THE PRODUCT OF THE EIGENVALUES OF A SYMMETRIC TENSOR

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Abstract. We study E-eigenvalues of a symmetric tensor f of degree d on a finite-dimensional Euclidean vector space V, and their relation with the E-characteristic polynomial of f. We show that the leading coefficient of the E-characteristic polynomial of f, when it has maximum degree, is the $(d-2)$-th power (respectively the $((d-2)/2)$-th power) when d is odd (respectively when d is even) of the $\widetilde{Q}$-discriminant, where $\widetilde{Q}$ is the d-th Veronese embedding of the isotropic quadric $Q \subseteq \mathbb{P}(V)$. This fact, together with a known formula for the constant term of the E-characteristic polynomial of f, leads to a closed formula for the product of the E-eigenvalues of f, which generalizes the fact that the determinant of a symmetric matrix is equal to the product of its eigenvalues.

1. Introduction

Let $(V, \langle\cdot\,,\cdot\rangle)$ be a real $(n+1)$-dimensional Euclidean space and denote with $\|\cdot\|$ the norm induced by $\langle\cdot\,,\cdot\rangle$. Our object of study is the vector space $\operatorname{Sym}^d V$ of degree d symmetric tensors on V. An excellent reference for the algebraic geometry for spaces of tensors is [12]. Any element $f \in \operatorname{Sym}^d V$ can be treated in coordinates as an element of $\mathbb{R}[x_1, \ldots, x_{n+1}]_d$, namely a degree d homogeneous polynomial in the indeterminates $x_1, \ldots, x_{n+1}$. The projective hypersurface defined by the vanishing of f is denoted by $[f]$.

The notions of *E-eigenvalue* and *E-eigenvector* of a symmetric tensor were proposed independently by Lek-Heng Lim and Liqun Qi in [14, 19] in the more general setting of $(n+1)$-dimensional tensors of order d, namely elements of $V^{\otimes d}$. There are different types of eigenvectors and eigenvalues in the literature, see [2, 9, 16, 18, 20]. Although the notions of E-eigenvalues and E-eigenvectors of tensors arise mainly in the context of approximation of tensors, which deals usually with real tensors, for our investigations we need to extend the Euclidean space $(V, \langle\cdot\,,\cdot\rangle)$ to its complexification $(V^{\mathbb{C}}, \langle\cdot\,,\cdot\rangle_{\mathbb{C}})$. In the following, we will use the same notation in V and $V^{\mathbb{C}}$ for the bilinear form $\langle\cdot\,,\cdot\rangle$ and the norm $\|\cdot\|$.

Definition 1.1. Given $f \in \operatorname{Sym}^d V$, a non-zero vector $x \in V^{\mathbb{C}}$ such that $\|x\| = 1$ is called an *E-eigenvector* of f (where the "E" stands for "Euclidean") if there exists $\lambda \in \mathbb{C}$ such that x is a solution of the equation

$$\frac{1}{d}\nabla f(x) = \lambda x. \tag{1}$$

The scalar λ corresponding to x is called an *E-eigenvalue* of f, while the pair (λ, x) is called an *E-eigenpair* of f. The corresponding power $x^d \in \operatorname{Sym}^d V^{\mathbb{C}}$ is called an *E-eigentensor* of f. In particular, for even order d, (λ, x) is an E-eigenpair of f if and only if $(\lambda, -x)$ is so; for odd order d, (λ, x) is an E-eigenpair of f if and only if $(-\lambda, -x)$ is so. If $x \in V^{\mathbb{C}}$ is a solution of (1) such that $\|x\| = 0$, we call x an *isotropic eigenvector* of f.

The factor $\frac{1}{d}$ appearing in (1) follows the notation in [19] conformed to the symmetric case. Observe that, if (λ, x) satisfies (1), then $(\alpha^{d-2}\lambda, \alpha x)$ satisfies (1) for any non-zero $\alpha \in \mathbb{C}$. This is why we impose the additional quadratic equation $\|x\| = 1$ in Definition 1.1. When $d = 2$, the definition of E-eigenvalue

2000 *Mathematics Subject Classification.* 14M20, 15A18, 15A69, 15A72, 65H17.

Key words and phrases. symmetric tensor; E-eigenvalue; E-characteristic polynomial; isotropic quadric.

and E-eigenvector is not the same as the standard definition of eigenvalue and eigenvector of a symmetric matrix, as a non-zero complex vector x satisfying $\|x\| = 0$ is excluded in the definition of E-eigenvector.

In this article we investigate a fundamental tool for computing the E-eigenvalues of a symmetric tensor, namely its *E-characteristic polynomial.* We recall its definition (for the definition of the resultant of m homogeneous polynomials in m variables see Section 2).

Definition 1.2. Given $f \in \operatorname{Sym}^d V$, when d is even the *E-characteristic polynomial* ψ_f of f is defined by $\psi_f(\lambda) := \operatorname{Res}(F_\lambda)$, where $\lambda \in \mathbb{C}$ and $\operatorname{Res}(F_\lambda)$ is the resultant of the $(n+1)$-dimensional vector

$$F_\lambda(x) := \frac{1}{d}\nabla f(x) - \lambda\|x\|^{d-2}x. \tag{2}$$

When d is odd, the E-characteristic polynomial is defined as $\psi_f(\lambda) := \operatorname{Res}(G_\lambda)$, where $\operatorname{Res}(G_\lambda)$ is the resultant of the $(n+2)$-dimensional vector

$$G_\lambda(x_0, x) := \begin{pmatrix} x_0^2 - \|x\|^2 \\ \frac{1}{d}\nabla f(x) - \lambda x_0^{d-2}x \end{pmatrix}. \tag{3}$$

For $d = 2$, the E-characteristic polynomial agrees with the characteristic polynomial of a symmetric matrix A, namely $\psi_A(\lambda) = \det(A - \lambda I)$. In this case, the roots of ψ_A are all the eigenvalues of A, and if the entries of A are real, then the roots of ψ_A are all real by the Spectral Theorem. Moreover, the leading coefficient of ψ_A is 1, implying that its constant term is equal to the product of the eigenvalues of A, that is the determinant of A.

The interesting fact is that this happens only for $d = 2$: as we will see throughout the paper, given $f \in \operatorname{Sym}^d V$ with $d > 2$, then some of the roots of the E-characteristic polynomial ψ_f may not be real even though the coefficients of f are real. However, there exist symmetric tensors with only real E-eigenvalues, as shown by Maccioni in [15] and Kozhasov in [11]. Moreover, the leading coefficient of ψ_f is a homogeneous polynomial over $\mathbb{Z}$ in the coefficients of f with positive degree for $d > 2$.

Our main result describes the product of the E-eigenvalues of a symmetric tensor f when ψ_f has maximum degree. In the following, $\widetilde{Q}$ denotes the Veronese embedding of the *isotropic quadric* $Q := \{\|x\|^2 = x_1^2 + \cdots + x_{n+1}^2 = 0\} \subseteq \mathbb{P}^n$, whereas $\Delta_{\widetilde{Q}}(f)$ is the $\widetilde{Q}$-*discriminant* of f. For the definition of the polynomial $\Delta_{\widetilde{Q}}(f)$ see Section 2. Moreover, for all $n \geq 1$ we define the integer $N := n+1$ for $d = 2$, whereas $N := ((d-1)^{n+1} - 1)/(d-2)$ for $d \geq 3$.

Main Theorem. *Consider a real symmetric tensor $f \in \operatorname{Sym}^d V$ for $d \geq 2$. If f admits the maximum number N of E-eigenvalues (counted with multiplicity), then their product is*

$$\lambda_1 \cdots \lambda_N = \pm\frac{\operatorname{Res}\left(\frac{1}{d}\nabla f\right)}{\Delta_{\widetilde{Q}}(f)^{\frac{d-2}{2}}}. \tag{4}$$

For the proof of the Main Theorem see Section 4. We note (see Lemma 4.11) that the assumption of the Main Theorem is satisfied for a general f, and it corresponds geometrically to the fact that the hypersurface $[f]$ is transversal to Q (see Remark 4.12). The formula (4) generalizes to the class of symmetric tensors the known fact that the determinant of a symmetric matrix is the product of its eigenvalues. In particular, we underline that the polynomial $\operatorname{Res}\left(\frac{1}{d}\nabla f\right)$ appearing in the numerator of (4) is equal to the classical *discriminant* $\Delta_d(f)$ of f times a constant factor (for the definition of discriminant of a homogeneous polynomial and a relation between the polynomials $\operatorname{Res}\left(\frac{1}{d}\nabla f\right)$ and $\Delta_d(f)$ see [6, Proposition XIII, 1.7]).

Among all the preliminary facts needed for the proof of the Main Theorem, we want to stress two of them in particular. First of all, the E-eigenvalues of $f \in \operatorname{Sym}^d V$ are roots of ψ_f, but the converse is true only for *regular* symmetric tensors (see Definition 3.5 and [18, Theorem 4]). The second fact is due to Cartwright and Sturmfels (see [2, Theorem 5.5]).

