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Cournot Equilibrium Uniqueness in Case of Concave Industry Revenue: a Simple Proof

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Abstract

A simple proof of an equilibrium uniqueness result by Murphy, Sherali and Soyster for homogeneous Cournot oligopolies with concave industry revenue function and convex cost functions is provided. Adapting this proof, a substantial improvement of their result is obtained; the improvement concerns capacity constraints, non-differentiable cost functions and industry revenue functions that are discontinuous at 0.

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1. Introduction

Various results in the literature guarantee the existence of a unique Cournot equilibrium in homogeneous Cournot oligopolies with continuous profit functions (see, e.g., Ewerhart (2014) for a review).[1] Uniqueness boils down to existence (i.e., the existence of at least one equilibrium) and semi-uniqueness (i.e., the existence of at most one equilibrium). To the best of our knowledge, the conditions assumed in all equilibrium uniqueness results with continuous profit functions that deal with an indefinite number of possibly non-symmetric firms allow a routine application of the well-known Nikaido-Isoda theorem (Nikaido and Isoda (1955)): in fact the equilibrium existence issue cannot be considered a real issue. Nevertheless equilibrium semi-uniqueness is a real issue in these results. This is particularly evident in the case of oligopolies with concave industry revenue function. The first result for this case is the following one in Murphy et al. (1982).

Theorem 1. *Suppose that no firm is capacity constrained, that the industry revenue function is concave and that cost functions are convex. Also, suppose that the price function is continuously differentiable and strictly decreasing and that the cost functions are continuously differentiable. Each of the following conditions is sufficient for the existence of at most one equilibrium: (I) The industry revenue function is strictly concave. (II) All cost functions are strictly convex.* $\diamond$

The proof of Theorem 1 in Murphy et al. (1982) and that of its variants, in Deneckere and Kovenock (1999), Hirai and Szidarovszky (2013) and Ewerhart (2014),[2] is not so simple. The proof in Murphy et al. (1982) is provided by means of a convex programming technique; those of the mentioned variants employ a technique based on the cumulative best reply correspondences (see Vives (2001)).[3] Using ideas in Quartieri (2008) and von Mouche and Quartieri (2013), we shall provide a very simple proof of Theorem 1, and a generalization thereof in Theorem 2.

2. Setting

A (homogeneous Cournot) *oligopoly* is a game in strategic form with a set $N := \{1, \ldots, n\}$ of players whose elements are called *firms*. Each firm i has a strategy set X_i which is a proper interval of $\mathbb{R}_+$ containing 0; the elements of X_i are referred to as *production levels*, and those of $Y := \sum_{l \in N} X_l$ as *industry production levels*. Each firm i has a payoff function, called *profit function*, $f_i : X_1 \times \cdots \times X_n \to \mathbb{R}$ given by

$$f_i(\mathbf{x}) := p(\sum_{l \in N} x_l) x_i - c_i(x_i),$$

where $p : Y \to \mathbb{R}$ is called *price function* (also called *inverse demand function*),[4] and $c_i : X_i \to \mathbb{R}$ is called *firm i's (net) cost function.* A Nash equilibrium of an oligopoly

[1] Terminology, definitions and notations are provided in Section 2.

[2] Hirai and Szidarovszky (2013) consider rent-seeking games. However, as first shown in Szidarovszky and Okuguchi (1997), results for such games generally imply a corresponding result for oligopolies; on this, see also the mimeo by Prof. Okuguchi cited in Szidarovszky and Okuguchi (1997).

[3] There is a claim in Watts (1996) for a related result. See also Okuguchi (2010), where in a structurally equivalent context (with cost functions replaced by production functions) the case of a concave industry revenue function is dealt with.

[4] The price function associates with each positive industry production level y the (greatest) unit price at which y is entirely sold to the market. In the present article we allow p to assume negative values; in this context Monderer and Shapley (1996) speak of a *quasi-Cournot* game.

is called a *(Cournot) equilibrium*. The *industry (aggregate) revenue function* $r : Y \to \mathbb{R}$ is defined by $r(y) := p(y)y$; therefore $r(0) = 0$. Henceforth, we shall denote the set of equilibria of an oligopoly by E.

Note that either $X_i = [0, m_i]$ with $m_i > 0$, or $X_i = [0, m_i[$ with $m_i > 0$, or $X_i = \mathbb{R}_+$. In the first two cases we say that firm i is *capacity constrained*.[5] Henceforth, for all $\mathbf{x} \in \mathbf{X}$ and $A \subseteq N$ we shall denote $\sum_{i \in A} x_i$ by x_A. By an *equilibrium aggregate* we mean an element of $\{e_N \mid \mathbf{e} \in E\}$.

