Ground state solutions to nonlinear equations with p-Laplacian

Zuzana Došlá and Serena Matucci

Abstract

The existence of positive radial solutions is investigated for a nonlinear elliptic equation with p-Laplace operator and sign-changing weight, both in superlinear and sublinear case. We prove the existence of solutions u which are globally defined and positive outside of a ball of radius R, satisfy fixed initial conditions $u(R) = c > 0$, $u'(R) = 0$ and tends to zero at infinity. Our method is based on a fixed point result for boundary value problems on noncompact intervals and on asymptotic properties of suitable auxiliary half-linear differential equations. The results are new also for the classical Laplace operator and may be used for proving the existence of ground state solutions and decaying solutions with exactly k-zeros which are defined in all the space. Some examples illustrate our results.

Keywords: Second order nonlinear differential equation, Ground state solution, Boundary value problem on the half-line.

MSC: 34B40, 34B18

1 Introduction

Consider the nonlinear elliptic equation with p-Laplace operator

$$\operatorname{div}\left(r(\mathbf{x})\,|\nabla u|^{p-2}\nabla u\right) + q(\mathbf{x})\,F(u) = 0, \quad p > 1,$$

where r and q are smooth functions defined on $\mathbb{R}^d$, $d \geq 2$, r is positive, $F \in C(\mathbb{R})$. Solutions u, which are positive, minimize certain energy functional, and satisfy $\lim_{|\mathbf{x}|\to\infty} u(\mathbf{x}) = 0$, are usually called *ground state solutions*. The search of radially symmetric ground state solutions outside of a ball of radius

R, satisfying Neumann boundary conditions, leads to the one-dimensional problem

$$\begin{aligned}&\left(t^{d-1}r(t)\Phi(u')\right)' + t^{d-1}q(t)F(u) = 0, \quad t \geq R,\\ &u'(R) = 0, \quad u(t) > 0 \text{ for } t \geq R, \quad \lim_{t\to\infty} u(t) = 0,\end{aligned} \tag{NP}$$

where $t = |\mathbf{x}|$, see for instance [1]. Here and henceforth,

$$\Phi(u) = |u|^{\alpha-1}u = |u|^{\alpha}\operatorname{sgn} u, \quad \alpha = p - 1 > 0.$$

Thus, in this paper we consider the equation

$$(a(t)\Phi(x'))' + b(t)F(x) = 0, \qquad t \in [t_0, \infty),\ t_0 \geq 0 \tag{1}$$

with the boundary conditions

$$x(t_0) = c,\ x'(t_0) = 0,\ x(t) > 0 \text{ for } t \in [t_0, \infty),\ \lim_{t\to\infty} x(t) = 0, \tag{2}$$

where c is a positive constant. We always assume that:

(i) a is a positive continuous function on $[t_0, \infty)$ satisfying

$$A = \int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)}\, dt < \infty; \tag{3}$$

(ii) b is a continuous function on $[t_0, \infty)$, such that

$$\int_{t_0}^{\infty} \frac{1}{a^{1/\alpha}(t)} \left(\int_{t_0}^{t} b_+(s)\, ds\right)^{1/\alpha} dt = \infty, \tag{4}$$

$$B_- = \int_{t_0}^{\infty} b_-(t)\, dt < \infty, \tag{5}$$

where b_+, b_- are the positive and the negative part of b, respectively, i.e. $b_+(t) = \max\{b(t), 0\}$, $b_-(t) = -\min\{b(t), 0\}$;

(iii) F is a continuous function on $\mathbb{R}$ with $F(0) = 0$, F is continuously differentiable on $(0, 2c]$ and $F'(u) \geq 0$ for $u \in (0, 2c]$.

A prototype of (1) is the Emden-Fowler equation

$$\big(a(t)\Phi(x')\big)' + b(t)|x|^{\beta}\mathrm{sgn}\, x = 0, \quad \beta > 0. \tag{6}$$

Concerning the forcing term F, two cases are here considered, according to the function $F(u)/u^{\alpha}$ is bounded or unbounded in a right neighborhood of zero, respectively. These cases represent for (6) the superlinear/half-linear case ($\beta \geq \alpha$) and the sublinear one ($0 < \beta < \alpha$), and are very different from each other. Indeed, consider the particular equation

$$(t^2x')' + b_0|x|^{\gamma}\mathrm{sgn}\ x = 0 \qquad t \in [t_0, \infty,),\ t_0 > 0, \tag{7}$$

where b_0 is a positive constant. The change of variable

$$y(t) = tx(t) \tag{8}$$

transforms (7) into

$$y'' + \frac{b_0}{t^{\gamma+1}}|y|^{\gamma}\ \mathrm{sgn}\ y = 0,$$

the well-known Emden-Fowler equation, which is oscillatory if and only if $\gamma \in (0, 1)$, see, e.g., [16]. Since the transformation (8) maintains the oscillation, the BVP (1), (2) is not solvable for (7) in the sublinear case $\gamma < 1$. On the other hand, in the linear case ($\gamma = 1$) equation (7) is nonoscillatory if $b_0 \leq 1/4$, and any its solution goes to zero as $t \to \infty$. This fact shows that the solvability of (1), (2) in the sublinear case is a more difficult problem, and the nonlinearity has to be, roughly speaking, "very close" to the power function $\Phi(u)$.

Notice that, in virtue of (3) and (4), we have

$$\int_{t_0}^{\infty} b_+(t)\, dt = \infty, \tag{9}$$

and the function b cannot be identically zero in a neighborhood of infinity. On the other hand, if b is identically zero on $[t_0, \infty)$, then the BVP (1), (2) is not solvable, as a direct computation shows.

It is well-known that the continuability at infinity of solutions of (1) is a serious matter, see, e.g., [4]. For instance, the Emden-Fowler equation (6), where $\alpha < \beta$ and b is allowed to take negative values, has solutions which tend to infinity in finite time, see [4, 5]. Moreover, again in the superlinear case

$\alpha < \beta$, if b is non-negative with isolated zeros, (6) may have solutions which change sign infinitely many times in the left neighborhood of some $\bar{t} > t_0$, and so these solutions are not continuable to infinity, see [7]. Further, even if global solutions exist, (i.e., solutions which are defined in the whole half-line $[t_0, \infty)$), their positivity is not guaranteed in general. Indeed, (1) may exhibit coexistence of nonoscillatory and oscillatory solutions; further, nonoscillatory solutions may have an arbitrary large number of zeros. On the other hand, the solutions of the boundary value problem (1),(2) are necessarily defined and positive on the whole half-line and, for this reason, we call them *globally positive solutions*.

Recently, boundary value problems associated to differential equations with indefinite weight have attracted an increasing interest, both in ODE and in PDE case, but generally on bounded domains. At our knowledge, [19] is one of the first works in which indefinite weight is considered; other significant results in compact intervals can be found, e.g. in [3, 23] and references therein. The existence of globally positive solutions on a half-line, satisfying different types of boundary conditions, has been studied, for instance, in [10, 11, 12, 14, 24]. Observe that, in case of indefinite weight, the behavior of solutions can be more varied with respect to the case of positive (negative) weight. For instance, (1) may have nonoscillatory solutions whose derivative change sign infinitely many times as $t \to \infty$, see, e.g., [8].

