Stack sorting with restricted stacks*

Giulio Cerbai[†]   Anders Claesson[‡]   Luca Ferrari[†]

Abstract

The (classical) problem of characterizing and enumerating permutations that can be sorted using two stacks connected in series is still largely open. In the present paper we address a related problem, in which we impose restrictions both on the procedure and on the stacks. More precisely, we consider a *greedy* algorithm where we perform the rightmost legal operation. Moreover, the first stack is required to be σ-avoiding, for some permutation σ, meaning that the elements maintained in the stack avoid the pattern σ when read from top to bottom. Since the set of permutations which can be sorted by such a device, which we call a *σ-machine*, is not always a class, it would be interesting to understand when it happens. We prove that the set of σ-machines whose associated sortable permutations are not a class is counted by Catalan numbers. Moreover, we analyze two specific σ-machines in full detail (namely $\sigma = 321$ and $\sigma = 123$), providing for each of them a complete characterization and enumeration of the sortable permutations.

1   Introduction

The birth of *stack sorting disciplines* can be traced back to a series of exercises proposed by Knuth [6]. Consider the problem of sorting a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ using a stack. More specifically, scan the permutation from left to right and, when π_i is read, either push π_i onto the stack or pop the top entry of the stack into the output (at the first available position). Call the two operations S and O, respectively. Knuth has showed that there is an optimal algorithm, called *Stacksort*, which is able to sort every sortable permutation. It consists of performing S whenever π_i is smaller than the current top of the stack, otherwise performing O (see Listing 1).

It is easy to realize that *Stacksort* has two key properties:

1. The stack is *increasing*; the elements inside the stack are maintained in increasing order, from top to bottom.

2. The algorithm is *right greedy*; it always chooses to perform S as long as the stack remains increasing. Here the expression "right greedy" refers to the usual pictorial representation, in which the input permutation is on the right, the stack is in the middle and the output permutation is on the left (see Figure 1).

Using *Stacksort*, it can be shown that the sortable permutations are precisely those avoiding the pattern 231; and it is well known that 231-avoiding permutations of length n are counted by Catalan numbers.

> **Algorithm 1:** Stacksort ($Stack$ is the stack, $TOP(Stack)$ is the current top of the stack, $\pi = \pi_1 \cdots \pi_n$ is the input permutation).
>
> $Stack := \emptyset$;
> $i := 1$;
> **while** $i \leq n$ **do**
>   **if** $Stack = \emptyset$ *or* $\pi_i < TOP(Stack)$ **then**
>   execute S;
> $i := i + 1$;
>   **end**
>   **else**
>   execute O;
>   **end**
> **end**
> **while** $Stack \neq \emptyset$ **do**
>   execute O;
> **end**

Though the above problem is rather simple, things become considerably more complicated if one allows more stacks connected in series. For the machine consisting of just two stacks in series we know at present very few results. We know, for instance, that sortable permutations can be characterized in terms of an *infinite* set of avoided patterns, but we do not have any explicit description of such a set; see Murphy [8]. The enumeration of sortable permutations is also unknown.

Since the general problem of sorting with two stacks is too difficult, several special cases have been considered. The so-called *West-2-stack-sortable permutations* (see West [11]) are those permutations which can be sorted by making two passes through a stack. Equivalently, they are the permutations that can be sorted by 2 stacks connected in series using a *right greedy algorithm*. West-2-stack-sortable permutations do not form a class, nevertheless it is possible to characterize them using some kind of generalized patterns (called *barred patterns*).

Another possible variation on the two-stacks problem is to impose some restrictions on the content of the stack. Note that to obtain the identity permutation it is necessary for the second stack to be increasing. Atkinson, Murphy and Ruškuc [1] studied the case in which the first stack is also required to be increasing. They characterized the sortable permutations in terms of avoiding an infinite set of patterns. They also proved that a left greedy algorithm is optimal in this case and derived a generating function for the number of sortable permutations.

Smith [10] studied the case in which the first stack is required to be *decreasing*. She described an optimal sorting algorithm, thanks to which she was able to completely characterize (in terms of two avoided patterns) and enumerate the sortable permutations.

In the present paper we will deal with similar sorting machines consisting of two stacks connected in series (see Figure 1).

Recalling the key properties of the *Stacksort* algorithm, we will consider machines obeying certain constraints, which are described below.

1. The stacks must obey some *restrictions*, which are expressed by saying that, at each step of the execution, the elements into each stack (read from top to bottom) must avoid certain forbidden configurations. In particular, in analogy with *Stacksort*, we require the second stack to be increasing. Notice that this can be equivalently expressed as follows: at every step, the sequence of numbers contained in the stack (read from top to bottom)
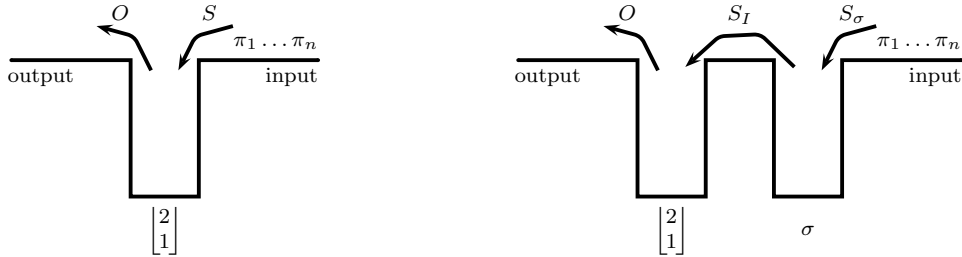
Figure 1: Sorting with one stack (on the left) and sorting with two stacks, where the first one is σ-restricted (on the right).

has to avoid the pattern 21. We will express this by saying that the stack is $\lfloor\begin{smallmatrix}2\\1\end{smallmatrix}\rfloor$-avoiding. Moreover, we will be interested in machines in which the first stack is σ-*avoiding*, for some pattern σ.

2. The algorithm we perform on the two stacks connected in series is *right greedy*. As already observed, this is equivalent to making two passes through a stack, performing the right greedy algorithm at each pass. However, due to the restriction described above, during the first pass the stack is σ-avoiding, whereas during the second pass it is $\lfloor\begin{smallmatrix}2\\1\end{smallmatrix}\rfloor$-avoiding.

We will use the term σ-*machine* to refer to the right greedy algorithm performed on two stacks in series, such that the first stack is σ-avoiding and the second stack is $\lfloor\begin{smallmatrix}2\\1\end{smallmatrix}\rfloor$-avoiding. Formally, the algorithm we are going to analyze is described in Listing 2.

