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S 410

Quantiles and stabilizing constants

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0. Introduction and Summary. A sequence of distribution functions $\{F_n\}$ belongs to the domain of attraction of a non-degenerate distribution function G (notation $\{F_n\} \in D(G)$) when it is possible to choose sequences $\{a_n\}$ ($a_n > 0$, $n = 1,2,3,\ldots$) and $\{b_n\}$ such that

$$(1) \qquad F_n(a_n x + b_n) \to G(x)$$

in the weak sense. A well-known theorem of Gnedenko states to which extent we may change the sequences of stabilizing constants. We give the theorem in its extended form (see [1], p.246).

Theorem 1. If (1) holds, we have

$$(2) \qquad F_n(\alpha_n x + \beta_n) \to G^*(x)$$

weakly (where G^* is non-degenerate) iff

$$(3) \qquad \alpha_n \sim A.a_n \ , \ a_n^{-1} (\beta_n - b_n) \to B \qquad \text{for } n \to \infty$$

and

$$G^*(x) = G(Ax + B).$$

In this report we give an explicite expression of the constants a_n and b_n as functions of the given distribution functions F_n when G is one-to-one. As an example we consider the case $F_n = F^n$ where F is a given distribution function; then Gnedenko's expression for stabilizing constants for maxima of independent random variables is seen to be a special case of theorem 2. There is also an application concerning stabilization by moments.

Finally we give a connection between quantiles and centering constants used with the weak law of large numbers.

1. <u>Choice of stabilizing constants</u>. For a sequence of distribution functions $\{F_n\}$ satisfying (1) it is not true in general that (1) holds with

$$b_n = \mu_n = \int_{-\infty}^{\infty} x \, d\, F_n(x)$$

and

$$a_n^{\,2} = \sigma_n^{\,2} = \int_{-\infty}^{\infty} x^2 \, dF_n(x) - \{ \int_{-\infty}^{\infty} x \, dF_n(x)\}^2,$$

even if μ_n and σ_n exist for every n. Defining

$$F_n(x) = (1 - \frac{1}{n}) \, F(x) + \frac{1}{n} \, \epsilon(x - n)$$

where F(x) is an arbitrary distribution function with $\mu = 0$ and $\sigma = 1$, we have

$$F_n(x) \longrightarrow F(x)$$

weakly, so (1) holds with $a_n = 1$ but

$$\frac{\sigma_n}{a_n} = \sqrt{n - \frac{1}{n}} \rightarrow \infty$$

We formulate a theorem giving the stabilizing constants as functions of F_n for a class of limit distributions. The formulation involves quantiles; the precise definition of a quantile does not matter. However, it is convenient to have a definition which determines the quantile uniquely. For each α $(0 < \alpha < 1)$ we define for the distribution function F_n

(4) $$\xi_\alpha^{(n)} = \inf \{x \mid F_n(x) \geq \alpha\} \quad .$$

Then

(5) $$F_n(\xi_\alpha^{(n)} - 0) \leq \alpha \leq F_n(\xi_\alpha^{(n)}).$$

<u>Theorem 2</u>. Suppose that the distribution function G is continuous on the whole real line and strictly increasing on $\{x \mid 0 < G(x) < 1\}$. If $\{F_n\} \in D(G)$ then

(6) $$F_n(a_n x + b_n) \to G(ax + b)$$

weakly (hence for all x), with

(7) $$b_n = \xi_\alpha^{(n)} \quad , \quad b = G^{-1}(\alpha)$$
$$a_n = \xi_\beta^{(n)} - \xi_\alpha^{(n)} \ , \qquad a = G^{-1}(\beta) - b$$

and α and β arbitrary (provided $0 < \alpha < \beta < 1$).

<u>Proof</u>.

Given

(8) $$F_n(a_n' x + b_n') \to G(x)$$

weakly with sequences $a_n' > 0$ end b_n' , for every pair of positive numbers ε_1 and ε_2 there is a positive integer n_0 such that for $n \geq n_0$

(9) $$F_n(a_n' (b - \varepsilon_1) + b_n') < G(b - \varepsilon_1) + \varepsilon_2$$
$$F_n(a_n' (b + \varepsilon_1) + b_n') > G(b + \varepsilon_1) - \varepsilon_2.$$

Choosing

$$\varepsilon_2 = \min \{G(b) - G(b - \varepsilon_1), G(b + \varepsilon_1) - G(b)\} > 0,$$

we have

(10) $$F_n(a_n'(b - \varepsilon_1) + b_n') < F_n(a_n'(b + \varepsilon_1) + b_n').$$