Theorem 1.3 (Cartwright-Sturmfels)**.** *Every symmetric tensor $f \in \mathrm{Sym}^d V$ has at most N distinct E-eigenvalues when d is even, and at most N pairs $(\lambda, -\lambda)$ of distinct E-eigenvalues when d is odd. This bound is attained for general symmetric tensors.*

This fact was previously conjectured in [16]. In [17], Oeding and Ottaviani review Cartwright and Sturmfels' formula and propose an alternative geometric proof based on Chern classes, with various generalizations. However, this result had already essentially been known in complex dynamics due to Fornæss and Sibony, who in [5] discuss global questions of iteration of rational maps in higher dimension.

These two results combined together show that the degree of the E-characteristic polynomial ψ_f is equal to N (or $2N$, depending on d even or odd), whereas it is smaller than the "expected" one exactly when f admits at least an isotropic eigenvector. This in particular motivated our research on the geometric meaning of the vanishing of the leading coefficient of ψ_f. Therefore the Main Theorem describes that, if the coefficients of f annihilate the polynomial $\Delta_{\widetilde{Q}}(f)$, then some of the E-eigenvalues of f have gone "to infinity": in practice, f admits at least an isotropic eigenvector whose corresponding eigenvalue does not appear as a root of the E-characteristic polynomial ψ_f. We stress that both numerator and denominator in (4) are orthogonal invariants of f, namely polynomials in the coefficients of f that are invariant under the orthonormal linear changes of coordinates in f. The product of the E-eigenvalues of f is *a priori* equal to the right-hand side of (4) times a constant factor depending only on n and d. Using the definitions of resultant and $\widetilde{Q}$*-discriminant*, we prove that this constant factor is (in absolute value) 1 by specializing to the family of *scaled Fermat polynomials.* However, the identity (4) is given up to sign since the definition of E-eigenvalue has this sign ambiguity.

This paper is organized as follows. After setting the notation, in Section 2 we give some basic notions on resultants and on the dual of a hypersurface. In Section 3, we recall the properties of the E-eigenvectors and the isotropic eigenvectors of a symmetric tensor f, and the first known properties of its E-characteristic polynomial ψ_f. In particular we point out that the coefficients of ψ_f are orthogonal invariants of f, and recall the fact proved in [13] that the constant term of ψ_f is equal (up to a constant factor) to the resultant of $\frac{1}{d}\nabla f$, for d even, or equal to the square of the resultant of $\frac{1}{d}\nabla f$, for d odd. Section 4 is devoted to the proof of the Main Theorem, before restating some useful facts from [8, 13]. Finally, in the first part of Section 5 we focus on the case of binary forms and rephrase some remarkable results in [13], whereas in the second part we stress with a concrete example how the presence of an isotropic eigenvector of $f \in \mathrm{Sym}^d V$ affects the geometry of the hypersurface $[f]$.

2. Preliminaries

Consider the a real $(n+1)$-dimensional Euclidean space $(V, \langle\cdot\,,\cdot\rangle)$, where $\langle\cdot\,,\cdot\rangle : V \times V \to \mathbb{R}$ is a positive definite symmetric bilinear form on V. The quadratic form associated to $\langle\cdot\,,\cdot\rangle$ is $q : V \to \mathbb{R}$ defined by $q(x) \coloneqq \langle x, x\rangle = \|x\|^2$. The set of automorphisms $A \in \mathrm{Aut}(V)$ that preserve $\langle\cdot\,,\cdot\rangle$, i.e., such that $\langle Ax, Ay\rangle = \langle x, y\rangle$ for all $x, y \in V$, forms the orthogonal group $\mathrm{O}(V)$ and is a subgroup of $\mathrm{Aut}(V)$. The special orthogonal group $\mathrm{SO}(V)$ is defined as the set of all A in $\mathrm{O}(V)$ with determinant 1. An $\mathrm{SO}(V)$-invariant (or orthogonal invariant) for $f \in \mathrm{Sym}^d V$ is a polynomial in the coefficients of f that does not vary under the action of $\mathrm{SO}(V)$ on the coefficients of f, where the above-mentioned action is the one induced by the linear action of $\mathrm{SO}(V)$ on the coordinates of V. If we fix a basis on V, we identify V with $\mathbb{R}^{n+1}$ and we consider the bilinear form defined by $\langle x, y\rangle = x_1y_1 + \cdots + x_{n+1}y_{n+1}$ for all $x = (x_1, \ldots, x_{n+1})$, $y = (y_1, \ldots, y_{n+1}) \in \mathbb{R}^{n+1}$: in this case, the associated quadratic form is $q(x) = \|x\|^2 = x_1^2 + \cdots + x_{n+1}^2$ for all $x = (x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}$.

The bilinear symmetric form $\langle\cdot,\cdot\rangle$ can be extended to a bilinear symmetric form on $\operatorname{Sym}^d V$. Given $x, y \in V$ and the corresponding d-th powers x^d, y^d, we set $\langle x^d, y^d\rangle := \langle x, y\rangle^d$. By linearity this defines $\langle\cdot,\cdot\rangle$ on the whole $\operatorname{Sym}^d V$. In particular, given $f, g \in \operatorname{Sym}^d V$, we define the *norm* of f by $\|f\| := \sqrt{\langle f, f\rangle}$ and the distance function between f and g by $d(f,g) := \|f-g\| = \sqrt{\langle f-g, f-g\rangle}$.

We denote by $v_{n,d}\colon \mathbb{P}(V) \to \mathbb{P}(\operatorname{Sym}^d V)$ the Veronese map, which sends $[x] \in \mathbb{P}(V)$ to $[x^d] \in \mathbb{P}(\operatorname{Sym}^d V)$. The image of $v_{n,d}$ is denoted by $V_{n,d}$ and is the projectivization of the subset of $(\operatorname{Sym}^d V)^\vee$ consisting of d-th powers of linear polynomials or, in other words, rank one symmetric tensors on V. By definition, a *critical rank one symmetric tensor* for $f \in \operatorname{Sym}^d V$ is a critical point of the distance function from f to the affine cone over $V_{n,d}$.

The following proposition is well known and defines the notion of *resultant* of a set of m homogeneous polynomials in m variables (see [3, 6]).

Proposition 2.1. *Let $f_1, \ldots, f_m$ be m homogeneous polynomials of positive degrees $d_1, \ldots, d_m$ respectively in the variables $x_1, \ldots, x_m$. Then there is a unique polynomial $\operatorname{Res}(f_1, \ldots, f_m)$ over $\mathbb{Z}$ in the coefficients of $f_1, \ldots, f_m$ such that*

i) $\operatorname{Res}(f_1, \ldots, f_m) = 0$ *if and only if the system $f_1 = \cdots = f_m = 0$ has a solution in $\mathbb{P}^{m-1}$.*

ii) $\operatorname{Res}\left(\frac{1}{d}\nabla f\right) = (a_1 \cdots a_m)^{(d-1)^{m-1}}$, *where $f(x_1, \ldots, x_m) = a_1 x_1^d + \cdots + a_m x_m^d$, $a_1, \ldots, a_m \in \mathbb{C}$, is the scaled Fermat polynomial.*

iii) $\operatorname{Res}(f_1, \ldots, f_m)$ *is irreducible, even when regarded as a polynomial over $\mathbb{C}$ in the coefficients of $f_1, \ldots, f_m$.*

The normalization assumption of *ii)* coincides with the classical definition made in [3, Theorem III, 2.3 and Theorem III, 3.5] and in [6, p. 427].

The degree of the resultant is known in general.

Proposition 2.2. $\operatorname{Res}(f_1, \ldots, f_m)$ *is a homogeneous polynomial of degree $d_1 \cdots d_{i-1} d_{i+1} \cdots d_m$ with respect to the coefficients of f_i for all $i = 1, \ldots, m$. Hence the total degree of* $\operatorname{Res}(f_1, \ldots, f_m)$ *is*

$$\deg \operatorname{Res}(f_1, \ldots, f_m) = \sum_{i=1}^{m} d_1 \cdots d_{i-1} d_{i+1} \cdots d_m.$$

In particular, when all the forms $f_1, \ldots, f_m$ have the same degree d, the resultant has degree d^{m-1} in the coefficients of each f_i, namely $\deg \operatorname{Res}(f_1, \ldots, f_m) = m d^{m-1}$.

The notion of resultant is closely related to the classical notion of *discriminant* of a homogeneous polynomial of degree d in m variables, as pointed out in [6]. The problem of computing the discriminant of a homogeneous polynomial is a particular case of a more general geometric problem, that is, finding the equations of the dual of a variety (see [6, 8, 23]).

Definition 2.3. Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety and denote by X_{sm} its smooth locus. The *dual variety* of X is

$$X^\vee := \overline{\{H \in (\mathbb{P}^n)^\vee \mid T_P X \subseteq H \text{ for some } P \in X_{sm}\}},$$

where the closure is taken with respect to the Zariski topology.

Definition 2.4. Let $X \subseteq \mathbb{P}^n$ be an irreducible projective variety. The *conormal variety* of X is

$$Z(X) := \overline{\{(P, H) \in \mathbb{P}^n \times (\mathbb{P}^n)^\vee \mid P \in X_{sm} \text{ and } T_P X \subseteq H\}}.$$

Consider the projections $\pi_1\colon Z(X) \to X_{sm}$ and $\pi_2\colon Z(X) \to (\mathbb{P}^n)^\vee$ of $Z(X)$. In particular π_1, π_2 are the restrictions of the canonical projections $pr_1\colon \mathbb{P}^n \times (\mathbb{P}^n)^\vee \to \mathbb{P}^n$, $pr_2\colon \mathbb{P}^n \times (\mathbb{P}^n)^\vee \to (\mathbb{P}^n)^\vee$. Note that, by Definition 2.3, $X^\vee$ coincides with the image of π_2 and is an irreducible variety. Moreover, since $\dim(Z(X)) = n-1$, it follows that $\dim(X^\vee) \le n-1$ and we expect that in "typical" cases $X^\vee$ is a hypersurface.