The set of permutations which are sortable by the σ-machine is denoted $\mathrm{Sort}(\sigma)$ and its elements are the σ-*sortable permutations*. The set of σ-sortable permutations of length n is denoted $\mathrm{Sort}_n(\sigma)$. In the present paper we initiate the study of the combinatorics of σ-machines. In particular, we aim at characterizing and enumerating σ-sortable permutations. After necessary preliminaries (contained in Section 2), we see that the set of σ-sortable permutations is a permutation class for some choices of σ, while it is not a permutation class for other choices of σ. In Section 3 we find an explicit characterization of those σ such that σ-sortable permutations constitute a class. We prove the striking fact that σ-machines whose σ-sortable permutations are not a class are counted by Catalan numbers with respect to the length of σ. Then we focus on a couple of specific σ-machines: Section 4 studies the 321-machine, giving a complete characterization and enumeration of sortable permutations, which happen to constitute a class. Our result is actually stronger, being stated for a decreasing permutation σ of any length. Section 5 is devoted to the analysis of the 123-machine. Also in this (considerably more challenging) case we are able to provide complete structural and enumerative results for sortable permutations (which do not form a class), describing in particular a bijection with a set of pattern-avoiding Schröder paths. The last section suggests some directions for further research.

2 Preliminaries and notations

Given a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ of length n, the *k-inflation of π at π_i* is the permutation of length $n + (k-1)$ obtained from π by replacing π_i with the consecutive increasing sequence $\pi_i(\pi_i+1)\cdots(\pi_i+(k-1))$ and suitably rescaling the remaining elements. For instance, the 3-inflation of the permutation $451\underline{3}2$ at 3 is $671\underline{345}2$.

The element π_i of π is called a *left-to-right maximum* (briefly, LTR maximum) when it is bigger than all the elements preceding it, i.e. $\pi_i > \max(\pi_1,\ldots,\pi_{i-1})$. The permutation

Algorithm 2: The σ-machine ($Stack_\sigma$ is the σ-avoiding stack, $Stack_I$ is the increasing stack, S_σ means pushing into $Stack_\sigma$, S_I means pushing into $Stack_I$, O means moving $TOP(Stack_I)$ into the output, $\circ$ is the concatenation operation).

> $Stack_I := \emptyset$;
> $Stack_\sigma := \emptyset$;
> $i := 1$;
> **while** $i \leq n$ **do**
> > **if** $\sigma \not\leq Stack_\sigma \circ \pi_i$ **then**
> > > execute S_σ;
> > > $i := i + 1$;
> >
> > **end**
> > **else if** $Stack_I = \emptyset$ or $TOP(Stack_\sigma) < TOP(Stack_I)$ **then**
> > > execute S_I;
> >
> > **end**
> > execute O;
> >
> **end**
> **while** $Stack_\sigma \neq \emptyset$ **do**
> > **if** $Stack_I = \emptyset$ or $TOP(Stack_\sigma) < TOP(Stack_I)$ **then**
> > > execute S_I;
> >
> > **else**
> > > execute O;
> >
> > **end**
> >
> **end**
> **while** $Stack_I \neq \emptyset$ **do**
> > execute O;
> >
> **end**

$\underline{315}\underline{5}762\underline{4}9\underline{8}$ has four LTR maxima, which are the elements underlined.

The symmetries of a permutation are the *reverse*, *inverse*, and *complement* operations as well as compositions of these operations. Given $\pi = \pi_1 \pi_2 \cdots \pi_n$, we define its reverse $\pi^r = \pi_n \pi_{n-1} \cdots \pi_1$, its complement $\pi^c = (n+1-\pi_1)(n+1-\pi_2)\cdots(n+1-\pi_n)$ and its inverse π^{-1} as the usual group-theoretic inverse.

A *Dyck path* is a path in the discrete plane $\mathbb{Z} \times \mathbb{Z}$ starting at the origin of a fixed Cartesian coordinate system, ending on the x-axis, never falling below the x-axis and using two kinds of steps (of length 1), namely up steps $\mathtt{U} = (1,1)$ and down steps $\mathtt{D} = (1,-1)$. The *length* of a Dyck path is its final abscissa, which coincides with the total number of its steps. For instance, $\mathtt{UUDUUDDDUD}$ is a Dyck path of length 10. According to their semilength, Dyck paths are counted by Catalan numbers (sequence A000108 in [9]). The n-th *Catalan number* is $C_n = \frac{1}{n+1}\binom{2n}{n}$ and the associated generating function is $C(x) = (1 - \sqrt{1-4x})/(2x)$.

There is a well known bijection between 213-avoiding permutations (of length k) and Dyck paths (of semilength k), which can be succinctly described as follows: given a Dyck path P of semilength k, label its down steps from right to left with positive integers 1 to k in increasing way, then label each up step with the same label as the down step it is matched with, finally read the labels of the up steps from left to right, so to obtain a 213-avoiding permutation. For instance, the above Dyck path $\mathtt{UUDUUDDDUD}$ corresponds to the permutation 25341, which in fact avoids 213 (see Figure 2). Krattenthaler [7] gives an equivalent version of the above bijection using 132-avoiding permutations.
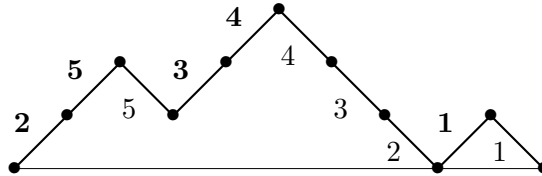

Figure 2: A Dyck path and its associated 213-avoiding permutation (read the bold labels from left to right).

Another classical family of lattice paths is that of Schröder paths. A *Schröder path* is defined exactly like a Dyck path, except that it uses one more kind of step, namely a double horizontal step $\mathtt{H_2} = (2,0)$. The length of a Schröder path does not coincide with the number of its steps; it can be rather obtained as the sum of the number of its up steps and down steps with twice the number of its double horizontal steps. As an example, $\mathtt{UH_2UDDH_2UD}$ is a Schröder path of length 10 .

Our main goal is to study the sorting power of σ-machines. We remark that, due to the specificity of our setting, a permutation π is σ-sortable if and only if the output $s_\sigma(\pi)$ resulting from the first pass (through the σ-avoiding stack) avoids 231. This fact (as well as the notation $s_\sigma(\pi)$) will be frequently used throughout the paper, especially in Section 5.

We close this section by analyzing the σ-machine when σ has length 2. If $\sigma = 21$, the σ-machine is precisely West's right greedy algorithm performed on two stacks in series, sbao we refer to [11]. If $\sigma = 12$, we have the following result, which completely characterize and enumerate 12-sortable permutations. We remark that the 12-machine is different from the one considered in [10]; Smith's device does not require the use of a right greedy algorithm.

