# ON ZEROS OF CHARACTERS OF FINITE GROUPS

SILVIO DOLFI, EMANUELE PACIFICI, AND LUCIA SANUS

Abstract. We survey some results concerning the distribution of zeros in the character table of a finite group and its influence on the structure of the group itself.

## 1. Introduction

Let $G$ be a finite group. If the character table of $G$ is known, then some very deep structural information on $G$ can be deduced; in fact, an important problem in character theory is to determine which structural features of $G$ can be detected by the knowledge of the character table of $G$ and, on the other hand, which aspects of the table are significant for this purpose.

Many results in the literature show that the *distribution of zeros* in the character table is relevant in this contex. Our aim in this paper is to present an outline of this research topic. We will discuss several aspects of the subject, from classical results to recent developments, and point out some open problems that could be of interest. (For the convenience of the reader, questions and conjectures are emphasized in slanted text.)

A large number of the results quoted in this paper rely on the classification of finite simple groups. This holds for virtually all the statements discussed from Section 4 to Section 9, except in some obvious situations (for instance, when the analysis involves only solvable groups). As for Sections 2 and 3, we indicate explicitly the cases in which the classification comes into play.

In what follows, every group is assumed to be finite and, for the notation, we refer to [16].

## 2. A theorem by W. Burnside

As one of the triggers for the research concerning zeros of characters, we recall a classical result by W. Burnside (Theorem (3.15) of [16]).

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**Theorem 2.1.** *Let $G$ be a group, and $\chi$ an irreducible character of $G$ which is nonlinear (i.e., whose degree is larger than 1). Then there exists $g \in G$ such that $\chi(g) = 0$.*

This important theorem has been extended in several directions. The following result by G. Navarro yields Burnside's theorem if the subgroup $N$ is chosen to be the trivial subgroup of $G$.

**Theorem 2.2** ([30], Theorem A)**.** *Let $G$ be a group, $\chi$ an irreducible character of $G$, and $N$ a normal subgroup of $G$. Then the restriction $\chi_N$, which is a character of $N$, is not irreducible if and only if there exists $g \in G$ such that $\chi(x) = 0$ for every $x \in gN$.*

As another kind of extension for Burnside's theorem, G. Malle, G. Navarro and J.B. Olsson investigated the relationship between the "arithmetical structure" of the degree of an irreducible character and that of an element on which the character vanishes.

**Theorem 2.3** ([24], Theorem B)**.** *Let $G$ be a group, and $\chi$ an irreducible character of $G$ which is nonlinear. Then there exists a prime number $p$ and a $p$-element $g \in G$ such that $\chi(g) = 0$.*

Now, let $p$ be a prime number. Recalling that, whenever a character $\chi$ of the group $G$ vanishes on a $p$-element of $G$, then the degree of $\chi$ is a multiple of $p$ ([6, Corollary 2.2]), the above theorem immediately yields the following nice corollary.

**Corollary 2.4** ([24], Theorem A)**.** *Let $G$ be a group, and $\chi$ an irreducible character of $G$ which is nonlinear. Assume that the degree of $\chi$ is a $\pi$-number, where $\pi$ is a set of primes. Then there exists a $\pi$-element $g \in G$ such that $\chi(g) = 0$.*

The two aforementioned results of [24] rely on the classification of finite simple groups.

In view of the previous statements, one may wonder whether it is true that if $\chi(1)$ is a $p$-power, then $\chi$ does not vanish on $p'$-elements. This is false in general; for instance, the Mathieu group $\mathrm{M}_{11}$ has an irreducible character of degree 11 which takes value 0 on an element of order 6. Moreover, a solvable example can be obtained considering the wreath product $G = C_6 \wr C_5$ of a cyclic group of order 6 with a cyclic group of order 5; it is not hard to check that $G$ has an irreducible character of degree 5 (induced from the base group) which vanishes on an element of order 6.

Nevertheless, in the solvable context and for *primitive* characters, the situation is quite neat.

**Theorem 2.5** ([29], Corollary B)**.** *Let $G$ be a solvable group, and $\chi$ a primitive character of $G$ which is nonlinear. Assume that the degree of $\chi$ is a $\pi$-number,* *where $\pi$ is a set of primes. Then, for $x \in G$, we have $\chi(x) = 0$ if and only if $\chi(x_\pi) = 0$, where $x_\pi$ denotes the $\pi$-part of the element $x$.*

Finally, another question that may arise looking at Theorem 2.3 is the following: does a nonlinear irreducible character always have zeros of prime order? The answer turns out to be affirmative for simple groups, as shown in [24] (in which simple groups of Lie type and sporadic simple groups are treated) and in [2] (where the authors consider alternating groups). On the other hand, the answer is negative in general: it is enough to consider the quaternion group $Q_8$, in which the unique element of prime order is central and therefore not a zero for any irreducible character. The next result yields some information in this context.

**Theorem 2.6** ([25], Theorem A)**.** *Let $\chi$ be a faithful irreducible character of $G$, and assume that $\chi(1)$ is a power of a prime $p$. If $\chi(x) \neq 0$ for every element $x \in G$ of order $p$, then the Sylow $p$-subgroups of $G$ are either cyclic or generalized quaternion groups.*

Note that if the degree of a faithful irreducible character $\chi$ is not a $p$-power, the condition that every element of order $p$ is not a zero for $\chi$ does not imply that the Sylow $p$-subgroups of $G$ are cyclic or generalized quaternion groups. In fact, consider $G = \mathrm{PSL}(2, 7)$; then $G$, which has dihedral Sylow 2-subgrops, also has an irreducible character $\chi$ of degree 6, such that $\chi(x) \neq 0$ for every involution $x \in G$.

## 3. Vanishing elements

Another way of stating Theorem 2.1 is the following.

*Let $\mathcal{R}$ be a row in the character table of a group $G$. Then $\mathcal{R}$ contains zeros if and only if $\mathcal{R}$ corresponds to a nonlinear character.*

(In fact, Theorem 2.1 provides the "if" part, whereas the "only if" part is an elementary fact in character theory.) So, the problem of determining which rows in the character table of a group actually contain zeros is completely solved.

