ASYMPTOTIC DECAY UNDER NONLINEAR AND NONCOERCIVE DISSIPATIVE EFFECTS FOR ELECTRICAL CONDUCTION IN BIOLOGICAL TISSUES

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ABSTRACT. We study the asymptotic convergence to a periodic steady state of the solution of a nonlinear system of equations modelling electric conduction in biological tissues. Such model keeps into account both the resistive behaviour of the intracellular and extracellular domain and the capacitive/resistive behaviour of the lipidic cellular membrane. Because of the large number of cells involved in the model an homogenized version of the problem is also available. Also for the homogenized problem asymptotic convergence to a periodic steady is proved. The rate of convergence is analyzed, moreover the systems of equations satisfied by the limits are exhibited.

KEYWORDS: Asymptotic decay, stability, nonlinear homogenization, two-scale techniques, electrical impedance tomography.
AMS-MSC: 35B40, 35B27, 45K05, 92C55

1. INTRODUCTION

s:introduction

Composite materials have widespread applications in science and technology and, for this reason, have been extensively studied especially using homogenization techniques. In this framework the authors have deeply investigated a problem arising in electric conduction in biological tissues (with the purpose of obtaining some useful results for applications in electrical tomography), see [?], [?], [?], [?], [?], [?], [?], [?], [?].

From a physical point of view the problem consists in the study of the electric currents crossing a living tissue when an electrical potential is applied at the boundary (see [?], [?], [?], [?], [?]). Here the living tissue is regarded as a composite periodic, domain made of extracellular and intracellular materials (both assumed to be conductive, possibly with different conductivities) separated by a lipidic membrane which experiments prove to exhibit both conductive (due to ionic channels in the membrane) and capacitive behavior. In this regard the large number of cells contained in the biological sample allows us and even imposes to use an homogenization technique. Such technique yields the system of partial differential equations satisfied by the macroscopic electric potential u, i.e. the limit of the electric potential u_ε in the tissue as ε (the characteristic length of the cell) tends to zero.

Naturally, if we want the capacitive and the conductive behavior of the membranes to be maintained when $\varepsilon \to 0$ we must properly rescale the capacity and the conductivity of such membranes with respect to ε. In [?] and [?] the authors have shown that, essentially, only three scalings are physically sensible. One of these scalings seems to be the more suitable to describe the behavior of the membranes at radiofrequency range (which is the standard frequency used in electric tomography) and for this reason it has been widely studied by the authors in [?], [?], [?], [?], [?], [?]. In this peculiar model the magnetic field is neglected (as suggested by experimental evidence) and the potential u_ε is assumed to satisfy an elliptic equation both in the intracellular and in the extracellular domain while, on the membranes it satisfies the equation

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u_\varepsilon] + f\left(\frac{[u_\varepsilon]}{\varepsilon}\right) = \sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon$$

where $[u_\varepsilon]$ denotes the jump of the potential across the membranes and $\sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon$ is the current crossing the membranes. Note how the scaling parameter ε appears in our model. The previous condition on the membranes was rigorously obtained by the authors by means of a concentrated capacity technique in [?].

From a mathematical point of view a big difference does exist between the case of linear f and the nonlinear case. Homogenization limits have been rigorously found in both cases. In the linear case the result has been obtained in [?], [?] and [?], via asymptotic expansion in ε and it has been proved that the limit potential u satisfies an elliptic equation with memory for which an existence and uniqueness theorem has been proved in [?]. In the nonlinear case the approach is much more complicated and relies on the two-scale convergence technique. The final result is that, in this case, a memory effect is still present but it is not possible to find a single partial, differential equation satisfied by u (see [?]). Indeed, in this situation, u is coupled with another function u_1 and together they satisfy a system of partial differential equations where u_1 keeps into account the microscopic properties of the material and depends both on the macroscopic variable x and the microscopic variable y (also for this case existence and uniqueness results have been proved in [?]).

Going back to the technical applications of bioimpedence tomography it must be noticed that usually a time harmonic boundary data is applied and it is assumed that the resulting potential inside the biological material is time harmonic too. Under this assumption the behavior of the biological tissue is modeled by means of complex elliptic equations (one for every harmonic frequency). The correctness of this model has been proved by the authors in the linear case in [?], [?] and [?], investigating the time limit, as $t \to +\infty$, of the solution u of the homogenized problem. The authors were able to prove that the equations presently used in electric tomography can be rigorously obtained by means of an asymptotic limit with respect to t (when time periodic boundary data are assigned). They proved that u tends exponentially to a limit u^p and such a limit satisfies the partial differential equation currently used in applications. Also, as a new input, they were able to find the relation linking the complex admittivity of the limiting partial differential equation to the frequency of the assigned boundary data.

It is remarkable that an elliptic equation with memory has no, in principle, asymptotic stability even if the memory kernel decreases to zero exponentially when $t \to +\infty$ (see [?]). For this reason, in [?], [?] and [?], the result is obtained proving an asymptotic exponential convergence in t for the problem of level ε (i.e. before homogenization) and observing that such a convergence is stable with respect to ε, so that it passes to the limit and holds also for u. The proof relies on an appropriate eigenvalue theorem for the problem of level ε satisfied in a periodic unitary cell. See, for instance, [?] and [?] for an alternative approach relying on some extra-assumptions on the structures of the kernel. Nevertheless the problem at level ε is not in general asymptotically stable in t. In fact, if f is identically equal to zero, u_ε does not tend to zero exponentially in t even if a homogeneous boundary condition is assigned. To get such a result the initial jumps of the potential across the cellular membranes must have zero mean value on each membrane. However this fact will have major consequences when the nonlinear case will be treated. Indeed, in the linear case we can always assume that the initial jumps have zero mean value by subtracting from the initial potential a piecewise constant function, while in the nonlinear one this is not possible and we must proceed in a different way. However such a pathology does not appear in the homogenized problem.

Motivated by the previous considerations, in this paper we investigate the behavior as $t \to +\infty$ of the nonlinear problem introduced in [?]. We will prove that, if periodic boundary data are assigned and f is coercive in the sense of (**??**), then the solution of the ε-problem converges as $t \to +\infty$ to a periodic function solving a suitable system of equations. In this case then such a convergence is exponential. Moreover, if we have homogeneous boundary data, then at least the homogenized solution tends exponentially to zero for a much more general class of functions f; for example f may be not increasing, provided it is Lipschitz-continuous and

$$f(s_1) - f(s_2) > -L_-(s_1 - s_2), \qquad \forall s_1, s_2 \in \boldsymbol{R}$$

with L_- sufficiently small (see Remark **??**).

Analogously, the asymptotic convergence of the solution (u, u_1) of the homogenized problem to a periodic solution (u^p, u^{1p}) solving a suitable system of equations is proved when periodic boundary data is assigned. Such convergence is exponential if the boundary data is identically equal to zero. If this is not the case, as before, f must be assumed to satisfy (**??**).

It is important to note that in [?], [?] and [?] our approach was based on eigenvalue estimates which made it possible to keep into account (as far as the asymptotic rate of convergence is concerned) both the dissipative properties of the intra/extra cellular phases and the dissipative properties of the membranes. Here, for technical reasons, we are able to use this method only in the case of homogeneous boundary data. Instead, in the general case, we proceed by exploiting the coercivity of f, hence the electrical properties of the intra/extra cellular phases do not appear in the rate of convergence.