Theorem 2.1. *A permutation π is 12-sortable if and only if $\pi \in \mathrm{Av}(213)$. As a consequence, $|\mathrm{Sort}_n(12)| = C_n$, the n-th Catalan number.*

Proof. Write π as $\pi = L1R$, where L is the prefix of π preceding 1 and R is the suffix of π following 1. It is easy to see that $s_{12}(\pi) = s_{12}(L1R) = s_{12}(L)s_{12}(R)1$. This is because 1 enters the stack only when the stack itself is empty, and exits the stack only at the end. Now, using induction on the length of π and a simple case by case analysis, it is not difficult to show that, if π contains 213, then $s_{12}(\pi)$ contains 231, and so π is not sortable.

On the other hand, suppose that π is not sortable, and so that $s_{12}(\pi)$ contains an occurrence of 231. Let b, c be the first two elements of this occurrence. Then necessarily b comes before c also in π (since a non-inversion in the output necessarily comes from a non-inversion in the input). However, b exits the stack before c enters it, and this must be due to the presence of an element x, located between b and c in π, which is smaller than b. The three elements b, x and c are thus an occurrence of 213 in π. $\qquad\square$

The above results implies that Sort(21) is not a class, whereas Sort(12) is.

3 Classes and nonclasses of σ-sortable permutations

Given a permutation σ, it is natural to ask if σ-sortable permutations constitute a permutation class. Concerning permutations of length 2, as we have already observed, 12-sortable permutations are a class, whereas 21-sortable permutations are not. Concerning permutations of length 3, some computations suggest that 321-sortable permutations might be a class, whereas in the five remaining cases σ-sortable permutations are not a class, as can be deduced from the table below:

σ	σ-sortable permutation	non-σ-sortable pattern
123	4132	132
132	2413	132
213	4132	132
231	361425	1324
312	3142	132

Looking at more data, by taking longer permutations, suggests a rather surprising conjecture: the number of permutations σ such that σ-sortable permutations are not a class is the n-th Catalan number! The rest of this section is devoted mainly to prove this conjecture, as well as to provide some related results.

We start by showing how the sortability of a permutation is affected by its connections with the pattern defining the constraint of the stack.

Lemma 3.1. *Let $\sigma = \sigma_1\sigma_2\cdots\sigma_k$ and suppose we use the σ-machine on the permutation π.*

1. *If $\pi \in \text{Av}(\sigma^r)$, then the output of the first stack is π^r. Therefore π is σ-sortable if and only if $\pi \in \text{Av}(132)$.*

2. *If π contains the pattern σ^r, then the output of the first stack contains the pattern $\hat{\sigma} = \sigma_2\sigma_1\sigma_3\sigma_4\cdots\sigma_k$, obtained by interchanging the first two elements of σ. Therefore, if $\hat{\sigma}$ contains the pattern 231, then π is not σ-sortable.*

Proof. 1. If $\pi \in \text{Av}(\sigma^r)$, then the restriction of the first stack never triggers, so it outputs π^r. Therefore π is sortable if and only if it avoids 132.

2. Suppose that π contains σ^r. Let $s_k s_{k-1} \cdots s_1$ be the (lexicographically) leftmost occurrence of σ^r in π. Then every element of π is pushed into the first stack until s_1 is scanned. Before pushing s_1 into the stack, the element s_2 has to be popped out due to the σ-restriction. Moreover, the element s_3 is not popped before s_1 enters the stack, otherwise in π there would be an occurrence of σ^r to the left of $s_k s_{k-1} \cdots s_1$, which is a contradiction. Therefore s_1 is pushed into the first stack when s_3 is still inside: this is enough to conclude that the output of the first stack will contain $s_2 s_1 s_3 \cdots s_k$, which is an occurrence of $\hat{\sigma}$. Therefore, if $\hat{\sigma}$ contains 231, then π is not σ-sortable.

$\square$

Theorem 3.2. *Let $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$ and $\hat{\sigma}$ as in the above lemma. If $\hat{\sigma}$ contains the pattern 231, then $\mathrm{Sort}(\sigma) = \mathrm{Av}(132, \sigma^r)$. In such a case, $\mathrm{Sort}(\sigma)$ is a class with basis either $\{132, \sigma^r\}$ (if $\sigma^r \in \mathrm{Av}(132)$) or $\{132\}$ (otherwise).*

Proof. Given any permutation π, suppose that π contains σ^r. Then the previous lemma implies that the output of the first stack contains $\hat{\sigma}$. Since $\hat{\sigma}$ contains 231 by hypothesis, π is not σ-sortable. Now suppose that π avoids σ^r, but that it contains 132. Again as a consequence of the previous lemma, the output of the first stack is then π^r, and π^r contains 231, so π is not σ-sortable. We have thus proved that $\mathrm{Sort}(\sigma) \subseteq \mathrm{Av}(132, \sigma^r)$.

Conversely, suppose that π avoids both 132 and σ^r. Then the previous lemma implies that the output of the first stack is π^r, which avoids $132^r = 231$ by hypothesis, therefore π is σ-sortable. Thus we also have that $\mathrm{Av}(132, \sigma^r) \subseteq \mathrm{Sort}(\sigma)$, and so the equality holds. $\square$

Corollary 3.3. *For all $k \geq 3$, $\mathrm{Sort}(k(k-1) \cdots 1) = \mathrm{Av}(132, 12 \cdots k)$. In particular, the set of 321-sortable permutations is a class.*

Theorem 3.2 provides a sufficient condition for a permutation σ in order to have that $\mathrm{Sort}(\sigma)$ is a class. It turns out that this condition is also necessary.

Theorem 3.4. *If $\hat{\sigma}$ avoids the pattern 231, then $\mathrm{Sort}(\sigma)$ is not a permutation class.*

Proof. The above corollary and the table at the beginning of this section tells that, if σ has length at most 3, the theorem holds.

Now suppose that σ has length at least 4. It is not hard to realize that the permutation 132 is not σ-sortable, since the output of the first stack is 231. We now show that, if $\hat{\sigma}$ avoids 231, then it is always possible to construct a permutation α such that α contains 132 and α is σ-sortable, thus proving that $\mathrm{Sort}(\sigma)$ is not a class. Suppose, as usual, that $\sigma = \sigma_1 \sigma_2 \cdots \sigma_k$. We distinguish two cases, depending on the relative order of the elements σ_1 and σ_2.

