New convolutions and their applicability to integral equations of Wiener-Hopf plus Hankel type

L. P. Castro*[1] | R. C. Guerra[1] | N. M. Tuan[2]

[1]Center for Research and Development in Mathematics and Applications (CIDMA), Department of Mathematics, University of Aveiro, Aveiro, Portugal

[2]Department of Mathematics, College of Education, Viet Nam National University, Hanoi, Viet Nam

Correspondence
*L. P. Castro. Email: castro@ua.pt

Summary

We propose four new convolutions exhibiting convenient factorization properties associated with two finite interval integral transformations of Fourier-type together with their norm inequalities. Moreover, we study the solvability of a class of integral equations of Wiener-Hopf plus Hankel type (on finite intervals) with the help of the factorization identities of such convolutions. Fourier-type series are used to produce the solution formula of such equations and a Shannon-type sampling formula is also obtained.

KEYWORDS:
Convolution, integral equation, factorization, Fourier integral operator, Wiener-Hopf operator, Hankel operator.

1 | INTRODUCTION AND PRELIMINARIES

Introducing new convolutions has a direct impact in the areas of Integral Equations and Operator Theory, and different convolutions are as much important as many properties and applications they will be able to exhibit[1,2,3,4,5]. Although the study of convolutions can also be seen as a classical subarea of Mathematical Analysis, the reason why it continues to attract the interest of several researchers is due to its high potential in exhibiting new properties and concrete applications. In fact, such convolutions can be seen as integral transforms that we can study from a theoretical point of view[6,7,8]. Moreover, they can also be applied to problems modelled by certain integral equations, as we can see in different contexts[9].

In the present paper, we will propose four new convolutions which will be written based on two parameters. Such parameters will enable these convolutions with extra levels of flexibility that are useful to enlarge the number of properties of their associated integral operators, as well as the number of possible applications (e.g., some classes of wave diffraction problems formulated as boundary value problems[10,11,12,13,14] can be analyzed and solved with the help of convolution operators of Wiener-Hopf plus Hankel type).

We will apply those new convolutions to the solvability analysis of a large class of integral equations. Such class of integral equations have brought about a great number of results in Mathematical Analysis, assembled in a huge number of works along the last decades[15,2,9,16,17,18,19,20,21]. We are speaking about the class of Wiener-Hopf plus Hankel integral equations. Namely, in the present work, we will obtain the L^2-solution of a Wiener-Hopf plus Hankel equation, in terms of a Fourier-type series[22]. For

that, we identify two finite integral transforms of Fourier-type which are associated with the set of our four new convolutions. The resulting realization will be the solution of that class of integral equations in terms of a Fourier-type series.

Then we will briefly present the main machinery that will be in use. We begin with the basic definition of the Fourier series.

Definition 1 (see [7,8]). Let f be a Lebesgue integrable function on a finite interval $[0, 2\pi]$. The infinite sum

$$(F_f)(x) := \frac{\hat{f}_c(0)}{2} + \sum_{n=1}^{\infty} \left[\hat{f}_c(n) \cos(nx) + \hat{f}_s(n) \sin(nx) \right] \tag{1}$$

is called the Fourier series of f on $[0, 2\pi]$, where $\hat{f}_c(n)$ and $\hat{f}_s(n)$ are the Fourier coefficients of f defined by

$$\hat{f}_c(n) = \frac{1}{\pi} \int_0^{2\pi} \cos(nx) f(x) dx, \ n \in \mathbb{N}, \tag{2}$$

$$\hat{f}_s(n) = \frac{1}{\pi} \int_0^{2\pi} \sin(nx) f(x) dx, \ n \in \mathbb{N}. \tag{3}$$

In the framework of this paper, the convergence is of the Lebesgue space $L^2([0, 2\pi])$ [7,23,8]. Let us write

$$\mathscr{E}(x) := \frac{b \cos(x) + c \sin(x)}{\sqrt{(|b|^2 + |c|^2)\pi}}, \tag{4}$$

with $bc \neq 0$. The series (1) may be rewritten as follows

$$(F_f)(x) = \frac{\hat{f}_c(0)}{2} + \sum_{n=1}^{\infty} \left\{ \frac{\sqrt{(|b|^2 + |c|^2)\pi}}{2} \left(\frac{1}{b} \hat{f}_c + \frac{1}{c} \hat{f}_s \right)(n) \mathscr{E}(nx) \right.$$

$$\left. + \frac{\sqrt{(|b|^2 + |c|^2)\pi}}{2} \left(\frac{1}{b} \hat{f}_c - \frac{1}{c} \hat{f}_s \right)(n) \mathscr{E}(-nx) \right\} = \frac{\hat{f}_c(0)}{2} + \sum_{0 \neq n=-\infty}^{+\infty} a_n \mathscr{E}(nx), \tag{5}$$

where

$$a_n = \begin{cases} \frac{\sqrt{(|b|^2 + |c|^2)\pi}}{2} \left(\frac{1}{b} \hat{f}_c(n) - \frac{1}{c} \hat{f}_s \right)(n) & \text{if} \quad n < 0 \\ \frac{\sqrt{(|b|^2 + |c|^2)\pi}}{2b} \hat{f}_c(n) & \text{if} \quad n = 0 \\ \frac{\sqrt{(|b|^2 + |c|^2)\pi}}{2} \left(\frac{1}{b} \hat{f}_c(n) + \frac{1}{c} \hat{f}_s \right)(n) & \text{if} \quad n > 0. \end{cases} \tag{6}$$

We will be using the usual inner product in $L^2([0, 2\pi])$. Namely, for any f and g in $L^2([0, 2\pi])$ their inner product is given by

$$\langle f, g \rangle_2 = \int_0^{2\pi} f(x) \overline{g}(x) \, dx. \tag{7}$$

The expression (5) leads us to the following result, which is also a consequence of the properties of the complete orthonormal basis $\left\{ e^{inx} \right\}_{n \in \mathbb{Z}}$ of the Hilbert space $L^2([0, 2\pi])$.