The existence of radial ground states or nodal solutions for elliptic equations with the classical Laplacian operator or with the p-Laplacian operator are problems that have attracted much attention in recent years, especially in case of positive weight. Among the extensive bibliography in this field, we refer to [1], [9] and the references therein. Ground state solutions of some superlinear elliptic equations with classical Laplacian ($p = 2$) and weight having one change of sign have been studied in [17, 18], see also references therein. To our knowledge, the case of indefinite weight has been not treated till now for the equation under our consideration.

A wide literature is devoted to the study of the asymptotic properties of solutions for (1), but few results deals with the problem of existence of positive global solutions in a given unbounded interval, and in particular, as far as we know, no results are known for the existence of positive global solutions of a Neumann type problem on a half-line. The problem of the existence of positive global solutions has been considered, for instance, in [10, 11, 12, 14, 24], with different initial and/or asymptotic conditions. In particular the results here presented extend [14, Theorem 3.2], in which the existence of decreasing solutions (the so called Kneser solutions) is proved for (1), but only in case $F(u)/u^\alpha$ is bounded. Our results extend also to the p-Laplacian operator case some results proved for the curvature operator (see, e.g., [2, Theorem 0.1], [13, Theorem 3.1]). The existence results here presented are new also if b is nonnegative (case of positive weight). In this easier case, the main results become more simple, see Corollary 1 and Corollary 2. At our knowledge, apart from the results of the present authors, until now the only known results for (1) deal with existence of solutions which are positive for large t, but not with the existence of positive solutions on an *a-priori* fixed interval. Thus, from this point of view, our results extend, e.g., [22, Theorem 3.2, Theorem 3.7], [21, Theorem 1.2]. Moreover our results are new also for equations with the classical Sturm-Liouville operator ($\alpha = 1$).

The problem of the existence of positive global solutions to (1), satisfying a Neumann type initial condition and zero asymptotic condition, is here solved by developing a new approach based on a comparison result between principal and nonprincipal solutions of Sturm majorant and minorant of auxiliary half-linear equations. The existence of a solution is then obtained by a fixed point approach for operators defined on noncompact intervals. One of the advantages of this approach is that the explicit form of the operator and its topological properties are not needed.

The paper is organized as follows. Sections 3 and 4 are devoted to the main results, according to either the boundedness or unboundedness of $F(u)/u^\alpha$ holds in a right neighborhood of zero, respectively. Section 2 contains some preliminaries on properties of half-linear equations and on a fixed point result that will be used in our main theorems. In Section 5 some sufficient conditions for the application of the main results are given. These lead to existence conditions that can be more easily check (see Corollaries 3 and 4); the applicability of the main results to some elliptic problem is also discussed. Some examples complete this section, illustrating the applicability of the main results and the corresponding conditions for (NP). Further, the methods used in [12, 14] are compared with the new method developed here, and possible applications of the results to the existence of ground state solutions and of solution with a prescribed number of zeros, defined on the whole real line, are explained.

2 Preliminaries: properties of half-linear equations and a fixed point result

Consider the half-linear equation

$$(a(t)\Phi(w'))' + \gamma(t)\Phi(w) = 0, \quad t \geq t_0, \tag{10}$$

where γ is a continuous function. This equation has many similarities with the corresponding linear equation

$$(a(t)w')' + \gamma(t)w = 0, \quad t \geq t_0,$$

and has been widely studied in the literature, see [15] and references therein. In particular, the continuability of solutions over $[t_0, \infty)$ and the uniqueness with respect to the initial data hold for (10). As concerns the asymptotic properties, Sturm theory remains to hold for (10). In consequence, all solutions of (10) have the same behavior with respect to the oscillation, and (10) is said to be nonoscillatory if it has a nonoscillatory solution, or equivalently, if all its nontrivial solutions are nonoscillatory.

In case of nonoscillation, the notion of principal solution, introduced in 1936 by W. Leighton and M. Morse for the linear case, see [20, Chapter XI. 6.], has been extended to (10) by J.D. Mirzov or A. Elbert and T. Kusano following the Riccati approach, see [15, Section 4.2]. In more details, denote by Φ^* the inverse operator of Φ, i.e., $\Phi^*(u) = |u|^{1/\alpha} \operatorname{sgn} u$, and let w be a solution of (10), different from zero in an interval I. Then $\xi(t) = a(t)\Phi(w'(t))/\Phi(w(t))$ is a solution of the Riccati type differential equation

$$\xi' + \gamma(t) + \alpha\, \xi\, \Phi^* \left(\frac{\xi}{a(t)} \right) = 0. \tag{11}$$

If (10) is nonoscillatory, then among all eventually different from zero solutions of (11), there exists one, say ξ_∞, which is continuable to infinity and is minimal in the sense that any other solution ξ of (11), which is continuable to infinity, satisfies $\xi_\infty(t) < \xi(t)$ for t large. Then, by definition, the *principal solution* w_0 of (10) is a nontrivial solution of the equation

$$w' = \Phi^* \Big(\frac{\xi_\infty(t)}{a(t)} \Big) w.$$

Notice that for any nontrivial solution w of (10), linearly independent of w_0, it holds

$$\frac{w_0'(t)}{w_0(t)} < \frac{w'(t)}{w(t)} \tag{12}$$

for t large, and w is sometimes called *nonprincipal solution* to (10).

Jointly with (10), consider the half-linear equation

$$(a_1(t)\Phi(z'))' + \gamma_1(t)\Phi(z) = 0, \tag{13}$$

where a_1, γ_1 are continuous functions such that

$$0 < a_1(t) \le a(t), \quad \gamma_1(t) \ge \gamma(t) \text{ for } t \ge t_0, \quad \gamma - \gamma_1 \not\equiv 0. \tag{14}$$

Equation (13) is called a Sturm majorant of (10), and if (13) is nonoscillatory, then (10) is nonoscillatory, too.

The next two lemmas state some known comparison results between principal solutions of (10) and (13), see [15, Theorems 4.2.2, 4.2.3].

Lemma 1. *Assume that (14) holds,* (10) *is nonoscillatory and its principal solution* w_0 *has a zero point, and let* $t_1 \ge t_0$ *be the largest of them. Then any solution* z *of the Sturm majorant (13) has a zero in* (t_1, ∞)*.*

Lemma 2. *Assume that (14) holds,* (13) *is nonoscillatory and let* w_0 *and* z_0 *be principal solutions of* (10) *and* (13)*, respectively, such that* $w_0(t) > 0$ *and* $z_0(t) > 0$ *for* $t \ge t_0$*. Then*

$$a^{1/\alpha}(t)\,\frac{w_0'(t)}{w_0(t)} \le a_1^{1/\alpha}(t)\,\frac{z_0'(t)}{z_0(t)}, \quad t \ge t_0.$$

The following comparison results for principal and nonprincipal solutions are new and play a key role in our approach.