Definition 2.5. Let $X \subseteq \mathbb{P}^n$ be a projective variety. If $X^\vee$ is a hypersurface, then it is defined by the vanishing of a homogeneous polynomial, denoted by Δ_X and called the X-*discriminant*. We assume the X-discriminant to have relatively prime integer coefficients: in this way, Δ_X is defined up to sign. If $X^\vee$ is not a hypersurface, then we set $\Delta_X := 1$.

If $X \subseteq \mathbb{P}^n$ is an irreducible variety such that $X^\vee$ is a hypersurface, then Δ_X is an irreducible homogeneous polynomial over the complex numbers. When X is the Veronese variety $V_{n,d}$, then it is known that, for all $d > 1$, $X^\vee$ is a hypersurface and its equation, the $V_{n,d}$-discriminant, coincides up to a constant factor with the discriminant $\Delta_d(h)$ of a homogeneous polynomial h of degree d in $n+1$ variables.

3. The E-characteristic polynomial of a symmetric tensor

In this section we recall the main properties of E-eigenvectors and isotropic eigenvectors of a symmetric tensor f. After this, we treat more in detail the properties of the E-characteristic polynomial of f.

Consider again the Definition 1.1. The first consequence is the following property.

Proposition 3.1. *Let $f \in \mathrm{Sym}^d V$. If (λ, x) is an E-eigenpair of f, then $\lambda = f(x)$.*

Proof. Apply the operator $\langle \cdot\,, x\rangle$ on both sides of equation (1). Then we have

$$\left\langle \frac{1}{d}\nabla f(x), x \right\rangle = \langle \lambda x, x\rangle.$$

Using Euler's identity, the left hand side of last identity is equal to $f(x)$, whereas by linearity and the fact that x has norm 1 the right-hand side is equal to λ. $\square$

A remarkable fact observed in [14, 19] is that the E-eigenvectors of $f \in \mathrm{Sym}^d V$ correspond to the critical points of the function $f(x)$ restricted on the unit sphere $S^n := \{x \in \mathbb{R}^{n+1} \mid \|x\| = 1\}$. Hence the E-eigenvectors of f are the normalized solutions x, in orthonormal coordinates, of:

$$\operatorname{rank}\begin{pmatrix} \nabla f(x) \\ x \end{pmatrix} \le 1.$$

Moreover, we recall an alternative interpretation of the eigenvectors of a symmetric form, meaning that an *eigenvector* of f is any solution of equation (1), whether it has unit norm or not.

Theorem 3.2 (Lim, variational principle)**.** *Given $f \in \mathrm{Sym}^d V$, the critical rank one symmetric tensors for f are exactly of the form x^d, where x is an eigenvector of f.*

This interpretation is used by Draisma, Ottaviani and Tocino in [4], where they deal more in general with the *best rank k approximation problem* for tensors.

Looking at Definition 1.1, a natural question is whether E-eigenvalues could change under an orthonormal linear change of coordinates in V (see [18, Theorem 1] and [20, Theorem 2.20]).

Theorem 3.3. *Given $f \in \mathrm{Sym}^d V$, the set of the E-eigenvalues of f is a $\mathrm{SO}(V)$-invariant of f.*

In particular Theorem 3.3 states that the symmetric functions of the E-eigenvalues of f are orthogonal invariants of f, giving rise to the following corollary (see [13, Theorem 3.3]):

Corollary 3.4. *Given $f \in \operatorname{Sym}^d V$, all the coefficients of the E-characteristic polynomial ψ_f are* $\mathrm{SO}(V)$*-invariants of f.*

Given $f \in \operatorname{Sym}^d V$, we observe that for d even there exists a non-zero constant $c \in \mathbb{Z}$ such that

$$\psi_f(\lambda) := \operatorname{Res}(F_\lambda(x)) = c \cdot \Delta_d \left(f(x) - \lambda \|x\|^d \right), \tag{5}$$

where the $(n+1)$-dimensional vector $F_\lambda(x)$ has been introduced in (2) (see again [6, Proposition XIII, 1.7]). On the other hand, a relation equivalent to (5) is no longer possible for d odd: in (3), an additional variable x_0 is required to make the polynomial ψ_f well defined.

In the study of the E-characteristic polynomial ψ_f, a crucial role is played by a family of particular symmetric tensors, the ones admitting at least a singular point on the isotropic quadric Q.

Definition 3.5. A symmetric tensor $f \in \operatorname{Sym}^d V$ is *irregular* if there exists a non-zero vector $x \in V^{\mathbb{C}}$ such that $\|x\| = 0$ and $\nabla f(x) = 0$. Otherwise f is called *regular.*

The first fact on irregular symmetric tensors is that, when $d > 2$, their E-characteristic polynomial is identically zero.

Proposition 3.6. *Given $f \in \operatorname{Sym}^d V$ with $d > 2$, if f is irregular then ψ_f is the zero polynomial.*

Proof. Suppose that f is irregular. Then, by Definition 3.5 there exists a non-zero vector $x \in V$ such that $\|x\| = 0$ and $\nabla f(x) = 0$. Looking at Definition 1.2, this implies that, for $d > 2$ even, x is a solution of the system $F_\lambda(x) = 0$ for all $\lambda \in \mathbb{C}$, whereas for $d > 2$ odd $(0, x)$ is a solution of the system $G_\lambda(x_0, x) = 0$ for all $\lambda \in \mathbb{C}$. By resultant theory, this means that $\psi_f(\lambda) = 0$ for all $\lambda \in \mathbb{C}$, namely ψ_f is identically zero. □

Remark 3.7. The statement of Proposition 3.6 is no longer true for $d = 2$. In fact, for $d = 2$ and any $n \geq 1$ there exist irregular symmetric tensors $f \in \operatorname{Sym}^d \mathbb{C}^{n+1}$ such that ψ_f is not identically zero. For example, the polynomial $f(x) = (x_1 + \sqrt{-1}x_2)^2 + x_3^2 + \cdots + x_{n+1}^2$ is irregular because the vector $(1, \sqrt{-1}, 0, \ldots, 0)$ is a solution of $\nabla f(x) = 0$, whereas one can easily check that $\psi_f(\lambda) = \lambda^2(1-\lambda)^{n-1}$, hence it is not identically zero.

The notion of regularity of a symmetric tensor plays a crucial role in the following result.

Theorem 3.8. *Suppose that $d \geq 3$. Given $f \in \operatorname{Sym}^d V$, every E-eigenvalue of f is a root of the E-characteristic polynomial ψ_f. If f is regular, then every root of ψ_f is an E-eigenvalue of f.*

Proof. For completeness we recover and adapt the proofs in [18, Theorem 4] and in [20, Theorem 2.23]. Suppose that $x \in V^{\mathbb{C}}$ is an E-eigenvector of f and $\lambda \in \mathbb{C}$ is the E-eigenvalue associated with λ. Then looking at Definition 1.2, when d is even we get that x and $-x$ are non-zero solutions of the system $F_\lambda(x) = 0$; when d is odd, $(1, x)$ and $(-1, -x)$ are non-zero solutions of the system $G_\lambda(x_0, x) = 0$. Therefore λ is a root of ψ_f by Proposition 2.1.

On the other hand, suppose that f is regular and let $\lambda \in \mathbb{C}$ be a root of ψ_f. By Definition 1.2 and Proposition 2.1, when d is even there exists a non-zero vector $x \in V$ such that $F_\lambda(x) = 0$ for that λ; when d is odd, there exists a non-zero vector $x \in V$ and $x_0 \in \mathbb{C}$ such that $G_\lambda(x_0, x) = 0$ for that λ. If $\|x\| = 0$, both $F_\lambda(x) = 0$ and $G_\lambda(x_0, x) = 0$ yield the condition $\nabla f(x) = 0$, which cannot be satisfied because of the regularity of f. Hence $\|x\| \neq 0$ and we consider $\tilde{x} = x/\|x\|$. Therefore, when d is even the equation (1) is satisfied by $(\lambda, \tilde{x})$ and $(\lambda, -\tilde{x})$, while for d odd it is satisfied by $(\lambda, \tilde{x})$ and $(-\lambda, -\tilde{x})$. This implies that λ is an E-eigenvalue of f. □

Example 3.9. Let us consider the case in which d is even and $f = \|x\|^d$. Then equation (1) becomes $\|x\|^{d-2}x = \lambda x$: this means, if $d = 2$, that every non-zero vector $x \in V$ such that $\|x\| = 1$ is an E-eigenvector of f with E-eigenvalue $\lambda = 1$ (and in fact the E-characteristic polynomial of f is $\psi_f(\lambda) = (\lambda - 1)^{n+1}$). Instead for $d > 2$ every non-zero vector $x \in V$ such that $\|x\| = 1$ is an E-eigenvector of f with corresponding E-eigenvalue $\lambda = 1$, and every non-zero vector $x \in V$ such that $\|x\| = 0$ is an isotropic eigenvector of f. In particular f is irregular for $d > 2$, and in fact in this case the E-characteristic polynomial of f is identically zero by Proposition 3.6.