1. If $\sigma_1 < \sigma_2$, define $\alpha = \sigma'_k \sigma'_{k-1} \cdots \sigma'_3 \, z \, \sigma'_2 \sigma'_1$, where

   - $z = \sigma_1$,
   - $\sigma'_i = \begin{cases} \sigma_i, & \text{if } \sigma_i < \sigma_1; \\ \sigma_i + 1, & \text{otherwise.} \end{cases}$

   Note that $z \sigma'_2 \sigma'_1$ is an occurrence of 132. We will show that α is σ-sortable by means of a detailed analysis of the behavior of the σ-machine on input α. The elements of α are pushed into the first stack until σ'_1 is scanned (it is the first element that triggers the restriction $\mathrm{Av}(\sigma)$). In particular, both the additional element z and σ'_2 can be pushed into the stack, because $\sigma'_2 z \cdots \sigma'_{k-1} \sigma'_k$ is not an occurrence of σ (since $\sigma_1 < \sigma_2$ and $\sigma'_2 > z$). Now, before σ'_1 enters the first stack, the element σ'_2 is extracted and pushed into the second stack. At this point, σ'_1 can enter without violating the restriction, again because

$\sigma_2 > \sigma_1$, whereas $z < \sigma_1'$, so that $\sigma_1' z \sigma_3' \cdots \sigma_k'$ is not an occurrence of σ. As a result, the output of the first stack is $out = \sigma_2' \sigma_1' z \sigma_3' \cdots \sigma_k'$, so it will be enough to show that out does not contain an occurrence of the pattern 231. Since $\hat{\sigma}$ avoids 231 by hypothesis, a potential occurrence of 231 necessarily involves the new element z. In particular, it is easy to observe that z can be neither the smallest nor the biggest element of such a pattern, because $z < \sigma_1' < \sigma_2'$ and z is the third element of out. Finally, if z were the first element of an occurrence $z\, \sigma_j'\, \sigma_l'$ of 231 in out, then $\sigma_1\, \sigma_j\, \sigma_l$ would be an occurrence of 231 in $\hat{\sigma}$, against the hypothesis.

2. If $\sigma_1 > \sigma_2$, define $\alpha = \sigma_k' \sigma_{k-1}' \cdots \sigma_3' \sigma_2' \sigma_1'\, z$, where

   - $z = \sigma_2 + 1$.
   - $\sigma_i' = \begin{cases} \sigma_i, & \text{if } \sigma_i \leq \sigma_2; \\ \sigma_i + 1, & \text{otherwise.} \end{cases}$

   Observe that $\sigma_2' \sigma_1' z$ is an occurrence of 132. As for the previous case, we now describe what happens when α is processed by the σ-machine. The first element that cannot be pushed into the first stack is σ_1', which forces σ_2' to be extracted. Successively both σ_1' and z can enter the first stack, since $z\sigma_1' \sigma_3' \cdots \sigma_k'$ is not an occurrence of σ (indeed $\sigma_1 > \sigma_2$ and $z < \sigma_1'$). Therefore the output of the first stack is $out = \sigma_2' z \sigma_1' \sigma_3' \cdots \sigma_k'$, and again a potential occurrence of 231 in out must involve the new element z. However z cannot be the smallest element of a pattern 231, because it is the second element of out. Moreover, if z were the first element of a 231, then σ_2 would be the first element of an occurrence of 231 in $\hat{\sigma}$, which is forbidden. Finally, if z were the largest element of a 231, then σ_2' would be the first element of such an occurrence, so also σ_1', which is greater than both σ_2' and z, would be the largest element of an occurrence of 231 which does not involve z, giving again a contradiction. Thus we have showed that out does not contain the pattern 231, which means that α is σ-sortable.

   $\square$

Corollary 3.5. *For every permutation σ, the set $\mathrm{Sort}(\sigma)$ of the permutations sortable using the σ-machine is not a permutation class if and only if $\hat{\sigma}$ avoids the pattern 231.*

Corollary 3.6. *The permutations σ for which $\mathrm{Sort}(\sigma)$ is not a permutation class are enumerated by Catalan numbers.*

Proof. Such permutations are in bijection with $\mathrm{Av}(231)$, which is known to be enumerated by Catalan numbers. $\square$

We have thus shown that $\mathrm{Sort}(\sigma)$ is a permutation class if and only if $\hat{\sigma}$ contains the pattern 231. In this case, $\mathrm{Sort}(\sigma) = \mathrm{Av}(132, \sigma^r)$, hence the basis of $\mathrm{Sort}(\sigma)$ has exactly two elements if and only if σ avoids 231 and $|\sigma| > 2$. We next give exact enumerative results concerning $\mathrm{Sort}(\sigma)$ when its basis has two elements.

Proposition 3.7. *Suppose that σ^r avoids the pattern 132. Then $\hat{\sigma}$ contains the pattern 231 if and only if $\sigma_1 \sigma_2 \sigma_3$ is an occurrence of the pattern 321.*

Proof. By hypothesis, σ avoids 231. If $\hat{\sigma} = \sigma_2 \sigma_1 \sigma_3 \cdots \sigma_k$ contains the pattern 231, then there can be only one occurrence of 231 and it has to involve both σ_1 and σ_2, respectively as the first and the second element of the pattern, with $\sigma_2 < \sigma_1$. Let $\sigma_2 \sigma_1 \sigma_i$ be such an occurrence,

with $i \geq 3$. If $\sigma_3 > \sigma_2$, then $i \geq 4$, and $\sigma_2 \sigma_3 \sigma_i$ would be an occurrence of 231 in σ, which is impossible. Therefore it must be $\sigma_3 < \sigma_2$, hence $\sigma_1 > \sigma_2 > \sigma_3$, as desired.

Conversely, if $\sigma_1 \sigma_2 \sigma_3$ is an occurrence of the pattern 321, then clearly $\sigma_2 \sigma_1 \sigma_3$ is an occurrence of 231 in $\hat{\sigma}$. $\qquad\square$

Proposition 3.8. *Let* $a_n = |\{\pi \in \mathrm{Av}_n(231) \,|\, \pi_1 \pi_2 \pi_3 \simeq 321\}|$; *then, for each* $n \geq 2$, *we have* $a_n = C_n - 2C_{n-1}$. *In particular, the generating function of the sequence* $(a_n)_{n \geq 0}$ *is*

$$A(x) = \frac{1 - 4x + 2x^2 - (1 - 2x)\sqrt{1 - 4x}}{2x}.$$

Proof. Suppose that $n \geq 2$. It is well known that $|\mathrm{Av}_n(231)| = C_n$, hence we have $a_n = C_n - (f_n + g_n)$, where

$$\begin{cases} \mathcal{F}_n = \{\pi \in \mathrm{Av}_n(231) \,|\, \pi_1 < \pi_2\}, & f_n = |\mathcal{F}_n|; \\ \mathcal{G}_n = \{\pi \in \mathrm{Av}_n(231) \,|\, \pi_1 > \pi_2, \ \pi_2 < \pi_3\}, & g_n = |\mathcal{G}_n|. \end{cases}$$

We now show that $f_n = g_n = C_{n-1}$ by explicitly finding bijections between $\mathcal{F}_n$ and $\mathrm{Av}_{n-1}(231)$ as well as between $\mathcal{G}_n$ and $\mathrm{Av}_{n-1}(231)$, thus obtaining the desired enumeration.