3. The simple proof

By way of contradiction suppose that either I or II holds and that $\mathbf{a}$ and $\mathbf{b}$ are distinct equilibria with $a_N \leq b_N$. Let $J := \{l \in N \mid a_l < b_l\}$ and $s := |J|$. Note that $\mathbf{b} \neq \mathbf{0}$, $s \geq 1$, $b_N > 0$, $a_J \leq a_N$, $b_J \leq b_N$, $b_J - a_J \geq b_N - a_N$ and $a_J < b_J$.

For all $i \in J$ we have $a_i < b_i$ and therefore, as $\mathbf{a}, \mathbf{b} \in E$, $D_i f_i(\mathbf{a}) \leq 0 \leq D_i f_i(\mathbf{b})$, i.e.,

$$Dp(a_N)a_i + p(a_N) - Dc_i(a_i) \leq 0 \leq Dp(b_N)b_i + p(b_N) - Dc_i(b_i). \tag{1}$$

Suppose for a moment that the following fundamental observation holds:

$$\sum_{i \in J}(Dp(a_N)a_i + p(a_N)) \geq \sum_{i \in J}(Dp(b_N)b_i + p(b_N)) \; (> \text{ in case I}). \tag{2}$$

As each c_i is convex (and strictly convex in case II), (2) implies

$$\sum_{i \in J}(Dp(a_N)a_i + p(a_N) - Dc_i(a_i)) > \sum_{i \in J}(Dp(b_N)b_i + p(b_N) - Dc_i(b_i)). \tag{3}$$

But we obtain a contradiction with (1), because (3) implies that for some $i \in J$

$$Dp(a_N)a_i + p(a_N) - Dc_i(a_i) > Dp(b_N)b_i + p(b_N) - Dc_i(b_i).$$

Let us prove the fundamental observation (2). By the decreasingness of p, (2) holds if $a_N = b_N =: \overline{y}$ since $Dp(\overline{y}) \leq 0$ and even[6] $Dp(\overline{y}) < 0$ in case I. Suppose that $a_N < b_N$. As r is (strictly) concave, the function $\tilde{r} : \mathbb{R}_+ \to \mathbb{R}$ defined by $\tilde{r}(x) := p(x + a_N - a_J)x$ is (strictly) concave.[7] Thus $D\tilde{r}(a_J) \geq D\tilde{r}(a_J + b_N - a_N)$ ($>$ in case I), i.e.,

$$Dp(a_N)a_J + p(a_N) \geq Dp(b_N)(a_J + b_N - a_N) + p(b_N) \; (> \text{ in case I}).$$

As $a_J + b_N - a_N \leq b_J$ and $Dp(b_N) \leq 0$, it follows that

$$Dp(a_N)a_J + p(a_N) \geq Dp(b_N)b_J + p(b_N) \; (> \text{ in case I}). \tag{4}$$

As $(s-1)p(a_N) \geq (s-1)p(b_N)$ by the decreasingness of p, we have $Dp(a_N)a_J + sp(a_N) \geq Dp(b_N)b_J + sp(b_N)$ ($>$ in case I). Thus (2) holds. $\square$

4. A counter-example

Theorem 1 provides two additional sufficient conditions for equilibrium semi-uniqueness: conditions I and II. Lemma 5 in Murphy et al. (1982) states that also the following additional condition is sufficient: p is convex. Example 1 below shows that even the strict convexity of p is not a sufficient additional condition.

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[5] See Laye and Laye (2008), and some references therein, about the importance of capacity constraints.

[6] Indeed, in this case r is strictly concave and $\overline{y} > 0$. This implies $Dr(\overline{y}) < (r(\overline{y}) - r(0))/(\overline{y} - 0)$. So $Dp(\overline{y})\overline{y} + p(\overline{y}) < p(\overline{y})$, and hence $Dp(\overline{y}) < 0$.

[7] If wished, see Lemma 1 in Murphy et al. (1982).