The greatest difference among eigenvectors of a symmetric matrix and eigenvectors of a symmetric tensor of degree $d > 2$ is related to the presence or not of isotropic eigenvectors. Suppose that $f \in \operatorname{Sym}^d V$ admits an isotropic eigenvector x and let $P := [x]$ be the corresponding point of the isotropic quadric Q. In the same fashion of Proposition 3.1, this time we have that $f(x) = 0$, that is, $P \in [f]$. As we will see in Section 4, equation (1) acquires a new interesting meaning: the isotropic eigenvectors of f are all the non-zero vectors x such that $[x] =: P \in [f] \cap Q$ and P is singular for $[f]$ (and hence f is irregular) or P is smooth for $[f]$ and $[f]$ is tangent to Q at P.

We study more in detail the coefficients of the E-characteristic polynomial of a symmetric tensor. Given a general $f \in \operatorname{Sym}^d V$, from Theorem 1.3 and Theorem 3.8 we have that $\deg(\psi_f) \leq N$ for d even, where $N := n+1$ for $d = 2$, whereas $N := ((d-1)^{n+1} - 1)/(d-2)$ for $d \geq 3$. Thus ψ_f can be written as

$$\psi_f(\lambda) = \sum_{j=0}^{N} c_j \lambda^j, \tag{6}$$

where for all $j = 0, \ldots, N$ the coefficient $c_j = c_j(n,d)$ is a homogeneous polynomial in the coefficents of f. Otherwise if d is odd and (λ, x) is an E-eigenpair of f, then $(-\lambda, -x)$ is an E-eigenpair of f as well. This means that for d odd the E-characteristic polynomial ψ_f has maximum degree N in λ^2 and in particular it contains only even power terms of λ. Hence ψ_f can be written explicitly as

$$\psi_f(\lambda) = \sum_{j=0}^{N} c_{2j} \lambda^{2j}. \tag{7}$$

Now we focus on the constant term of the E-characteristic polynomial ψ_f. In particular we recover the fact that, when non-zero, the constant term of ψ_f is a power of $\operatorname{Res}\left(\frac{1}{d}\nabla f\right)$ times a constant factor, as in the proof of [13, Theorem 3.5].

Theorem 3.10. *Let $f \in \operatorname{Sym}^d V$. Then for even d we have that*

$$c_0 = c \cdot \operatorname{Res}\left(\frac{1}{d}\nabla f\right), \tag{8}$$

while for odd d we have that

$$c_0 = c \cdot \operatorname{Res}\left(\frac{1}{d}\nabla f\right)^2 \tag{9}$$

for some constant $c \in \mathbb{Z}$ depending on n and d.

Proof. The relations (8) and (9) are trivially satisfied when f is irregular (compare with Proposition 3.6), so we can assume f regular. When d is even, from relation (2) we have that

$$c_0 = \psi_f(0) = \operatorname{Res}(F_\lambda)|_{\{\lambda=0\}} = c \cdot \operatorname{Res}(F_0) = c \cdot \operatorname{Res}\left(\frac{1}{d}\nabla f\right)$$

for some constant $c = c(n,d) \in \mathbb{Z}$.

Now suppose that d is odd. From relation (3) we have that

$$c_0 = \psi_f(0) = \mathrm{Res}(G_\lambda)|_{\{\lambda=0\}} = c \cdot \mathrm{Res}(G_0), \quad G_0(x_0, x) = \begin{pmatrix} x_0^2 - \|x\|^2 \\ \frac{1}{d}\nabla f(x) \end{pmatrix}$$

for some constant $c = c(n,d) \in \mathbb{Z}$. In order to prove relation (9), it is sufficient to prove that

$$\mathrm{Res}(G_0) = \mathrm{Res}\left(\frac{1}{d}\nabla f\right)^2. \tag{10}$$

First of all, we prove that the system

$$\begin{cases} x_0^2 - \|x\|^2 = 0 \\ \frac{1}{d}\nabla f(x) = 0 \end{cases} \tag{11}$$

has a nonzero solution if and only if $\mathrm{Res}\left(\frac{1}{d}\nabla f\right) = 0$. Let (x_0, x) be a non-zero solution of (11). In particular, x is a non-zero solution of $\nabla f(x) = 0$. Thus, $\mathrm{Res}\left(\frac{1}{d}\nabla f\right) = 0$. On the other hand, suppose that $\mathrm{Res}\left(\frac{1}{d}\nabla f\right) = 0$. Then $\nabla f(x) = 0$ admits a non-zero solution x and $(\|x\|, x)$ is a non-zero solution of (11).

Hence the equations $\mathrm{Res}(G_0) = 0$ and $\mathrm{Res}\left(\frac{1}{d}\nabla f\right) = 0$ define the same variety. By definition $\mathrm{Res}\left(\frac{1}{d}\nabla f\right)$ is an irreducible polynomial over $\mathbb{Z}$ in the coefficients of f. Therefore

$$\mathrm{Res}(G_0) = \mathrm{Res}\left(\frac{1}{d}\nabla f\right)^k$$

for some positive integer k. Since the polynomial $x_0^2 - \|x\|^2$ is quadratic, from Proposition 2.2 we have that $\mathrm{Res}(G_0)$ is a homogeneous polynomial in the coefficients of $\partial f/\partial x_1, \ldots, \partial f/\partial x_{n+1}$ of degree $2(d-1)^n$. On the other hand, the degree of $\mathrm{Res}\left(\frac{1}{d}\nabla f\right)^k$ is $k(d-1)^n$. Therefore relation (11) is satisfied only if $k = 2$. This completes the proof. □

We apply the following result when we study the degree of the leading coefficient and of the constant term of ψ_f, viewed as polynomials in the coefficients of f (see [13, Proposition 3.6]).

Proposition 3.11. *Consider $f \in \mathrm{Sym}^d V$ and its E-characteristic polynomial ψ_f written as in (6), (7).*

i) When d is even, c_i is a homogeneous polynomial in the coefficients of f with degree $(n+1)(d-1)^n - i$. In particular $\deg(c_N) = (n+1)(d-1)^n - N =: \varphi_n(d)$, where the integer N has been introduced in the Main Theorem. In particular $\varphi_n(2) = 0$ for all $n \geq 1$.

ii) When d is odd, c_{2i} is a homogeneous polynomial in the entries of f with degree $2(n+1)(d-1)^n - 2i$. In particular $\deg(c_{2N}) = 2(n+1)(d-1)^n - 2N = 2\varphi_n(d)$.

Remark 3.12. It can be easily shown that the polynomial $\varphi_n(d)$ defined in Proposition 3.11 is a strictly increasing function in the variable d. This fact, together with Proposition 3.11, implies that c_N (respectively c_{2N}) has positive degree in the coefficients of f for all $n \geq 1$ and $d > 2$.

We have this natural question: is there a geometric meaning for the vanishing of the leading coefficient of ψ_f? The answer is positive and will be stated in Proposition 4.10.

4. Proof of the Main Theorem

In this section we give the proof of the Main Theorem. The proof starts with an example: in fact, the next lemma studies the product of the E-eigenvalues of a particular class of symmetric tensors, the scaled Fermat polynomials $f(x_1, \ldots, x_{n+1}) = a_1 x_1^d + \cdots + a_{n+1} x_{n+1}^d$, where $a_1, \ldots, a_{n+1} \in \mathbb{C}$. This result is important to prove the identity (4) up to sign in the statement of the Main Theorem.

Lemma 4.1. *Let $d \geq 2$ and consider the scaled Fermat polynomial $f = a_1x_1^d + \cdots + a_{n+1}x_{n+1}^d$, where $a_1, \ldots, a_{n+1} \in \mathbb{C}$. The product $\lambda_1 \cdots \lambda_N$ of the E-eigenvalues of f, where N is the number defined in Theorem 1.3, can be written as*

$$\lambda_1 \cdots \lambda_N = \frac{\mathrm{Res}\left(\frac{1}{d}\nabla f\right)}{h^{\frac{d-2}{2}}}, \tag{12}$$

where $h = h(a_1, \ldots, a_{n+1})$ is a homogeneous polynomial of degree $2\varphi_n(d)/(d-2)$ and the polynomial $\varphi_{n,d}$ has been defined in Proposition 3.11. Moreover, the leading term of h with respect to the lexicographic term order is monic and it is equal to

$$LT_{Lex}(h) = \prod_{s=1}^{n+1} a_s^{2\frac{(d-1)^n-(d-1)^{s-1}}{d-2}}.$$

Proof. In this case, rewriting the number N of E-eigenvalues as $N = \sum_{j=1}^{n+1}\binom{n+1}{j}(d-2)^{j-1}$, the binomial $\binom{n+1}{j}$ denotes the number of E-eigenvalues for f whose corresponding E-eigenvectors have exactly j non-zero coordinates, while the factor $(d-2)^{j-1}$ corresponds to the number of $(j-1)$-arrangements (allowing repetitions) of the elements of $\{0, 1, \ldots, d-3\}$, for all $j = 1, \ldots, n+1$. Let $x = (x_1, \ldots, x_{n+1})$, $\|x\| = 1$ be an E-eigenvector of f. We have