- If $\pi \in \mathcal{F}_n$, then it must be $\pi_1 = 1$, otherwise $\pi_1 \pi_2 1$ would be an occurrence of 231 in π. Thus we can define $f : \mathcal{F}_n \to \mathrm{Av}_{n-1}(231)$ such that $f(\pi)$ is obtained from π by removing $\pi_1 = 1$ and rescaling the remaining elements. It is clear that $f(\pi) \in \mathrm{Av}_n(231)$ and that f is an injection. Moreover, if $\tau \in \mathrm{Av}_{n-1}(231)$, then adding a new minimum at the beginning cannot create any occurrence of 231, so f is also surjective.

- If $\pi \in \mathcal{G}_n$, then it must be $\pi_2 = 1$, otherwise the elements $\pi_2 \pi_3 1$ would form an occurrence of 231 in π. We thus define $g : \mathcal{F}_n \to \mathrm{Av}_{n-1}(231)$ such that $g(\pi)$ is obtained from π by removing $\pi_2 = 1$ and rescaling the remaining elements. Again it is clear that $g(\pi) \in \mathrm{Av}_n(231)$ and that g is an injection. Finally, if $\tau \in \mathrm{Av}_{n-1}(231)$, then the permutation π obtained from τ by adding a new minimum in the second position avoids 231, because a potential occurrence of 231 in π should involve the added element π_2, and so π_2 would be either the first or the second element of such an occurrence, which cannot be since $\pi_2 = 1$. Therefore g is a bijection between $\mathcal{F}_n$ and $\mathrm{Av}_{n-1}(231)$, as desired.

We can now compute the generating function $A(x) = \sum_{n \geq 0} a_n x^n$ as follows:

$$A(x) = \sum_{n \geq 0} a_{n+2} x^{n+2} = \sum_{n \geq 0} C_{n+2} x^{n+2} - 2x \sum_{n \geq 0} C_{n+1} x^{n+1}$$
$$= C(x) - x - 1 - 2x(C(x) - 1) = C(x)(1 - 2x) + x - 1,$$

from which we get $A(x) = (1 - 4x + 2x^2 - (1 - 2x)\sqrt{1 - 4x})/(2x)$, as desired. $\qquad\square$

Sequence $(a_n)_{n \geq 0}$ starts $0, 0, 1, 4, 14, 48, 165, 572, 2002, \ldots$ and is recorded as sequence A002057 in [9] (with offset 2). Observe that $A(x) = x^2 C(x)^4$, a fact for which we do not have a combinatorial explanation.

In Figure 3 we report some enumerative results concerning classes of σ-sortable permutations with basis of cardinality 2. Each case can be proved with a direct combinatorial argument.

length	Pattern σ	G.F.	Sequence	OEIS
3	321	$\frac{1-x}{1-2x}$	1,1,2,4,8,16,32,64,128,256,512,...	A000079
4	3214 4213 4312 4321	$\frac{1-2x}{1-3x+x^2}$	1,1,2,5,13,34,89,233,610,1597,4181,...	A001519
5	32145	$\frac{-3x^4+9x^3-12x^2+6x-1}{(x-1)(x^2-3x+1)^2}$	1,2,5,14,41,121,355,1032,2973,8496,...	A116845
	52134	$\frac{(1-x)(2x-1)^2}{x^4-9x^3+12x^2-6x+1}$	1,2,5,14,41,121,355,1033,2986,8594,...	not in [9]
	54123	$\frac{1-4x+5x^2-3x^3}{x^4-6x^3+8x^2-5x+1}$	1,2,5,14,41,121,356,1044,3057,8948, ...	not in [9]
	32154 42135 43125 43215 52143 53124 53214 54132 54213 54312 54321	$\frac{x^2-3x+1}{3x^2-4x+1}$	1,2,5,14,41,122,365,1094,3281,9842,...	A124302

Figure 3: Classes of σ-sortable permutations whose basis has two elements.

4   The 321-machine

Let $\rho_k = k(k-1)\cdots21$ be the reverse of the identity permutation of length k. Recall from Corollary 3.3 that $\mathrm{Sort}(\rho_k) = \mathrm{Av}(12\cdots k, 132)$. For small values of k, we have the following table, where the row labelled k records the number of permutations of length n sortable by the ρ_k-machine:

$k \backslash n$	0	1	2	3	4	5	6	7	8	9	10	11	OEIS
3	1	1	2	4	8	16	32	64	128	256	512	1024	A011782
4	1	1	2	5	13	34	89	233	610	1597	4181	10946	A001519
5	1	1	2	5	14	41	122	365	1094	3281	9842	29525	A124302
6	1	1	2	5	14	42	131	417	1341	4334	14041	45542	A080937
7	1	1	2	5	14	42	132	428	1416	4744	16016	54320	A024175

Since $|\mathrm{Av}_n(132)| = C_n$, it is clear that the rows tend to the sequence of Catalan numbers. For $k = 3$, we have that $|\mathrm{Sort}_n(321)| = 2^{n-1}$ (for $n \geq 1$); this is sequence A011782 in the OEIS [9]. Looking at the OEIS references (reported in the above table), we observe that, for any given k, the associated sequence counts the number of Dyck paths of height at most $k-1$. This can be proved by using the mentioned bijection between Dyck paths and 132-avoiding permutations described in [7], observing that the maximum length of an increasing sequence corresponds to the height of the path. Dyck paths of bounded height are rather well studied objects, see for example Bousquet-Mélou [2] or Gessel and Xin [5].

Exploiting this connection, we can give a description of the generating function of the sequence recorded in the k-th row. Using the usual "first-return" decomposition of Dyck paths, it is possible to find a recursive description of the generating function $F_k(x)$ of Dyck paths of height at most k with respect to the semilength: $F_0(x) = 1$ and, for $k \geq 1$,

$$F_k(x) = 1 + xF_{k-1}(x)F_k(x).$$

From the above recurrence it is immediate to see that $F_k(x)$ is rational, for all k; more specifically, we have $F_k(x) = G_k(x)/G_{k+1}(x)$, where $G_k(x)$ satisfies the recurrence $G_{k+1}(x) = G_k(x) - xG_{k-1}(x)$, with initial conditions $G_0(x) = G_1(x) = 1$. Solving this recurrence yields $G_k(x) = \sum_{i\geq0} \binom{n-i}{i}(-x)^i$. The polynomials $G_k(x)$ are sometimes called *Catalan polynomials*, see for instance Clapperton, Larcombe and Fennessey [4]; the table of their coefficients is sequence A115139 in [9].

5   The 123-machine

Now suppose that the first stack is $\left\lfloor \begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix} \right\rfloor$-avoiding. This machine is considerably more challenging than the previous one. The first thing we observe is that, unlike the previous case, the set of 123-sortable permutation is not a class, as we already knew. For instance, the permutation 4132 is 123-sortable, whereas its pattern 132 is not.