Example 1. Put $n = 2$, $X_1 = X_2 := \mathbb{R}_+$, $c_1(x_1) := 1000x_1$, $c_2(x_2) := \frac{x_2}{4}$ and

$$p(y) := \begin{cases} \frac{81}{16y+16} - \frac{5}{16} & \text{if} \quad y \in [0, 2], \\ \frac{9}{4y} + \frac{1}{4} & \text{if} \quad y \in [2, 3], \\ \frac{4}{y+1} & \text{if} \quad y \in [3, +\infty[. \end{cases}$$

It is straightforward to verify that the equilibrium set is $\{(0, x_2) \mid 2 \leq x_2 \leq 3\}$, that p is strictly convex and that the example satifies *all* conditions of Theorem 1 *but* additional conditions I and II. $\diamond$

5. Improvement

Theorem 2 below provides a substantial improvement of Theorem 1. Its proof, contained in the Appendix, is along the same lines of that of Theorem 1, but more technical.

Theorem 2. *Suppose that the industry revenue function is concave and that cost functions are convex. Besides suppose that one of the following three conditions holds:*

I. The industry revenue function is strictly concave.
II. All cost functions are strictly convex.
III. The industry revenue function is decreasing on the interior of its domain and cost functions are strictly increasing.

Finally, suppose there exists at least one equilibrium. Then:

(1) There is a unique equilibrium aggregate, say Ψ.
(2) There is a unique equilibrium if the price function is differentiable at Ψ.
(3) There is a unique equilibrium if the price function is differentiable on the interior of its domain. $\diamond$

Admittedly, the statement of Theorem 2 does not allow for a non-concave industry revenue function which is concave on the set where it is positive (such as, e.g., in the pedagogical case of an industry revenue function associated to a price function defined by $p(y) = \max(a - by, 0)$, with $a > 0$ and $b > 0$). The following simple observation clarifies that many of these industry revenue functions are in fact allowed for by Theorem 2.

Observation Consider an oligopoly with strictly increasing cost functions and with a price function $p^+ : Y \to \mathbb{R}$ such that $p^+ = \max(p, 0)$ for some function $p : Y \to \mathbb{R}$. The set of equilibria is unchanged if p^+ is replaced by p. $\diamond$

6. Conclusions

In the first part of this article, with the proof of Theorem 1, we have provided a simple proof of a known result (and we have pointed out an error in the literature related to this theorem). In the second part, with Theorem 2, we have proved a more general result employing more technical arguments. The novelties of Theorem 2 concern both cost functions and industry revenue functions.

Novelties on cost functions. Theorem 2 allows for non-differentiable cost functions.[8] For instance, part (I) of Theorem 2 allows for cost functions which are piecewise linear and convex, part (III) allows for convex piecewise linear strictly increasing cost functions and part (II) allows for cost functions which are the sum of a piecewise linear convex function and a strictly convex function. Besides, capacity constrained firms are allowed for and, more generally, cost functions that are defined on (possibly half-open) intervals are allowed for (e.g., the cost function c_i on $[0, 1[$ defined by $x/(1-x)$).

Novelties on industry revenue functions. Theorem 2 allows for industry revenue functions which are discontinuous at 0. For instance, an industry revenue r defined by $r(y) = 1 + by^c + dy$ at all y>0 is compatible with part (I) of Theorem 2 if $b > 0$ and $0 < c < 1$, with part (III) of Theorem 2 if $b, d \leq 0$ and $c \geq 1$, and with part (II) of Theorem 2 if $b \geq 0$ and $0 \leq c \leq 1$.

Note that in all numerical examples of industry revenue functions illustrated above, profit functions are not continuous. This fact alone does not in the least imply that equilibria do not exist. Indeed, several equilibrium existence results in case of discontinuous industry revenue functions can be developed; however, we shall not examine such results in the present article on the semi-uniqueness of equilibria.

Appendix. Proof of Theorem 2

Preliminaries about case I, II and III. Henceforth, we shall denote $Y \setminus \{0\}$ by Y^+. The concavity of r implies that p is semi-differentiable at each interior point of its domain with $D^+p \leq D^-p \leq 0$ and when $Y = [0, m]$ also that the left derivative of p at m exists as an element of $\mathbb{R} \cup \{-\infty\}$ with $D^-p(m) \leq 0$. If r is strictly concave, the last two inequalities are even strict.[9] This in turn implies that p is strictly decreasing on Y^+ in case I and that p is decreasing on Y^+ in case II. Note that, in case III, r must be decreasing on Y^+ because r is concave by assumption and decreasing on $\mathrm{Int}(Y)$.