If f is not coercive it must be assumed to be monotone increasing and we can proceed in to different ways: with the first one the asymptotic convergence of (u, u^1) as $t \to +\infty$ is proved via a Liapunov-style technique and the rate of convergence is not quantified, with the second one &SPERIAMO CHE FUNZIONI the convergence of (u, u^1) is, again, proved to be exponential.

The paper is organized as follows: in Section 2 we present the geometrical setting and the nonlinear differential model governing our problem at the microscale ε. In Section 3 we prove the exponential decay in time of the solution of the microscopic problem, both in the case of homogeneous boundary data and in the case of time-periodic boundary data. In this last case, we need the additional coercivity assumption on the nonlinear function f, governing the dissipative properties of the cell membrane. Finally, in Section 4, we recall the definition and some useful results concerning the two-scale convergence on bulks and on surfaces and we prove the exponential decay in time of the solution of the macroscopic (or homogenized) problem, providing also the differential system satisfied by the limit function.

2. Preliminaries

s:prel

Let Ω be an open bounded subset of $\boldsymbol{R}^N$. In the sequel γ or $\widetilde{\gamma}$ will denote constants which may vary from line to line and which depend on the characteristic parameters of the problem, but which are independent of the quantities tending to zero, such as ε, δ and so on, unless explicitly specified.

ss:geometry

2.1. **The geometrical setting.** The typical geometry we have in mind is depicted in Figure 1. In order to be more specific, assume $N \geq 3$ and let us introduce a

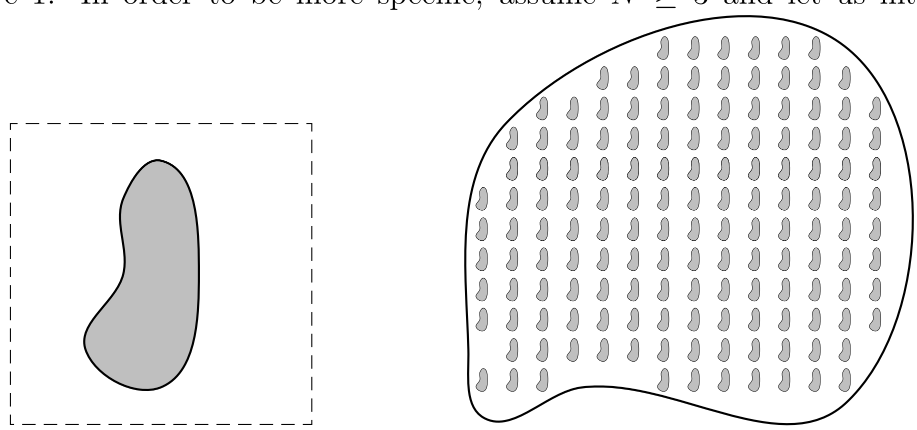

FIGURE 1. On the left: an example of admissible periodic unit cell $Y = E_1 \cup E_2 \cup \Gamma$ in $\boldsymbol{R}^2$. Here E_1 is the shaded region and Γ is its boundary. The remaining part of Y (the white region) is E_2. On the right: the corresponding domain $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon \cup \Gamma^\varepsilon$. Here Ω_1^ε is the shaded region and Γ^ε is its boundary. The remaining part of Ω (the white region) is Ω_2^ε. fig:omega

periodic open subset E of $\boldsymbol{R}^N$, so that $E + z = E$ for all $z \in \boldsymbol{Z}^N$. For all $\varepsilon > 0$ define $\Omega_1^\varepsilon = \Omega \cap \varepsilon E$, $\Omega_2^\varepsilon = \Omega \setminus \overline{\varepsilon E}$. We assume that Ω, E have regular boundary, say of class $\mathcal{C}^\infty$ for the sake of simplicity, that Ω_2^ε is a connected subset of Ω and $dist(\Gamma^\varepsilon, \partial\Omega) \geq \gamma\varepsilon$, where $\Gamma^\varepsilon = \partial\Omega_1^\varepsilon$. We also employ the notation $Y = (0,1)^N$, and

$E_1 = E \cap Y$, $E_2 = Y \setminus \overline{E}$, $\Gamma = \partial E \cap \overline{Y}$. As a simplifying assumption, we stipulate that $\Gamma \cap \partial Y = \emptyset$. We denote by ν the normal unit vector to Γ pointing into E_2, so that $\nu_\varepsilon(x) = \nu(\varepsilon^{-1}x)$.
For later use, we denote also

$$\sigma(y) = \begin{cases} \sigma_1 & \text{if } y \in E_1, \\ \sigma_2 & \text{if } y \in E_2, \end{cases} \qquad \text{and} \qquad \sigma_0 = |E_1|\sigma_1 + |E_2|\sigma_2 \,,$$

where σ_1, σ_2 are positive constants, and we also set $\sigma^\varepsilon(x) = \sigma(\varepsilon^{-1}x)$. Moreover, let us set

$$\mathfrak{C}^1_\#(Y) := \{u : Y \to \boldsymbol{R} \mid u_{|E_1} \in \mathcal{C}^1(\overline{E_1}) \,,\ u_{|E_2} \in \mathcal{C}^1(\overline{E_2}) \,,\ \text{and } u \text{ is } Y-\text{periodic}\} \,,$$

$$\mathcal{X}^1_\#(Y) := \{u \in L^2(Y) \mid u_{|E_1} \in H^1(E_1) \,,\ u_{|E_2} \in H^1(E_2) \,,\ \text{and } u \text{ is } Y-\text{periodic}\} \,,$$

and

$$\mathcal{X}^1(\Omega_\varepsilon) := \{u \in L^2(\Omega) \mid u_{|\Omega_1^\varepsilon} \in H^1(\Omega_1^\varepsilon),\ u_{|\Omega_2^\varepsilon} \in H^1(\Omega_2^\varepsilon)\} \,.$$

We note that, if $u \in \mathcal{X}^1_\#(Y)$ then the traces of $u_{|E_i}$ on Γ, for $i = 1, 2$, belong to $H^{1/2}(\Gamma)$, as well as $u \in \mathcal{X}^1(\Omega_\varepsilon)$ implies that the traces of $u_{|\Omega_i^\varepsilon}$ on Γ^ε, for $i = 1, 2$, belong to $H^{1/2}(\Gamma^\varepsilon)$.