Lemma 3. *Assume that (14) holds,* (13) *is nonoscillatory and has a solution* z *satisfying* $z(t) > 0$ *on* $[t_0, \infty)$ *and* $z'(t_0) < 0$*. Then (10) is nonoscillatory and:*

(i) the principal solution w_0 *of (10) is positive on* $[t_0, \infty)$ *and* $w_0'(t_0) < 0$*;*

(ii) any solution w *of (10) satisfying the initial data* $w(t_0) > 0$, $w'(t_0) \ge 0$ *is positive on* $[t_0, \infty)$*.*

Proof. Claim (i). At first we show that, if a solution z of (13) exists, which is positive for $t \geq t_0$ and satisfies $z'(t_0) < 0$, then the same properties hold for the principal solution z_0, i.e., $z_0(t) > 0$ on $[t_0, \infty)$, $z_0'(t_0) < 0$. If z is principal, the assertion follows. Thus, let z be a nonprincipal solution. Assume by contradiction that z_0 has zero points, and let $t_1 \geq t_0$ be the largest of them. Then z should have a zero point in (t_1, ∞) [15, Theorem 4.2.3], which is a contradiction, and $z_0 > 0$ on $[t_0, \infty)$ follows. Now, since z_0, z are both positive in $[t_0, \infty)$, inequality (12) holds on $[t_0, \infty)$. Thus,

$$\frac{z_0'(t_0)}{z_0(t_0)} < \frac{z'(t_0)}{z(t_0)} < 0$$

implies $z_0'(t_0) < 0$.

In virtue of (14), equation (13) is a Sturm majorant of (10). Thus, equation (10) is nonoscillatory and the principal solution w_0 of (10) is positive on $[t_0, \infty)$. Indeed, if w_0 has the last zero in $[t_1, \infty)$, $t_1 \geq t_0$, by Lemma 1 the principal solution z_0 of (13) should have a zero in $[t_1, \infty)$, a contradiction. Using Lemma 2 with z_0 principal solution of (10), we get $w_0'(t_0) < 0$.

Claim (ii). For any solution w of (10) such that $w(t_0) > 0, w'(t_0) \geq 0$, the Wronskian

$$W(w, w_0)(t) = w'(t)w_0(t) - w_0'(t)w(t)$$

is positive at $t = t_0$, i.e.,

$$W(w, w_0)(t_0) > 0.$$

Since, similarly to the linear case, the Wronskian of two solutions of a half-linear equation is either identically zero or always nonzero (see [15, Lemma 1.3.1]), we obtain that $W(w, w_0)(t) > 0$ for all $t \geq t_0$. Then we have for $t \geq t_0$

$$\left(\frac{w(t)}{w_0(t)}\right)' = \frac{W(w, w_0)(t)}{w_0(t)^2} > 0,$$

i.e., w/w_0 is increasing on $[t_0, \infty)$. Since $w_0(t) > 0$ for $t \geq t_0$ and $w(t_0)/w_0(t_0) > 0$, then w is positive for $t > t_0$. □

Lemma 4. *Assume that (10) is nonoscillatory and its principal solution is positive on $[t_0, \infty)$. Let z_0 be the positive principal solution of*

$$(a(t)\Phi(z'))' - \gamma_2(t)\Phi(z) = 0, \tag{15}$$

where γ_2 *is a continuous function,* $\gamma_2(t) \geq 0$*, and*

$$\gamma(t) \geq -\gamma_2(t) \text{ for } t \geq t_0. \tag{16}$$

Then any positive solution w *of (10) with the initial condition* $w(t_0) \geq z_0(t_0) > 0$ *satisfies* $w(t) \geq z_0(t)$ *on* $[t_0, \infty)$*.*

Proof. Let w_0 be the positive principal solution of (10), and let w be any positive nonprincipal solution such that $w_0(t_0) = w(t_0)$. Since $w(t) > 0, w_0(t) > 0$ on $[t_0, \infty)$, then (12) holds for all $t \geq t_0$. By integrating (12) on $[t_0, \infty)$ and taking into account that $w(t_0) = w_0(t_0)$, we get $w(t) > w_0(t)$ on (t_0, ∞). From (16), equation (15) is a Sturm minorant of (10). Then, if z_0 is the positive principal solution of (15) satisfying $z_0(t_0) \leq w_0(t_0)$, Lemma 2 gives

$$\frac{w_0'(t)}{w_0(t)} \geq \frac{z_0'(t)}{z_0(t)}, \quad t \geq t_0,$$

and by integration we get $w_0(t) \geq z_0(t)$ on $[t_0, \infty)$. Thus $w(t) \geq w_0(t) \geq z_0(t)$ for $t \geq t_0$, and the proof is complete. □

The following lemma describes the properties of the principal solution for (10) in case γ is nonpositive.

Lemma 5. *Assume that* $\gamma(t) \leq 0$ *for all* $t \geq t_0$ *and*

$$\int_{t_0}^{\infty} |\gamma(t)|\, dt < \infty.$$

Then equation (10) is nonoscillatory and the principal solution w_0*, with* $w_0(t_0) > 0$*, is positive decreasing on* $[t_0, \infty)$ *with* $\lim_{t\to\infty} w_0(t) = 0$*.*

Proof. The proof can be found, for instance, in [15] when γ is not identically zero in a neighborhood of infinity. If $\gamma(t) = 0$ for t large, then the principal solution is

$$\int_{t}^{\infty} a^{-1/a}(s) ds$$

and the assertion follows. □

To prove our existence results, we will use a fixed point theorem given in [6, Theorem 1.3] for operators $\mathcal{T}$ defined in the Frechét space $C(J, \mathbb{R}^n)$

of the continuous vectors defined on a (possibly unbounded) real interval J, endowed with the topology of uniform convergence on compact subsets of J. In view of this result, no topological properties of the operator $\mathcal{T}$ are needed to be checked, since they are a direct consequence of *a-priori* bounds.

The fixed point theorem is stated in [6] for boundary value problems on (non)compact intervals associated to nonlinear systems of the form

$$\dot{\mathbf{x}} = f(t, \mathbf{x})$$

where $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous map. Here we formalize this result for scalar second order equation in the form that is needed in the sequel. Denote by $C^1(J, \mathbb{R})$ the Frechét space of the continuously differentiable functions defined on J, with the topology of uniform convergence of functions and their derivatives on compact intervals contained in J. Recall that a subset $\Omega \subset C^1(J, \mathbb{R})$ is bounded if and only if there exists a positive continuous function $\theta : J \to \mathbb{R}$ such that $|u(t)| + |u'(t)| \le \theta(t)$ for all $t \in J$ and $u \in \Omega$.

Theorem 1. ([6, Theorem 1.3]) *Let $J = [t_0, \infty)$. Consider the boundary value problem*

$$\begin{aligned} &(a(t)\Phi(x'))' + h(t, x) = 0, \quad t \in J, \\ &x \in S, \end{aligned} \tag{17}$$

where h is a continuous function on $J \times \mathbb{R}$ and S is a subset of $C^1(J, \mathbb{R})$. Let g be a continuous function on $J \times \mathbb{R}^2$ such that

$$g(t, c, c) = h(t, c) \quad \text{for all } (t, c) \in J \times \mathbb{R},$$

and assume that there exist a closed convex subset Ω of $C^1(J, \mathbb{R})$ and a bounded closed subset S_1 of $S \cap \Omega$ which make the problem

$$\begin{aligned} &(a(t)\Phi(y'))' + g(t, y, u) = 0, \quad t \in J, \\ &y \in S_1 \end{aligned} \tag{18}$$

uniquely solvable for all $u \in \Omega$. Then the BVP (17) has at least a solution in Ω.