$$a_i x_i^{d-1} = \lambda x_i \quad \forall i = 1, \ldots, n+1. \tag{13}$$

Suppose that exactly j coordinates of x are non-zero, call them $x_{k_1}, \ldots, x_{k_j}$ with indices $1 \leq k_1 < \cdots < k_j \leq n+1$. Moreover, we write $a_i = \xi_i^{d-2}$ for all $i = 1, \ldots, n+1$. Looking at (13), if $x_i \neq 0$ we obtain that $\lambda = a_i x_i^{d-2} = (\xi_i x_i)^{d-2}$ for all $i = 1, \ldots, n+1$. Moreover, considering (13) with respect to the indices $i_1 < i_2$, we get the relations $a_{i_1}x_{i_1}^{d-1} = \lambda x_{i_1}$, $a_{i_2}x_{i_2}^{d-1} = \lambda x_{i_2}$, from which we obtain the equation $x_{i_1}x_{i_2}\prod_{k=0}^{d-3}(\xi_{i_1}x_{i_1} - \varepsilon^k\xi_{i_2}x_{i_2}) = 0$, where ε is a $(d-2)$-th root of unity. This means that, for any indices $i_1 < i_2$ it could be that $x_{i_1} = 0$, $x_{i_2} = 0$ or $\xi_{i_1}x_{i_1} = \varepsilon^k\xi_{i_2}x_{i_2}$ for some $k \in \{0, \ldots, d-3\}$. Therefore the coordinates of x, when non-zero, can be always written as $x_{k_l} = \xi_{k_1}\cdots\widehat{\xi_{k_l}}\cdots\xi_{k_j}\varepsilon^{\alpha_{k_l}}/\|x\|$, where $\alpha_{k_l} \in \{0, 1, \ldots, d-3\}$ for all $l = 1, \ldots, j$. Since $|\varepsilon| = 1$, we can assume $\alpha_{k_1} = 0$. In addition to this, the norm of x can be written as $\|x\| = \left(\xi_{k_2}^2\cdots\xi_{k_j}^2 + \sum_{l=2}^{j}\xi_{k_1}^2\cdots\widehat{\xi_{k_l}^2}\cdots\xi_{k_j}^2\varepsilon^{2\alpha_{k_l}}\right)^{1/2}$ and the E-eigenvalue corresponding to x is $\lambda = (\xi_{k_l}x_{k_l})^{d-2} = a_{k_1}\cdots a_{k_j}/\|x\|^{d-2}$. From this argument we obtain that the product of the E-eigenvalues of the scaled Fermat polynomial f is equal to $\lambda_1\cdots\lambda_N = g/h^{(d-2)/2}$, where

$$g = g(a_1, \ldots, a_{n+1}) \coloneqq \prod_{j=1}^{n+1}\prod_{1\leq k_1<\cdots<k_j\leq n+1}\prod_{\alpha_{k_2},\ldots,\alpha_{k_j}=0}^{d-3} a_{k_1}\cdots a_{k_j}, \tag{14}$$

$$h = h(a_1, \ldots, a_{n+1}) \coloneqq \prod_{j=1}^{n+1}\prod_{1\leq k_1<\cdots<k_j\leq n+1}\prod_{\alpha_{k_2},\ldots,\alpha_{k_j}=0}^{d-3}\left(\xi_{k_2}^2\cdots\xi_{k_j}^2 + \sum_{l=2}^{j}\xi_{k_1}^2\cdots\widehat{\xi_{k_l}^2}\cdots\xi_{k_j}^2\varepsilon^{2\alpha_{k_l}}\right). \tag{15}$$

Now consider in particular the polynomial g defined in (14). We have that

$$g = \prod_{j=1}^{n+1}\prod_{1\leq k_1<\cdots<k_j\leq n+1}(a_{k_1}\cdots a_{k_j})^{(d-2)^{j-1}} = \prod_{j=1}^{n+1}(a_1\cdots a_{n+1})^{\binom{n}{j-1}(d-2)^{j-1}} = (a_1\cdots a_{n+1})^{(d-1)^n},$$

where the last polynomial coincides exactly with $\mathrm{Res}\left(\frac{1}{d}\nabla f\right)$ by Proposition 2.1. On the other hand, having fixed Lex as term order in $\mathbb{Z}[a_1, \ldots, a_{n+1}]$, the leading term of h is equal to

$$LT_{Lex}(h) = \prod_{j=2}^{n+1}\prod_{1\leq k_1<\cdots<k_j\leq n+1}\prod_{\alpha_{k_2},\ldots,\alpha_{k_j}=0}^{d-3}\xi_{k_1}^2\cdots\xi_{k_{j-1}}^2\varepsilon^{2\alpha_{k_j}} = \prod_{j=2}^{n+1}\prod_{1\leq k_1<\cdots<k_j\leq n+1}(\xi_{k_1}^2\cdots\xi_{k_{j-1}}^2)^{(d-2)^{j-1}}.$$

Observe that in the last product (with j fixed) the factors $\xi_1^{2(d-2)^{j-1}}, \ldots, \xi_{j-1}^{2(d-2)^{j-1}}$ appear $\binom{n}{j-1}$ times, while $\xi_s^{2(d-2)^{j-1}}$ appears $\binom{n}{j-1} - \binom{s-1}{j-1}$ times for $s = j, \ldots, n+1$. Hence (assuming $\binom{a}{b} = 0$ for $a < b$)

$$LT_{Lex}(h) = \prod_{j=2}^{n+1} \prod_{s=1}^{n+1} \xi_s^{2\left[\binom{n}{j-1} - \binom{s-1}{j-1}\right](d-2)^{j-1}} = \prod_{s=1}^{n+1} \xi_s^{2\left[(d-1)^n - (d-1)^{s-1}\right]} = \prod_{s=1}^{n+1} a_s^{2\frac{(d-1)^n - (d-1)^{s-1}}{d-2}}. \qquad \square$$

Remark 4.2. Observe that in Lemma 4.1 the degree of $LT_{Lex}(h)$, that is the total degree of h, is

$$\frac{2}{d-2}\sum_{s=1}^{n+1}[(d-1)^n - (d-1)^{s-1}] = \frac{2}{d-2}\left((n+1)(d-1)^n - \frac{(d-1)^{n+1} - 1}{d-2}\right) = \frac{2}{d-2}\varphi_n(d),$$

where $\varphi_n(d)$ has been introduced in Proposition 3.11 and $d \geq 3$. This value is the one expected as shown in the sequel.

Now we move to the general case. We recall the definition of the *polar classes* $\delta_j(X)$ associated to a projective variety $X \subseteq \mathbb{P}^n$ of dimension r.

Consider the conormal variety $Z(X)$ of X introduced in Definition 2.4 and the *Chow cohomology class*

$$[Z(X)] \in A^*(\mathbb{P}^n \times (\mathbb{P}^n)^\vee) = \mathbb{Z}[u, v],$$

where $A^*(\mathbb{P}^n \times (\mathbb{P}^n)^\vee)$ denotes the Chow (cohomology) ring of $\mathbb{P}^n \times (\mathbb{P}^n)^\vee$, $u = pr_1^*([H])$, $v = pr_2^*([H'])$ and H, H' denote hyperplanes in $\mathbb{P}^n$ and $(\mathbb{P}^n)^\vee$, respectively. Then $[Z(X)]$ can be written as

$$[Z(X)] = \sum_{j=0}^{r-1} \delta_j(X) u^{n-j} v^{j+1},$$

where $\delta_j(X)$ is a non-negative integer for all $j = 0, \ldots, r-1$. If X is smooth, we have the following formulas for the invariants $\delta_j(X)$ (see [8]):

$$\delta_j(X) = \sum_{k=j}^{r} (-1)^{r-k} \binom{k+1}{j+1} \deg(c_{r-k}(TX)), \tag{16}$$

where $\deg(c_{r-k}(TX))$ is the degree of the $(r-k)$-th Chern class of the tangent bundle of X.

We will use the following result (see [8, Theorem 3.4]).

Theorem 4.3. *If* $\delta_j(X) = 0$ *for all* $j < l$ *and* $\delta_l(X) \neq 0$*, then* $\dim(X^\vee) = n-1-l$ *and* $\delta_l(X) = \deg(X^\vee)$.