ss:statment

2.2. **Statement of the problem.** We write down the model problem:

$$-\operatorname{div}(\sigma_1 \nabla u_\varepsilon) = 0 \,, \qquad \text{in } \Omega_1^\varepsilon \times (0,T); \tag{2.1}$$
eq:PDEin

$$-\operatorname{div}(\sigma_2 \nabla u_\varepsilon) = 0 \,, \qquad \text{in } \Omega_2^\varepsilon \times (0,T); \tag{2.2}$$
eq:PDEout

$$[\sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon] = 0 \,, \qquad \text{on } \Gamma^\varepsilon \times (0,T); \tag{2.3}$$
eq:FluxCont

$$\frac{\alpha}{\varepsilon} \frac{\partial}{\partial t}[u_\varepsilon] + f\left(\frac{[u_\varepsilon]}{\varepsilon}\right) = \sigma^\varepsilon \nabla u_\varepsilon \cdot \nu_\varepsilon \,, \qquad \text{on } \Gamma^\varepsilon \times (0,T); \tag{2.4}$$
eq:Circuit

$$[u_\varepsilon](x,0) = S_\varepsilon(x) \,, \qquad \text{on } \Gamma^\varepsilon; \tag{2.5}$$
eq:InitData

$$u_\varepsilon(x) = \Psi(x,t) \,, \qquad \text{on } \partial\Omega \times (0,T), \tag{2.6}$$
eq:BoundDat

where σ_1, σ_2 are defined in the previous subsection and $\alpha > 0$ is a constant; moreover, we note that, by the definition already given in the previous section, ν_ε is the normal unit vector to Γ^ε pointing into Ω_2^ε. Since u_ε is not in general continuous across Γ^ε we set

$$u_\varepsilon^{(1)} := \text{ trace of } u_{\varepsilon|\Omega_1^\varepsilon} \text{ on } \Gamma^\varepsilon \times (0,T); \quad u_\varepsilon^{(2)} := \text{ trace of } u_{\varepsilon|\Omega_2^\varepsilon} \text{ on } \Gamma^\varepsilon \times (0,T).$$

Indeed we refer conventionally to Ω_1^ε as to the *interior domain*, and to Ω_2^ε as to the *outer domain*. We also denote

$$[u_\varepsilon] := u_\varepsilon^{(2)} - u_\varepsilon^{(1)} \,.$$

Similar conventions are employed for other quantities, for example in (2.3). In this framework we will assume that

$$S_\varepsilon \in H^{1/2}(\Gamma^\varepsilon) \,, \qquad \int_{\Gamma^\varepsilon} S_\varepsilon^2(x) \, \mathrm{d}\sigma \le \gamma\varepsilon \,. \tag{2.7}$$
eq:assumpt2

Moreover, $f : \boldsymbol{R} \to \boldsymbol{R}$ satisfies

$$f \text{ is a Lipschitz-continuous function with Lipschitz constant } L\,, \tag{2.8}$$
eq:assumpt1

$$f \text{ is a strictly monotone function and } f'(s) \geq 0\,,\ \forall s \in \boldsymbol{R}\,; \tag{2.9}$$
eq:a37

$$f(0) = 0\,, \tag{2.10}$$
eq:assumpt1

$$f'(s) \geq \delta_0\,, \qquad \text{for a suitable } \delta_0 > 0 \text{ and } \forall s \text{ sufficiently large.} \tag{2.11}$$
eq:a38

Previous assumptions imply also

$$f(s)s \geq \lambda_1 s^2 - \lambda_2 |s|\,, \qquad \text{for some constants } \lambda_1 > 0 \text{ and } \lambda_2 \geq 0. \tag{2.12}$$
eq:a36

Finally, $\Psi : \Omega \times \boldsymbol{R} \to \boldsymbol{R}$ is a function satisfying the following assumption

$$\begin{aligned} &i) \quad \Psi \in L^2_{loc}\big(\boldsymbol{R}; H^2(\Omega)\big)\,;\\ &ii) \quad \Psi_t \in L^2_{loc}\big(\boldsymbol{R}; H^1(\Omega)\big)\,;\\ &iii) \quad \Psi(x,\cdot) \text{ is 1-periodic} \qquad \text{for a.e. } x \in \Omega. \end{aligned} \tag{2.13}$$
eq:h1

These assumptions will guarantee the time asymptotic decay of the solution u_ε, when ε is fixed. The set of equations (2.1)–(2.6) models electrical conduction in a biological tissue. It is important to notice that the first term in the left hand side of (2.4) models the behavior of the lipidic cell membrane which acts mainly as a capacitor, while the second term in the left hand side keeps into account the resistive behavior of the membrane which is caused by channels allowing charged molecules to go through. Here the resistive behavior is assumed to be nonlinear and it is relevant that the small parameter ε, which is of the order of magnitude of the cell width, appears inside the argument of f. Existence of such solution has been proved in [**?**]. Moreover, by [**?**, Lemma 3.6 and Remark 3.8] it follows that $u_\varepsilon \in \mathcal{C}^0\big((0,T]; \mathcal{X}^1(\Omega_\varepsilon)\big)$ and $[u_\varepsilon] \in \mathcal{C}^0\big((0,T]; L^2(\Gamma^\varepsilon)\big)$.

s:asymptotic1

3. Asymptotic convergence to a periodic solution of the ϵ-problem

The purpose of this section is to prove the asymptotic convergence of the solution of problem (2.1)–(2.6) to a periodic function $u^\#_\varepsilon$ when $t \to +\infty$. The function $u^\#_\varepsilon$ is, in turn, a solution of the system

$$-\operatorname{div}(\sigma^\varepsilon \nabla u^\#_\varepsilon) = 0\,, \qquad \text{in } (\Omega^\varepsilon_1 \cup \Omega^\varepsilon_2) \times \boldsymbol{R}; \tag{3.1}$$
eq:per_PDEb

$$[\sigma^\varepsilon \nabla u^\#_\varepsilon \cdot \nu_\varepsilon] = 0\,, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{3.2}$$
eq:per_Flux

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u^\#_\varepsilon] + f\left(\frac{[u^\#_\varepsilon]}{\varepsilon}\right) = (\sigma^\varepsilon \nabla u^\#_\varepsilon \cdot \nu_\varepsilon)\,, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{3.3}$$
eq:per_Circ

$$u^\#_\varepsilon(x,t) = \Psi(x,t)\,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}; \tag{3.4}$$
eq:per_Boun

$$u^\#_\varepsilon(x,\cdot) \text{ is 1-periodic,} \qquad \text{in } \Omega. \tag{3.5}$$
eq:per_peri

Indeed, this problem is derived from (2.1)–(2.6) replacing equation (2.5) with (3.5). As a first step we will prove the following result.

p:prop6

Proposition 3.1. *Under the assumptions* (2.9)–(2.12)*, problem* (3.1)–(3.4) *admits a* 1*-periodic solution* $u^\#_\varepsilon \in \mathcal{C}^0([0,1]; \mathcal{X}^1(\Omega_\varepsilon))$.