Proposition 1. The set of functions

$$\left\{ \frac{1}{\sqrt{2\pi}}, \mathscr{E}(nx), n \in \mathbb{Z} \backslash \{0\} \right\} \tag{8}$$

is a complete orthonormal basis of $L^2([0, 2\pi])$.

Proof. By (1) and (5), it is sufficient to prove the orthogonality of the system $\{\mathscr{E}(nx)\}_{n \in \mathbb{Z}}$. In view of this, we compute

$$\langle \mathscr{E}(nx), \mathscr{E}(mx) \rangle_2 = \frac{1}{(|b|^2 + |c|^2)\pi} \int_0^{2\pi} (b \cos(nx) + c \sin(nx))(\overline{b} \cos(mx) + \overline{c} \sin(mx)) \, dx$$

$$= \begin{cases} 0, & n \neq m \\ 1, & n = m, \end{cases}$$

for $n, m \neq 0$.

For $n = m = 0$, we have

$$\langle \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}} \rangle_2 = \frac{1}{2\pi} \int_0^{2\pi} 1 \, dx = 1.$$

For $n = 0$ and $m \neq 0$, we have

$$\langle \frac{1}{\sqrt{2\pi}}, \mathscr{E}(mx) \rangle_2 = \frac{1}{\pi \sqrt{2(|b|^2 + |c|^2)}} \int_0^{2\pi} (\bar{b} \cos(mx) + \bar{c} \sin(mx)) \, dx = 0.$$

□

In what follows, the notation $(F_f)(x)$ of the Fourier series will be replaced by $\mathcal{E}_f(x)$ given by (5). Furthermore, the representation in the form (5) naturally suggests us to define a finite Fourier-type transformation as follows.

Definition 2 (Finite two parameters Fourier-type transformation). (a) Let f be a Lebesgue integrable function on $[0, 2\pi]$. The finite (two parameters) Fourier-type transformation of f corresponding to $\mathscr{E}(nx)$ is defined by

$$(T_1 f)(n) := \frac{1}{2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left(\frac{1}{b} \cos(nx) + \frac{1}{c} \sin(nx) \right) f(x) dx, \ n \in \mathbb{Z}. \tag{9}$$

(b) The infinite sum

$$(\mathcal{E}_f)(x) := \sum_{n \in \mathbb{Z}} (T_1 f)(n) \mathscr{E}(nx) \tag{10}$$

is called finite Fourier-type series of f on $[0, 2\pi]$ corresponding to $\mathscr{E}(nx)$.

We observe that the constant before the integral sign in (9) is chosen for later purposes.

Let us recall that

$$\ell_2 := \left\{ \{s_n\}_n : s_n \in \mathbb{C} \text{ with } \sum_{n \in \mathbb{Z}} |s_n|^2 < \infty \right\},$$

stands for the Hilbert space of all square summable complex series.

Let us also recall the known Riesz-Fischer theorem, which states that a measurable function on $[-\pi, \pi]$ is square integrable if and only if the corresponding Fourier series converges (in the space L^2). In other words, for any $f \in L^2([-\pi, \pi])$ and its n-th Fourier coefficient given by

$$F_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx \quad (n \in \mathbb{Z}), \tag{11}$$

we have

$$f(x) = \sum_{n \in \mathbb{Z}} F_n e^{inx} \quad \text{for almost every } x \in (-\pi, \pi); \tag{12}$$

conversely, for any $\{a_n\}_{n=-\infty}^{n=+\infty} \in \ell_2$, there exists a function $f \in L^2([-\pi, \pi])$ such that the values a_n are the Fourier coefficients of f. In view of this, the integral transform (11) defines an invertible bounded linear operator between two Hilbert spaces $L^2([-\pi, \pi])$ and ℓ_2 in which the inversion formula is given by (12).

Remark 1. The above consideration of T_1 is natural for the extension (10), as $\frac{1}{b} \cos(nx) + \frac{1}{c} \sin(nx)$ is the inversion kernel of the basis $b \cos(nx) + c \sin(nx) = \sqrt{(|b|^2 + |c|^2)\pi} \mathscr{E}(nx)$.

Put

$$T_1 : L^2([0, 2\pi]) \longrightarrow \quad \ell_2$$
$$f \quad \longmapsto \quad \{T_1 f\}_{n \in \mathbb{Z}}.$$

By (5), (9), and (10) we can state the following theorem.

Theorem 1. The integral transform (9) defines a bounded linear map from $L^2(-\pi, \pi)$ to ℓ_2 whose inversion map is given by (10).

The following propositions are immediate consequences of the finite Fourier integral transformation case.