We apply the previous facts in our particular case. Let $v_{n,d}$ be the Veronese embedding defined in the introduction, and denote by $\widetilde{Q}$ the isotropic quadric Q embedded in $\mathbb{P}^N$ via the map $v_{n,d}$, where $N+1 = \binom{n+d}{d}$. In particular, $\widetilde{Q}$ is smooth, hence we can apply the relations (16). Moreover, it is known that

$$\widetilde{Q}^\vee = \overline{\{[f] \in (\mathbb{P}^N)^\vee \mid [f] \text{ is tangent to } Q \text{ at some smooth point}\}}.$$

Lemma 4.4. *In the hypotheses above,* $\delta_0(\widetilde{Q}) = 2\sum_{k=0}^{n-1} \alpha_k d^k$*, where*

$$\alpha_k \coloneqq (k+1) \sum_{j=0}^{n-1-k} \binom{n+1}{j} (-1)^j 2^{n-1-k-j}. \tag{17}$$

Proof. First of all we compute the Chern polynomial of TQ:

$$c(TQ) = \frac{(1+t)^{n+1}}{1+2t} = \sum_{i,j=0}^{n-1} \binom{n+1}{i} (-2)^j t^{i+j} = \sum_{s=0}^{n-1} \left(\sum_{i=0}^{s} \binom{n+1}{i} (-2)^{s-i} \right) t^s.$$

Then we compute the polar class $\delta_0(\widetilde{Q})$ using (26) with $r = n-1$ and taking into account that $\widetilde{Q} = v_{n,d}(Q)$.

$$\begin{aligned}
\delta_0(\widetilde{Q}) &= \sum_{k=0}^{n-1}(-1)^{n-1-k}(k+1)\deg(c_{n-1-k}(T\widetilde{Q})) \\
&= \sum_{k=0}^{n-1}(-1)^{n-1-k}(k+1)\deg\left[\left(\sum_{j=0}^{n-1-k}\binom{n+1}{j}(-2)^{n-1-k-j}\right)t^{n-1-k}(dt)^k\right] \\
&= \sum_{k=0}^{n-1}(k+1)\deg\left[\left(\sum_{j=0}^{n-1-k}\binom{n+1}{j}(-1)^j 2^{n-1-k-j}\right)d^k t^{n-1}\right] \\
&= 2\sum_{k=0}^{n-1}(k+1)\left[\sum_{j=0}^{n-1-k}\binom{n+1}{j}(-1)^j 2^{n-1-k-j}\right]d^k. \qquad \square
\end{aligned}$$

In the following technical Lemma, we rewrite the polynomial $\varphi_n(d)$ defined in Proposition 3.11 in a useful way for the sequel.

Lemma 4.5. *Let $\varphi_n(d)$ be the polynomial defined in Proposition 3.11. Then $\varphi_n(d) = (d-2)\sum_{k=0}^{n-1}\beta_k d^k$, where*

$$\beta_k := (k+1)\sum_{l=0}^{n-1-k}\binom{k+l+1}{l}(-1)^l. \tag{18}$$

Proof. With a bit of work, the polynomial $\varphi_n(d)$ can be rewritten as

$$\varphi_n(d) = (d-2)\sum_{k=0}^{n-1}(k+1)(d-1)^k = (d-2)\sum_{k=0}^{n-1}\beta_k d^k,$$

where

$$\begin{aligned}
\beta_k &= \sum_{i=k}^{n-1}(i+1)\binom{i}{k}(-1)^{i-k} \\
&= \sum_{l=0}^{n-1-k}(l+k+1)\binom{l+k}{k}(-1)^l \\
&= \sum_{l=0}^{n-1-k}\left[\frac{l(l+k)!}{l!k!} + \frac{(k+1)(l+k)!}{l!k!}\right](-1)^l \\
&= (k+1)\sum_{l=0}^{n-1-k}\left[\binom{k+l}{k+1} + \binom{k+l}{k}\right](-1)^l \\
&= (k+1)\sum_{l=0}^{n-1-k}\binom{k+l+1}{l}(-1)^l. \qquad \square
\end{aligned}$$

Now we prove that the degree of the leading coefficient of ψ_f is a multiple of the polar class $\delta_0(\widetilde{Q})$ computed in Lemma 4.4.

Proposition 4.6. *For any $n \geq 1$ and for any $k = 0, \ldots, n-1$, $\alpha_k = \beta_k$. In particular,*

$$\varphi_n(d) = \frac{d-2}{2}\delta_0(\widetilde{Q}). \tag{19}$$

Proof. From the identities (17) and (18) we see that both α_k and β_k are multiples of $k+1$. In particular, we have to prove that

$$\sum_{j=0}^{n-1-k}\binom{n+1}{j}(-1)^j 2^{n-1-k-j} = \sum_{j=0}^{n-1-k}\binom{k+j+1}{j}(-1)^j. \tag{20}$$

The proof is by induction on n. If $n=1$, both the sides of the equality are equal to 1. Suppose now that the equality is true at the n-th step. At the $(n+1)$-th step, the right-hand side of the equality is

$$\sum_{j=0}^{n-k}\binom{k+j+1}{j}(-1)^j = \sum_{j=0}^{n-1-k}\binom{k+j+1}{j}(-1)^j + (-1)^{n-k}\binom{n+1}{n-k},$$

while the left-hand side at the $(n+1)$-th step is equal to

$$\begin{aligned}
\sum_{j=0}^{n-k}\binom{n+2}{j}(-1)^j 2^{n-k-j} &= 2^{n-k} + \sum_{j=1}^{n-k}\binom{n+2}{j}(-1)^j 2^{n-k-j}\\
&= 2^{n-k} + \sum_{j=1}^{n-k}\left[\binom{n+1}{j}+\binom{n+1}{j-1}\right](-1)^j 2^{n-k-j}\\
&= \sum_{j=0}^{n-k}\binom{n+1}{j}(-1)^j 2^{n-k-j} + \sum_{j=1}^{n-k}\binom{n+1}{j-1}(-1)^j 2^{n-k-j}\\
&= 2\sum_{j=0}^{n-1-k}\binom{n+1}{j}(-1)^j 2^{n-1-k-j} + (-1)^{n-k}\binom{n+1}{n-k} +\\
&\quad + \sum_{j=0}^{n-1-k}\binom{n+1}{j}(-1)^{j+1} 2^{n-1-k-j}\\
&= \sum_{j=0}^{n-1-k}\binom{n+1}{j}(-1)^j 2^{n-1-k-j} + (-1)^{n-k}\binom{n+1}{n-k}
\end{aligned}$$

By applying the induction hypothesis we conclude the proof of (20). □

Remark 4.7. Matteo Gallet suggested an alternative proof of the identity (20), applying the so-called *"Zeilberger's Algorithm"* (see [24, 25]). For example, using the Mathematica package `HolonomicFunctions`, developed by Cristoph Koutschan (see [10]), the code

```
Annihilator[Sum[Binomial[n+1,j]*(-1)^j*2^(n-1-k-j),{j,0,n-1-k}],{S[k],S[n]}]
Annihilator[Sum[Binomial[k+j+1,j]*(-1)^j,{j,0,n-1-k}],{S[k],S[n]}]
```

provides the operators that annihilate the left-hand and right-hand side in (20), respectively, thus showing that (20) holds true.

Corollary 4.8. *Consider the isotropic quadric $Q \subseteq \mathbb{P}^n$ and its Veronese embedding $\widetilde{Q} \subseteq \mathbb{P}^N$ with the same notations as before. Then $\widetilde{Q}^\vee$ is a hypersurface of $(\mathbb{P}^n)^\vee$ of degree* $\deg(\widetilde{Q}^\vee) = \delta_0(\widetilde{Q})$.

Proof. From Remark 3.12 and Proposition 4.6 we have that $\delta_0(\widetilde{Q})$ is a positive integer for all $n \geq 1$ and $d > 2$. Applying Theorem 4.3 we conclude the proof. □

Summing up, there is an explicit formula for the degree of the leading coefficient of ψ_f in terms of the degree of the dual variety of Q embedded in $\mathbb{P}^N$ via the Veronese map, stated in the following corollary.

Corollary 4.9. *Given $f \in \mathrm{Sym}^d V$, if f is general then*

$$deg(c_N) = \frac{d-2}{2}\deg(\widetilde{Q}^\vee)$$

when d is even, while

$$deg(c_{2N}) = (d-2)\deg(\widetilde{Q}^\vee)$$

when d is odd.

In the following, we prove that the leading coefficient of ψ_f is a power of the discriminant $\Delta_{\widetilde{Q}}(f)$, where the exponent has been obtained in Corollary 4.9. The next two lemmas will clarify the geometrical meaning of the vanishing of the polynomial c_N (respectively c_{2N}).

Lemma 4.10. *Assume that $d > 2$ and let $f \in \mathrm{Sym}^d V$. Then the leading coefficient of ψ_f vanishes if and only if the system*

$$\begin{cases} \frac{1}{d}\nabla f(x) = \lambda x \\ \|x\| = 0\ , \end{cases} \tag{21}$$

called deficit system *in* [13]*, has a nontrivial solution.*

Proof. If f is irregular, from Definition 3.5 we have that the system (21) has a non trivial solution when $\lambda = 0$, while from Proposition 3.6 we have that ψ_f is identically zero.