Proof. For $\delta > 0$, let us denote by $f_\delta(s) := f(s) + \delta s$, for every $s \in \boldsymbol{R}$, and consider the problem

$$-\operatorname{div}(\sigma^\varepsilon \nabla u^\#_{\varepsilon,\delta}) = 0\,, \qquad \text{in } (\Omega_1^\varepsilon \cup \Omega_2^\varepsilon) \times \boldsymbol{R}; \tag{3.6}$$
eq:per_PDEb

$$[\sigma^\varepsilon \nabla u^\#_{\varepsilon,\delta} \cdot \nu_\varepsilon] = 0\,, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{3.7}$$
eq:per_Flux

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[u^\#_{\varepsilon,\delta}] + f_\delta\left(\frac{[u^\#_{\varepsilon,\delta}]}{\varepsilon}\right) = \sigma^\varepsilon \nabla u^\#_{\varepsilon,\delta} \cdot \nu_\varepsilon\,, \qquad \text{on } \Gamma^\varepsilon \times \boldsymbol{R}; \tag{3.8}$$
eq:per_Circ

$$u^\#_{\varepsilon,\delta}(x,t) = \Psi(x,t)\,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}; \tag{3.9}$$
eq:per_Boun

$$u^\#_{\varepsilon,\delta}(x,\cdot) \text{ is 1-periodic,} \qquad \text{in } \Omega. \tag{3.10}$$
eq:per_peri

For any positive ε and δ, the preceding problem admits an unique periodic solution because of the results already proved in [**?**].
On the other hand, denoting with $u_{\varepsilon,\delta}$ such a solution, multiplying equation (3.6) by $u^\#_{\varepsilon,\delta} - \Psi$, integrating by parts on $\Omega \times [0,1]$, using the periodicity and taking into account equations (3.7)–(3.9), we get

$$\int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla u^\#_{\varepsilon,\delta}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_{\Gamma^\varepsilon} f_\delta\left(\frac{[u^\#_{\varepsilon,\delta}]}{\varepsilon}\right)[u^\#_{\varepsilon,\delta}]\,\mathrm{d}\sigma\,\mathrm{d}t \le \int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}t\,. \tag{3.11}$$
eq:a39

Finally, using (2.12), we obtain

$$\int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla u^\#_{\varepsilon,\delta}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_{\Gamma^\varepsilon} \frac{\lambda_1}{2\varepsilon}[u^\#_{\varepsilon,\delta}]^2\,\mathrm{d}\sigma\,\mathrm{d}t \le \int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2\lambda_1}\lambda_2^2|\Gamma^\varepsilon|\,. \tag{3.12}$$
eq:a40

Multiplying now equation (3.6) by $u^\#_{\varepsilon,\delta,t} - \Psi_t$, integrating by parts on $\Omega \times [0,1]$, using the periodicity and taking into account equations (3.7)–(3.9), we get

$$\begin{aligned}
&\frac{\alpha}{\varepsilon}\int_0^1\int_{\Gamma^\varepsilon} [u^\#_{\varepsilon,\delta,t}]^2\,\mathrm{d}\sigma\,\mathrm{d}t + \int_0^1\int_{\Gamma^\varepsilon} f_\delta\left(\frac{[u^\#_{\varepsilon,\delta}]}{\varepsilon}\right)[u^\#_{\varepsilon,\delta,t}]\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\quad \le \int_0^1\int_\Omega \sigma^\varepsilon \nabla u^\#_{\varepsilon,\delta}\nabla\Psi_t\,\mathrm{d}x\,\mathrm{d}t \le \int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla u^\#_{\varepsilon,\delta}|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla\Psi_t|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad \le \int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}t + \frac{\varepsilon}{2\lambda_1}\lambda_2^2|\Gamma^\varepsilon| + \int_0^1\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla\Psi_t|^2\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned} \tag{3.13}$$
eq:a41

where we used (3.12). Hence

$$
\begin{aligned}
&\frac{\alpha}{2\varepsilon}\int_0^1\int_{\Gamma^\varepsilon}[u^{\#}_{\varepsilon,\delta,t}]^2\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\le \int_0^1\int_{\Gamma^\varepsilon}\frac{(L+\delta)^2}{2\alpha\varepsilon}[u^{\#}_{\varepsilon,\delta}]^2\,\mathrm{d}\sigma\,\mathrm{d}t+\int_0^1\int_{\Omega}\frac{\sigma^\varepsilon}{2}|\nabla\Psi_t|^2\,\mathrm{d}x\,\mathrm{d}t+\int_0^1\int_{\Omega}\frac{\sigma^\varepsilon}{2}|\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{\varepsilon}{2\lambda_1}\lambda_2^2|\Gamma^\varepsilon| \\
&\le \left(\int_0^1\int_{\Omega}\frac{\sigma^\varepsilon}{2}|\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{\varepsilon}{2\lambda_1}\lambda_2^2|\Gamma^\varepsilon|\right)\left(\frac{(L+\delta)^2\lambda_1}{\alpha}+1\right)+\int_0^1\int_{\Omega}\frac{\sigma^\varepsilon}{2}|\nabla\Psi_t|^2\,\mathrm{d}x\,\mathrm{d}t\,,
\end{aligned}
\tag{3.14}
$$

eq:a42

where we used again (3.12). Inequalities (3.12) and respectively (3.14), for $\varepsilon>0$ fixed, yield the weak convergence of $u^{\#}_{\varepsilon,\delta}$ and $\nabla u^{\#}_{\varepsilon,\delta}$ in $L^2(\Omega^\varepsilon_i\times(0,1))$, $i=1,2$, and respectively the strong convergence of $[u^{\#}_{\varepsilon,\delta}]$ in $L^2(\Gamma^\varepsilon\times(0,1))$, for $\delta\to 0$. Since all the functions $u^{\#}_{\varepsilon,\delta}$ are 1-periodic, denoting as usual with $u^{\#}_\varepsilon$ the limit of $u^{\#}_{\varepsilon,\delta}$ we have that the same periodicity holds true for $u^{\#}_\varepsilon$. Moreover we can pass to the limit, as $\delta\to 0$, in problem (3.6)–(3.9), thus obtaining that $u^{\#}_\varepsilon$ is a 1-periodic solution of problem (3.1)–(3.4), under the assumptions (2.9)–(2.11).
Since the estimates above, and the ones for $\nabla u_{\varepsilon,\delta,t}$ are uniform in δ, we have that $u^{\#}_\varepsilon$ belongs to the class claimed in the statement. □

r:rem1 *Remark* 3.2. The uniformity of the above estimates is a result similar to the one obtained in Lemma 3.6 of [?]. □

Given $\varepsilon>0$, it remains to prove the asymptotic convergence of a solution u_ε of (2.1)–(2.6) to $u^{\#}_\varepsilon$, for $t\to+\infty$, also in this case, as stated in the following theorem.

t:t5 **Theorem 3.3.** *Let $\varepsilon>0$ be fixed and let u_ε be the solution of problem* (2.1)–(2.6). *Then, for $t\to+\infty$, $u_\varepsilon\to u^{\#}_\varepsilon$ in the following sense:*

$$\lim_{t\to+\infty}\|u_\varepsilon(\cdot,t)-u^{\#}_\varepsilon(\cdot,t)\|_{L^2(\Omega)}=0\,;\tag{3.15}$$ eq:decayper

$$\lim_{t\to+\infty}\|\nabla u_\varepsilon(\cdot,t)-\nabla u^{\#}_\varepsilon(\cdot,t)\|_{L^2(\Omega)}=0\,;\tag{3.16}$$ eq:decayper

$$\lim_{t\to+\infty}\frac{1}{\varepsilon}\|[u_\varepsilon](\cdot,t)-[u^{\#}_\varepsilon](\cdot,t)\|_{L^2(\Gamma^\varepsilon)}=0\,.\tag{3.17}$$ eq:decayper