Proposition 2 (Uniqueness theorem)**.** If $f \in L^1([0, 2\pi])$, with $(T_1 f)(n) = 0$, for all $n \in \mathbb{Z}$, then $f \equiv 0$ (in L^1-norm).

Proposition 3 (Riemann-Lebesgue lemma)**.** If $f \in L^1([0, 2\pi])$, then

$$\lim_{n \to \infty} (T_1 f)(n) = 0.$$

We will also consider

$$(T_2 f)(n) := \frac{1}{2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left(\frac{1}{b} \cos(nx) - \frac{1}{c} \sin(nx) \right) f(x) dx, \ n \in \mathbb{Z}.$$

Since $T_2(n) = T_1(-n)$ for every $n \in \mathbb{Z}$, the above propositions also obviously work for the integral transform T_2. Due to the circumstance that $T_2(n) = T_1(-n)$, we can also say that T_2 corresponds to the composition of T_1 with the reflection operator in frequency domain.

2 | FOUR NEW CONVOLUTIONS AND NORM INEQUALITIES

In this section, we construct four new generalized convolutions associated with T_1 and T_2, and prove some norm inequalities.

Definition 3. For any two functions f, g that are Lebesgue integrable on $[0, 2\pi]$, and where f is a function defined on $\mathbb{R}$ and 2π-periodic, we define four (two parameter) finite Fourier-type convolutions as follows:

$$(f \underset{T_1, T_1, T_1}{*} g)(x) := \frac{1}{8bc^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left[(3c^2 - b^2) f(x - v) + (b^2 + c^2) f(x + v) \right.$$
$$\left. + (b^2 + c^2) f(-x + v) - (b^2 + c^2) f(-x - v) \right] g(v) \, dv; \tag{13}$$

$$(f \underset{T_1, T_1, T_2}{*} g)(x) := \frac{1}{8bc^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left[(b^2 + c^2) f(x - v) + (3c^2 - b^2) f(x + v) \right.$$
$$\left. - (b^2 + c^2) f(-x + v) + (b^2 + c^2) f(-x - v) \right] g(v) \, dv; \tag{14}$$

$$(f \underset{T_1, T_2, T_1}{*} g)(x) := \frac{1}{8bc^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left[(b^2 + c^2) f(x - v) - (b^2 + c^2) f(x + v) \right.$$
$$\left. + (3c^2 - b^2) f(-x + v) + (b^2 + c^2) f(-x - v) \right] g(v) \, dv; \tag{15}$$

$$(f \underset{T_1, T_2, T_2}{*} g)(x) := \frac{1}{8bc^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left[-(b^2 + c^2) f(x - v) + (b^2 + c^2) f(x + v) \right.$$
$$\left. + (b^2 + c^2) f(-x + v) + (3c^2 - b^2) f(-x - v) \right] g(v) \, dv. \tag{16}$$

The next theorem clarifies the relation of those convolutions with the previously considered operators.

Theorem 2 (Convolution theorem)**.** If f, g are Lebesgue integrable on $[0, 2\pi]$ and f is a function defined on $\mathbb{R}$ and 2π-periodic, then each one of the convolutions introduced in Definition 3 has two factorization identities associated with the integral operators T_1 and T_2, and the corresponding norm inequality:

$$(T_j(f \underset{T_1, T_1, T_1}{*} g))(n) = (T_j f)(n)(T_j g)(n), \ for \ n \in \mathbb{Z},$$

$$\|f \underset{T_1, T_1, T_1}{*} g\|_1 \leq \frac{2|b|^2 + 3|c|^2}{4|b||c|^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \|f\|_1 \|g\|_1; \tag{17}$$

$$(T_j(f \underset{T_1, T_1, T_2}{*} g))(n) = (T_j f)(x)(T_k g)(n), \ for \ n \in \mathbb{Z},$$

$$\|f \underset{T_1, T_1, T_2}{*} g\|_1 \leq \frac{2|b|^2 + 3|c|^2}{4|b||c|^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \|f\|_1 \|g\|_1; \tag{18}$$

$$(T_j(f \underset{T_1, T_2, T_1}{*} g))(n) = (T_k f)(x)(T_j g)(n), \ for \ n \in \mathbb{Z},$$

$$\|f \underset{T_1,T_2,T_1}{*} g\|_1 \le \frac{2|b|^2 + 3|c|^2}{4|b||c|^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \|f\|_1 \|g\|_1;\tag{19}$$

$$(T_j(f \underset{T_1,T_2,T_2}{*} g))(n) = (T_k f)(x)(T_k g)(n), \quad for \ \ n \in \mathbb{Z},$$

$$\|f \underset{T_1,T_2,T_2}{*} g\|_1 \le \frac{2|b|^2 + 3|c|^2}{4|b||c|^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \|f\|_1 \|g\|_1,\tag{20}$$

with $j = 1$ and $k = 2$, or $j = 2$ and $k = 1$.