Suppose instead that f is regular. By Proposition 3.8 the roots of ψ_f are exactly the E-eigenvalues of f and for d even $\deg(\psi_f) \leq N$, whereas for d odd $\deg(\psi_f) \leq 2N$. However, we know by Theorem 1.3 that a general f has N distinct E-eigenvalues when d is even, and N pairs $(\lambda, -\lambda)$ of distinct E-eigenvalues when d is odd, which means that ψ_f would have exactly N distinct roots when d is even, and $2N$ distinct roots when d is odd. On the other hand, E-eigenvalues are the normalized solutions x of equation (1), and by definition ψ_f is the resultant of the homogeneization of the system whose equations are (1) and the condition $\|x\| = 1$. The solutions at infinity of this system are precisely the solution of the system (21). Hence a symmetric tensor f such that ψ_f has not the maximal degree provides a nontrivial solution of the system (21), or equivalently admits an isotropic eigenvector. □

Lemma 4.11. *Given $f \in \mathrm{Sym}^d V$, the system (21) has a nontrivial solution if and only if the coefficients of f annihilate the polynomial $\Delta_{\widetilde{Q}}(f)$, namely f is represented by a point of $\widetilde{Q}^\vee$.*

Proof. Suppose that x is a solution of (21). By regularity of f we have that $\lambda \neq 0$. Moreover, $P = [x]$ is a smooth point of f, and f is tangent to Q at P. This means that f, thought as a point of $\mathbb{P}(\mathrm{Sym}^d V)^\vee$, belongs to $\widetilde{Q}^\vee$, namely its coefficients annihilate the polynomial $\Delta_{\widetilde{Q}}(f)$. The converse is true by reversing the implications. □

Remark 4.12. One could ask if the condition on f to have the maximum number of E-eigenvalues imposed in the Main Theorem has a geometric counterpart. For example, this condition is not the same as requiring $[f]$ to be regular: although any symmetric tensor f having the maximum number of E-eigenvalues is necessarily regular, there exist regular symmetric tensors f admitting at least one isotropic eigenvector. The right property to consider is revealed by Lemma 4.11, which shows that $f \in \mathrm{Sym}^d V$ admits an isotropic eigenvector if and only if the hypersurface $[f]$ and the isotropic quadric Q are tangent. This means that the condition on f in the Main Theorem is satisfied if and only if $[f]$ is transversal to Q.

Remark 4.12 is even more interesting when considering the following result (see [1, Claim 3.2]):

Proposition 4.13. *If two smooth hypersurfaces of degree d_1, d_2 in projective space are tangent along a positive dimensional set, then $d_1 = d_2$.*

An immediate consequence of Proposition 4.13 is the following

Corollary 4.14. *Given $f \in \operatorname{Sym}^d V$ with $d > 2$, if $[f]$ is smooth then f has always a finite number of isotropic eigenvectors.*

A detailed example of a symmetric tensor f admitting an isotropic eigenvector, with a study of the tangency of the variety defined by f with the isotropic quadric Q, is given in Section 5.2.

Returning to the proof of the Main Theorem, an immediate consequence of Corollary 4.9 and Lemmas 4.10 and 4.11 is the following formula for the leading coefficient of the E-characteristic polynomial of a symmetric tensor.

Theorem 4.15. *Given $f \in \operatorname{Sym}^d V$ and $d > 2$, if f does not admit isotropic eigenvectors, then*

$$c_N = e \cdot \Delta_{\widetilde{Q}}(f)^{\frac{d-2}{2}} \tag{22}$$

when d is even, while

$$c_{2N} = e \cdot \Delta_{\widetilde{Q}}(f)^{d-2} \tag{23}$$

when d is odd, for some integer constant $e = e(n, d)$.

Proof. Applying Lemma 4.10 and Lemma 4.11 we obtain that the varieties $\{c_N = 0\}$ and $\{\Delta_{\widetilde{Q}}(f) = 0\}$ coincide. The proof for the case $n = 1$ is postponed to Section 5.1, where we treat more in detail binary forms. If $n > 1$, then $\widetilde{Q}$ is an irreducible hypersurface and the variety $\widetilde{Q}^\vee$ is irreducible as well. Corollary 4.8 tells us that $\widetilde{Q}^\vee$ is in fact a hypersurface. Hence, for d even, $c_N = e \cdot \Delta_{\widetilde{Q}}(f)^j$, whereas for d odd $c_{2N} = e \cdot \Delta_{\widetilde{Q}}(f)^k$ for some integer constant $e = e(n, d)$ and positive integers j, k. Moreover, from Corollary 4.9 we have that $j = (d-2)/2$ and $k = d - 2$. $\square$

Proof of the Main Theorem. Theorems 3.10 and 4.15 describe respectively the constant term c_0 and the leading coefficient c_N (or c_{2N}) of the E-characteristic polynomial ψ_f of a generic symmetric tensor f, up to a constant integer factor. Moreover, the product of the E-eigenvalues of f is c_0/c_N (respectively c_0/c_{2N}). If we restrict to the class of scaled Fermat polynomials, as in Lemma 4.1, we notice that the integers c and e of Theorems 3.10 and 4.15 have to coincide, for the leading term of the denominator in (12) is monic and by definition $\Delta_{\widetilde{Q}}(f)$ has relatively prime integer coefficients. This concludes the proof. $\square$

5. Examples with binary and ternary symmetric tensors

In this section we give two examples to understand better the statement and the proof of the Main Theorem. The first one deals with the case of binary forms: in particular, we show that in this particular case equation (4) can be rewritten more explicitly. The second is an example of a cubic ternary form f which admits one isotropic eigenvector: we compute explicitly its E-characteristic polynomial ψ_f, observe that $\deg(\psi_f) < N$ and visualize its tangency with the isotropic quadric Q.

5.1. The case of binary forms.

In this example we focus on the case $n = 1$ and recover the results of Li, Qi and Zhang in [13]. An element of $\operatorname{Sym}^d \mathbb{C}^2$ is represented by the binary form

$$f(x_1, x_2) = \sum_{j=0}^{d} \binom{d}{j} a_j x_1^{d-j} x_2^j\,, \quad a_0, \ldots, a_d \in \mathbb{C}. \tag{24}$$

According to Theorem 1.3, a general binary form f of degree d admits $N = d$ E-eigenvectors. As one can easily see from relation (1), the E-eigenvectors of f are the normalized solutions (x_1, x_2) of the equation

$D(f)=0$, where the *discriminant operator* D is defined by $D(f) := x_1(\partial f/\partial x_2) - x_2(\partial f/\partial x_1)$. The operator D is well-known and its properties are collected in [15].

We are interested in the E-characteristic polynomial ψ_f of a regular binary form f. We know that $\deg(\psi_f)=d$ in the even case, while $\deg(\psi_f)=2d$ in the odd case. A remarkable formula for the leading coefficient of the E-characteristic polynomial of a 2-dimensional tensor of order d is given in [13]. We show that this formula can be simplified a lot in the symmetric case.

Following the argument used in [13], the isotropic eigenvectors of f are the solutions of the following simplified version of the system (21):

$$\begin{cases} \sum_{j=1}^{d}\binom{d-1}{j-1}a_{j-1}x_1^{d-j}x_2^{j-1} & = & \lambda x_1 \\ \sum_{j=1}^{d}\binom{d-1}{j-1}a_{j}x_1^{d-j}x_2^{j-1} & = & \lambda x_2 \\ x_1^2+x_2^2 & = & 0\ . \end{cases} \tag{25}$$

We observe that all the non trivial solutions (x_1,x_2) of (25) are non-zero multiples of $(1,\sqrt{-1})$ or $(1,-\sqrt{-1})$. Substituting $(1,\sqrt{-1})$ to (25) and eliminating λ we obtain the condition

$$\sum_{j=0}^{d}\binom{d}{j}a_j\sqrt{-1}^{\,j}=0. \tag{26}$$

In the same manner, considering instead the vector $(1,-\sqrt{-1})$ we obtain the condition

$$\sum_{j=0}^{d}\binom{d}{j}a_j(-\sqrt{-1})^{j}=0. \tag{27}$$

Therefore, if the binary form f has at least one isotropic eigenvector, then the product of the left-hand sides of equations (26) and (27) vanishes. On the other hand, if this product is zero, then $(1,\sqrt{-1})$ or $(1,-\sqrt{-1})$ is a solution of the system (25) and is in turn an isotropic eigenvector of f.

We observe that the left-hand sides in (26) and (27) have an interesting interpretation. Consider in general the linear change of coordinates defined by the equations

$$x_1=\gamma_{11}z_1+\gamma_{12}z_2,\quad x_2=\gamma_{21}z_1+\gamma_{22}z_2.$$

Applying this change of coordinates, the binary form $f(x_1,x_2)$ is transformed into the binary form $\widetilde{f}(z_1,z_2)$ in the new variables z_1, z_2 defined by

$$\widetilde{f}(z_1,z_2)=\sum_{j=0}^{d}\binom{d}{j}a_j(\gamma_{11}z_1+\gamma_{12}z_2)^{d-j}(\gamma_{21}z_1+\gamma_{22}z_2)^j=\sum_{j=0}^{d}\binom{d}{j}\widetilde{a}_jz_1^{d-j}z_2^j,$$

where (see [22, Proposition 3.6.1])

$$\widetilde{a}_j=\sum_{k=0}^{d}\left[\sum_{l=\max(0,k-j)}^{\min(k,d-j)}\binom{d-j}{l}\binom{j}{k-l}\gamma_{11}^{l}\gamma_{12}^{k-l}\gamma_{21}^{d-j-l}\gamma_{22}^{j-k+l}\right]a_k,\quad j=0,\ldots,d. \tag{28}$$

In particular consider the new coordinates

$$z_1=-\frac{\sqrt{-1}}{2}(x_1+\sqrt{-1}\,x_2),\quad z_2=-\frac{\sqrt{-1}}{2}(x_1-\sqrt{-1}\,x_2).$$

The inverse change of coordinates has equations

$$x_1=\sqrt{-1}(z_1+z_2),\quad x_2=z_1-z_2.$$

With this choice, applying formula (28) the coefficients $\widetilde{a}_j$ of the transformed binary form $\widetilde{f}(z_1,z_2)$ are

$$\widetilde{a}_j=\sum_{k=0}^{d}\left[\sum_{l=\max(0,k-j)}^{\min(k,d-j)}\binom{d-j}{l}\binom{j}{k-l}\sqrt{-1}^{2(j+l)-k}\right]a_k,\quad j=0,\dots,d.$$