Further preliminaries about case III. In case III, if $r \leq 0$ on Y, then $E = \{\mathbf{0}\}$ by the strict increasingness of c_i, and hence parts 1 and 2 of the theorem hold. Henceforth suppose that in case III the function r is positive somewhere. In case III, as r is decreasing on Y^+, we have that $V := \{y \in Y^+ \mid r(y) > 0\}$ must be a proper interval, $\lim_{x \downarrow 0} r(x) \geq r(y)$ $(y \in Y^+)$ and $\lim_{x \downarrow 0} r(x) > 0$. Besides in case III we have:

$$\mathbf{e} \in E \ \Rightarrow \ \sum_{j \in N \setminus \{i\}} e_j \neq 0 \ (i \in N); \tag{5}$$

$$\mathbf{e} \in E \ \Rightarrow \ e_N \in V; \tag{6}$$

$$p \text{ is strictly decreasing on } V \text{ and } D^-p < 0 \text{ on } V. \tag{7}$$

[8]Note that, like in Murphy et al. (1982) but unlike a good part of the literature, we have not assumed that cost functions are increasing and non-negative. Therefore (net) cost functions can include production costs and subsidies to production. E.g., the function $c_i(x) := x^2 - x$ is firm i's net cost function when x^2 is the cost of producing x units of the good and x is the subsidy received for the production of x units of the good.

[9]This can be proven as in footnote 6 but considering now left and right derivatives.

Proof of (5). Suppose instead that $\sum_{j\in N\setminus\{i\}} e_j = 0$ for some i. Then $e_j = 0$ $(j \neq i)$. As c_i is convex and increasing, c_i is continuous at 0 and we have

$$\lim_{x_i \downarrow 0} f_i(x_i; \mathbf{e}_{\hat{\imath}}) = \lim_{x_i \downarrow 0} (r - c_i)(x_i) = \lim_{x_i \downarrow 0} r(x_i) - c_i(0).$$

If $e_i > 0$ then $\lim_{x_i \downarrow 0} r(x_i) - c_i(0) \geq r(e_i) - c_i(0) > r(e_i) - c_i(e_i) = f_i(\mathbf{e})$ and if $e_i = 0$ then $\lim_{x_i \downarrow 0} r(x_i) - c_i(0) > -c_i(0) = f_i(\mathbf{e})$: a contradiction with $\mathbf{e} \in E$.

Proof of (6). By (5), $e_N \neq 0$. As c_i is strictly increasing and $\mathbf{e} \in E$ it follows that $p(e_N) > 0$. So $r(e_N) = p(e_N)e_N > 0$ and hence $e_N \in V$.

Proof of (7). As $p(y) = r(y)\frac{1}{y}$ $(y \in Y^+)$, r is decreasing on Y^+ and positive on V, it follows that p is strictly decreasing on V. Besides, $0 \geq D^- r(y) = D^- p(y) y + p(y)$ $(y \in V)$. It follows that $D^- p(y) < 0$ $(y \in V)$.

We are now ready for the proof of the theorem.

Proof of part 1. By way of contradiction suppose that $\mathbf{a}$,$\mathbf{b} \in E$ and $a_N < b_N$. Let $J := \{l \in N \mid a_l < b_l\}$ and $s := |J|$. Note that $s \geq 1$, $b_N > 0$, $a_J \leq a_N$, $b_J \leq b_N$, $a_J < b_J$ and

(8) $$b_J - a_J \geq b_N - a_N.$$

Suppose for a moment that

(9) $$a_N \neq 0.$$

As $\mathbf{a}$, $\mathbf{b} \in E$ and $a_N > 0$, for all $i \in J$ we have $D_i^+ f_i(\mathbf{a}) \leq 0 \leq D_i^- f_i(\mathbf{b})$, i.e.,[10]

(10) $$D^+ p(a_N) a_i + p(a_N) - D^+ c_i(a_i) \leq 0 \leq D^- p(b_N) b_i + p(b_N) - D^- c_i(b_i).$$

Suppose for a moment that also the following fundamental observation holds:

(11) $$D^+ p(a_N) a_J + s p(a_N) \geq D^- p(b_N) b_J + s p(b_N) \ (> \text{in cases I and III}).$$

As each c_i is convex (and strictly convex in case II), (11) implies that

$$D^+ p(a_N) a_J + s p(a_N) - \sum_{i\in J} D^+ c_i(a_i) > D^- p(b_N) b_J + s p(b_N) - \sum_{i \in J} D^- c_i(b_i)$$

and hence that $D^+ p(a_N) a_i + p(a_N) - D^+ c_i(a_i) > D^- p(b_N) b_i + p(b_N) - D^- c_i(b_i)$ for some $i \in J$, in contradiction with (10).