Proof. We first prove the factorization identity (17). In this case, the factorization identity just includes one of the operators, T_1 or T_2, and we will prove it for $j = 1$, this is, for the operator T_1. So, considering the kernel of the operator T_1, we can rewrite the product of two kernels of this type as a linear combination of four terms of the same type in suitable variables, as follows

$$\left[\frac{1}{b}\cos(nu) + \frac{1}{c}\sin(nu)\right]\left[\frac{1}{b}\cos(nv) + \frac{1}{c}\sin(nv)\right]$$

$$= \frac{1}{4bc^2}\left\{(3c^2 - b^2)\left[\frac{1}{b}\cos(n(u+v)) + \frac{1}{c}\sin(n(u+v))\right]\right.$$

$$+ (b^2 + c^2)\left[\frac{1}{b}\cos(n(u-v)) + \frac{1}{c}\sin(n(u-v))\right]$$

$$+ (b^2 + c^2)\left[\frac{1}{b}\cos(n(-u+v)) + \frac{1}{c}\sin(n(-u+v))\right]$$

$$\left. - (b^2 + c^2)\left[\frac{1}{b}\cos(n(-u-v)) + \frac{1}{c}\sin(n(-u-v))\right]\right\}.\tag{21}$$

Using this identity, changing appropriately the variables and noting the 2π-periodicity of f, we have

$$(T_1 f)(n)(T_1 g)(n) = \frac{|b|^2 + |c|^2}{4\pi} \int_0^{2\pi}\int_0^{2\pi} \left[\frac{1}{b}\cos(nu) + \frac{1}{c}\sin(nu)\right]$$

$$\left[\frac{1}{b}\cos(nv) + \frac{1}{c}\sin(nv)\right] f(u)g(v)du\,dv$$

$$= \frac{|b|^2 + |c|^2}{4bc^2(4\pi)} \int_0^{2\pi}\int_0^{2\pi} \left\{(3c^2 - b^2)\left[\frac{1}{b}\cos(n(u+v)) + \frac{1}{c}\sin(n(u+v))\right]\right.$$

$$+ (b^2 + c^2)\left[\frac{1}{b}\cos(n(u-v)) + \frac{1}{c}\sin(n(u-v))\right]$$

$$+ (b^2 + c^2)\left[\frac{1}{b}\cos(n(-u+v)) + \frac{1}{c}\sin(n(-u+v))\right]$$

$$\left. - (b^2 + c^2)\left[\frac{1}{b}\cos(n(-u-v)) + \frac{1}{c}\sin(n(-u-v))\right]\right\} f(u)g(v)du\,dv$$

$$= \frac{|b|^2 + |c|^2}{4bc^2(4\pi)} \int_0^{2\pi} g(v)\left\{\int_v^{2\pi+v}(3c^2 - b^2)\left[\frac{1}{b}\cos(nx) + \frac{1}{c}\sin(nx)\right] f(x-v)dx\right.$$

$$+ \int_{-v}^{2\pi-v}(b^2 + c^2)\left[\frac{1}{b}\cos(nx) + \frac{1}{c}\sin(nx)\right] f(x+v)dx$$

$$- \int_v^{-2\pi+v}(b^2 + c^2)\left[\frac{1}{b}\cos(nx) + \frac{1}{c}\sin(nx)\right] f(-x+v)dx$$

$$\left. + \int_{-v}^{-2\pi-v}(b^2 + c^2)\left[\frac{1}{b}\cos(nx) + \frac{1}{c}\sin(nx)\right] f(-x-v)dx\right\} dv$$

$$= \frac{|b|^2 + |c|^2}{4bc^2(4\pi)} \int_0^{2\pi} g(v) \left\{ \int_v^{2\pi+v} (3c^2 - b^2) \left[\frac{1}{b} \cos(nx) + \frac{1}{c} \sin(nx) \right] f(x - v) dx \right.$$

$$+ \int_{-v}^{2\pi-v} (b^2 + c^2) \left[\frac{1}{b} \cos(nx) + \frac{1}{c} \sin(nx) \right] f(x + v) dx$$

$$+ \int_{-2\pi+v}^{v} (b^2 + c^2) \left[\frac{1}{b} \cos(nx) + \frac{1}{c} \sin(nx) \right] f(-x + v) dx$$

$$\left. - \int_{-2\pi-v}^{-v} (b^2 + c^2) \left[\frac{1}{b} \cos(nx) + \frac{1}{c} \sin(nx) \right] f(-x - v) dx \right\} dv$$

$$= \frac{1}{2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left[\frac{1}{b} \cos(nx) + \frac{1}{c} \sin(nx) \right]$$

$$\left\{ \frac{1}{8bc^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \left[(3c^2 - b^2) f(x - v) + (b^2 + c^2) f(x + v) \right. \right.$$

$$\left. + (b^2 + c^2) f(-x + v) - (b^2 + c^2) f(-x - v) \right] g(v) dv \right\} dx$$

$$= T_1 (f \underset{T_1, T_1, T_1}{*} g)(n).$$

Now, let us prove the corresponding norm inequality

$$\int_0^{2\pi} \left| (f \underset{T_1, T_1, T_1}{*} g)(x) \right| dx \leq \frac{1}{8|b||c|^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \int_0^{2\pi} \left[(3|c|^2 + |b|^2) |f(x - v)| \right.$$

$$+ (|b|^2 + |c|^2) |f(x + v)| + (|b|^2 + |c|^2) |f(-x + v)|$$

$$\left. + (|b|^2 + |c|^2) |f(-x - v)| \right] |g(v)| \, dv \, dx$$

$$= \frac{4|b|^2 + 6|c|^2}{8|b||c|^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \int_0^{2\pi} \int_0^{2\pi} |f(y)||g(u)| \, dv \, dy$$

$$= \frac{2|b|^2 + 3|c|^2}{4|b||c|^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \|f\|_1 \|g\|_1.$$