Proof. The equation in (17) can be equivalently written as the system

$$x_1' = \frac{1}{a^{1/\alpha}(t)} \Phi^*(x_2), \quad x_2' = -h(t, x_1),$$

and the equation in (18) is equivalent to the system

$$x_1' = \frac{1}{a^{1/\alpha}(t)}\Phi^*(x_2), \quad x_2' = -g(t, x_1, u).$$

Taking into account that $C^1(J, \mathbb{R})$ can be embedded in a closed subset of $C(J, \mathbb{R}^2)$ via the map $v \to (v, v')$, the assertion follows immediately from [6, Theorem 1.3]. □

3 The case $F(u)/u^\alpha$ bounded

In this section we study the existence of solutions for the BVP (1), (2) in case the nonlinear term F satisfies the condition

$$\limsup_{u\to 0^+} \frac{F(u)}{u^\alpha} < \infty. \tag{19}$$

Denote

$$M = \sup_{(0,2c]} \frac{F(u)}{u^\alpha}. \tag{20}$$

The following existence result holds.

Theorem 2. *Assume (19). Let the half-linear equation*

$$(a(t)\Phi(z'))' + Mb_+(t)\Phi(z) = 0, \tag{21}$$

be nonoscillatory and have a solution z which is positive for $t \ge t_0$ and satisfies $z'(t_0) < 0$. If

$$B_- \le \left(\frac{1}{A}\log 2\right)^\alpha \frac{1}{M}, \tag{22}$$

where A, B_- are defined in (3),(5), then the BVP (1), (2) has at least one solution x such that

$$0 < x(t) \le 2c \text{ for } t \in [t_0, \infty), \; x'(t) < 0 \text{ for large } t. \tag{23}$$

Proof. At first, we consider the auxiliary problem of the existence of a globally positive solution of the equation (1), satisfying fixed initial data, i.e. the BVP

$$\begin{cases} (a(t)\Phi(x'))' + b(t)F(x) = 0, & t \in [t_0, \infty) \\ x \in S, \end{cases} \tag{24}$$

where

$$S = \{u \in C^1[t_0, \infty) : \ u(t_0) = c, \ u'(t_0) = 0, \ u(t) > 0 \text{ for } t \geq t_0\}.$$

The existence of a solution of the BVP (24) will be proved by using Theorem 1.

Consider the half-linear equation

$$(a(t)\Phi(w'))' - Mb_{-}(t)\Phi(w) = 0. \tag{25}$$

Using (5) and Lemma 5, equation (25) is nonoscillatory and its principal solution is positive decreasing on $[t_0, \infty)$, with zero limit. Let w_0 be the principal solution of (25) satisfying $w_0(t_0) = c$. Let

$$\Omega = \left\{u \in C^1[1, \infty) : \ u(t_0) = c, \, u'(t_0) = 0, \, w_0(t) \leq u(t) \leq 2c\right\}.$$

Since $w_0(t)$ is positive for $t \geq t_0$, we have $u(t) > 0$ and so $\Omega \subset S$ for any $u \in \Omega$. Hence

$$S \cap \Omega = \Omega.$$

Let $\mathcal{T} : \Omega \to C^1([t_0, \infty)$ be the operator which maps every $u \in \Omega$ into the unique solution $y = \mathcal{T}(u)$ of the Cauchy problem

$$\begin{cases} (a(t)\Phi(y'))' + b(t)\dfrac{F(u(t))}{u^{\alpha}(t)}\Phi(y) = 0, \\ y(t_0) = c, \, y'(t_0) = 0. \end{cases} \tag{26}$$

Let $S_1 = \overline{\mathcal{T}(\Omega)}$. Then, the problem

$$\begin{cases} (a(t)\Phi(y'))' + b(t)\dfrac{F(u(t))}{u^{\alpha}(t)}\Phi(y) = 0, \\ y \in S_1, \end{cases} \tag{27}$$

has a unique solution for any $u \in \Omega$.

We show that

$$S_1 \subset S \cap \Omega = \Omega. \tag{28}$$

Since Ω is closed, for proving (28), it is sufficient to show that $\mathcal{T}(\Omega) \subset \Omega$. Any $y \in \mathcal{T}(\Omega)$ satisfies $y(t_0) = c$, $y'(t_0) = 0$. Thus, we only need to show that $w_0(t) \leq y(t) \leq 2c$ for all $t \geq t_0$.

In virtue of (20), for $u \in \Omega$

$$-Mb_-(t) \le b(t)\frac{F(u(t))}{u^\alpha(t)} \le Mb_+(t).$$

Thus, (21) is a Sturm majorant of the equation in (27) and (25) is a Sturm minorant. Applying Lemma 3 to equation in (27), we get that any solution y, with initial conditions $y(t_0) = c, y'(t_0) = 0$, is positive on $[t_0, \infty)$. Applying Lemma 4, we get that $y(t) \ge w_0(t)$ for all $t \ge t_0$.

To obtain a upper bound for y, we integrate (27) on $[t_0, t]$, $t > t_0$. From (5) and (20), in view of the positivity of y, we have for $t \ge t_0$

$$\begin{aligned}
y(t) &= c - \int_{t_0}^t \Phi^*\left(\frac{1}{a(s)}\int_{t_0}^s b(r)\frac{F(u(r))}{u^\alpha(r)} y^\alpha(r)\,dr\right) ds \\
&\le c + \int_{t_0}^t \left(\frac{1}{a(s)}\int_{t_0}^s b_-(r)\frac{F(u(r))}{u^\alpha(r)} y^\alpha(r)\,dr\right)^{1/\alpha} ds \\
&\le c + (MB_-)^{1/\alpha}\int_{t_0}^t \frac{Y(s)}{a^{1/\alpha}(s)}\,ds,
\end{aligned}$$

where $Y(t) = \max_{[t_0,t]} y(s)$. Thus

$$Y(t) \le c + (MB_-)^{1/\alpha}\int_{t_0}^t \frac{Y(s)}{a^{1/\alpha}(s)}\,ds,$$

and, using (22), the Gronwall's lemma gives

$$y(t) \le Y(t) \le c\exp\left((MB_-)^{1/\alpha}A\right) \le c\exp(\ln 2) = 2c,$$

and so $\mathcal{T}(\Omega) \subseteq \Omega$. Thus, in order to apply Theorem 1, it remains to prove that S_1 is bounded in $C^1[t_0, \infty)$.

Since $\mathcal{T}(\Omega) \subseteq \Omega$ and Ω is bounded in $C[t_0, \infty)$, for any $u \in \Omega$ the functions $T(u(t))$ are equibounded on every compact interval $K \subset [t_0, \infty)$. Then, from (20) and (27), the functions $(a(t)\Phi(\mathcal{T}(u(t))'))'$ are equibounded on K. Since $(\mathcal{T}(u))'(t_0) = 0$, the mean value theorem gives the equiboundedness of the functions $a(t)\Phi(\mathcal{T}(u(t))')$ on K, i.e., $\mathcal{T}(\Omega)$ is bounded in $C^1[t_0, \infty)$. Thus S_1 is bounded in $C^1[t_0, \infty)$.