Now, if one considers the "dual" question of *which columns* in the character table of a group may contain zeros, the situation is much more complicated. In this context the relevant objects are the so-called "vanishing elements", introduced in an important paper by I.M. Isaacs, G. Navarro and T.R. Wolf ([17]): an element $g \in G$ is a vanishing element if there exists an irreducibe character $\chi$ of $G$ such that $\chi(g) = 0$. The question we are considering is therefore related to understanding which elements of a group are vanishing elements.

Given the standard duality between results concerning rows (i.e., irreducible characters) and columns (i.e., conjugacy classes) in the character table of a group, one might naively ask whether the following holds.

*Let $\mathcal{C}$ be a column in the character table of a group $G$. Is it then true that $\mathcal{C}$ contains zeros if and only if $\mathcal{C}$ corresponds to a noncentral conjugacy class?*

It is immediately clear that the "only if" part is true by elementary arguments, but the "if" part fails in general. In order to see it, we can just consider a 3-cycle in the symmetric group $S_3$: such an element is obviously noncentral and also nonvanishing in that group.

Certainly there are special situations in which the "if" part is also true (for instance, it holds for nilpotent groups, as shown in Theorem B of [17]), but in general a nonvanishing element of $G$ can even fail to lie in any *abelian* normal subgroup of $G$. Actually, in Section 5 of [17], the authors provide the following family of examples: for every prime $p$, they construct a solvable group $G$ having nonvanishing $p$-elements (also, elements of order $p$ when $p \neq 2$) which do not lie in any abelian normal subgroup of $G$.

However, under some suitable assumptions, a nonvanishing element of $G$ is forced to lie in a *nilpotent* normal subgroup of $G$ (i.e., it lies in the Fitting subgroup $\mathbf{F}(G)$). In fact, the main result of [17] is as follows.

**Theorem 3.1** ([17], Theorem D)**.** *Let $G$ be a solvable group. If $g$ is a nonvanishing element of $G$, then the image of $g$ under the natural homomorphism onto $G/\mathbf{F}(G)$ has 2-power order.*

In particular, in a solvable group $G$, the nonvanishing elements *of odd order* lie in $\mathbf{F}(G)$.

*In* [17]*, the authors actually conjecture that* every *nonvanishing element of a solvable group $G$ lies in* $\mathbf{F}(G)$. They point out that their methods would prove this claim, if it can be proved that every nonvanishing element *of order* 2 of a solvable group $G$ lies in an abelian normal subgroup of $G$ (recall that, in [17, Section 5], the authors provide a counterexample to a similar statement where 2 is replaced by any odd prime). However, in a recent paper ([15]), M. Grüninger constructs an example of a solvable group having a nonvanishing involution which fails to lie in any abelian normal subgroup, thus showing that the prime 2 is not an exception. In any case, at the time of this writing, the conjecture by Isaacs, Navarro and Wolf is still an open problem.

On the other hand, the assumption of solvability is certainly crucial in Theorem 3.1. If we look, for instance, at the character table of the alternating group $A_7$ (whose Fitting subgroup is of course trivial), we see that there are nonvanishing elements of order 2 and 6, but also of odd order (namely, of order 3).

In fact, the primes 2 and 3 do play a distinguished role in this context.

**Theorem 3.2** ([8], Theorem A)**.** *Let $G$ be a group, and $g \in G$ an element whose order is coprime to 6. If $g$ is a nonvanishing element of $G$, then $g \in \mathbf{F}(G)$.*

(The proof of the above theorem uses the classification of finite simple groups.) It seems natural to think that, for any group $G$, the nonvanishing elements of $G$ should always lie in the generalized Fitting subgroup $\mathbf{F}^*(G)$, but this is not true.

The group $G = 2^{11} : \mathrm{M}_{24}$ has nonvanishing elements of order 2 and 4 not lying in $\mathbf{F}(G) = \mathbf{F}^*(G)$. *However, we conjecture that any nonvanishing element of odd order of a group $G$ lies in $\mathbf{F}^*(G)$.*

In order to give an idea of some methods that are relevant in the present context, we close this section with two easy remarks and a last theorem.

**Proposition 3.3.** *Let $N$ be a normal subgroup of $G$, and let $\theta$ be an irreducible character of $N$. Then every element of $G$ not lying in $\bigcup_{g \in G} I_G(\theta^g)$ is a vanishing element of $G$.*

(In the above statement, $I_G(\theta)$ denotes the inertia subgroup of $\theta$ in $G$, i.e., the stabilizer of $\theta$ in the natural action of $G$ on $\mathrm{Irr}(N)$.)

**Proposition 3.4.** *Let $N$ be a normal subgroup of $G$, and $p$ a prime. If there exists an irreducible character of $p$-defect zero of $N$ (i.e., a character $\theta \in \mathrm{Irr}(N)$ such that $p$ does not divide $|N|/\theta(1)$), then every $g \in N$ with $p \mid o(g)$ is a vanishing element of $G$.*

Proposition 3.3 (whose proof is an immediate application of Clifford's Theory) is particularly useful in the case when $N$ is an elementary abelian $p$-group for some prime $p$ (for instance, when $N$ is an abelian minimal normal subgroup of $G$). In this situation, the set $\mathrm{Irr}(N)$ is an elementary abelian $p$-group as well, and the natural action of $G$ on this set can be regarded as a module action. By Proposition 3.3 an element $g \in G$ is a vanishing element provided, under this natural action, there exists a *deranged orbit* for $g$, i.e., an orbit in which no element is fixed by $g$. So, the study of certain orbit properties in module actions turns out to be crucial when dealing with vanishing elements.

Also Proposition 3.4 can be proved by means of elementary character theory, taking into account that an irreducible character of $p$-defect zero takes value zero on every element of the group whose order is divisible by $p$. This proposition comes into play when $N$ is a *nonabelian* minimal normal subgroup of $G$. In this case, in fact, $N$ is a direct product $S_1 \times \cdots \times S_k$ of pairwise isomorphic nonabelian simple groups and, given a prime divisor $p$ of $|N|$, irreducible characters of $p$-defect zero of $N$ very often exist (this happens in particular whenever the $S_i$ are simple groups of Lie type).