We will omit the proofs for the cases of the other convolutions, because they are similar to this one, using, of course, the corresponding different identities. The following identities help us to obtain the factorizations for the other convolutions, when we consider $j = 1$ and $k = 2$, through the decomposition presented for the factorization (17). Namely, we have:

$$\left[\frac{1}{b} \cos(nu) + \frac{1}{c} \sin(nu) \right] \left[\frac{1}{b} \cos(nv) - \frac{1}{c} \sin(nv) \right]$$

$$= \left[\frac{1}{b} \cos(nu) + \frac{1}{c} \sin(nu) \right] \left[\frac{1}{b} \cos(-nv) + \frac{1}{c} \sin(-nv) \right]; \tag{22}$$

$$\left[\frac{1}{b} \cos(nu) - \frac{1}{c} \sin(nu) \right] \left[\frac{1}{b} \cos(nv) + \frac{1}{c} \sin(nv) \right]$$

$$= \left[\frac{1}{b} \cos(-nu) + \frac{1}{c} \sin(-nu) \right] \left[\frac{1}{b} \cos(nv) + \frac{1}{c} \sin(nv) \right]; \tag{23}$$

$$\left[\frac{1}{b} \cos(nu) - \frac{1}{c} \sin(nu) \right] \left[\frac{1}{b} \cos(nv) - \frac{1}{c} \sin(nv) \right]$$

$$= \left[\frac{1}{b} \cos(-nu) + \frac{1}{c} \sin(-nu) \right] \left[\frac{1}{b} \cos(-nv) + \frac{1}{c} \sin(-nv) \right]. \tag{24}$$

We will also omit the proofs of the factorization identities when we consider $j = 2$ and $k = 1$, since they are very similar to the proofs of the factorization identities with $j = 1$ and $k = 2$. In fact, after proving the factorization identity for the first convolution, with $j = 2$, we can use the following identities to obtain the corresponding decompositions associated with the other convolutions, when considering $j = 2$ and $k = 1$. Namely, we have:

$$\left[\frac{1}{b}\cos(nu) - \frac{1}{c}\sin(nu)\right]\left[\frac{1}{b}\cos(nv) + \frac{1}{c}\sin(nv)\right]$$
$$= \left[\frac{1}{b}\cos(nu) - \frac{1}{c}\sin(nu)\right]\left[\frac{1}{b}\cos(nv) - \frac{1}{c}\sin(-nv)\right];$$
$$\left[\frac{1}{b}\cos(nu) + \frac{1}{c}\sin(nu)\right]\left[\frac{1}{b}\cos(nv) - \frac{1}{c}\sin(nv)\right]$$
$$= \left[\frac{1}{b}\cos(nu) - \frac{1}{c}\sin(-nu)\right]\left[\frac{1}{b}\cos(nv) - \frac{1}{c}\sin(nv)\right];$$
$$\left[\frac{1}{b}\cos(nu) + \frac{1}{c}\sin(nu)\right]\left[\frac{1}{b}\cos(nv) + \frac{1}{c}\sin(nv)\right]$$
$$= \left[\frac{1}{b}\cos(nu) - \frac{1}{c}\sin(-nu)\right]\left[\frac{1}{b}\cos(nv) - \frac{1}{c}\sin(-nv)\right].$$

$\square$

Remark 2. Looking at the four convolutions, we see that:

(i) If $b^2 + c^2 = 0$, for instance $b = 1$ and $c = i$, T_1 and T_2 become the finite Fourier and inverse Fourier transformations and all the constructed convolutions are unified in the unique well-known Fourier convolution.

(ii) If $3c^2 - b^2 = b^2 + c^2$ or, equivalently, $b = \pm c$, then the considered transforms are exactly the finite Hartley transformations, and all the mentioned convolutions turn out the recently published Hartley convolutions[2,24].

(iii) If $3c^2 - b^2 = 0$, each convolution has just three terms. For instance, if $b = \sqrt{3}$ and $c = 1$ then T_1 and T_2 are symmetric (Hermitian) transformations, and the convolution (13) yields

$$(f \underset{T_1,T_1,T_1}{*} g)(x) := \frac{1}{\sqrt{3\pi}} \int_0^{2\pi} [f(x + v) + f(-x + v) - f(-x - v)] g(v)\, dv. \tag{25}$$

For convenience, let us write $\widetilde{f}(t) := f(-t)$ which corresponds to the reflection of the function f in time domain. It is surprising that the convolution (25) does not contain the so-called *convolution term*. Actually, it is constituted by three terms: two cross-correlation terms, that are $f(x + v)$ and $\widetilde{f}(x + v)$, together with the convolution term of the reflection $\widetilde{f}$, that is $f(-x + v)$. The form of this convolution is notably different from the previously published convolutions. Of course, the three other convolutions (14), (15), and (16) have the usual convolution term $f(x - v)$. From our point of view, the convolution (25) may be used for processing reflective periodic signals in practical problems.

In summary, the well-known Fourier and Hartley convolutions, and the convolution given by (25) can be considered as special cases among the studied family of convolutions, in some sense.

