>. It follows from the remark after Lemma 3 that $u_{nx} \geq \delta$ in $\bar{Q}_T$ for all $n \geq 1$. Hence $u_x \geq \delta$ in $\mathcal{D}$.

Let $\tau \in [0,T]$, and $x_1,x_2 \in (0,\zeta(\tau))$ such that $x_1 > x_2$. Then

$$u(x_1,\tau) - u(x_2,\tau) \geq \delta(x_1-x_2),$$

and therefore

$$u(x_2,\tau) \leq -\delta(x_1-x_2) + u(x_1,\tau) < -\delta(x_1-x_2),$$

or

$$v(x_2,\tau) < c(-\delta(x_1-x_2)).$$

Hence, if we let x_1 tend to $\zeta(\tau)$, we obtain in view of the continuity of c.

$$v(x_2,\tau) < c(-\delta(\zeta(\tau)-x_2)) \qquad 0 < x_2 < \zeta(\tau). \tag{5.10}$$

Now suppose ζ is discontinuous at $t = t_0$. Then there exist a constant $\varepsilon > 0$ and a sequence $\{t_k\} \subset (0,T)$ such that $t_k \to t_0$ as $k \to \infty$ with the property

$$|\zeta(t_k) - \zeta(t_0)| \geq \varepsilon > 0 \qquad \text{for all } k \geq 1.$$

Let $\{t_{k'}\}$ and $\{t_{k''}\}$ denote the elements of $\{t_k\}$ such that

$$\zeta(t_{k'}) \leq \zeta(t_0) - \varepsilon \tag{5.11}$$

and

$$\zeta(t_{k''}) \geq \zeta(t_0) + \varepsilon. \tag{5.12}$$

Suppose $t_{k'} \to t_0$ as $k' \to \infty$. In view of (5.11)

$$v(\zeta(t_0) - \varepsilon, t_{k'}) = 1 \qquad \text{for all } k'$$

and hence

$$v(\zeta(t_0) - \varepsilon, t_0) = 1$$

which is impossible in view of the definition of $\zeta(t_0)$.

Next, suppose that $t_{k''} \to t_0$ as $k'' \to \infty$. By (5.10) and (5.12) we have for each k''

$$v(\zeta(t_0), t_{k''}) < c(-\delta(\zeta(t_{k''}) - \zeta(t_0)))$$

$$\leq c(-\delta\varepsilon).$$

Hence, letting k'' tend to infinity we obtain

$$v(\zeta(t_0), t_0) \leq c(-\delta\varepsilon) < 1$$

which is again incompatible with the definition of $\zeta(t_0)$.

6. BEHAVIOUR AS $t \to \infty$

Consider the stationary problem corresponding to Problem I:

$$\begin{cases} u_{xx} = 0 & 0 < x < 1 \\ u(0) = -1; \quad u(1) = 1. \end{cases}$$

Plainly the unique solution $\bar{u}$ of this problem is given by

$$\bar{u}(x) = 2x - 1 \qquad 0 \le x \le 1.$$

In this section we shall show, that if u is a weak solution of Problem I, then

$$c(u(x,t)) \to c(\bar{u}(x)) \qquad \text{as } t \to \infty,$$

uniformly with respect to $x \in [0,1]$. In addition we shall derive estimates for the rate of convergence, first in terms of a weighted L^1-norm and then in terms of the supremum norm. Finally, if $u_0' \ge \delta > 0$ for some $\delta > 0$, we shall show that

$$\zeta(t) - \tfrac{1}{2} = O(|c^{-1}(1-Ke^{-\lambda t})|) \qquad \text{as } t \to \infty,$$

where λ and K are constants defined in Theorem 13.

LEMMA 7. *Let* u_1 *and* u_2 *be weak solutions of Problem* I, *corresponding to the initial values* v_{01}, *respectively* v_{02}. *Suppose* $v_{01} \ge v_{02}$ *and the conditions of the maximum principle (Theorem* 3*) are satisfied. Then*

$$\int_0^1 \eta(x)\{c(u_1(x,t))-c(u_2(x,t))\}dx \le \int_0^1 \eta(x)\{v_{01}(x)-v_{02}(x)\}dx.e^{-\pi^2 t/K} \qquad t \ge 0,$$

where $\eta(x) = \sin \pi x$.

PROOF. Let u_{n1} and u_{n2} be the solutions of Problem I(n) with initial values u_{0n1} and u_{0n2}. As we saw in the proof of Theorem 3 it is possible to choose u_{01} and u_{02} such that $u_{0n1} \ge u_{0n2}$ for all $n \ge 1$. Subtracting the equation for u_{n1} from the one for u_{n2} we obtain

$$\{c_n(u_{n1})-c_n(u_{n2})\}_t = (u_{n1}-u_{n2})_{xx} \qquad \text{in } Q_T. \tag{6.1}$$

We multiply (6.1) by $\eta(x) = \sin \pi x$ and we integrate over (0,1). This yields

$$\frac{d}{dt}\int_0^1 \eta\{c_n(u_{n1})-c_n(u_{n2})\}dx = -\pi^2\int_0^1 \eta(u_{n1}-u_{n2})dx \qquad (6.2)$$

However, because by construction $u_{0n1} \geq u_{0n2}$, and hence $u_{n1} \geq u_{n2}$ we have for each $n \geq 1$,

$$c_n(u_{n1}) - c_n(u_{n2}) \leq K(u_{n1}-u_{n2}).$$

Hence, writing

$$\Phi(t) = \int_0^1 \eta\{c_n(u_{n1})-c_n(u_{n2})\}dx$$

we obtain

$$\Phi'(t) \leq -\frac{\pi^2}{K}.\Phi(t)$$

and therefore

$$\Phi(t) \leq \Phi(0)e^{-\pi^2 t/K}.$$

Passing to the limit we obtain, in view of Lemma 5 the desired estimate.