Proof. Setting $r_\varepsilon:=u^{\#}_\varepsilon-u_\varepsilon$, we obtain that r_ε satisfies

$$-\operatorname{div}(\sigma^\varepsilon\nabla r_\varepsilon)=0\,,\qquad \text{in } (\Omega^\varepsilon_1\cup\Omega^\varepsilon_2)\times(0,+\infty);\tag{3.18}$$ eq:PDEboth1

$$[\sigma^\varepsilon\nabla r_\varepsilon\cdot\nu_\varepsilon]=0\,,\qquad \text{on } \Gamma^\varepsilon\times(0,+\infty);\tag{3.19}$$ eq:FluxCont

$$\frac{\alpha}{\varepsilon}\frac{\partial}{\partial t}[r_\varepsilon]+g_\varepsilon(x,t)\frac{[r_\varepsilon]}{\varepsilon}=\sigma^\varepsilon\nabla r_\varepsilon\cdot\nu_\varepsilon\,,\qquad \text{on } \Gamma^\varepsilon\times(0,+\infty);\tag{3.20}$$ eq:Circuit1

$$[r_\varepsilon](x,0)=[u^{\#}_\varepsilon(x,0)]-S_\varepsilon(x)=:\widehat{S}_\varepsilon(x)\,,\qquad \text{on } \Gamma^\varepsilon;\tag{3.21}$$ eq:InitData

$$r_\varepsilon(x)=0\,,\qquad \text{on } \partial\Omega\times(0,+\infty);\tag{3.22}$$ eq:BoundDat

where $g_\varepsilon(x,t) := \dfrac{f\Big(\frac{[u_\varepsilon^\#]}{\varepsilon}(x,t)\Big) - f\big(\frac{[u_\varepsilon]}{\varepsilon}(x,t)\big)}{\frac{[u_\varepsilon^\#]}{\varepsilon}(x,t) - \frac{[u_\varepsilon]}{\varepsilon}(x,t)} \geq 0$, and $\widehat{S}_\varepsilon(x)$ still satisfies assumption (2.7) because of the energy inequality satisfied by $u_\varepsilon^\#$ and by (3.12). Multiplying equation (3.18) by r_ε and integrating by parts we have

$$\int_\Omega \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x + \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_{\varepsilon,t}][r_\varepsilon] \,\mathrm{d}\sigma + \int_{\Gamma^\varepsilon} \frac{g_\varepsilon(x,t)}{\varepsilon} [r_\varepsilon]^2 \,\mathrm{d}\sigma = 0\,. \tag{3.23}$$

eq:a43

Equation (3.23) implies that the function $t \mapsto \frac{\alpha}{\epsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma$ is a positive, decreasing function of t; hence, it tends to a limit value $\overline{r}_\varepsilon \geq 0$ as $t \to +\infty$. We claim that the value $\overline{r}_\varepsilon$ must be zero. Otherwise, there exists $\overline{t} > 0$, such that, for $t \geq \overline{t}$, $\frac{\alpha}{\epsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \geq \frac{\overline{r}_\varepsilon}{2}$. On the other hand, given $t > \overline{t}$ and setting $\Gamma^\varepsilon_{\overline{r}_\varepsilon}(t) := \{x \in \Gamma^\varepsilon : [r_\varepsilon(x,t)]^2 \leq \frac{\overline{r}_\varepsilon \varepsilon}{4\alpha|\Gamma^\varepsilon|}\}$, it is evident that

$$\frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \geq \frac{\overline{r}_\varepsilon}{4}\,, \qquad \forall t \geq \overline{t}\,. \tag{3.24}$$

eq:a44

Indeed, for every $t \geq \overline{t}$, by definition,

$$\begin{aligned}
\frac{\overline{r}_\varepsilon}{2} &\leq \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \leq \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma + \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \\
&\leq \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma + \frac{\alpha}{\varepsilon} \frac{\overline{r}_\varepsilon}{4\alpha|\Gamma^\varepsilon|} |\Gamma^\varepsilon_{\overline{r}_\varepsilon}| \leq \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma + \frac{\overline{r}_\varepsilon}{4}\,,
\end{aligned}$$

which implies (3.24). Moreover, for $t \geq \overline{t}$, we have that, on $\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)$, $g_\varepsilon(x,t) \geq \chi > 0$, where χ is a suitable positive constant depending only on $(\overline{r}_\varepsilon, \varepsilon, \alpha, |\Gamma^\varepsilon|)$ (this last result follows from assumption (2.9)–(2.12)). Hence, using (3.23), it follows

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\alpha}{2\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \right) &\leq - \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} \frac{g_\varepsilon(x,t)}{\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \\
&\leq -\chi \int_{\Gamma^\varepsilon \setminus \Gamma^\varepsilon_{\overline{r}_\varepsilon}(t)} \frac{1}{\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma \leq -\frac{\overline{r}_\varepsilon}{4\alpha} \chi < 0\,.
\end{aligned} \tag{3.25}$$

eq:a45

Inequality (3.25) clearly contradicts the asymptotic convergence in t of the function $t \mapsto \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma$, hence

$$\lim_{t \to +\infty} \frac{\alpha}{\varepsilon} \int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma = 0\,. \tag{3.26}$$

eq:a46

In particular, this gives (3.17). Integrating (3.23) in $[t,\infty)$ and taking into account (3.26), we get

$$\int_t^{+\infty}\int_\Omega \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \frac{\alpha}{2\varepsilon}\int_{\Gamma^\varepsilon} [r_\varepsilon(x,t)]^2 \,\mathrm{d}\sigma\,, \tag{3.27}$$

eq:a47

which implies

$$\lim_{t\to+\infty}\int_t^{+\infty}\int_\Omega \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = 0\,. \tag{3.28}$$

eq:a48

Condition (3.28) guarantees that for every positive η there exists a $\widehat{t}(\eta) > 0$, such that

$$\int_{\widehat{t}}^{+\infty}\int_\Omega \sigma^\varepsilon |\nabla r_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t \leq \eta\,,$$

which, in turn implies that, for every natural number n, there exists a $t_n \in (\widehat{t}+n, \widehat{t}+(n+1))$, such that

$$\int_\Omega \sigma^\varepsilon |\nabla r_\varepsilon(x,t_n)|^2 \,\mathrm{d}x \leq \eta\,. \tag{3.29}$$

eq:a49

Now, we multiply (3.18) by $r_{\varepsilon,t}$ and integrate in Ω, so that

$$\int_\Omega \sigma^\varepsilon \nabla r_\varepsilon \nabla r_{\varepsilon,t}(x,t)\,\mathrm{d}x + \frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[r_{\varepsilon,t}(x,t)]^2\,\mathrm{d}\sigma + \int_{\Gamma^\varepsilon}\frac{g_\varepsilon(x,t)}{\varepsilon}[r_\varepsilon(x,t)]\,[r_{\varepsilon,t}(x,t)]\,\mathrm{d}\sigma = 0\,, \tag{3.30}$$

eq:a50

which implies

$$\int_\Omega \sigma^\varepsilon \nabla r_\varepsilon \nabla r_{\varepsilon,t}\,\mathrm{d}x \leq \int_{\Gamma^\varepsilon}\frac{g_\varepsilon^2(x,t)}{2\alpha\varepsilon}[r_\varepsilon]^2\,\mathrm{d}\sigma\,. \tag{3.31}$$

eq:a51

Moreover, integrating (3.31) in $[t_n,t^*]$ with $t^* \in [t_n, t_n+2]$ and using (3.29), we have

$$\sup_{t\in[t_n,t_n+2]}\left(\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla r_\varepsilon(x,t)|^2\,\mathrm{d}x\right) \leq \frac{\eta}{2} + \frac{L^2}{2\alpha^2}\sup_{t\in[t_n,+\infty)}\left(\frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[r_\varepsilon(x,t)]^2\,\mathrm{d}\sigma\right)\,,\quad \forall n\in\boldsymbol{N}\,.$$