First, we show that a necessary condition for a permutation to be 123-sortable is that the first two elements do not form a large ascent.

Lemma 5.1. *If $\pi = \pi_1\pi_2\cdots\pi_n$ is 123-sortable, then $\pi_2 \leq \pi_1 + 1$.*

Proof. Suppose that $\pi_2 \geq \pi_1 + 2$. This implies that there exists an index $i \in \{3, 4, \ldots, n\}$ such that $\pi_2 > \pi_i > \pi_1$. During the sorting process, the first two elements π_1 and π_2 enters the stack and reach the output only when the stack is emptied at the end of the process (since $\pi_1 < \pi_2$).

Thus the three elements π_i, π_2 and π_1 show an occurrence of 231 in $s_{123}(\pi)$, hence π is not 123-sortable. $\qquad\square$

As a consequence of the above lemma, we can partition the set of 123-sortable permutations into two classes: Those starting with an ascent consisting of two consecutive values and those starting with a descent. We now wish to show that, given a 123-sortable permutation, we can add in front of it an arbitrary number of consecutive ascents and the resulting permutation is still 123-sortable.

Lemma 5.2. *Let $\pi = \pi_1\pi_2\cdots\pi_n$ and let π' the permutation (of length $n+1$) obtained from π by 2-inflating its first element π_1. Then π is 123-sortable if and only if π' is 123-sortable.*

Proof. Observe that, by hypothesis, the first two elements of π' are consecutive in value (x and $x+1$, say) and the first one is smaller than the second one. Therefore, during the sorting process, such two elements remain at the bottom of the stack (with $x+1$ above x) until all the other elements of the input permutations have exited the stack. Moreover, the behavior of the stack is not affected by the presence of $x+1$, meaning that x and $x+1$ can be considered as a single element. As a consequence, the last two elements of $s_{123}(\pi')$ are $x+1$ and x, and then that $s_{123}(\pi)$ contains 231 if and only $s_{123}(\pi')$ contains 231. $\qquad\square$

Corollary 5.3. *Given $\pi \in S_n$, let π' be obtained from π by k-inflating the first element of π (with $k \geq 1$). Then π is 123-sortable if and only if π' is 123-sortable.*

The above results tell us that, up to "deflating" the prefix of consecutive ascents (if there is one), we can restrict to investigate sortability of permutations starting with a descent. Denote by $\mathrm{Sort}_n^{\downarrow}(123)$ this subset of $\mathrm{Sort}_n(123)$; that is, $\mathrm{Sort}_n^{\downarrow}(123) = \{\pi \in \mathrm{Sort}_n(123) \mid \pi_1 > \pi_2\}$. Our goal is now to characterize and enumerate $\mathrm{Sort}_n^{\downarrow}(123)$.

Lemma 5.4. *Let $\pi \in \mathrm{Sort}_n^{\downarrow}(123)$, with $\pi_1 = k$. Then we have $s_{123}(\pi) = n(n-1)\cdots(k+1)(k-1)\cdots 21k$.*

Proof. Let $\gamma = s_{123}(\pi) = \gamma_1\gamma_2\cdots\gamma_n$. Clearly $\gamma_n = k$. Now suppose that the two elements x and y constitute an ascent in γ, with $x < y$ and $y \neq k$.

We first show that y comes before x in π. Suppose in fact that this is not the case, and focus on the instant when x exits the first stack. We distinguish two cases.

- x exits the first stack because it is the second element of a pattern 321 in π. More specifically, there is an element c in the stack such that $c > x$ and the next element a of π is such that $x > a$. This implies, in particular, that $a \neq y$, and so that y follows a in π. Therefore $s_{123}(\pi)$ contains either the subword xay, which is impossible since x and y are supposed to be consecutive in $s_{123}(\pi)$, or the subword xya, which is impossible too since otherwise $s_{123}(\pi)$ would contain the pattern 231.

- x exits the first stack because the next element a of π is smaller than two elements b and c in the stack, with $b < c$ and c deeper than b. In this case, $s_{123}(\pi)$ would contain the subword xby, which is impossible, again because x and y would not be consecutive.

Thus we can write π as $\pi = k\pi_2\cdots y\cdots x\cdots$. Since x and y are consecutive in $s_{123}(\pi)$, x must enter the stack just above y. This implies, in particular, that $y \geq \pi_1$, otherwise π_1, y and x would constitute a forbidden pattern inside the stack.

We also notice that, when x enters the first stack, at the bottom of the stack there is at least one element $w < \pi_1$ just above π_1. Indeed, either π_2 is still in the stack (and in this case

$w = \pi_2$) or π_2 has been forced to exit the stack by an element $\tilde{w} < \pi_2 < \pi_1$; in this case, $\tilde{w}$ replaces π_2 just above π_1. Iterating this argument, we get the desired property.

Summing up, when x enters the first stack, the stack itself contains the elements (from bottom to top) π_1, w, y, x. Now, we have that $x > w$, otherwise π_1, w and x would constitute a forbidden pattern in the stack. Hence $s_{123}(\pi)$ must contain the subword xyw, which is isomorphic to the pattern 231; this means that π is not 123-sortable. $\square$

Corollary 5.5. *Let $\pi \in \mathrm{Sort}_n^\downarrow(123)$ and suppose that $\pi_1 \neq n$. Also, suppose that $\pi_i = n$. Then either $\pi_{i-1} = n - 1$ (if $\pi_1 \neq n - 1$) or $\pi_{i-1} = n - 2$ (if $\pi_1 = n - 1$).*

Proof. Notice that $i \in \{3, 4, \ldots, n\}$ (indeed $i \neq 1$ by hypothesis and $i \neq 2$ since π starts with a descent). The element n enters the first stack immediately above π_{i-1}, since pushing n into the stack can never generate a forbidden pattern. Moreover, n and π_{i-1} exit the stack together, since n cannot play the role of the second element in a forbidden pattern inside the stack. Therefore, $s_{123}(\pi)$ contains the factor $n\pi_{i-1}$. By Lemma 5.4, this implies the result. $\square$

Corollary 5.6. *The set of permutations of $\mathrm{Sort}_n^\downarrow(123)$ starting with n is the set of 213-avoiding permutations of length n starting with n, for all $n \geq 2$.*

Proof. Let $\pi \in \mathrm{Sort}_n^\downarrow(123)$, and suppose that π starts with n. As soon as n enters the stack, it makes the stack act as a $\lfloor \frac{1}{2} \rfloor$-avoiding stack for the rest of the permutation. Therefore, by Theorem 2.1, π is 123-sortable if and only if the permutation obtained from π by removing the first element avoids 213, which is in turn equivalent to the fact that π avoids 213. $\square$

Since it is well known that 213-avoiding permutations are counted by Catalan numbers, an immediate consequence of the previous corollary is that the number of permutations of $\mathrm{Sort}_n^\downarrow(123)$ starting with n is the $(n-1)$-th Catalan number C_{n-1}.