As we will see in Section 3, the factorization identities in Theorem 2 are essential for solving Fredholm integral equations with mixed kernels. Before going to Section 3 we should confirm the L^2-boundedness of the constructed convolution type operators.

Due to the fact of $L^2([0, 2\pi]) \subset L^1([0, 2\pi])$, we can state the following theorem.

Theorem 3. Let f be a function defined on $\mathbb{R}$ and 2π-periodic. If $f, g \in L^2([0, 2\pi])$, then the following norm inequality holds

$$\|f \underset{T}{*} g\|_2 \leq \frac{\sqrt{2(|b|^2 + |c|^2)(|3c^2 - b^2|^2 + 3|b^2 + c^2|^2)}}{4|b||c|^2}\|f\|_2\|g\|_2, \tag{26}$$

where, for short, the notation $\underset{T}{*}$ represents any of the convolution type operators (13)–(16).

Proof. We will just prove the theorem for the convolution (13), since the other ones can be proceeded in a similar way. By the Cauchy-Schwarz inequality, we have $|a + b + c + d|^2 \leq 4(|a|^2 + |b|^2 + |c|^2 + |d|^2)$ and so,

$$
\left| (f \underset{T_1,T_1,T_1}{*} g)(x) \right|^2 = \left| \frac{1}{8bc^2} \sqrt{\frac{|b|^2 + |c|^2}{\pi}} \right|^2 \left| \int_0^{2\pi} \left[(3c^2 - b^2)f(x - v) + (b^2 + c^2)f(x + v) \right. \right.
$$

$$
\left. \left. + (b^2 + c^2)f(-x + v) - (b^2 + c^2)f(-x - v) \right] g(v)dv \right|^2
$$

$$
\leq \frac{|b|^2 + |c|^2}{64|b|^2|c|^4\pi} \int_0^{2\pi} |g(v)|^2 dv \int_0^{2\pi} \left| (3c^2 - b^2)f(x - v) + (b^2 + c^2)f(x + v) \right.
$$

$$
\left. + (b^2 + c^2)f(-x + v) - (b^2 + c^2)f(-x - v) \right|^2 dv
$$

$$
\leq \frac{|b|^2 + |c|^2}{64|b|^2|c|^4\pi} \int_0^{2\pi} |g(v)|^2 dv \int_0^{2\pi} 4 \left[|3c^2 - b^2|^2|f(x - v)|^2 + |b^2 + c^2|^2|f(x + v)|^2 \right.
$$

$$
\left. + |b^2 + c^2|^2|f(-x + v)|^2 + |b^2 + c^2|^2|f(-x - v)|^2 \right] dv
$$

$$
\leq \frac{4(|b|^2 + |c|^2)(|3c^2 - b^2|^2 + 3|b^2 + c^2|^2)}{64|b|^2|c|^4\pi} \|f\|_2^2 \|g\|_2^2.
$$

This implies that

$$
\|f \underset{T_1,T_1,T_1}{*} g\|_2 = \left(\int_0^{2\pi} \left| (f \underset{T_1,T_1,T_1}{*} g)(x) \right|^2 dx \right)^{\frac{1}{2}} \leq \frac{\sqrt{2(|b|^2 + |c|^2)(|3c^2 - b^2|^2 + 3|b^2 + c^2|^2)}}{4|b||c|^2} \|f\|_2 \|g\|_2,
$$

which proves the proposition. □

3 | ON THE SOLVABILITY OF WIENER-HOPF PLUS HANKEL EQUATIONS

In this section, we will study the solvability of the following Fredholm integral equation

$$
\lambda\varphi(x) + \int_0^{2\pi} K(x, u)\varphi(u) \, du = f(x), \tag{27}
$$

where φ is unknown, $\lambda \in \mathbb{C}$, $K(x, y) := p(x - u) + q(x + u)$, with p, q and f being given functions in which p and q are 2π-periodic one-variable functions.

We will use the above introduced convolutions to analyze the solvability of this equation, and to obtain the explicit solution in the form of a Fourier-type series.

We start by introducing some auxiliary functions. Let

$$
A(n) := \lambda + \frac{1}{2b} \sqrt{\frac{\pi}{|b|^2 + |c|^2}} \left[(3b^2 - c^2)(T_1 p)(n) + (b^2 + c^2)(T_2 p)(n) + (b^2 + c^2)(T_1 q)(n) \right.
$$

$$
\left. - (b^2 + c^2)(T_2 q)(n) \right]; \tag{28}
$$

$$
B(n) := \frac{1}{2b} \sqrt{\frac{\pi}{|b|^2 + |c|^2}} \left[(b^2 + c^2)(T_1 p)(n) - (b^2 + c^2)(T_2 p)(n) + (3b^2 - c^2)(T_1 q)(n) \right.
$$

$$
\left. + (b^2 + c^2)(T_2 q)(n) \right]; \tag{29}
$$

$$
D(n) := A(n)A(-n) - B(n)B(-n); \tag{30}
$$

$$
D_1(n) := A(-n)(T_1 f)(n) - B(n)(T_2 f)(n); \tag{31}
$$

$$
D_2(n) := A(n)(T_2 f)(n) - B(-n)(T_1 f)(n). \tag{32}
$$

Theorem 4. Assume that the functions p and q are 2π-periodic. Moreover, assume that they are piecewise continuous and $f \in L^2([0, 2\pi])$.