Since $t_{n+1}-t_n < 2$, the intervals of the form $[t_n, t_n+2]$, when n varies in $\boldsymbol{N}$, are overlapping; hence, we obtain

$$\sup_{t\in[\widehat{t}+1,+\infty)}\left(\int_\Omega \frac{\sigma^\varepsilon}{2}|\nabla r_\varepsilon(x,t)|^2\,\mathrm{d}x\right) \leq \frac{\eta}{2} + \frac{2L^2}{\alpha^2}\sup_{t\in[\widehat{t},+\infty)}\left(\frac{\alpha}{\varepsilon}\int_{\Gamma^\varepsilon}[r_\varepsilon]^2\,\mathrm{d}\sigma\right)\,. \tag{3.32}$$

eq:a52

Because of (3.26) the integral in the right-hand side of (3.32) can be made smaller than $\frac{\eta}{2}\left(\frac{L^2}{\alpha^2}\right)^{-1}$, provided $\widehat{t}$ is chosen sufficiently large in dependence of η. This means that

$$\sup_{t\in[\hat{t}+1,+\infty)}\left(\int_{\Omega}\frac{\sigma^{\varepsilon}}{2}|\nabla r_{\varepsilon}(x,t)|^{2}\,\mathrm{d}x\right)\leq\eta\,, \tag{3.33}$$ eq:a53

hence

$$\lim_{t\to+\infty}\int_{\Omega}\sigma^{\varepsilon}|\nabla r_{\varepsilon}(x,t)|^{2}\,\mathrm{d}x=0\,. \tag{3.34}$$ eq:a54

In particular, this gives (3.16). Finally, Poincare's inequality together with (3.26) and (3.34) yield

$$\lim_{t\to+\infty}\int_{\Omega}|r_{\varepsilon}(x,t)|^{2}\,\mathrm{d}x=0\,, \tag{3.35}$$ eq:a55

which gives (3.15). □

r:rem10bis *Remark* 3.4. Observe that this asymptotic convergence result implies uniqueness of the periodic solution (in the class of functions specified above). □

asymptotic_hom 4. Exponential decay of the solution of the homogenized problem

The aim of this section is to prove a result similar to the one proved in the previous section; i.e., the convergence of the solution to a periodic steady state, for the homogenized problem. We will employ a slightly different technique which is maybe simpler than the one displayed above. To this purpose we will make use of some fundamental properties of two-scale convergence, which we recall in the following.

ss:twoscale 4.1. **Two-scale convergence.** In this subsection we recall some definitions and properties concerning two-scale convergence in the time-dependent case (for a survey in this topic see, for instance, [**?**, Section 4]).
We firstly recall the following definition ([**?**, Definition 2.1]).

d:2scale_test2 **Definition 4.1.** A function $\varphi\in L^{2}\big((0,T);L^{2}(\Omega\times Y)\big)$, which is Y-periodic in y and which satisfies

$$\lim_{\varepsilon\to0}\int_{0}^{T}\int_{\Omega}\varphi^{2}\left(x,\frac{x}{\varepsilon},t\right)\,\mathrm{d}x\,\mathrm{d}t=\int_{0}^{T}\int_{\Omega\times Y}\varphi^{2}(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,, \tag{4.1}$$ eq:test3

is called *admissible* test function for the two-scale convergence on $L^{2}\big((0,T);L^{2}(\Omega)\big)$.

r:test1bis *Remark* 4.2. We recall that any function $\varphi\in\mathcal{C}^{0}\big(\overline{\Omega}\times[0,T];\mathcal{C}^{0}_{\#}(Y)\big)$ is an admissible test function as well as any function $\varphi\in L^{2}_{\#}\big(Y;\mathcal{C}^{0}(\overline{\Omega}\times[0,T])\big)$ (see [**?**, Remark 1.5]). □

d:2scale_new **Definition 4.3.** Given a sequence $\{h_{\varepsilon}\}\in L^{2}\big((0,T);L^{2}(\Omega)\big)$ and a function $h_{0}\in L^{2}\big((0,T);L^{2}(\Omega\times Y)\big)$, we say that h_{ε} two-scale converges to h_{0} in $L^{2}\big((0,T);L^{2}(\Omega\times Y)\big)$ for $\varepsilon\to0$ (and we write $h_{\varepsilon}\overset{2-sc}{\to}h_{0}$) if

$$\lim_{\varepsilon\to0}\int_{0}^{T}\int_{\Omega}h_{\varepsilon}(x,t)\varphi\left(x,\frac{x}{\varepsilon},t\right)\,\mathrm{d}x\,\mathrm{d}t=\int_{0}^{T}\int_{\Omega\times Y}h_{0}(x,y,t)\varphi(x,y,t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$

for any admissible test function.

th:2scale_new **Theorem 4.4.** (See [?], [?])

- *From any bounded sequence in* $L^2\big((0,T);L^2(\Omega)\big)$, *it is possible to extract a two-scale converging subsequence.*
- *If* $h_\varepsilon \stackrel{2-sc}{\to} h_0$ *then, setting* $h(x,t)=\int_Y h_0(x,y,t)\,\mathrm{d}y$, *it follows that* $h_\varepsilon \rightharpoonup h$ *weakly in* $L^2\big((0,T);L^2(\Omega)\big)$.
- *If* $h_\varepsilon \stackrel{2-sc}{\to} h_0$ *then, setting* $h(x,t)=\int_Y h_0(x,y,t)\,\mathrm{d}y$, *it follows that*
$$\liminf_{\varepsilon\to 0}\|h_\varepsilon\|_{L^2\left((0,T);L^2(\Omega)\right)} \geq \|h_0\|_{L^2\left((0,T);L^2(\Omega\times Y)\right)} \geq \|h\|_{L^2\left((0,T);L^2(\Omega)\right)}\,.$$

p:dafare1 **Proposition 4.5.** *Let* $\{h_\varepsilon\}\subseteq L^2\big(\Omega\times(0,T)\big)$ *be a sequence of functions converging to a function* $h\in L^2\big(\Omega\times(0,T)\big)$ *strongly in* $L^2_{loc}\big((0,T);L^2(\Omega)\big)$. *Assume also that there exists a constant* $\gamma>0$ *such that* $\|h_\varepsilon\|_{L^2(\Omega\times(0,T))}\leq\gamma$. *Then* $h_\varepsilon \stackrel{2-sc}{\to} h$.

Recalling [?], we extend the notion of two-scale convergence to sequences of functions defined on periodic surfaces and depending on the time t.