In order to completely characterize the set $\mathrm{Sort}_n^\downarrow(123)$, what we need to do is to analyze the subset of $\mathrm{Sort}_n^\downarrow(123)$ consisting of permutations which do not start with n. In other words, these are the permutations of $\mathrm{Sort}_n^\downarrow(123)$ having at least two LTR maxima. Denote this set by $\mathrm{Sort}_n^\downarrow(\geq 2; 123)$. Moreover, the set of permutations of $\mathrm{Sort}_n^\downarrow(123)$ having precisely i LTR maxima will be denoted $\mathrm{Sort}_n^\downarrow(i; 123)$.

Theorem 5.7. *Let $n \geq 3$. There exists a bijection $\varphi : \mathrm{Sort}_{n-1}^\downarrow(123) \to \mathrm{Sort}_n^\downarrow(\geq 2, 123)$. Moreover, the restriction of φ to $\mathrm{Sort}_{n-1}^\downarrow(i; 123)$ is a bijection between $\mathrm{Sort}_{n-1}^\downarrow(i; 123)$ and $\mathrm{Sort}_n^\downarrow(i+1; 123)$.*

Proof. Let $\pi = \pi_1 \cdots \pi_{n-1} \in \mathrm{Sort}_{n-1}^\downarrow(123)$. Let $\varphi(\pi) = \tilde{\pi}$ be obtained from π by inserting n:

- either after $n - 1$, if $\pi_1 \neq n - 1$, or

- after $n - 2$, if $\pi_1 = n - 1$.

First we show that φ is well defined, i.e. that $\tilde{\pi} \in \mathrm{Sort}_n^\downarrow(\geq 2; 123)$. We analyze the two cases in the definition of φ separately.

If $\pi \in \mathrm{Sort}_{n-1}^\downarrow(1; 123)$ (that is $\pi_1 = n - 1$), then, by Lemma 5.4, $s_{123}(\pi) = (n-2)(n-3)\cdots 1(n-1)$. Now we analyze what happens with input $\tilde{\pi}$ after the first pass through the (restricted) stack. Remember that the first element of $\tilde{\pi}$ is $n - 1$ and that n immediately follows $n - 2$; moreover, suppose that $n - 2$ is the i-th element of $\tilde{\pi}$. Therefore, the first i elements of π and $\tilde{\pi}$ are equal, and so they are processed exactly in the same way by the stack. In particular, since $n - 2$ is the first element of $s_{123}(\pi)$, when $n - 2$ enters the stack, all the previous elements of $\tilde{\pi}$ are still inside the stack. Immediately after $n - 2$ enters the stack, n enters the stack as

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well, since it cannot produce a forbidden pattern inside the stack. Now we claim that n and $n-2$ exit the stack together. In fact, if n is not the last element of $\tilde{\pi}$, consider the next element π_{i+1}. Such an element cannot enter the stack, otherwise $n-1$ (which is at the bottom of the stack), $n-2$ and π_{i+1} would constitute a forbidden pattern. Thus $n-2$ must exit the stack before π_{i-1} enters it, and this forces n to exit as well. As a consequence of this fact, we have that $s_{123}(\tilde{\pi}) = n(n-2)(n-3)\cdots1(n-1)$, and such a permutation does not contain the pattern 231. Hence $\tilde{\pi}$ is 123-sortable.

If $\pi \in \mathrm{Sort}_{n-1}^{\downarrow}(\geq 2; 123)$ (that is $\pi_1 = k \neq n-1$), then $s_{123}(\pi) = (n-1)(n-2)\cdots(k+1)(k-1)\cdots21k$, and an analogous argument proves that $s_{123}(\tilde{\pi}) = n(n-1)(n-2)\cdots(k+1)(k-1)\cdots21k$, and so that $\tilde{\pi}$ is 123-sortable.

To complete the proof we now have to show that φ is a bijection. The fact that φ is injective is trivial. To show that φ is surjective, consider the map $\psi : \mathrm{Sort}_n^{\downarrow}(\geq 2; 123) \to \mathrm{Sort}_{n-1}^{\downarrow}(123)$ which removes n from $\alpha = \alpha_1\alpha_2\cdots\alpha_n \in \mathrm{Sort}_n^{\downarrow}(\geq 2; 123)$. Set $\psi(\alpha) = \hat{\alpha}$. Let $i \in \{3, 4, \ldots n\}$ such that $\alpha_i = n$. From Corollary 5.5 we have that either $\alpha_{i-1} = n-1$ (if $\alpha_1 \neq n-1$) or $\alpha_{i-1} = n-2$ (if $\alpha_1 = n-1$). Moreover, Lemma 5.4 implies that $s_{123}(\pi) = n(n-1)\cdots(k+1)(k-1)\cdots21k$, with $k = \alpha_1 \geq 2$. Therefore, when n enters the stack, all the previous elements are still inside the stack. In particular, at the top of the stack there are n and α_{i-1}. Now notice that, if n is forced to exit the stack, this is due to the fact that there exist j, h, l, with $j < h \leq i$ and $l > i$, such that α_j, α_h and α_l form an occurrence of 321. However, it cannot be $h = i$, since n cannot play the role of the 2 in a 321. Similarly, it cannot be $h = i-1$: in fact, if $\alpha_{i-1} = n-1$, then n and $n-1$ are consecutive in the stack and so they play the same role in any pattern; if instead $\alpha_{i-1} = n-2$, then $\alpha_1 = n-1$ is at the bottom of the stack, and so n and $n-2$ play the same role in any forbidden pattern. As a consequence, $h < i-1$, and so n and α_{i-1} are forced to leave the stack together. This means that basically n does not modify the behavior of the machine, and so $s_{123}(\hat{\alpha}) = (n-1)(n-2)\cdots(k+1)(k-1)\cdots21k$, that is $\hat{\alpha}$ is 123-sortable, as desired. $\qquad\square$

Corollary 5.8. *For all $n \geq 3$, $|\mathrm{Sort}_n^{\downarrow}(\geq 2; 123)| = |\mathrm{Sort}_{n-1}^{\downarrow}(123)|$.*

What we have proved so far, and in particular Corollary 5.3, Corollary 5.6 and Theorem 5.7, completely determine the structure of 123-sortable permutations. Indeed, any $\pi \in \mathrm{Sort}_n(123)$ which is not the identity permutation can be uniquely constructed as follows:

- choose $\alpha = \alpha_1\alpha_2\cdots\alpha_k \in \mathrm{Av}_k(213)$, with $\alpha_1 = k \geq 2$;

- add h new maxima, $k+1, \ldots k+h$, one at a time, using the bijection φ of Theorem 5.7;

- add $n-k-h$ consecutive ascents at the beginning, by inflating the first element of the permutation, according to Corollary 5.3.