Theorem 1 can therefore be applied to (24), and the existence of a solution x to (24) follows. Since $x \in \Omega$, we have

$$\max_{t\in[t_0,\infty)} x(t) \le 2c.$$

In order to prove that x is also a solution of problem (1), (2), it remains to show that $\lim_{t\to\infty} x(t) = 0$.

Since $0 < x(t) \le 2c$, $x'(t_0) = 0$, integrating by part on $[t_0, t]$, $t \ge t_0$, the equality

$$\frac{(a(t)\Phi(x'(t)))'}{F(x)} + b(t) = 0$$

we obtain

$$\frac{a(t)\Phi(x'(t))}{F(x(t))} + \int_{t_0}^{t} \frac{a(s)\Phi(x'(s))F'(x(s))x'(s)}{F^2(x(s))}\, ds = -\int_{t_0}^{t} b(s)\, ds.$$

Taking into account that $\Phi(u)u = |u|^{\alpha+1} \ge 0$ and $F'(x(t)) \ge 0$, we have

$$\frac{a(t)\Phi(x'(t))}{F(x(t))} \le -\int_{t_0}^{t} b(s)\, ds.$$

Since assumptions (4), (5) imply $\int_{t_0}^{\infty} b(t)\, dt = +\infty$, we get

$$\lim_{t\to\infty} \frac{a(t)\Phi(x'(t))}{F(x(t))} = -\infty.$$

Hence, t_1 sufficiently large exists such that $x'(t) < 0$ for all $t > t_1$.

Assume by contradiction that $\lim_{t\to\infty} x(t) > 0$. Then there exist two positive constants $c_1 < c_2$ such that $c_1 \le x(t) \le c_2$ for all $t \ge t_1$. Let

$$d_1 = \min_{u\in[c_1,c_2]} F(u), \quad d_2 = \max_{u\in[c_1,c_2]} F(u).$$

By integrating the equation in (24) on $[t_1, t]$, $t \ge t_1 \ge t_0$, we obtain

$$\begin{aligned} x'(t) &= x'(t_1) - \Phi^*\left(\frac{1}{a(t)}\int_{t_1}^{t} b(s)F(x(s))\, ds\right) \\ &\le \Phi^*\left(-\frac{d_1}{a(t)}\int_{t_1}^{t} b_+(s)\, ds + \frac{d_2}{a(t)}\int_{t_1}^{t} b_-(s)\, ds\right). \end{aligned}$$

Since $\int_{t_1}^{\infty} b_+(t)\, dt = \infty$, $\int_{t_1}^{\infty} b_-(t)\, dt < \infty$, then $t_2 > t_1$ exists such that

$$\int_{t_1}^{t_2} b_+(t)\, dt \ge \frac{d_2 B_-}{d_1},$$

where B_- was defined in (5). We have

$$\begin{aligned} x'(t) &\le \Phi^*\left(-\frac{d_1}{a(t)}\int_{t_1}^{t_2} b_+(s)\,ds - \frac{d_1}{a(t)}\int_{t_2}^{t} b_+(s)\,ds + \frac{d_2 B_-}{a(t)}\right) \\ &\le -\left(\frac{d_1}{a(t)}\int_{t_2}^{t} b_+(s)\,ds\right)^{1/\alpha}, \end{aligned}$$

which gives

$$x(t) - x(t_2) \le -d_1^{1/\alpha}\int_{t_2}^{t}\left(\frac{1}{a(s)}\int_{t_2}^{s} b_+(r)\,dr\right)^{1/\alpha} ds,$$

and, from (4), the above integral is divergent for $t \to \infty$, which implies $\lim_{t\to\infty} x(t) = -\infty$, a contradiction with the positivity of x. □

Remark 1. A closer examination of the proof of Theorem 2 shows that if the BVP (1), (2) has a solution for some fixed c, then it has solution for any $\bar{c}$, $0 < \bar{c} \le c$. In particular, since $M = M(c)$ in (20) is nondecreasing, then (22) can be understood as a upper bound for the values c for which (1), (2) has solution. Notice that, if $F(u) = u^\beta$ for $u \ge 0$, $\beta > \alpha$, then $M = (2c)^{\beta-\alpha}$ and (22) can be written as

$$c \le \frac{1}{2B_-^{1/(\beta-\alpha)}}\left(\frac{\log 2}{A}\right)^{\alpha/(\beta-\alpha)},$$

while, if $F(u) = u^\alpha$ for $u \ge 0$, then $M = 1$ for all $c > 0$ and so, if assumptions in Theorem 2 are satisfied, then (1), (2) has solution for all $c > 0$.

In case $b \ge 0$, Theorem 2 has a more simple form, since (22) is trivially satisfied and every solution of (27) is nonincreasing. Thus the following holds.

Corollary 1. *Assume (19) and $b(t) \ge 0$ for $t \ge t_0$. Let the half-linear equation* (21) *be nonoscillatory and have a solution z which is positive for $t \ge t_0$ and satisfies $z'(t_0) < 0$. Then the BVP (1), (2) has at least one solution x which is nonincreasing on $[t_0, \infty)$ and decreasing for t large.*

4 The case $F(u)/u^\alpha$ unbounded

In this section we consider the case

$$G(u) = \frac{F(u)}{u^\alpha} \quad \text{nonincreasing for } u \in (0, 2c). \tag{29}$$

Clearly, (29) does not require the unboundedness of G in a right neighborhood of zero, even if the unboundedness of G represents the more interesting situation when (29) is valid. This case, as mentioned in the Introduction, requires some additional assumptions with respect to the previous one.

Our main result is the following.

Theorem 3. *Assume* (29). *Let* $\theta \in L^1[t_0, \infty)$ *be a positive function such that*

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)} \int_t^{\infty} \theta(s)\, ds \right)^{1/\alpha} dt = c, \quad \theta(t) G(\varphi(t)) \in L^1([t_0, \infty)),$$

where

$$\varphi(t) = \int_t^{\infty} \left(\frac{1}{a(s)} \int_s^{\infty} \theta(r)\, dr \right)^{1/\alpha} ds. \tag{30}$$

Assume that the half-linear equation

$$(a(t)\Phi(z'))' + b_+(t) G(\varphi(t)) \Phi(z) = 0 \tag{31}$$

is nonoscillatory and has a solution z *which is positive for* $t \geq t_0$ *and satisfies* $z'(t_0) < 0$.

If

$$b_-(t) \leq \frac{\theta(t)}{N}, \tag{32}$$

where

$$N = \max \left\{ F(2c),\ \Theta \Big(\frac{A}{\ln 2} \Big)^{\alpha} \right\}, \quad \Theta = \int_{t_0}^{\infty} \theta(t) G(\varphi(t))\, dt,$$

then the BVP (1), (2) has at least a solution satisfying (23).