(i) If $\lambda \neq 0$, then there exists an integer K^* such that $D(n) \neq 0$, for every $n \geq K^*$.

(ii) If $D(n) \neq 0$, for every $n \in \mathbb{N}$, then equation (27) has a unique solution for every $f \in L^2([0, 2\pi])$, which is given by

$$\varphi(x) = \sum_{n \in \mathbb{Z}} \frac{D_1(n)}{D(n)} \mathscr{E}(nx). \tag{33}$$

Proof. (i) By the Riemann-Lebesgue lemma for T_1 and T_2, we deduce that

$$\lim_{n \to \infty} D(n) = \lambda^2 \neq 0.$$

Hence, there exists an integer $K^* \in \mathbb{N}$ such that $D(n) \neq 0$, for all $n \geq K^*$.

(ii) The coefficients $3c^2 - b^2, b^2 + c^2, b^2 + c^2, -(b^2 + c^2)$ that appear in the right-hand sides of the convolutions (13)-(16) constitute a symmetric matrix as

$$M := \begin{pmatrix} 3c^2 - b^2 & b^2 + c^2 & b^2 + c^2 & -(b^2 + c^2) \\ b^2 + c^2 & 3c^2 - b^2 & -(b^2 + c^2) & b^2 + c^2 \\ b^2 + c^2 & -(b^2 + c^2) & 3c^2 - b^2 & b^2 + c^2 \\ -(b^2 + c^2) & b^2 + c^2 & b^2 + c^2 & 3c^2 - b^2 \end{pmatrix}, \tag{34}$$

with $\det(M) = -256b^2 c^6 \neq 0$, since $bc \neq 0$ as assumed above.

Therefore, replacing g with φ and f with p in (13)-(16) and converting those expressions (the fact of $\det(M) = -256b^2 c^6 \neq 0$ allows these conversions), we obtain

$$\int_0^{2\pi} p(x - u)\varphi(u)du = \frac{1}{2b}\sqrt{\frac{\pi}{|b|^2 + |c|^2}} \left[(3b^2 - c^2)(p \underset{T_1}{*} \varphi)(x) + (b^2 + c^2)(p \underset{T_1,T_1,T_2}{*} \varphi)(x) \right.$$
$$\left. + (b^2 + c^2)(p \underset{T_1,T_2,T_1}{*} \varphi)(x) - (b^2 + c^2)(p \underset{T_1,T_2,T_2}{*} \varphi)(x) \right].$$

Again, replacing g with φ and f with q and converting those expressions as above, we get

$$\int_0^{2\pi} q(x + u)\varphi(u)du = \frac{1}{2b}\sqrt{\frac{\pi}{|b|^2 + |c|^2}} \left[(b^2 + c^2)(q \underset{T_1}{*} \varphi)(x) + (3b^2 - c^2)(q \underset{T_1,T_1,T_2}{*} \varphi)(x) \right.$$
$$\left. - (b^2 + c^2)(q \underset{T_1,T_2,T_1}{*} \varphi)(x) + (b^2 + c^2)(q \underset{T_1,T_2,T_2}{*} \varphi)(x) \right].$$

Applying T_1 to both sides of these identities and using the factorization identities of the convolutions appeared on the right-hand side, we obtain

$$T_1\left(\int_0^{2\pi} p(x - u)\varphi(u)du \right)(n) = \frac{1}{2b}\sqrt{\frac{\pi}{|b|^2 + |c|^2}} \left[(3b^2 - c^2)(T_1 p)(n)(T_1 \varphi)(n) \right.$$
$$\left. + (b^2 + c^2)(T_1 p)(n)(T_2 \varphi)(n) + (b^2 + c^2)(T_2 p)(n)(T_1 \varphi)(n) - (b^2 + c^2)(T_2 p)(n)(T_2 \varphi)(n) \right]; \tag{35}$$

$$T_1\left(\int_0^{2\pi} q(x + u)\varphi(u)du \right)(n) = \frac{1}{2b}\sqrt{\frac{\pi}{|b|^2 + |c|^2}} \left[(b^2 + c^2)(T_1 q)(n)(T_1 \varphi)(n) \right.$$
$$\left. + (3b^2 - c^2)(T_1 q)(n)(T_2 \varphi)(n) - (b^2 + c^2)(T_2 q)(n)(T_1 \varphi)(n) + (b^2 + c^2)(T_2 q)(n)(T_2 \varphi)(n) \right]. \tag{36}$$

Analogously, applying the operator T_2 as above, we have

$$T_2\left(\int_0^{2\pi} p(x - u)\varphi(u)du \right)(n) = \frac{1}{2b}\sqrt{\frac{\pi}{|b|^2 + |c|^2}} \left[(3b^2 - c^2)(T_2 p)(n)(T_2 \varphi)(n) \right.$$
$$\left. + (b^2 + c^2)(T_2 p)(n)(T_1 \varphi)(n) + (b^2 + c^2)(T_1 p)(n)(T_2 \varphi)(n) - (b^2 + c^2)(T_1 p)(n)(T_1 \varphi)(n) \right]; \tag{37}$$

$$T_2\left(\int_0^{2\pi} q(x+u)\varphi(u)du\right)(n) = \frac{1}{2b}\sqrt{\frac{\pi}{|b|^2+|c|^2}}\left[(b^2+c^2)(T_2q)(n)(T_2\varphi)(n)\right.$$

$$\left.+(3b^2-c^2)(T_2q)(n)(T_1\varphi)(n) - (b^2+c^2)(T_1q)(n)(T_2\varphi)(n) + (b^2+c^2)(T_1q)(n)(T_1\varphi)(n)\right]. \tag{38}$$