From (5) it follows in view of the continuity of G

(11) $$F_n(b_n - 0) \leq G(b) \leq F_n(b_n) \qquad n = 1,2,3,\ldots \ .$$

Combining (10) and (11) we obtain

$$a_n'(b - \varepsilon_1) + b_n' < b_n \leq a_n'(b + \varepsilon_1) + b_n' \qquad n = 1,2,3, \ldots ,$$

so

(12) $$\frac{b_n - b_n'}{a_n'} \longrightarrow b \qquad n \to \infty$$

Starting in (9) with a+b instead of b we obtain

(13) $$\frac{a_n}{a_n'} + \frac{b_n - b_n'}{a_n'} \to a+b \quad n \to \infty.$$

Application of theorem 1 gives the statement of the theorem.

Remark. A slight adaptation of the proof shows that the requirements on G can be weakened to the following ones:

a. $\xi_\alpha < \xi_\beta$; here $\xi_\alpha = \inf < x \mid G(x) \geq \alpha \}$.
b. There exists no $\varepsilon > 0$ such that G is constant on $[\xi_\alpha, \xi_\alpha + \varepsilon)$ and $[\xi_\beta, \xi_\beta + \varepsilon)$.

As an application we prove Gnedenko's theorem about choosing stabilizing constants for sequences $F_n = F^n$ attracted by the double-exponential law ([2] p.446).

Corollary 1. If for a distribution function F the sequence $\{F^n\}$ is in the domain of

(14) $$G(x) = \exp\{-e^{-x}\}$$

then

(15) $$F^n(a_n x + b_n) \to G(x) \qquad \text{for x all x, } n \to \infty$$

with

(16) $$b_n = \inf \{x \mid F(x) \geq 1 - \tfrac{1}{n}\}$$

and $$a_n = \inf \{x \mid F(x) \geq 1 - \tfrac{1}{ne}\} - b_n \qquad n = 1,2,3, \ldots$$

Proof. It is not difficult to see that theorem 2 holds with

(17) $$b_n = \xi_{\alpha_n}^{(n)}$$

and

$$a_n = \xi_{\beta_n}^{(n)} - b_n \quad ,$$

when $\alpha_n \longrightarrow \alpha$ and $\beta_n \longrightarrow \beta$ $(n \to \infty)$. Applying the adapted theorem 2 with

$$\alpha_n = (1 - \frac{1}{n})^n \text{ and } \beta_n = (1 - \frac{1}{ne})^n$$

we obtain (16).

Remark. Obviously (15) and (16) are also true for the two other possible types of limit laws G.

Another application concerns stabilization by moments.

Corollary 2. If

(18) $$F_n(\sigma_n x + b_n) \longrightarrow G(x)$$

for all x, where G is one-to-one and b_n arbitrary, then there exists a B such that

(19) $$F_n(\sigma_n x + \mu_n) \longrightarrow G(x + B).$$

Proof. Let $\{X_n\}$ be random variables with distribution functions $\{F_n\}$ and

(20)

$$Y_n = \frac{X_n - \xi_\alpha^{(n)}}{\xi_\beta^{(n)} - \xi_\alpha^{(n)}} \qquad (n = 1,2,3, \ldots).$$

Then

(21) $$\mu(Y_n) = \frac{\mu_n - \xi_\alpha^{(n)}}{\xi_\beta^{(n)} - \xi_\alpha^{(n)}} \quad \text{and} \quad \sigma(Y_n) = \frac{\sigma_n}{\xi_\beta^{(n)} - \xi_\alpha^{(n)}} \quad .$$

From (18) and theorem 2 it follows that for some positive A

$$(22) \qquad \sigma(Y_n) \longrightarrow A.$$

The inequality

$$-(1-\alpha)^{-\frac{1}{2}} \leq \frac{\mu_n - \xi_\alpha^{(n)}}{\sigma_n} \leq \alpha^{-\frac{1}{2}}$$

(see [3] p.244) gives that the sequence $\{\mu(Y_n)\}$ is bounded so that (from (22)) the sequence $\{E\, Y_n^2\}$ is bounded; hence the sequence

$$\mu(Y_n) = \frac{\mu_n - \xi_\beta^{(n)}}{\xi_\alpha^{(n)} - \xi_\beta^{(n)}}$$

converges. Application of theorem 1 gives (19).