Proof. Similarly to the proof of Theorem 2, at first we prove the existence of a solution to (24) by using Theorem 1. Let Ω be the set given by

$$\Omega = \left\{ u \in C^1[t_0, \infty) :\ u(t_0) = c,\ u'(t_0) = 0,\ \varphi(t) \leq u(t) \leq 2c \right\},$$

where φ is defined in (30). Since $\varphi(t)$ is positive for $t \geq t_0$, we have $\Omega \subset S$ and $u(t) > 0$ for any $u \in \Omega$. Let $\mathcal{T} : \Omega \to C^1[t_0, \infty)$ be the operator which maps every $u \in \Omega$ into the unique solution $y = \mathcal{T}(u)$ of the Cauchy problem (26), and let $S_1 = \overline{\mathcal{T}(\Omega)}$. Then the problem

$$\begin{aligned} &(a(t)\Phi(y'))' + b(t)G(u(t))\Phi(y) = 0, \\ &y \in S_1, \end{aligned} \tag{33}$$

has a unique solution for all $u \in \Omega$. In order to apply Theorem 1, we have to show that $S_1 \subset S \cap \Omega = \Omega$, and S_1 is bounded in $C^1([t_0, \infty))$.

As before, to prove that $S_1 \subset \Omega$, it is sufficient to show that $\mathcal{T}(\Omega) \subset \Omega$ since Ω is closed. By definition of $\mathcal{T}$, every $y \in \mathcal{T}(\Omega)$ satisfies $y(t_0) = c$, $y'(t_0) = 0$. Thus we only need to show that $\varphi(t) \le y(t) \le 2c$ for all $t \ge t_0$. In virtue of (29), the inequality $b(t)G(u(t)) \le b_+(t)G(\varphi(t))$ holds for $u \in \Omega$. Thus, equation (31) is a Sturm majorant of (33). By applying Lemma 3 to the equation in (33), we get that any solution y with initial conditions $y(t_0) = c, y'(t_0) = 0$, is positive on $[t_0, \infty)$. Let $W(t) = y(t) - \varphi(t)$. Since $y(t_0) = \varphi(t_0) = c$, then $W(t_0) = 0$. Further, since $\lim_{t\to\infty} \varphi(t) = 0$, we get $\liminf_{t\to\infty} W(t) \ge 0$. Thus, to prove that W is nonnegative on $[t_0, \infty)$, it is sufficient to show that W does not have negative minima. By contradiction, let $T > t_0$ be a point of negative minimum for W, and let $t_1 > T$ be such that $W'(t_1) > 0$, $W(t) < 0$ on $[T, t_1]$. Then, since $W'(T) = 0$, we obtain

$$\begin{aligned} 0 < a(t_1)[\Phi(y'(t_1)) &- \Phi(\varphi'(t_1))] \\ &= -\int_T^{t_1} [b(t)G(u(t))\Phi(y(t)) + (a(t)\Phi(\varphi'(t)))']\, dt \\ &\le \int_T^{t_1} b_-(t)G(u(t))\big(\Phi(y(t)) - \Phi(\varphi(t))\big)\, dt \\ &- \int_T^{t_1} (a(t)\Phi(\varphi'(t)))' - b_-(t)G(u(t))\Phi(\varphi(t))\, dt \\ &\le -\int_T^{t_1} (a(t)\Phi(\varphi'(t)))' - b_-(t)G(u(t))\Phi(\varphi(t))\, dt, \end{aligned}$$

where we used $y(t) < \varphi(t)$ on $[T, t_1]$. Now, we have $(a(t)\Phi(\varphi'(t)))' = \theta(t)$. In view of (32), for every $u \in \Omega$ we obtain

$$\begin{aligned} -\theta(t) + b_-(t)G(u(t))\Phi(\varphi(t)) &= -\theta(t) + b_-(t)F(u(t))\Big(\frac{\varphi(t)}{u(t)}\Big)^{\alpha} \\ &\le -\theta(t) + Nb_-(t) \le 0. \end{aligned}$$

Thus $a(t_1)[\Phi(y'(t_1)) - \Phi(\varphi'(t_1))] \le 0$, a contradiction. This fact proves $W(t) = y(t) - \varphi(t) \ge 0$ on $[t_0, \infty)$. To obtain the upper bound for y, similarly to the case treated in Theorem 2, we integrate (33) on $[t_0, t]$, $t \ge t_0$. From (29) and (32), in view of the positivity of y, we have

$$
\begin{aligned}
y(t) &\le c + \int_{t_0}^{t} \left(\frac{1}{a(s)} \int_{t_0}^{s} b_-(r) G(u(r)) y^{\alpha}(r)\, dr \right)^{1/\alpha} ds \\
&\le c + \frac{1}{N^{1/\alpha}} \int_{t_0}^{t} \left(\frac{1}{a(s)} \int_{t_0}^{s} \theta(r) G(\varphi(r)) y^{\alpha}(r)\, dr \right)^{1/\alpha} ds \\
&\le c + \left(\frac{\Theta}{N} \right)^{1/\alpha} \int_{t_0}^{t} \frac{Y(s)}{a^{1/\alpha}(s)}\, ds,
\end{aligned}
$$

where $Y(t) = \max_{[t_0,t]} y(s)$. Thus the Gronwall's lemma gives

$$
y(t) \le Y(t) \le c \exp \left[\left(\frac{\Theta}{N} \right)^{1/\alpha} A \right] \le 2c.
$$

The argument for showing that S_1 is bounded in $C^1[t_0, \infty)$ is analogous to the one done in Theorem 2. Thus, Theorem 1 can be applied to (24), and the existence of the solution y to (24) follows. The monotonicity of y on a neighborhood of infinity and $\lim_{t\to\infty} y(t) = 0$ may be proved in the same way as in Theorem 2. Thus y is also a solution of problem (1), (2) and satisfies (23). □

Analogously to Theorem 2, the statement of Theorem 3 is more simple if $b \ge 0$, since (32) is trivially satisfied and every solution of (33) is nonincreasing. The following holds.

Corollary 2. *Assume* (29) *and* $b(t) \ge 0$ *for* $t \ge t_0$. *Let* φ, θ *be the functions defined in Theorem 3 and assume that the half-linear equation* (31) *is nonoscillatory and has a solution* z *which is positive for* $t \ge t_0$ *and satisfies* $z'(t_0) < 0$. *Then the BVP (1), (2) has at least one solution* x *which is nonincreasing on* $[t_0, \infty)$ *and decreasing for* t *large.*

5 Applications and concluding remarks

In the first part of this section, we give effective criteria for the solvability of problem (1), (2) together with some examples.

Theorem 2 [Theorem 3] requires the existence of a positive solution z of the associated half-linear equation (21), [(31)] which satisfies $z'(t_0) < 0$. To check this property, we may use Lemma 3 and, as a Sturmian majorant, the generalized Euler equation

$$(t^n\Phi(y'))' + \left(\frac{n-\alpha}{\alpha+1}\right)^{\alpha+1} t^{n-\alpha-1}\Phi(y) = 0, \quad n > \alpha,\ t \geq t_0 > 0. \tag{34}$$

Denote

$$\delta = \frac{n-\alpha}{\alpha+1}.$$

It is easy to check that $y(t) = t^{-\delta}$ is a solution of (34) and satisfies $y'(t_0) < 0$.