We note that, as Proposition 3.4 may suggest, nonsolvable groups tend to have a large number of vanishing elements (for instance, by the above remarks about the existence of characters of $p$-defect zero, every nontrivial element of a simple group of Lie type is vanishing); in other words, a small ratio of vanishing elements in the group should imply solvability. *In fact, we conjecture that the smallest value of this ratio among nonsolvable groups is attained by the alternating group $\mathrm{A}_7$, in which the vanishing elements are 2134 out of 2520 ($\sim 85\%$).*

The two propositions above, together with some other techniques and ideas (and the classification of finite simple groups), are used in order to prove the following theorem, which is in turn very useful for locating vanishing elements.

**Theorem 3.5** ([3], Corollary 4.4). *Let $A$ be an abelian minimal normal subgroup of $G$. Let $N/M$ be a chief factor of $G$ such that $|N/M|$ is coprime with $|A|$ and $\mathbf{C}_N(A) = M$. Then every element of $N \setminus M$ is a vanishing element of $G$.*

## 4. Ito-Michler Theorem and vanishing elements

An important object that can be "extracted" from the character table of a group $G$ is the set $\mathrm{cd}(G)$, whose elements are the degrees of the irreducible characters of $G$. Even this relatively small set of positive integers, as shown by many results in the literature, encodes nontrivial information about the structure of $G$; in particular, there is a significant interplay between the group structure and the arithmetical structure of $\mathrm{cd}(G)$ (i.e., the way in which the numbers in this set decompose into prime factors). As a famous example of this relationship, we recall the celebrated Ito-Michler Theorem.

**Theorem 4.1** (Ito-Michler). *Let $G$ be a group, and $p$ a prime. Then every number in $\mathrm{cd}(G)$ is not divisible by $p$ if and only if $G$ has an abelian normal Sylow $p$-subgroup.*

The above statement can be regarded as a model for a certain kind of results that, following G. Navarro, we call "Ito-Michler type" theorems (see [31]). The question addressed in such theorems is the following (or a variation of it): consider a finite nonempty set $X$ of positive integers which is attached to a group $G$, and assume that a given prime $p$ does not divide any number in $X$; which structural properties of $G$ can be derived as a consequence of this assumption?

Many sets of positive integers, related with a finite group $G$, have been considered in the literature. Among them, some classical examples are the set $\mathrm{o}(G)$ of orders of the elements of $G$, and the set $\mathrm{cs}(G)$ of conjugacy class sizes of $G$. Now, these sets can be "filtered" by means of the irreducible characters of $G$, in terms of the zeros appearing in the character table of $G$: namely, our following discussion will focus on the sets

$$\mathrm{vo}(G) = \{o(g) \mid g \text{ is a vanishing element of } G\}$$

and

$$\mathrm{vcs}(G) = \{|g^G| \mid g \text{ is a vanishing element of } G\},$$

where by $g^G$ we denote the conjugacy class of the element $g$ in $G$.

## 5. Ito-Michler type theorems: the set vo($G$)

In this section we survey some Ito-Michler type theorems concerning the first of the two sets introduced above (or theorems that, however, relate some arithmetical properties of this set to the group structure). We start by considering the situation when, given a prime $p$, the set $\mathrm{vo}(G)$ does not contain any $p$-power.

**Theorem 5.1** ([12], Theorem A)**.** *Let $G$ be a group, $p$ a prime number, and $P$ a Sylow $p$-subgroup of $G$. Assume that, for every $\chi \in \mathrm{Irr}(G)$ and $x \in P$, we have $\chi(x) \neq 0$ (i.e., assume that $\mathrm{vo}(G)$ does not contain any $p$-power). Then $G$ has a normal Sylow $p$-subgroup.*

The above statement (which is a consequence of Theorem 3.2 if $p$ is larger than 3) is actually a bit stronger than a classical Ito-Michler type theorem, as the assumption that $p$ does not divide any number in $\mathrm{vo}(G)$ clearly implies the hypothesis of Theorem 5.1.

Also the original Ito-Michler assumption that $p$ does not divide any number in $\mathrm{cd}(G)$ implies the hypothesis of Theorem 5.1, because, as recalled in the paragraph following Theorem 2.3, an irreducible character of $G$ which vanishes on a $p$-element has a degree divisible by $p$. On the other hand, the converse is not true. For example, let $G$ be the normalizer of a Sylow 2-subgroup in the Suzuki group Suz(8); then $G$ is a Frobenius group with a Frobenius complement of order 7 and a nonabelian Frobenius kernel of order $2^6$. It turns out that $\mathrm{vo}(G) = \{7\}$, and $\mathrm{cd}(G) = \{1, 7, 14\}$. More generally, [6, Example 1] shows that there is no bound on the derived length of the Sylow $p$-subgroup of a group $G$ such that $\mathrm{vo}(G)$ does not contain any $p$-power.

As an immediate consequence of Theorem 5.1, we get the following refinement of another famous result by Burnside, the so-called $p^\alpha q^\beta$ Theorem.

**Theorem 5.2** ([12], Corollary B)**.** *Let $G$ be a group, and let $p$, $q$ be prime numbers. If every vanishing element of $G$ is a $\{p, q\}$-element, then $G$ is solvable.*

In the next result the hypothesis of Theorem 5.1 is relaxed, assuming only that $\mathrm{vo}(G)$ does not contain the prime $p$.

**Theorem 5.3** ([12], Theorem 4.3)**.** *Let $G$ be a group, and $p$ a prime divisor of $|G|$. Assume that either $p$ is odd, or that $p = 2$ and $G$ has no composition factor isomorphic to $\mathrm{M}_{22}$, $\mathrm{A}_7$ or $\mathrm{A}_{15}$. If $\mathrm{vo}(G)$ does not contain $p$, then $\mathbf{O}_p(G) \neq 1$.*

Before we proceed in our discussion related to Ito-Michler type theorems, we take some time to consider the opposite situation in which the set $\mathrm{vo}(G)$ only contains $p$-powers, or it even reduces to a single prime number.