In particular the extreme coefficients become

$$\widetilde{a}_0=\sum_{j=0}^{d}\binom{d}{j}a_j\sqrt{-1}^j,\quad \widetilde{a}_d=(-1)^d\sum_{j=0}^{d}\binom{d}{j}a_j(-\sqrt{-1})^j.$$

Therefore, if we define $b_0\coloneqq\widetilde{a}_0$ and $b_d\coloneqq(-1)^d\widetilde{a}_d$, then the left-hand sides of equations (26) and (27) are equal to b_0 and b_d, respectively. Moreover, we observe that the product b_0b_d has integer coefficients even though some of the coefficients of b_0 and b_d have non-zero imaginary part: in fact we see that

$$b_0b_d=\sum_{j,k=0}^{d}\binom{d}{j}\binom{d}{k}a_ja_k(-1)^j\sqrt{-1}^{j+k}=\sum_{s=0}^{d}\left[\sum_{j=0}^{s}\binom{d}{j}\binom{d}{s-j}a_ja_{s-j}(-1)^j\right]\sqrt{-1}^s,\tag{29}$$

where in the last relation all summands corresponding to odd indices s vanish. Since the coefficient of a_0 in the expression of b_0b_d is 1, we conclude that $b_0b_d=\Delta_{\widetilde{Q}}(f)$ up to sign. In particular $\widetilde{Q}^\vee=\{b_0b_d=0\}$: in fact in this case $\widetilde{Q}$ is the union of two distinct points (more precisely, the classes of the rank one symmetric tensors $(x_1+\sqrt{-1}x_2)^d$ and $(x_1-\sqrt{-1}x_2)^d$), while the variety $\widetilde{Q}^\vee$ is the quadric union of the hyperplanes $\{b_0=0\}$, $\{b_d=0\}$. In particular, the hyperplane $\{b_0=0\}$ parametrizes the binary forms having $(1,\sqrt{-1})$ as isotropic eigenvector, while $\{b_d=0\}$ parametrizes the binary forms having $(1,-\sqrt{-1})$ as isotropic eigenvector.

Regarding the leading coefficient of the E-characteristic polynomial ψ_f, the previous argument suggests that it must coincide with $c\cdot b_0^ib_d^j$ for some $c=c(d)\in\mathbb{Z}$. Since ψ_f is a polynomial in the indeterminates $a_0,\dots,a_d$ with integer coefficients, it follows that $i=j$. Hence, for d even, $c_d=e\cdot\Delta_{\widetilde{Q}}(f)^p$, whereas for d odd $c_{2d}=e\cdot\Delta_{\widetilde{Q}}(f)^q$ for some $e=e(n,d)\in\mathbb{Z}$ and positive integers p,q. From Corollary 4.9 we have that $p=(d-2)/2$ and $q=d-2$, thus completing the proof of Theorem 4.15 in the case $n=1$.

Remark 5.1. If we specialize to the class of scaled Fermat binary forms $f(x_1,x_2)=\alpha x_1^d+\beta x_2^d$, $\alpha,\beta\in\mathbb{C}$, from relation (29) we confirm the statement of Lemma 4.1 by observing that

$$\Delta_{\widetilde{Q}}(f)=\alpha^2+(1+(-1)^d)\sqrt{-1}^d\alpha\beta+\beta^2.$$

5.2. **A plane cubic admitting an isotropic eigenvector.** The following example has the goal to explain better Lemma 4.11. First of all, we recall that, due to Theorem 1.3, a general ternary form has $N=d^2-d+1$ E-eigenvalues. Consider the cubic ternary form

$$\begin{aligned}f(x_1,x_2,x_3)={}&342\sqrt{-1}\,x_1^3-522\sqrt{-1}\,x_1x_2^2-389\sqrt{-1}\,x_1^2x_3+79\sqrt{-1}\,x_2^2x_3-474\sqrt{-1}\,x_1x_3^2+\\&+95\sqrt{-1}\,x_3^3-773x_1^2x_2+191x_2^3-48x_1x_2x_3+175x_2x_3^2.\end{aligned}$$

It can be easily verified that the vector $x=(0,1,-\sqrt{-1})$ is an isotropic eigenvector of f. In particular the projective curve $[f]$ is tangent to the isotropic quadric Q at $[x]\in\mathbb{P}^2$, and the common tangent line has equation $x_2-\sqrt{-1}\,x_3=0$. In order to represent graphically this situation, we consider the change of coordinates

$$z_1=-\sqrt{-1}\,x_1,\quad z_2=x_2+\sqrt{-1}\,x_3,\quad z_3=x_2-\sqrt{-1}\,x_3.$$

In the z_i's the quadric Q (the red curve in the affine representation of Figure 1) has equation $z_1^2 - z_2 z_3 = 0$. The image of the isotropic eigenvector x is $z = (0, 2, 0)$, while the image of the projective curve $[f]$ (the blue curve in Figure 1) is the projective curve of equation

$$g(z_1, z_2, z_3) = 342z_1^3 + 581z_1^2 z_2 + 192z_1^2 z_3 + 498z_1 z_2 z_3 + 139z_2^2 z_3 + 24z_1 z_3^2 + 48z_2 z_3^2 + 4z_3^3.$$

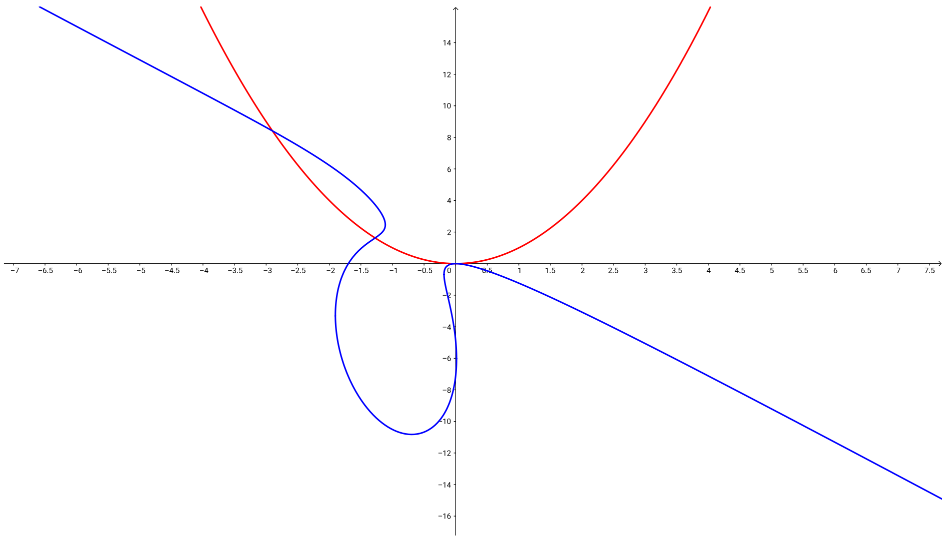

FIGURE 1. The isotropic quadric Q and the ternary cubic f in the affine plane $z_2 = 2$. They are not transversal at the origin.

The presence of an isotropic eigenvector can be detected by computing explicitly the E-characteristic polynomial of f as well. In order to compute $\psi_f(\lambda)$ we used the following Macaulay2 code [7] (for the package `Resultants` see [21]), taking into account Definition 1.2 modified according to the given change of coordinates:

```
loadPackage "Resultants"; KK=QQ[t]; R=KK[z_0..z_3];
f=342*z_1^3+581*z_1^2*z_2+192*z_1^2*z_3+498*z_1*z_2*z_3+139*z_2^2*z_3+24*z_1*z_3^2+48*z_2*z_3^2+4*z_3^3;
F_0=z_0^2-(-z_1^2+z_2*z_3); F_1=diff(z_1,f)/3+t*z_0*z_1;
F_2=diff(z_2,f)/3+diff(z_3,f)/3-t*z_0*(z_2+z_3)/2; F_3=diff(z_2,f)/3-diff(z_3,f)/3+t*z_0*(z_2-z_3)/2;
characteristic_polynomial=Resultant({F_0,F_1,F_2,F_3}, Algorithm=>Macaulay)
```

The output of `characteristic_polynomial` is

$$\begin{aligned}\psi_g(\lambda) = {} & 224053792039458000000\lambda^{12} + 1737672597491537284396875\lambda^{10} + 4568660944049253131212218187 5\lambda^8 \\ & + 5386198710022212712472131345526 25\lambda^6 + 2746031584320556852962647720783548350\lambda^4 \\ & + 21377525988865149579810902794140433910 31\lambda^2 + 1384380765990937946402742775323612027006919 6.\end{aligned}$$

Since a general cubic ternary form has seven E-eigenvalues, we expect that $\deg(\psi_f) = 14$, but in this case $\deg(\psi_f) = 12$. This confirms that f has one isotropic eigenvector and six E-eigenvectors (counted with multiplicity) up to sign.

Acknowledgement

Luca Sodomaco is member of INDAM-GNSAGA. This paper has been partially supported by the Strategic Project "Azioni di gruppi su varietà e tensori" of the University of Florence. The author is very grateful to his advisor Giorgio Ottaviani for valuable guidance. Moreover, he warmly thanks Matteo Gallet for Remark 4.7 and other appreciated suggestions.

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