2scale_all_new **Definition 4.6.** Given a sequence $\{\widehat{h}_\varepsilon\}\in L^2\big((0,T);L^2(\Gamma^\varepsilon)\big)$ and a function $\widehat{h}_0\in L^2\big(\Omega\times(0,T);L^2(\Gamma)\big)$, we say that $\widehat{h}_\varepsilon$ two-scale converges to $\widehat{h}_0$ in $L^2\big(\Omega\times(0,T);L^2(\Gamma)\big)$ for $\varepsilon\to 0$ (and we write $\widehat{h}_\varepsilon \stackrel{2-sc}{\to} \widehat{h}_0$) if
$$\lim_{\varepsilon\to 0}\varepsilon\int_0^T\int_{\Gamma^\varepsilon}\widehat{h}_\varepsilon(x,t)\widehat{\varphi}\left(x,\frac{x}{\varepsilon},t\right)\mathrm{d}\sigma\,\mathrm{d}t=\int_0^T\int_\Omega\int_\Gamma\widehat{h}_0(x,y,t)\widehat{\varphi}(x,y,t)\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t$$
for any test function $\widehat{\varphi}\in\mathcal{C}^0\big(\overline{\Omega}\times[0,T];\mathcal{C}^0_\#(Y)\big)$.

r:rem3 *Remark* 4.7. In some part of the paper we will choose as test function $\widehat{\varphi}\nu$, with $\widehat{\varphi}\in\mathcal{C}^0\big(\overline{\Omega}\times[0,T];\mathcal{C}^0_\#(Y)\big)$. In this regard, ν will denote a continuous extension of the normal vector to the whole Y. □

2scale_allnew **Theorem 4.8.** (See [?], [?])

- *From any sequence* $\{\widehat{h}_\varepsilon\}$ *in* $L^2\big((0,T);L^2(\Gamma^\varepsilon)\big)$ *bounded in the following sense*
$$\varepsilon\int_0^T\int_{\Gamma^\varepsilon}|\widehat{h}_\varepsilon(x,t)|^2\,\mathrm{d}\sigma\,\mathrm{d}t\leq\gamma\,,$$
where γ *is a positive constant, it is possible to extract a two-scale converging subsequence.*
- *If* $\{\widehat{h}_\varepsilon\}$ *is a sequence in* $L^2\big((0,T);L^2(\Gamma^\varepsilon)\big)$ *which two-scale converges to* $\widehat{h}_0\in L^2\big(\Omega\times(0,T);L^2(\Gamma)\big)$, *then the measure* $\varepsilon\widehat{h}_\varepsilon\,\mathrm{d}\sigma$ *converges in the sense of distribution in* $\Omega\times(0,T)$, *to the function* $\widehat{h}(x,t)=\int_\Gamma\widehat{h}_0(x,y,t)\,\mathrm{d}\sigma(y)$, *with* $\widehat{h}\in L^2\big(\Omega\times(0,T)\big)$.

- *If* $\{\widehat{h}_\varepsilon\}$ *is a sequence in* $L^2\big((0,T);L^2(\Gamma^\varepsilon)\big)$ *which two-scale converges to* $\widehat{h}_0 \in L^2\big(\Omega\times(0,T);L^2(\Gamma)\big)$ *then, setting* $\widehat{h}(x,t)=\int_\Gamma \widehat{h}_0(x,y,t)\,\mathrm{d}\sigma(y)$*, it follows that*

$$\liminf_{\varepsilon\to 0}\sqrt{\varepsilon}\|\widehat{h}_\varepsilon\|_{L^2\left((0,T);L^2(\Gamma^\varepsilon)\right)} \geq \|\widehat{h}_0\|_{L^2\left(\Omega\times(0,T);L^2(\Gamma)\right)} \geq \|\widehat{h}\|_{L^2\left((0,T);L^2(\Omega)\right)}\,.$$

2scale_humnew **Theorem 4.9.** *Assume that* $\{u_\varepsilon\}\subseteq L^2\big((0,T);\mathcal{X}^1(\Omega_\varepsilon)\big)$*, be a sequence of functions such that* $u_\varepsilon=\Psi$ *on* $\partial\Omega$ *and*

$$\int_0^T\int_\Omega |\nabla u_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{\varepsilon}\int_0^T\int_{\Gamma^\varepsilon}[u_\varepsilon]^2(x,t)\,\mathrm{d}\sigma\,\mathrm{d}t\leq\gamma\,, \tag{4.2}$$ eq:hum1bis3

then there exist two functions $u\in L^2\big((0,T);H^1(\Omega)\big)$*,* $u=\Psi$ *on* $\partial\Omega$*, and* $u^1\in L^2\big(\Omega\times(0,T);\mathcal{X}^1_\#(Y)\big)$ *such that, up to a subsequence,* $u_\varepsilon \overset{2-sc}{\to} u$*,* $1_{\Omega\setminus\Gamma^\varepsilon}\nabla u_\varepsilon \overset{2-sc}{\to} \nabla u+\nabla_y u^1$ *in* $L^2\big((0,T);L^2(\Omega\times Y)\big)$ *and* $\varepsilon^{-1}[u_\varepsilon]\nu_\varepsilon \overset{2-sc}{\to}[u^1]\nu$ *in* $L^2\big(\Omega\times(0,T);L^2(\Gamma)\big)$ *for* $\varepsilon\to 0$*.*

r:rem4 *Remark* 4.10. Since the normal ν_ε can be included in the test function for the two-scale convergence in $L^2\big(\Omega\times(0,T);L^2(\Gamma)\big)$ (see Remark 4.7), by Theorem 4.9 we obtain also that $\varepsilon^{-1}[u_\varepsilon]\overset{2-sc}{\to}[u^1]$ in $L^2\big(\Omega\times(0,T);L^2(\Gamma)\big)$.

□

ss:asympt1

4.2. **Asymptotic convergence to a periodic steady state.** Let $(u,u^1)\in L^2\big((0,T);H^1(\Omega)\big)\times L^2\big(\Omega\times(0,T);\mathcal{X}^1_\#(Y)\big)$ be the two-scale limit obtained in Theorem 4.9, when u_ε is the solution of problem (2.1)–(2.6) and the initial data S_ε satisfies the additional condition that $S_\varepsilon/\varepsilon$ two-scale converges in $L^2\big(\Omega;L^2(\Gamma)\big)$ to a function S_1 such that $S_1(x,\cdot)=S_{|\Gamma}(x,\cdot)$ for some $S\in\mathcal{C}\big(\overline{\Omega};\mathcal{C}^1_\#(Y)\big)$, and

$$\lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma^\varepsilon}\left(\frac{S_\varepsilon}{\varepsilon}\right)^2(x)\,\mathrm{d}\sigma=\int_\Omega\int_\Gamma S_1^2(x,y)\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,. \tag{4.3}$$ eq:init_asy