**Theorem 5.4** ([6], Theorem A)**.** *Let $G$ be a nonabelian group, and $p$ a prime. If every number in $\mathrm{vo}(G)$ is a $p$-power, then one of the following holds.*

(a) *$G$ is a $p$-group.*

(b) *$G/\mathbf{Z}(G)$ is a Frobenius group with a Frobenius complement of $p$-power order and $\mathbf{Z}(G) = \mathbf{O}_p(G)$.*

**Theorem 5.5** ([6], Theorem B)**.** *Let $G$ be a nonabelian group, and $p$ a prime. If $\mathrm{vo}(G) = \{p\}$, then one of the following holds.*

(a) *$G$ is a $p$-group of exponent $p$.*

(b) *$G = E \times F$, where $E$ is a (possibly trivial) elementary abelian $p$-group and $F$ is a Frobenius group with a Frobenius complement of order $p$.*

**Theorem 5.6** ([6], Theorem C)**.** *Let $G$ be a nonabelian group. Then $\mathrm{vo}(G) = \{2\}$ if and only if $G = E \times F$, where $E$ is an elementary abelian $2$-group and $F$ is a Frobenius group with a Frobenius complement of order $2$.*

Finally, we resume the discussion about Theorem 5.1 by observing that, as shown by any nonabelian $p$-group, the converse of that statement is false. In other words, Theorem 5.1 does not provide a characterization of normality for a Sylow $p$-subgroup in terms of the character table of $G$.

The problem of achieving such a characterization along this line was considered by G. Malle and G. Navarro in [23]. In that paper, the authors introduce one particular set of irreducible characters of a group: given a group $G$, a prime $p$ and a Sylow $p$-subgroup $P$ of $G$, they define

$$\mathrm{Irr}((1_P)^G) = \{\chi \in \mathrm{Irr}(G) \mid \langle \chi_P, 1_P \rangle \neq 0\},$$

i.e., the subset of $\mathrm{Irr}(G)$ whose elements are the irreducible constituent of the character of $G$ obtained by inducing the principal character of $P$.

Our discussion concerning Theorem 5.1 yields

$$p \nmid \chi(1) \text{ for every } \chi \in \mathrm{Irr}(G) \Rightarrow \chi(x) \neq 0 \text{ for every } \chi \in \mathrm{Irr}(G) \text{ and } x \in P \Rightarrow P \trianglelefteq G.$$

Now, if every occurence of $\mathrm{Irr}(G)$ in the previous line is replaced by $\mathrm{Irr}((1_P)^G)$, then both the implications are in fact "if and only if".

**Theorem 5.7** ([23], Theorem B)**.** *Let $G$ be a group, $p$ a prime number, and $P$ a Sylow $p$-subgroup of $G$. Then the following conditions are equivalent.*

(a) *For every $\chi$ in $\mathrm{Irr}((1_P)^G)$, the prime $p$ does not divide $\chi(1)$.*

(b) *For every $\chi$ in $\mathrm{Irr}((1_P)^G)$ and $x \in P$, we have $\chi(x) \neq 0$.*

(c) *$P \trianglelefteq G$.*

Therefore, while Ito-Michler Theorem yields a characterization of *normality and abelianity* of a Sylow $p$-subgroup in terms of the character table, the theorem above provides a neat characterization of *normality* for a Sylow $p$-subgroup in terms of the character table (namely, in terms of degrees and of the distribution of zeros in the character table).

## 6. Ito-Michler type theorems: the set vcs($G$)

In the same spirit as in the previous section, we now focus on the set of conjugacy class sizes of a group. First of all, we state the classical Ito-Michler type theorem on the whole set of class sizes, whose proof is an elementary exercise.

**Theorem 6.1.** *Let $G$ be a group, and $p$ a prime number. Then $p$ does not divide any number in* $\mathrm{cs}(G)$ *if and only if $G$ has a central Sylow $p$-subgroup (i.e., $G$ has a $p$-complement $H$ that is a direct factor, and $G/H$ is abelian).*

What if the Ito-Michler assumption is required only for the sizes of the vanishing conjugacy classes? In this case, the right idea is to focus on the *principal $p$-block* (see for instance [28, p. 49]). In fact, the following lemma turns out to be a crucial one.

**Lemma 6.2.** *Let $G$ be a group, $p$ a prime, and $B_0$ the principal $p$-block of $G$. If $\chi$ is an (ordinary) irreducible character of $G$ lying in $B_0$, and $x \in G$ is such that $\chi(x) = 0$, then $p$ divides $|x^G|$.*

*Proof.* Let $\mathbf{R}$ be the ring of algebraic integers in the complex field, and let $M$ be a fixed maximal ideal of $\mathbf{R}$ containing the ideal generated by $p$. Also, denote by $^*$ the natural homomorphism of $\mathbf{R}$ onto the field $\mathbf{R}/M$.

By definition, since the irreducible character $\chi$ of $G$ lies in $B_0$, we get

$$\left(\frac{|g^G| \cdot \chi(g)}{\chi(1)}\right)^* = |g^G|^*$$

for every $g \in G$. In particular, as $\chi(x) = 0$, we have that $|x^G|$ is an integer lying in $p\mathbf{R}$; it follows that $p$ divides $|x^G|$, as claimed. ∎

Define now

$$\mathrm{Van}(B_0) = \{x \in G \mid \chi(x) = 0 \text{ for some } \chi \in \mathrm{Irr}(B_0)\}.$$

As an immediate consequence of Lemma 6.2, we obtain the following result.

**Theorem 6.3.** *Let $G$ be a group, and $p$ a prime number. Then $p$ does not divide $|x^G|$ for every $x \in$ $\mathrm{Van}(B_0)$ if and only if $G$ has a normal $p$-complement $H$ and $G/H$ is abelian.*

*Proof.* If $G$ has a normal $p$-complement $H$, then [28, Theorem 6.10] yields that $H$ is the intersection of the kernels of all the irreducible ordinary characters in $B_0$. Therefore, if $G/H$ is assumed to be abelian, these characters are in fact linear, so that $\mathrm{Van}(B_0)$ is empty and nothing else needs to be proved.