Proof of (9). In case III, (9) holds by (5). Suppose now that either condition I or II holds. Fix $i \in N$ with $b_i > 0$ and let $\mathbf{b}_{\hat{\imath}}$ denote the production profile of the other firms. For a given production profile $\mathbf{z}$ of the other firms let $f_i^{(\mathbf{z})} : X_i \to \mathbb{R}$ denote the profit function of firm i as a function of its own production level. Every $f_i^{(\mathbf{z})}$ is strictly concave. As $\mathbf{b}$ is an equilibrium, this strict concavity implies that b_i is the unique maximiser of $f_i^{(\mathbf{b}_{\hat{\imath}})}$; so $f_i^{(\mathbf{b}_{\hat{\imath}})}(b_i) > f_i^{(\mathbf{b}_{\hat{\imath}})}(0)$. As p is decreasing on Y^+, we obtain $f_i^{(\mathbf{0})}(b_i) \geq f_i^{(\mathbf{b}_{\hat{\imath}})}(b_i) > f_i^{(\mathbf{b}_{\hat{\imath}})}(0) = f_i^{(\mathbf{0})}(0)$. This implies that $\mathbf{0}$ is not an equilibrium, and hence that $a_N \neq 0$.

Proof of (11). Let us finish the proof of part 1 showing that (11) holds. Clearly, (11) holds if $D^- p(b_N) = -\infty$. Henceforth suppose that $D^- p(b_N) \neq -\infty$. As r is concave (and strictly concave in case I), the function $\tilde{r} : Y \cap (Y - \{a_N - a_J\}) \to \mathbb{R}$ defined

[10]The right derivative $D^+ c_i(0)$ may be $-\infty$; if $X_i = [0, m_i]$, the left derivative $D^- c_i(m_i)$ may be $+\infty$.

by $\tilde{r}(x) := p(x + a_N - a_J)x$ is concave (and strictly concave in case I). So $D^+\tilde{r}(a_J) \geq D^-\tilde{r}(a_J + b_N - a_N)$ ($>$ in case I), i.e.,

$$D^+p(a_N)a_J + p(a_N) \geq D^-p(b_N)(a_J + b_N - a_N) + p(b_N) \text{ (> in case I).} \tag{12}$$

As $a_J + b_N - a_N \leq b_J$ and $D^-p(b_N) \leq 0$, it follows that

$$D^+p(a_N)a_J + p(a_N) \geq D^-p(b_N)b_J + p(b_N) \text{ (> in case I).} \tag{13}$$

The monotonicity properties of p imply that

$$(s-1)p(a_N) \geq (s-1)p(b_N), \tag{14}$$

which, together with (13), implies the validity of (11) in cases I and II. Consider now case III. In order to show that (11) holds, we prove by contradiction that either the inequality in (14) is strict or that in (12) is strict: consequently also the inequality in (13) must be strict. Suppose that neither the inequality in (14) is strict nor that in (12) is so. This implies that $s = 1$ (by (6) and (7)) and $a_N = a_J$ (as $a_N > a_J$ implies that $\tilde{r}$ is strictly concave). Let $m \in N$ be such that $J = \{m\}$. By (8), $b_J = b_N$. This in turn implies that $\sum_{l \in N \setminus \{m\}} a_l = 0$. But this is in contradiction with (5).

Proof of part 2. We prove by contradiction that g is injective on E; then part 2 follows from part 1. Suppose $\mathbf{a}, \mathbf{b} \in E$ with $\mathbf{a} \neq \mathbf{b}$ and $a_N = b_N = \Psi$. Then $\Psi \in \text{Int}(Y)$. Fix i such that $a_i < b_i$. As $\mathbf{a}, \mathbf{b} \in E$ and p is differentiable at Ψ,

$$Dp(\Psi)b_i + p(y) - D^-c_i(b_i) \geq 0 \geq Dp(\Psi)a_i + p(y) - D^-c_i(a_i).$$

Thus the inequality $Dp(\Psi)(b_i - a_i) \geq D^-c_i(b_i) - D^+c_i(a_i)$ holds. If either condition I or condition III holds, the left hand side of the last inequality is an element of $]-\infty, 0[$ and the right hand side is an element of $[0, +\infty]$; if condition II holds, the left hand side of the last inequality is an element of $]-\infty, 0]$ and the right hand side is an element of $]0, +\infty]$. Both cases are impossible.

Proof of part 3. A consequence of part 2, as $\Psi \in \text{Int}(Y)$ if $\#E \geq 2$. □

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