As an example to illustrate the above construction, consider the permutation $\pi = 567148923$. By deflating the starting consecutive ascents we get the permutation 5146723; we then observe that the last permutation is obtained by adding two new maxima to the permutation $51423 \in \mathrm{Av}(213)$ according to the bijection of Theorem 5.7. Since 51423 starts with its maximum, we can conclude that π is 123-sortable.

The above construction allows us to enumerate $\mathrm{Sort}_n(123)$.

Theorem 5.9. *For all $n \geq 1$,*

$$|\mathrm{Sort}_n(123)| = 1 + \sum_{h=1}^{n-1} (n-h)C_h,$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the n-th Catalan number.

Proof. A permutation $\pi \in \text{Sort}_n(123)$ is either the identity or it is obtained by choosing a permutation α of $\text{Av}_k(213)$ starting with its maximum k (with $k \geq 2$) and then (possibly) adding the remaining $n - k$ elements according to the above construction, i.e. adding new maxima and/or some consecutive ascents at the beginning. Concerning α, there are C_{k-1} possible choices, thanks to the observation following Corollary 5.6. Concerning the remaining elements, one has to choose, for instance, the number of new maxima to add, which runs from 0 to $n - k$, so that the total number of choices is $n - k + 1$. Summing on all possible values of k, we get:

$$|\text{Sort}_n(123)| = 1 + \sum_{k=2}^{n} C_{k-1} \cdot (n - k + 1) = 1 + \sum_{h=1}^{n-1} (n - h) C_h,$$

as desired. □

From the above closed form it is not difficult to find the generating function of $|\text{Sort}_n(123)|$. However, we prefer to use a bijective argument. The integer sequence $(|\text{Sort}_n(123)|)_{n \in \mathbb{N}}$ is sequence A294790 in [9]. A combinatorial interpretation of this sequence has been given by Cioni and Ferrari [3]: it counts the number of Schröder paths avoiding the pattern UH_2D.

We say that a Schröder path P avoids the pattern UH_2D when P does not contain three steps that, read from left to right, are U, H_2 and D, respectively. In [3] it is observed that Schröder paths avoiding UH_2D are precisely those of the form $\text{H}_2^\alpha Q \text{H}_2^\beta$, where Q is a Dyck path.

We now describe a bijection f between 123-sortable permutations of length n and UH_2D-avoiding Schröder paths of semilength $n-1$. Given $\pi \in \text{Sort}_n(123)$, we decompose it as $\pi = Lw$, where L is the (possibly empty) initial sequence of consecutive ascents of π deprived of the last element and w is the remaining suffix of π. Suppose that L has length r. Now repeatedly remove the maximum from w until the remaining word v starts with its maximum. Denote with s the number of elements removed in this way. The permutation obtained from v after rescaling is then a 213-avoiding permutation ρ of length $k + 1 = n - r - s$ that starts with its maximum. Removing the maximum from ρ results in another 213-avoiding permutation σ of length k. We can now describe the Schröder path $f(\pi)$ associated with π: it starts with r double horizontal steps and ends with s double horizontal steps; in the middle, there is the Dyck path of semilength k associated with the 213-avoiding permutation σ through the bijection described in Section 2. For instance, referring to the above notations, given the permutation $\pi = 567489132$, we have $L = 56$, $w = 7489132$, and so $r = s = 2$. Moreover, $\sigma = 4132$, and so the Dyck path associated with σ is UDUUDUDD. Finally, we thus have that $f(\pi) = \text{H}_2\text{H}_2\text{UDUUDUDDH}_2\text{H}_2$ (see Figure 4).
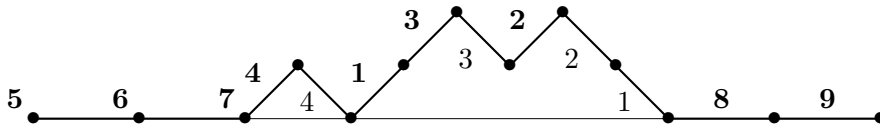

Figure 4: A Schröder path avoiding UH_2D; the associated permutation is obtained by reading the bold labels according to the bijection described above.

By exploiting the above bijection we now derive the generating function for the sequence $(|\text{Sort}_n(123)|)_{n \in \mathbb{N}}$. Let $C(x)$ be the generating function for the Catalan numbers. A generic UH_2D-avoiding Schröder path either consists of double horizontal steps only (with generating function $(1 - x)^{-1}$), or it can be obtained by concatenating an initial sequence of double horizontal steps (having generating function $(1 - x)^{-1}$) with a nonempty Dyck path (whose generating function is $(C(x) - 1) \cdot x$, where the additional factor x takes into account the removal

of the starting maximum from the permutation ρ above), finally adding a sequence of double horizontal steps (again with generating function $(1-x)^{-1}$). Summing up, we get:

$$\sum_{n \geq 0} |\text{Sort}_n(123)|x^n = \frac{1}{1-x} + \frac{1}{1-x}\big(x(C(x)-1)\big)\frac{1}{1-x} = \frac{(1-x)^2}{1-2x+xC(x)}.$$

6 Further work

Our paper just scratches the surface of the subject, and opens the way to the investigation of sorting permutations using restricted stacks in series, where the restrictions are expressed in terms of pattern avoidance.

Along the lines of this work, concerning patterns of length 3, there are four cases yet to study (corresponding to the four nonmonotone patterns). This is something we are presently working on. We have some data concerning the enumeration of σ-sortable permutations of length n, for small values of n. In the following table we report our data, including the (conjectured) references to [9], where applicable.

$\sigma \setminus n$	0	1	2	3	4	5	6	7	8	9	10	11	OEIS
132	1	1	2	5	15	51	188	731	2950	12235	51822	223191	A294790
213	1	1	2	5	16	62	273	1307	6626	35010	190862	1066317	unknown
231	1	1	2	6	23	102	496	2569	13934	78295	452439	2674769	unknown
312	1	1	2	5	15	52	201	843	3764	17659	86245	435492	A202062

Concerning longer patterns, it would be interesting to classify σ-machines in terms of the number of permutations they sort. This gives rise to a notion of *Wilf-equivalence* on σ-machines. This seems to be particularly interesting when the set of σ-sortable permutations constitute a class. For instance, for patterns of length 4, there are precisely two Wilf-equivalence classes, corresponding to two types of σ-machines, which depend on the number of sortable permutations: the resulting sequences are Catalan numbers and (essentially) odd-indexed Fibonacci numbers (sequence A001519 in [9]).

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