To prove the uniqueness of the solution of equation (27), we invoke the Fredholm alternative theorem. Suppose that the homogeneous equation corresponding to equation (27) has a solution $\varphi_* \in L^2([0,2\pi])$, this is,

$$\lambda\varphi_*(x) + \int_0^{2\pi} [p(x-u)+q(x+u)]\,\varphi_*(u)du = 0.$$

Applying T_1 and T_2 to both sides of this equation and using identities (35)–(38), we obtain the following system of two linear equations

$$\begin{cases} A(n)(T_1\varphi_*)(n) + B(n)(T_2\varphi_*)(n) = 0 \\ B(-n)(T_1\varphi_*)(n) + A(-n)(T_2\varphi_*)(n) = 0. \end{cases}$$

Since $D(n) \neq 0$, for every $n \in \mathbb{N}_0$, we obtain $(T_1\varphi_*)(n) = (T_2\varphi_*)(n) = 0$, for $n \geq 0$. Due to the "uniqueness theorems" of the operators T_1 and T_2, we obtain that $\varphi_* = 0$. Thus, the homogeneous equation associated with the equation (27) has only the trivial solution and so, by the Fredholm alternative theorem, equation (27) has a unique non-trivial solution, provided $0 \neq f \in L^2([0,2\pi])$.

Now, we will obtain the solution formula (33). Suppose that $\varphi \in L^2([0,2\pi])$ is fulfilling (27). Proceeding in a similar way as used for the homogeneous equation, we obtain the following system

$$\begin{cases} A(n)(T_1\varphi)(n) + B(n)(T_2\varphi)(n) = (T_1f)(n) \\ B(-n)(T_1\varphi)(n) + A(-n)(T_2\varphi)(n) = (T_2f)(n). \end{cases}$$

Since $D(n) \neq 0$ for every $n \geq 0$, this system has a unique solution given by

$$(T_1\varphi)(n) = \frac{D_1(n)}{D(n)}, \quad (T_2\varphi)(n) = \frac{D_2(n)}{D(n)} = \frac{D_1(-n)}{D(n)} = (T_1\varphi)(-n), \tag{39}$$

for $n = 0, 1, \ldots$. Taking the inverse transform of T_1, we obtain

$$\varphi(x) = \sum_{n\in\mathbb{Z}} \frac{D_1(n)}{D(n)}\mathscr{E}(nx),$$

which belongs to $L^2([0,2\pi])$ and fulfills (27) for almost every $x \in [0,2\pi]$, where

$$\|\varphi\|_2^2 = \frac{\pi}{2}|\hat{f}_c(0)|^2 + \sum_{0\neq n\in\mathbb{Z}} \left|\frac{D_1(n)}{D(n)}\right|^2 < \infty.$$

$\square$

Remark 3. The convolution functions given by (13), (14), (15), and (16) are linearly independent. In other words, each one of those four convolution functions cannot be expressed as a linear combination of the three others. This fact is an immediate consequence of $\det(M) \neq 0$ as shown by (34).

Remark 4. (i) It is known that T_1, T_2 and their inverses, T_1^{-1} and T_2^{-1}, are integral transformations from the time domain to frequency domain, and vice versa, respectively. Looking at the solution φ given by (33), we can state that the formula can be seen as a new version of the Shannon sampling theorem for periodic signals sampled in the frequency domain, in which we can recover all signal frequencies from the knowledge of the information at the frequency samples $\frac{D_1(n)}{D(n)}$. In practice, the above formula has some advantage when computing data directly in the frequency domain, instead of that in the time domain.

(ii) Rewriting the function D in terms of the finite cosine and sine Fourier transformations, considering

$$(\hat{f}_c)(n) = \frac{1}{\pi} \int\limits_0^{2\pi} \cos(nx) f(x) dx;$$

$$(\hat{f}_s)(n) = \frac{1}{\pi} \int\limits_0^{2\pi} \sin(nx) f(x) dx,$$

we obtain

$$D(n) = \lambda^2 + 2\lambda\pi(\hat{p}_c)(n) + \pi^2 \left[(\hat{p}_c)^2(n) - (\hat{q}_c)^2(n) + (\hat{p}_s)^2(n) - (\hat{q}_s)^2(n) \right]. \tag{40}$$

Having this form of D in mind, we can obtain the following proposition, whose proof is direct, just assuming that $D(n) \neq 0$ and solving the equation in terms of λ.

We end up by describing a criterion for $D(n) \neq 0$ (emphasizing, by this way, that proposition (ii) in Theorem 4 is only a sufficient condition).

Proposition 4. Suppose that the functions p and q in the equation (27) are Lebesgue integrable. The determinant $D(n) \neq 0$ if and only if

$$\lambda \neq \pi \left[-(\hat{p}_c)(n) \pm \sqrt{(\hat{q}_c)^2(n) + (\hat{q}_s)^2(n) - (\hat{p}_s)^2(n)} \right]. \tag{41}$$

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Conflict of interest

The authors declare that there are no conflicts of interest to this work.

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