Conversely, if $p$ does not divide $|x^G|$ for every $x \in \mathrm{Van}(B_0)$, then Lemma 6.2 (together with Burnside's Theorem 2.1) yields that the irreducible characters in $B_0$ are all linear. Now, again Theorem 6.10 of [28] implies that $G'$ lies in $\mathbf{O}_{p'}(G)$ (the maximal normal $p'$-subgroup of $G$), and the desired conclusion easily follows. ∎

Thus, *if $G$ is a group and $p$ is a prime which does not divide any number in* $\mathrm{vcs}(G)$*, then $G$ has a normal $p$-complement $H$ and $G/H$ is abelian.* In fact, in [5, Theorem C], the author obtains a strengthening of this statement: if $p$ does not divide the class size *of any vanishing $p'$-element of $G$*, then $G$ has a normal $p$-complement with abelian factor group.

Observe that the structural information which is lost in this context, with respect to the stronger assumptions of Theorem 6.1, concerns *the normality* of a Sylow $p$-subgroup of $G$. In fact, in the symmetric group $G = \mathrm{S}_3$, the class of transpositions is the unique vanishing conjugacy class. This class has size 3, therefore the prime 2 does not divide any number in $\mathrm{vcs}(G)$; nevertheless, $G$ does not have a normal Sylow 2-subgroup.

Assume now that, for a given prime $p$, the group $G$ has a $p$-complement $H$, and let us define

$$\mathrm{Van}(G \,|\, 1_H) = \{x \in G \ \mid \ \chi(x) = 0 \text{ for some } \chi \in \mathrm{Irr}(G) \text{ with } \langle \chi_H, 1_H \rangle \neq 0\}.$$

Taking into account that every irreducible constituent of the induced character $(1_H)^G$ lies in $B_0$ (see [28, Theorem 2.27]) and arguing along the line of Theorem 6.3, it is not difficult to prove the following result, that is very much in the spirit of the work by Malle and Navarro in [23].

**Theorem 6.4.** *Let $p$ be a prime, and $G$ a group having a $p$-complement $H$. Then $p$ does not divide $|x^G|$ for every $x \in \mathrm{Van}(G \,|\, 1_H)$ if and only if $H \trianglelefteq G$ and $G/H$ is abelian.*

We close this section remarking that, again in the spirit of the work by Malle and Navarro in [23], *it could be interesting to find a characterization of $p$-nilpotency for a group $G$ (i.e., the existence of a normal $p$-complement $H \leq G$, but without any extra condition on $G/H$) in terms of vanishing conjugacy classes.*

## 7. Vanishing graphs

Given a nonempty finite set $X$ of positive integers, a way to express the arithmetical properties of the integers in $X$ is as follows. Consider the so-called *prime graph* on $X$, that is the simple undirected graph $\Delta(X)$ with vertex set

$$\mathrm{V}(\Delta(X)) = \{p \text{ prime } \ \mid \ \text{there exists } x \in X \text{ divisible by } p\},$$

and define two vertices $p$, $q$ to be adjacent in $\Delta(X)$ if there exists an integer $x \in X$ such that $pq$ divides $x$.

(Similarly, another graph that comes naturally into consideration is the "common divisor graph" $\Gamma(X)$, whose vertex set is $X \setminus \{1\}$ and $x, y \in X \setminus \{1\}$ are connected if $\gcd(x, y) \neq 1$.)

The main general question in this context is how the group structure of $G$ is related to the structure of the corresponding graphs $\Delta(X)$, for various sets $X$ of invariants of a group $G$.

As mentioned in Section 3, one of the earliest instances is the set $X = \mathrm{o}(G)$ consisting of the orders of the elements of the group $G$. The corresponding graph $\Pi(G) = \Delta(\mathrm{o}(G))$ is called the *Gruenberg-Kegel graph* and it has been extensively studied both in the solvable as well as in the nonsolvable case.

Among the various graph properties, the most commonly studied in the present literature are related to the diameter and the number of connected components. In the following discussion, given a graph $\Delta$, we denote by $\mathrm{n}(\Delta)$ the number of connected components of $\Delta$ and by $\mathrm{diam}(\Delta)$ its diameter. Finally, we denote by $\iota(\Delta)$ the independence number of $\Delta$, that is the largest size of an independent set, i.e. a subset of pairwise nonadjacent vertices of $\Delta$.

We recall that a group $G$ is said to be a 2-Frobenius group if there exist two normal subgroups $F$ and $L$ of $G$ such that $L$ is a Frobenius group with kernel $F$, and $G/F$ is a Frobenius group with kernel $L/F$. For the Gruenberg-Kegel graph of solvable groups, we have:

**Theorem 7.1** ([21], [35])**.** *Let $G$ be a solvable group.*

(a) $\mathrm{n}(\Pi(G)) \leq 2$*, i.e. $\Pi(G)$ has at most two connected components.*
(b) *If $\Pi(G)$ is disconnected, then $G$ is either a Frobenius or a 2-Frobenius group and each connected component of $\Pi(G)$ is a complete graph.*
(c) *For any choice of three vertices of $\Pi(G)$, at least two of them are adjacent in $\Pi(G)$ (i.e. $\iota(\Pi(G)) \leq 2$).*

Aiming at filtering the elements of the set $\mathrm{o}(G)$ by properties related to character values, in Section 4 we introduced the set $\mathrm{vo}(G)$ consisting of the orders of the vanishing elements of $G$. Accordingly, one defines the *vanishing* Gruenberg-Kegel graph $\Pi_v(G) = \Delta(\mathrm{vo}(G))$ of $G$ as the prime graph on the set of the orders of the vanishing elements of $G$. Clearly, $\Pi_v(G)$ is a subgraph of $\Pi(G)$. Still, it is not an *induced* subgraph: as an example, consider $G = \mathrm{S}_3 \times D_{10}$, where 3 and 5 are vertices of $\Pi_v(G)$ which are linked in $\Pi(G)$, but not in $\Pi_v(G)$.