<u>THEOREM 12</u>. *Let the conditions of Theorem* 1 *be satisfied, and let* u *be the weak solution of Problem* I *with initial value* v_0. *Then there exists a constant* C, *which only depends on* v_0, *such that*

$$\int_0^1 \eta(x)\,|c(u(x,t)) - c(\bar{u}(x))|dx \leq C\, e^{-\pi^2 t/K} \qquad t \geq 0 \qquad (6.3)$$

where $\bar{u}(x) = 2x-1$.

<u>PROOF</u>. Define two initial values v_0^+ and v_0^- such that

$$v_0^+ \geq \max\{v_0, c(\bar{u})\}$$

$$v_0^- \leq \min\{v_0, c(\bar{u})\}$$

and such that there exist corresponding functions u_0^+ and u_0^- which satisfy H2a.b. This is clearly always possible. Let u^+ and u^- be the weak solutions of Problem I, emanating from v_0^+, respectively v_0^-. Then, by Theorem 3, we have in $\bar{Q}_T$

$$c(u^+) \geq \max\{c(u), c(\bar{u})\}$$

$$c(u^-) \leq \min\{c(u), c(\bar{u})\},$$

and hence

$$|c(u)-c(\bar{u})| \leq c(u^+)-c(u^-).$$

Lemma 7, applied to the solutions u^+ and u^- now yields (6.3).

The integral estimate obtained in Theorem 12 can readily be turned into a pointwise estimate by means of Lemma 3 and the following proposition.

PROPOSITION 2. *Let* $f \in H^1(0,1)$ *have the following properties:*

(i) $f(0) = f(1) = 0$ *and* $f(x) \geq 0$ *on* $(0,1)$;
(ii) $|f'| \leq A$ *a.e. on* $[0,1]$ *and*
(iii) $\int_0^1 f(x) \sin\pi x dx \leq \varepsilon$. *Then*

$$|f(x)| \leq (\tfrac{3}{4}A^2\varepsilon)^{1/3} \qquad \text{for } 0 \leq x \leq 1. \tag{6.4}$$

We leave the proof to the reader.

THEOREM 13. *Let the conditions of Theorem* 1 *be satisfied, and let* u *be the weak solution of Problem* I *corresponding to the initial value* v_0. *Then*

$$|c(u(x,t)) - c(\bar{u}(x))| \leq \mathcal{K}e^{-\lambda t} \qquad 0 \leq x \leq 1 \quad t \geq 0, \tag{6.5}$$

where $\mathcal{K} = (\frac{3}{4}K^2(L+2)^2C)^{1/3}$ *and* $\lambda = \pi^2/3K$.

PROOF. Define

$$w(x,t) = |c(u(x,t))-c(\bar{u}(x))|.$$

Then $w(\cdot,t)$ satisfies the hypotheses of Proposition 2, with $A = K(L+2)$ on $[0,\infty)$. Thus (6.5) follows from Theorem 12 and Proposition 2.

We conclude this section with an estimate for the behaviour of $\zeta(t)$ as $t \to \infty$.

THEOREM 14. *Let* H1a-c *and* H2a,b *be satisfied, and let* v_0 *be such that* u_0 *can be chosen so that* $u_0'(x) \geq \delta > 0$ *a.e. on* $(0,1)$. *Then there exists a* *constant* $\rho > 0$ *such that*

$$|\zeta(t)-\tfrac{1}{2}| \le \rho|c^{-1}(1-Ke^{-\lambda t})| \qquad t > 0.$$

PROOF. Fix $t > 0$. Suppose $\zeta(t) \ge \frac{1}{2}$. Then, because $u_x \ge \delta$ in $\mathcal{D}$ we obtain

$$v(\tfrac{1}{2},t) \le c(\delta(\tfrac{1}{2}-\zeta(t))). \tag{6.6}$$

On the other hand, by Theorem 13

$$v(\tfrac{1}{2},t) \ge \bar{v}(\tfrac{1}{2}) - Ke^{-\lambda t}, \tag{6.7}$$

where $\bar{v} = c(\bar{u})$. Thus (6.6) and (6.7) together imply

$$c(\delta(\tfrac{1}{2}-\zeta(t))) \ge 1 - Ke^{-\lambda t}$$

or

$$\zeta(t) - \tfrac{1}{2} \le -\frac{1}{\delta}\, c^{-1}(1-Ke^{-\lambda t}). \tag{6.8}$$

Next, assume that $\zeta(t) < \frac{1}{2}$. Then, by Theorem 13

$$v(\zeta(t),t) - \bar{v}(\zeta(t)) \le Ke^{-\lambda t}$$

or

$$\bar{v}(\zeta(t)) \ge 1 - Ke^{-\lambda t}$$

and therefore

$$\zeta(t)-\tfrac{1}{2} \ge \tfrac{1}{2}c^{-1}(1-Ke^{-\lambda t}). \tag{6.9}$$

Setting $\rho = \max\{\frac{1}{2}, \frac{1}{\delta}\}$ we obtain from (6.8) and (6.9) the desired estimate.

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