We get the following results.

Corollary 3. *Let $n > \alpha$. Assume (19), (22) and*

$$a(t) \geq t^n, \quad Mb_+(t) \leq \left(\frac{n-\alpha}{\alpha+1}\right)^{1+\alpha} t^{n-\alpha-1} \quad \text{for } t \geq t_0 > 0, \tag{35}$$

where M is given in (20). Then the BVP (1), (2) has at least one solution satisfying (23).

Proof. In view of (35), equation (34) is a Sturmian majorant of (21). Thus, by Lemma 3, the principal solution of (21) is positive decreasing on $[t_0, \infty)$. Appying Theorem 2, we get the assertion. □

When (29) holds, using a similar argument to the one given in the proof of Corollary 3, we obtain from Theorem 3 the following criterion.

Corollary 4. *Let $n > \alpha$. Assume* (29) *and let θ and φ satisfy conditions in Theorem 3. Let (32) hold and*

$$a(t) \geq t^n, \quad b_+(t)G(\varphi(t)) \leq \left(\frac{n-\alpha}{\alpha+1}\right)^{1+\alpha} t^{n-\alpha-1} \quad \text{for } t \geq t_0 > 0. \tag{36}$$

Then the BVP (1), (2) has at least one solution satisfying (23).

Corollaries 3 and 4 can be easily interpreted in terms of problem (NP), since $r(t) = a(t)t^{1-d}, q(t) = b(t)t^{1-d}$.

Other applications can be obtained using any half-linear equation having a positive decreasing solution. For instance, this happens for the classical Euler equation or the Riemann-Weber equation, for more details see [14].

The next examples illustrate our existence results (Theorem 2 and Theorem 3). In Examples 1 and 2 the case of weight with indefinite sign is considered, if $G(u) = F(u)/u^\alpha$ is bounded or unbounded near zero, respectively. In particular, if $G(u)$ is unbounded, as already claimed in the Introduction, the nonlinearity need to be "close" to the power function with exponent α, i.e. $G(u)$ for $u \to 0^+$ needs to be an infinite of order less than any power. In order to simplify the calculations, the case of linear operator ($\alpha = 1$) is considered in the both the examples. Finally, Example 3 deals with the simple case of positive weight and coefficients like power functions, in order to put in evidence the relations between the coefficients and the applicability of the result to radial solutions of elliptic equations.

Example 1. Consider the equation for $t \geq 1$

$$(t^2x')' + \Big(\lambda\, S(t) - \frac{\mu}{t^2}\, s(t)\Big)\, |x|^\beta \operatorname{sgn} x = 0, \quad \beta \geq 1, \tag{37}$$

where $\lambda > 0$, $\mu \geq 0$ and

$$S(t) = \max\{\sin t, 0\}, \quad s(t) = -\min\{\sin t, 0\} \quad \text{for } t \geq 1. \tag{38}$$

We look for solutions of (37) satisfying (2), with $t_0 = 1$ and $c = 1$. Obviously, (3) and (5) are satisfied. Standard calculations shows that also (4) holds. Moreover, $A = 1$, $M = 2^{\beta-1}$ and $B_- < \mu$. Thus, if $\mu \leq 2^{1-\beta}\log 2$, then (22) is satisfied. Since (19) holds, we apply Corollary 3. From $b_+(t) \leq \lambda$, taking $\alpha = 1$, $n = 2$ and $\lambda \leq 2^{-\beta-1}$, also the inequality (35) is satisfied. Therefore, equation (37) has a solution x satisfying the boundary conditions

$$x(1) = 1,\ x'(1) = 0,\ 0 < x(t) \leq 2 \text{ for } t \in [1, \infty),\ \lim_{t\to\infty} x(t) = 0, \tag{39}$$

if

$$0 < \lambda \leq 2^{-\beta-1}, \quad 0 \leq \mu \leq 2^{1-\beta}\log 2.$$

Example 2. Consider the equation for $t \geq 1$

$$(t^2x')' + \left(\frac{\lambda S(t)}{\log(2et^{3/2})} - \frac{\mu s(t)}{t^{3/2}}\right) F(x) = 0, \tag{40}$$

where S, s are defined by (38), and F is a continuous function on $\mathbb{R}$ such that $F(0) = 0$ and

$$F(u) = u \left| \log \frac{u}{2e} \right| \quad \text{if } u \in (0, 2].$$

Clearly, $F \in C^1(0, 2]$, $F'(u) > 0$ in $(0, 2)$ and $G(u) = |\log \frac{u}{2e}|$ is decreasing in $(0, 2)$. Further, (3) and (5) are satisfied, and some computations show that (4) holds too. Since (29) is fulfilled, for applying Corollary 4, take $\theta(t) = 3t^{-3/2}/4$. Thus we obtain $\varphi(t) = t^{-3/2}$, and inequalities (32), (36) are satisfied, respectively, if $\mu \le 1/14$ and $\lambda \le 1/4$. Therefore, by Corollary 4, equation (40) has a solution x satisfying (39) if

$$0 < \lambda \le \frac{1}{4}, \quad 0 \le \mu \le \frac{1}{14}.$$

Example 3. Consider the equation for $t \ge 1$

$$(t^n \Phi(x'))' + t^\gamma |x|^\beta \operatorname{sgn} x = 0, \quad \beta \ge \alpha, \tag{41}$$

where $n, \gamma \in \mathbb{R}$. Conditions (3), (4) are clearly satisfied if and only if $n > \alpha$ and $\gamma \ge n - 1 - \alpha$. Thus, by applying Corollary 1 and Corollary 3 we obtain the following results:
Case $\beta > \alpha$. If $n > \alpha$ and $\gamma = n - 1 - \alpha$, then there exists a solution of (41) satisfying (2) for every $c > 0$ such that

$$c \le \frac{1}{2} \left(\frac{n - \alpha}{\alpha + 1} \right)^{\frac{\alpha+1}{\beta-\alpha}}. \tag{42}$$

Case $\beta = \alpha$. If $n \ge 2\alpha + 1$ and $\gamma = n - 1 - \alpha$, then there exists a solution of (41) satisfying (2) for every $c > 0$.
Clearly, a similar result holds for a wide class of nonlinearity F, for instance:

$$F(u) = u^\beta + u^\sigma \quad \text{for } u \ge 0,$$

with $\sigma > \beta \ge \alpha$. The details are left to the reader.

Example 3 can be easily written in the notations deriving from the study of radial solutions of elliptic equations, that is, for problem (NP). For instance, in case of the Neumann problem outside the ball of radius 1 in $\mathbb{R}^3$, then $d-1 = 2$ and (NP) has solution if there exists $n > \alpha$ such that $r(t) = t^{n-2}$, $q(t) = t^{n-\alpha-3}$, $F(u) = |x|^{\beta} \operatorname{sgn} x$, $\beta > \alpha$, and c satisfies (42). Analogous result can be formulated if $\alpha = \beta$.

Concluding remarks.