In the process of comparing $\Pi(G)$ and $\Pi_v(G)$, one can first ask about the difference between the vertex sets $\mathrm{V}(\Pi(G))$ and $\mathrm{V}(\Pi_v(G))$.

**Theorem 7.2** ([13])**.** *Let $G$ be a nonabelian group, $p$ a prime number, and $P \in \mathrm{Syl}_p(G)$. If $p$ is a vertex of $\Pi(G)$ but not of $\Pi_v(G)$, then $P \trianglelefteq G$, $G/\mathbf{O}_{p'}(G)$ is a Frobenius group with kernel $P\mathbf{O}_{p'}(G)/\mathbf{O}_{p'}(G)$ and $\mathbf{O}_{p'}(G)$ is nilpotent.*

We say that a group $G$ is a *nearly 2-Frobenius group* if there exist two normal subgroups $F$ and $L$ of $G$ with the following properties: $F = F_1 \times F_2$ is nilpotent, where $F_1$ and $F_2$ are normal subgroups of $G$, $G/F$ is a Frobenius group with kernel

$L/F$, $G/F_1$ is a Frobenius group with kernel $L/F_1$, and $G/F_2$ is a 2-Frobenius group. The next result should be compared with Theorem 7.1.

**Theorem 7.3** ([14])**.** *Let $G$ be a solvable group. Then the following conclusions hold.*

(a) $\Pi_v(G)$ *has at most two connected components. If* $\Pi_v(G)$ *is disconnected, then each component is a complete graph, and $G$ is a Frobenius or a nearly* 2-*Frobenius group.*

(b) $\mathrm{diam}(\Pi_v(G)) \leq 4$.

We remark that the bound $\mathrm{diam}(\Pi_v(G)) \leq 4$ is sharp ([14, Example 5.2]).

By contrast, the similarity of the ordinary and vanishing Gruenberg-Kegel graphs breaks down when one considers independence numbers: while one has independent sets of maximal size two, the other can have arbitrarily large independent sets.

**Theorem 7.4** ([14], Theorem B)**.** *For every positive integer $k$, there exists a solvable group $G$ such that* $\Pi_v(G)$ *has an independent set of size $k$.*

Removing the assumption of solvability, from [19] and [35] it is possible do derive the following result.

**Theorem 7.5.**

(a) *If $S$ is a nonabelian simple group, then* $\mathrm{n}(\Pi(S)) \leq 6$.

(b) *Let $G$ be a nonsolvable group. If* $\Pi(G)$ *is disconnected, then $G$ has a unique nonabelian composition factor $S$, and* $\mathrm{n}(\Pi(G)) \leq \mathrm{n}(\Pi(S))$. *Hence,* $\mathrm{n}(\Pi(G)) \leq 6$.

Similarly, for the vanishing Gruenberg-Kegel graph:

**Theorem 7.6** ([13], Theorem A)**.** *Let $G$ be a finite group. Then the following conclusions hold.*

(a) $\Pi_v(G)$ *has at most six connected components.*

(b) *If* $\Pi_v(G)$ *is disconnected, then $G$ has a unique nonabelian composition factor $S$, and* $\mathrm{n}(\Pi_v(G)) \leq \mathrm{n}(\Pi(S))$ *unless $G$ is isomorphic to* $\mathrm{A}_7$.

In fact, it turns out that $\mathrm{A}_7$ is the unique nonabelian simple group $S$ such that $\Pi_v(S) \neq \Pi(S)$. Note that $\mathrm{n}(\Pi_v(\mathrm{A}_7)) = 4$, while $\mathrm{n}(\Pi(\mathrm{A}_7)) = 3$.

We stress that, notwithstanding the similarities among the two graphs, the edge set in the graph $\Pi_v(G)$ can be quite smaller than in the graph $\Pi(G)$; for any integer $k$, there exists a (nonsolvable) group $G$ such that $\Pi(G)$ has a complete subgraph on $k$ vertices, that instead induces an independent set in $\Pi_v(G)$ ([13, Example 6.5]).

*Other graph properties, like connectivity, chromatic number or girth, might be subjects for further investigation.*

Also the arithmetical properties of the sets $\mathrm{cd}(G)$ and $\mathrm{cs}(G)$, that have been introduced in Section 3, can be studied via the prime graph. Several properties of the graphs $\Delta(\mathrm{cd}(G))$ and $\Delta(\mathrm{cs}(G))$, as well as their connection to the algebraic structure of the group, have been studied in the last two decades. For an overview up to 2008, we refer to the survey paper [20].

In the same spirit, we now focus on the set $\mathrm{vcs}(G)$ of the sizes of vanishing classes. As we observed in Section 6, if a prime number $p$ is not a vertex of $\Delta(\mathrm{vcs}(G))$, then $G$ has a normal $p$-complement and abelian Sylow $p$-subgroups. We also observed that the vertex set of $\Delta(\mathrm{vcs}(G))$ can be smaller than that of $\Delta(\mathrm{cs}(G))$. Yet, if one assumes that $G$ has a nonabelian minimal normal subgroup, then the two vertex sets coincide, as proved in [3]. In this situation, the absence of an edge in the graph $\Delta(\mathrm{vcs}(G))$ reflects in the normal structure of $G$.