2. Weak law of large numbers.

A sequence of distribution functions $\{F_n\}$ is said to satisfy the weak law of large numbers if there is a sequence of real numbers $\{b_n\}$ such that

$$(23) \qquad F_n(x + b_n) \longrightarrow \epsilon(x)$$

for $x \neq 0$.

Theorem 3. For a sequence of distribution functions $\{F_n\}$ the following propositions are equivalent:

a. The sequence $\{F_n\}$ satisfies the weak law of large numbers.

b. For each α $(0 < \alpha < 1)$

$$(24) \qquad F_n(x + \xi_\alpha^{(n)}) \longrightarrow \epsilon(x)$$

for $x \neq 0$.

c. For each α and β $(0 < \alpha < \beta < 1)$

(25) $$\lim_{n\to\infty} (\xi_\beta^{(n)} - \xi_\alpha^{(n)}) = 0$$

<u>Proof.</u>

b $\Longrightarrow$ a: Trivial.

a $\Longrightarrow$ b: Choose α $(0 < \alpha < 1)$ and ε $(\varepsilon > 0)$; from

(26) $$\lim_{n\to\infty} F_n(x + b_n) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0 \end{cases}$$

it follows that when $n \geq n_0$

$$F_n(-\varepsilon + b_n) < c < F_n(\varepsilon + b_n).$$

Applying (5) we have

(27) $$b_n - \varepsilon < \xi_\alpha^{(n)} < b_n + \varepsilon,$$

hence for each x

$$F_n(x - \varepsilon + b_n) \leq F_n(x + \xi_\alpha^{(n)}) \leq F_n(x + \varepsilon + b_n).$$

From this and (26) we obtain (24).

a,b $\Longrightarrow$ c: relation (27) gives

$$\lim_{n\to\infty} \{\xi_\alpha^{(n)} - b_n\} = 0.$$

As α is arbitrary we have (25).

c $\Longrightarrow$ b: Choose α $(0 < \alpha < 1)$, $x > 0$ and $\varepsilon > 0$ arbitrary and ε_1 such that $0 < \varepsilon_1 < x$.

Relation (25) implies for $n \geq n_0$

$$\xi_{1-\varepsilon}^{(n)} - \varepsilon_1 < \xi_\alpha^{(n)} \quad ,$$

hence by (5)

$$1 - \varepsilon < F_n(\xi_{1-\varepsilon}^{(n)}) \le F_n(x - \varepsilon_1 + \xi_{1-\varepsilon}^{(n)}) \le F_n(x + \xi_\alpha^{(n)}) \le 1,$$

so for $x > 0$

$$\lim_{n\to\infty} F_n(x + \xi_\alpha^{(n)}) = 1.$$

Analogously one proves (24) for $x < 0$.

By a simple transformation we can restate the results of theorem 3 as conditions for a sequence of distribution functions $\{F_n\}$ concentrated on the non-negative half-axis which is relatively stable i.e. for which

$$F_n(a_n x) \longrightarrow \iota(x - 1)$$

weakly for suitably chosen positive constants $\{a_n\}$.

<u>Theorem 4</u>. For a sequence of distribution functions $\{F_n\}$ with $F_n(0-) = 0$ for $n = 1,2,3, \ldots$ the following propositions are equivalent:

a. The sequence $\{F_n\}$ is relatively stable.

b. For each α $(0 < \alpha < 1)$

$$F_n(x.\ \xi_\alpha^{(n)}) \longrightarrow \iota(x - 1)$$

for $x \neq 1$.

c. For each α and β $(0 < \alpha < \beta < 1)$

$$\lim_{n\to\infty} \frac{\xi_\beta^{(n)}}{\xi_\alpha^{(n)}} = 1.$$

<u>Proof</u>. A sequence $\{F_n\}$ satisfies the conditions of theorem 4 iff the sequence $\{G_n\}$ defined by

$$G_n(x) = F_n(e^x)$$

satisfies the conditions of theorem 3.

As in section 1 the results of Gnedenko ([2] p.426) concerning the law of large numbers and the relative stability of the sequence of maxima of independent identically distributed random variables can be seen as corollaries to the theorems 3 and 4.

References

[1] Feller, W. (1966). An introduction to probability theory and its applications 2. Wiley, New-York.

[2] Gnedenko, B.V. (1943). Sur la distribution limite du terme maximum d'une série aléatoire. Annals of Math. 44 423-453.

[3] Loève, M. (1963). Probability theory. van Nostrand, New-York.