1. As already claimed, in [14] the existence of positive solutions to (1), satisfying different initial and terminal conditions, has been proved using a different approach. Indeed, the solvability of the BVP considered in [14] is obtained by looking for the principal solution of a suitable auxiliary half-linear equation. For problem (1), (2), this method cannot be applied, due to the prescribed initial value of the derivative of the solution. Indeed, solutions of the half-linear equation in (26), starting at t_0 with a zero derivative, may be nonprincipal solutions. Therefore here we have derived a new approach that allows us to look for nonprincipal solutions.

2. Theorem 3 does not require that $F(u)/u^{\alpha}$ is unbounded in a neighborhood of zero. Thus, if $F(u)/u^{\alpha}$ is bounded and decreasing in an interval $(0, \delta)$, then both Theorem 2 and Theorem 3 may be applied. Of course, they may require different sufficient conditions for the existence of a solution to the BVP (1), (2). Indeed, in this case, equation (21) is a Sturm majorant for (31), and therefore if the conditions in Theorem 2 are satisfied for (21), then the conditions in Theorem 3 for (31) are satisfied too. However, assumption (32) may be stronger than (22), since (32) is a pointwise estimate, while (22) is an integral one.

3. Our results can be useful to derive the existence of ground state solutions, or solutions with a prescribed number of sign changes, defined in all the space. Indeed, if the Neumann problem in a ball

$$\begin{cases} \operatorname{div}\left(r(\mathbf{x})\,|\nabla u|^{p-2}\nabla u\right) + q(\mathbf{x})\,F(u) = 0, \quad |\mathbf{x}| \le R, \\ \frac{\partial u}{\partial n} = 0 \ \text{ if } |\mathbf{x}| = R \end{cases}$$

has a radial positive solution, and an estimate of its sup-norm is known, then our results assure that this solution can be extended outside the ball till infinity, it is globally positive and tends to zero. Similarly, if the above Neumann problem has a radial solution with one (or more) change of sign, and an estimate of its sup-norm is known, then our results assure that this solution can be extended outside the ball till infinity, it has no sign changes outside the ball and tends to zero.

Acknowledgements

The authors want to thank the anonymous referees for their valuable comments and suggestions which were precious to increase the value of the manuscript.
The first author is supported by Grant GA 17-03224S of the Czech Science Foundation. The second author is partially supported by Gnampa, National Institute for Advanced Mathematics (INdAM).

References

[1] Abreu R., Montenegro M., *Existence of a ground state solution for an elliptic problem with critical growth in an exterior domain*, Nonlinear Anal. **109** (2014), 341–349.

[2] Azzollini A., *Ground state solution for a problem with mean curvature operator in Minkowski space*, J. Funct. Anal. **266**(4) (2014), 2086–2095.

[3] Boscaggin A., Dambrosio W., Papini D., *Multiple positive solutions to elliptic boundary blow-up problems*, J. Differential Equations **262** (2017), 5990–6017.

[4] Butler G.J., *The existence of continuable solutions of a second order order differential equation,* Can. J. Math., **XXIX** (1977), 472–479.

[5] Burton T.A., Grimmer R.C., *On the continuability of the second order differential equation,* Proc. Amer. Math. Soc. **29**(1971), 277–283.

[6] Cecchi M., Furi M., Marini M., *On continuity and compactness of some nonlinear operators associated with differential equations in noncompact intervals,* Nonlinear Anal., T.M.A, **9** (1985), 171–180.

[7] Coffman C.V., Ullrich D.F., *On the continuation of solutions of a certain nonlinear differential equation,* Monatsh. Math. 71 (1967), 385–392.

[8] Cecchi M., Došlá Z., Marini M., *Half-linear differential equations with oscillating coefficient*, Diff. Integr. Equat. **18** (2005), 1243–1256.

[9] Cortázar C., Dolbeault J., Garcia-Huidobro M., Manásevich R., *Existence of sign changing solutions for an equation with a weighted p-Laplace operator*, Nonlinear Anal. **110** (2014), 1–22.

[10] Došlá Z., Marini M., Matucci S., *A boundary value problem on a half-line for differential equations with indefinite weight*, Commun. Appl. Anal. **15** (2011), 341–352.

[11] Došlá Z., Marini M., Matucci S., *Positive solutions of nonlocal continuous second order BVP's*, Dynamic Syst. Appl. **23** (2014), 431–446.

[12] Došlá Z., Marini M., Matucci S., *A Dirichlet problem on the half-line for nonlinear equations with indefinite weight*. Ann. Mat. Pura Appl. **196** (2017), 51–64. DOI, 10.1007/s10231-016-0562-y

[13] Došlá Z., Marini M., Matucci S., *Positive decaying solutions to BVPs with mean curvature operator*, Rend. Istit. Mat. Univ. Trieste **49** (2017), 147–164.

[14] Došlá Z., Marini M., Matucci S., *Global Kneser solutions to nonlinear equations with indefinite weight*, DCDS-B **23**(8) (2018), 3297–3308.

[15] Došlý O., Řehák P., Half-linear Differential Equations, North-Holland, Mathematics Studies **202**, Elsevier Sci. B.V., Amsterdam, 2005.

[16] Fowler R.H.,*Further studies of Emden's and similar differential equations*, Quart. J. Math. Oxford Ser. **2** (2) (1931), 259–288.

[17] Franca M., Sfecci A., *On a diffusion model with absorption and production*. Nonlinear Analysis, Real World Appl.**34** (2017), 41–60.

[18] Franca M., Sfecci A., *Entire solutions of superlinear problems with indefinite weights and Hardy potentials*, J. Dynam. Differential Equations **30**(3) (2018), 1081–1118.

[19] Gaudenzi M., Habets P., Zanolin F., *An example of a superlinear problem with multiple positive solutions*, Atti Sem. Mat. Fis. Univ. Modena **51** (2003), 259–272.

[20] Hartman P., Ordinary Differential Equations, 2 Ed., Birkäuser, Boston-Basel-Stuttgart 1982.

[21] Kusano T., Ogata A., Usami H., *Oscillation theory for a class of second order quasilinear ordinary differential equations with application to partial differential equations*. Japan. J. Math. (N.S.) **19**(1993), 131–147.

[22] Li W.T., Cheng S.S., *Limiting Behaviours of Non-oscillatory Solutions of a pair of Coupled Nonlinear differential Equations*, Proc. Edinb. Math. Soc. **43** (2000), 457–473.

[23] Lopez-Gomez J., Tellini A., Zanolin F., *High multiplicity and complexity of the bifurcation diagrams of large solutions for a class of superlinear indefinite problems*, Commun. Pure Appl. Anal. **13** (2014), 1, 1–73.

[24] Matucci S., *A new approach for solving nonlinear BVP's on the half-line for second order equations and applications,* Mathematica Bohemica **140** (2015), 153–169.

Authors' addresses:

Zuzana Došlá, Department of Mathematics and Statistics, Masaryk University, Kotlářská 2, CZ-61137 Brno, Czech Republic. E-mail: dosla@math.muni.cz

Serena Matucci, Department of Mathematics and Informatics "Ulisse Dini", University of Florence, I-50139 Florence, Italy. E-mail: serena.matucci@unifi.it