**Theorem 7.7** ([3], Theorem A)**.** *Let $G$ be a finite group, and suppose that $G$ has a nonabelian minimal normal subgroup. If $p$ and $q$ are vertices of $\Delta(\mathrm{vcs}(G))$, but there is no vanishing conjugacy class of $G$ whose size is divisible by $pq$, then $G$ is $\{p, q\}$-solvable.*

We remark that the assumption concerning the existence of a nonabelian minimal normal subgroup in $G$ is critical in the above statement. In fact, whenever $p$ and $q$ are primes such that $p \geq 7$ and $q \equiv 1 \pmod{5p}$, it is possible to construct a Frobenius group $H$ whose kernel is elementary abelian of order $q^2$ and whose complements are isomorphic to $C_p \times \mathrm{SL}(2,5)$; it is not difficult to see that $p$ is not a vertex in $\Delta(\mathrm{vcs}(H))$. Now, take $p = 7$, $q = 71$, and consider $G = \mathrm{D}_{10} \times H$ (where $\mathrm{D}_{10}$ is the dihedral group of order 10); clearly, 2 and 7 are nonadjacent vertices in $\Delta(\mathrm{vcs}(G))$ (although they are adjacent in $\Delta(\mathrm{cs}(G))$), nevertheless $G$ is not 2-solvable. (The authors whish to thank Victor Manuel Ortiz Sotomayor for pointing out this kind of examples; a solvable one is $G = \mathrm{D}_{10} \times \mathrm{A}_4$, in which 2 and 3 are nonadjacent vertices of $\Delta(\mathrm{vcs}(G))$ that are adjacent in $\Delta(\mathrm{cs}(G))$.)

A consequence of the previous theorem is that, still assuming the existence of a nonabelian minimal normal subgroup in $G$, if a vertex $p$ of $\Delta(\mathrm{vcs}(G))$ is not complete (i.e. adjacent to all other vertices), then the group $G$ is $p$-solvable.

Moreover, if the group has no abelian normal subgroup, then the graph $\Delta(\mathrm{vcs}(G))$ is complete.

**Theorem 7.8** ([3], Theorem B)**.** *Let $G$ be a finite group with trivial Fitting subgroup. Then every prime divisor of $|G|$ is a vertex of $\Delta(\mathrm{vcs}(G))$, and $\Delta(\mathrm{vcs}(G))$ is a complete graph.*

*We are not aware of any examples where, under the assumption that $G$ has a nonabelian minimal normal subgroup, two primes $p$ and $q$ are vertices of $\Delta(\mathrm{vcs}(G))$ that are not adjacent in this graph, but adjacent in $\Delta(\mathrm{cs}(G))$. In other words, it is an open question whether in this case $\Delta(\mathrm{vcs}(G)) = \Delta(\mathrm{cs}(G))$.*

## 8. The number of conjugacy classes of vanishing elements

Given an irreducible character $\chi$ of $G$, we define $v(\chi) = |\{x^G \;:\; \chi(x) = 0\}|$, the number of zero entries in the row corresponding to $\chi$ in the character table of $G$. By Burnside's theorem, $v(\chi) = 0$ if and only if $\chi$ is a linear character.

It is natural to ask how the largest number

$$M(G) = \max_{\chi \in \mathrm{Irr}(G)} v(\chi)$$

of zeros in a row of the character table of $G$ is related to the structure of $G$.

**Theorem 8.1** ([26], Theorem A)**.** *There exist two real numbers $c_1$ and $c_2$ such that, for every solvable group $G$ with $M(G) > 1$,*

$$h(G) \leq c_1 \log\log M(G) + c_2\,,$$

*where $h(G)$ is the Fitting height.*

In [26], Moreto and Sangroniz also prove that the index of suitable terms of the Fitting series of a solvable group $G$ can be bounded in terms of $M(G)$ ([26, Theorem B]). Furthermore, the order of a nilpotent group can be bounded by some function of $M(G)$. This is not true in general, as the dihedral groups show.

Similarly, one can consider the minimum number of zeros

$$m(G) = \min_{\chi \in \mathrm{Irr}(G), \chi(1) > 1} v(\chi)$$

appearing in the rows of the character table of a group $G$. Moreto and Sangroniz prove that the derived length of a $p$-group $P$ can be bounded by a function of $m(P)$ ([26, Theorem E]). They also propose the following conjectures.

**Conjecture 8.2** ([26], Conjectures F and G)**.** Let $G$ be a solvable group. Then

(a) the derived length $dl(G)$ and the index $|G : \mathbf{F}(G)|$ can be bounded in terms of $M(G)$;
(b) the Fitting height $h(G)$ can be bounded in terms of $m(G)$.

The finite groups whose irreducible characters vanish on "few" conjugacy classes have been classified.

**Theorem 8.3** ([1], Theorem 5; [7], Proposition 2.7)**.** *$M(G) = 1$ if and only if $G$ is a Frobenius group with complement of order 2.*

**Theorem 8.4** ([4], Theorem 1.1; [26], Theorem H)**.** *$M(G) = 2$ if and only if $G \simeq \mathrm{S}_4, \mathrm{A}_5, \mathrm{PSL}(2,7)$, or there is a normal subgroup $N$ with $M(G/N) = 1$ and $|N| = 2$ or $G$ is a Frobenius group with complement of order 3 and abelian kernel.*

Finally, a classification of the groups $G$ such that $M(G) = 3$ is given in [32].

Dually, looking at the columns of the character table of a group $G$, one defines $v^*(g) = |\{\chi \in \mathrm{Irr}(G) \;:\; \chi(g) = 0\}|$ and $M^*(G) = \max_{g \in G}\{v^*(g)\}$.

**Theorem 8.5** ([27, Theorem A])**.** *The number of nonlinear irreducible characters of $G$ is bounded in terms of $M^*(G)$. Hence, if $G$ is solvable, the derived length $dl(G)$ is bounded above by $M^*(G)$.*

In [34], one finds a complete classification of the groups $G$ such that $M^*(G) < p$, where $p$ is the smallest prime divisor of $|G|$; they are either isomorphic to $\mathrm{A}_5$ or they belong to one of seven families of solvable groups ([34, Theorem 1.1]).

*A natural question in this context is about groups that have "few" orbits of vanishing conjugacy classes, or of conjugacy classes that are zeros for single irreducible characters, under some natural actions (e.g. Galois conjugation).*

Finally, we mention a result that outlines a connection between rows and columns (from the point of view of zero entries) in a character table.

**Theorem 8.6** ([33])**.** *For any finite group $G$, the following conditions are equivalent.*

(a) *$v(\chi) \leq 1$ for all but one of the irreducible characters $\chi$ of $G$;*
(b) *$v^*(x^G) \leq 1$ for all but one of the conjugacy classes of $G$.*

*Moreover, $G$ satisfies one of the above condiditions if and only if $G$ is one of the following groups:*

- *an extra-special 2-group;*
- $\mathrm{SL}(2,3)$*,* $\mathrm{S}_4$ *or* $\mathrm{A}_8$*;*
- *a Frobenius group which is either 2-transitive with an abelian complement or it has a complement of order 2.*

## 9. Brauer characters

Unlike ordinary characters, it is possible that a nonlinear irreducible Brauer character does not vanish on any element. For instance, in characteristic 7, the irreducible Brauer characters of $\mathrm{PSL}(3,2)$ of degree 5 and 7 do not take the value 0.

Even more, there exist nonabelian groups $G$ whose Brauer character table, in some characteristic $p$, does not contain any zeros: consider for instance $G = \mathrm{S}_4$ and $p = 3$. However, for odd characteristic, this phenomenon can only happen when $G$ is a solvable group (see Theorem 9.2 below).

The next result shows that, for $p$ odd, all nonabelian simple groups have an irreducible $p$-Brauer character that vanishes on a full $\mathrm{Aut}(G)$-orbit of $p$-regular elements.

**Theorem 9.1** ([22, Theorem 1.1])**.** *Let $G$ be a nonabelian simple group and $p$ a prime. Then there exists a $\phi \in \mathrm{IBr}_p(G)$ and a $p$-regular $g \in G$ such that $\phi(g^\alpha) = 0$ for all $\alpha \in \mathrm{Aut}(G)$, unless $p = 2$ and*

- *$G = L_2(2^m)$, $m \geq 2$;*
- *$G = L_2(q)$, $q = 2^m + 1$ $m \geq 2$;*
- *$G = {}^2B_2(2^{2m+1})$, $m \geq 1$;*

- $G = S_4(2^m)$, $m \geq 2$.

*In these cases, the degrees of all irreducible 2-Brauer characters of $G$ are powers of 2.*

From the above result, one derives the following

**Theorem 9.2** ([22, Theorem 1.3]). *Assume that $G$ is not solvable and $p \neq 2$. Then there exists an irreducible $p$-Brauer character of $G$ which vanishes on some $p$-regular element of $G$.*

*An open question in this context is whether the degrees of the irreducible Brauer characters of a group $G$ are necessarily all 2-powers if the 2-Brauer character table of $G$ has no zeros.*

It is natural to guess that solvable groups whose $p$-Brauer character table has no zeros, must have a structure of somewhat restricted type relatively to the prime $p$. There are examples of such groups with both $p$-length ($l_p(G)$) and $p'$-length ($l_{p'}(G)$) equal to 2 ([9, Example 4.1]), but this is (for $p \neq 3$) the worst it can get.

**Theorem 9.3** ([9], [10]). *Let $p$ be prime and let $G$ be a finite group such that the $p$-Brauer character table of $G$ contains no zeros. Then*

(a) *If $p \geq 5$, then the Hall $p'$-subgroups of the factor group $G/\mathbf{F}(G)$ are abelian; so, $l_{p'}(G/\mathbf{F}(G)) \leq 1$, $l_{p'}(G) \leq 2$ and $l_p(G/\mathbf{O}_p(G)) \leq 2$.*

(b) *If $p = 3$, then then $G/\mathbf{F}(G)$ is a subgroup of a direct product $A \times B$, where $A$ is a $\{2,3\}$-group with elementary abelian Sylow 2-subgroups and $3'$-length at most 1 and $B \simeq (\mathrm{Sym}(3) \wr \mathrm{Sym}(3)) \wr P$, where $P$ is a 3-group. In particular, $l_{3'}(G) \leq 3$, $l_3(G/\mathbf{O}_3(G)) \leq 3$.*

(c) *If $p = 2$ and $G$ is solvable, then $G/\mathbf{F}(G)$ is a $\{2,3\}$-group with elementary abelian Sylow 3-subgroups; also, $l_{2'}(G) \leq 2$, $l_2(G/\mathbf{O}_2(G)) \leq 2$.*

(d) *If $p = 2$ and $G$ is nonsolvable, then there exist normal subgroups $R, N$ of $G$, $R \leq N$, with $R$ solvable, $l_{2'}(R) \leq 4$, $N/R$ a direct product of simple groups as listed in Theorem 9.1 and $G/N$ a group of 2-power order.*

*We remark that no examples are known of groups with no zeros in the 3-Brauer character table and with $3'$-length greater than 2. So, part (b) of the above theorem can possibly be improved.*

Let $p$ be a prime; a $p$-regular element of a group $G$ is called a $p$-nonvanishing element of $G$ if no irreducible $p$-Brauer character of $G$ takes value zero on it. The following statement, which strengthens Theorem 9.3 for $p > 7$, locates $p$-nonvanishing elements of a solvable group $G$ with respect to the $p$-series of $G$. It should be compared with Theorem 3.1.

**Theorem 9.4** ([11, Theorem A]). *Let $p$ be a prime number greater than 3, let $G$ be a finite solvable group with $\mathbf{O}_p(G) = 1$, and let $g$ be a $p$-regular element of $G$*

*that is p-nonvanishing. Then g lies in* $\mathbf{O}_{p'pp'}(G)$*, unless* $p \in \{5, 7\}$ *and the order of g is divisible by* 2 *or* 3*.*

*It is unknown, at the moment, whether the assumption* $p > 7$ *is really needed in the above statement.* The ideas used in [11] break down for small primes, but other methods could take over. *Another issue that is wide open concerns the distribution of p-nonvanishing elements in nonsolvable groups.*

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Silvio Dolfi, Dipartimento di Matematica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
*E-mail address*: `dolfi@math.unifi.it`

Emanuele Pacifici, Dipartimento di Matematica F. Enriques,
Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy.
*E-mail address*: `emanuele.pacifici@unimi.it`

Lucia Sanus, Departament de Matemàtiques, Facultat de Matemàtiques,
Universitat de València, 46100 Burjassot, València, Spain.
*E-mail address*: `lucia.sanus@uv.es`