We recall that, by [?, Theorem 2.1], the pair (u,u^1) is the solution of problem

$$-\operatorname{div}\left(\sigma_0\nabla u+\int_Y\sigma\nabla_y u^1\,\mathrm{d}y\right)=0\,,\qquad\text{in }\Omega\times(0,T); \tag{4.4}$$ eq:PDE_limi

$$-\operatorname{div}_y(\sigma\nabla u+\sigma\nabla_y u^1)=0\,,\qquad\text{in }\Omega\times(E_1\cup E_2)\times(0,T); \tag{4.5}$$ eq:PDEper_l

$$[\sigma(\nabla u+\nabla_y u^1)\cdot\nu]=0\,,\qquad\text{on }\Omega\times\Gamma\times(0,T); \tag{4.6}$$ eq:FluxCont

$$\alpha\frac{\partial}{\partial t}[u^1]+f\left([u^1]\right)=\sigma(\nabla u+\nabla_y u^1)\cdot\nu\,,\qquad\text{on }\Omega\times\Gamma\times(0,T); \tag{4.7}$$ eq:Circuit_

$$[u^1](x,y,0)=S_1(x,y)\,,\qquad\text{on }\Omega\times\Gamma; \tag{4.8}$$ eq:InitData

$$u(x,t)=\Psi(x,t)\,,\qquad\text{on }\partial\Omega\times(0,T). \tag{4.9}$$ eq:BoundDat

qui c'è una struttura

$$\int_{\Omega\times Y}\sigma\nabla u\nabla u+\int_{\Omega\times Y}\sigma\nabla_y u_1\nabla u+$$

$$\int_{\Omega \times Y} \sigma \nabla u \nabla u_1 + \int \sigma \nabla_y u_1 \nabla_y u_1$$

Note that the variational formulation of problem (4.4)–(4.9) is the following:

$$\int_0^T \int_\Omega \int_Y \sigma \left(\nabla u + \nabla_y u^1\right) (\nabla \phi + \nabla_y \Phi) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t + \int_0^T \int_\Omega \int_\Gamma f([u^1])[\Phi] \, \mathrm{d}x \, \mathrm{d}\sigma \, \mathrm{d}t$$
$$- \alpha \int_0^T \int_\Omega \int_\Gamma [u^1] \frac{\partial}{\partial t}[\Phi] \, \mathrm{d}x \, \mathrm{d}\sigma \, \mathrm{d}t - \alpha \int_\Omega \int_\Gamma [\Phi] S_1 \, \mathrm{d}x \, \mathrm{d}\sigma = 0 \,. \quad (4.10)$$ eq:a76

Moreover we assume that u satisfies the boundary condition on $\partial\Omega \times [0,T]$ in the trace sense (i.e. $u(x,t) = \Psi(x,t)$ a.e.) and u^1 is periodic in Y and has zero mean value in Y for every $(x,t) \in \Omega \times (0,T)$.
Here ϕ is any regular function depending on (x,t), with compact support in Ω, and Φ is any Y-periodic function on $\Omega \times Y \times [0,T]$, which may jump across Γ, is zero when $t = T$ and is regular elsewhere.
For later use, let us define

$$|||\big(h(\cdot,t), h^1(\cdot,t)\big)||| := \|h\|_{\mathcal{C}^0([0,1];L^2(\Omega))} + \|\nabla h\|_{\mathcal{C}^0([0,1];L^2(\Omega))}$$
$$+ \|h^1\|_{\mathcal{C}^0([0,1];L^2(\Omega\times Y))} + \|\nabla_y h^1\|_{\mathcal{C}^0([0,1];L^2(\Omega\times Y))} + \|[h^1]\|_{\mathcal{C}^0([0,1];L^2(\Omega\times \Gamma))} \,, \quad (4.11)$$ eq:a86

where $(h, h^1) \in \mathcal{C}^0([0,T]; H^1(\Omega)) \times \mathcal{C}^0([0,T]; L^2(\Omega; \mathcal{X}^1_\#(Y))$, and

$$||||(\widetilde{h}, \widetilde{h}^1)|||| := \|\widetilde{h}\|_{H^1(\Omega)} + \|\widetilde{h}^1\|_{L^2(\Omega\times Y)} + \|\nabla_y \widetilde{h}^1\|_{L^2(\Omega\times Y)} + \|[\widetilde{h}^1]\|_{L^2(\Omega\times\Gamma)} \,, \quad (4.12)$$ eq:a85

where $(\widetilde{h}, \widetilde{h}^1) \in H^1(\Omega) \times L^2(\Omega; \mathcal{X}^1_\#(Y))$.
As in the previous section, we firstly prove that there exists a periodic solution of the homogenized problem

$$- \operatorname{div} \left(\sigma_0 \nabla u^\# + \int_Y \sigma \nabla_y u^{1,\#} \, \mathrm{d}y \right) = 0 \,, \qquad \text{in } \Omega \times \boldsymbol{R}; \quad (4.13)$$ eq:PDE_limi

$$- \operatorname{div}_y (\sigma \nabla u^\# + \sigma \nabla_y u^{1,\#}) = 0 \,, \qquad \text{in } \Omega \times (E_1 \cup E_2) \times \boldsymbol{R}; \quad (4.14)$$ eq:PDEper_l

$$[\sigma(\nabla u^\# + \nabla_y u^{1,\#}) \cdot \nu] = 0 \,, \qquad \text{on } \Omega \times \Gamma \times \boldsymbol{R}; \quad (4.15)$$ eq:FluxCont

$$\alpha \frac{\partial}{\partial t}[u^{1,\#}] + f\left([u^{1,\#}]\right) = \sigma(\nabla u^\# + \nabla_y u^{1,\#}) \cdot \nu \,, \text{on } \Omega \times \Gamma \times \boldsymbol{R}; \quad (4.16)$$ eq:Circuit_

$$[u^{1,\#}](x,y,\cdot) \text{ is 1-periodic,} \qquad \text{on } \Omega \times \Gamma; \quad (4.17)$$ eq:InitData

$$u^\#(x,t) = \Psi(x,t) \,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}. \quad (4.18)$$ eq:BoundDat

p:prop7 **Proposition 4.11.** *Under the assumptions* (2.9)–(2.11)*, problem* (4.13)–(4.18) *admits a* 1*-periodic in time solution.*

Proof. For $\delta > 0$, let us denote by $f_\delta(s) := f(s) + \delta s$, for every $s \in \boldsymbol{R}$, and consider the problem

$$-\operatorname{div}\left(\sigma_0 \nabla u_\delta^\# + \int_Y \sigma \nabla_y u_\delta^{1,\#}\,\mathrm{d}y\right) = 0\,, \qquad \text{in } \Omega \times \boldsymbol{R}\,; \tag{4.19}$$

eq:PDE_limi

$$-\operatorname{div}_y(\sigma \nabla u_\delta^\# + \sigma \nabla_y u_\delta^{1,\#}) = 0\,, \qquad \text{in } \Omega \times (E_1 \cup E_2) \times \boldsymbol{R}\,; \tag{4.20}$$

eq:PDEper_l

$$[\sigma(\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#})\cdot\nu] = 0, \qquad \text{on } \Omega \times \Gamma \times \boldsymbol{R}\,; \tag{4.21}$$

eq:FluxCont

$$\alpha \frac{\partial}{\partial t}[u_\delta^{1,\#}] + f_\delta\left([u_\delta^{1,\#}]\right) = \sigma(\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#}) \cdot \nu\,, \quad \text{on } \Omega \times \Gamma \times \boldsymbol{R}\,; \tag{4.22}$$

eq:Circuit_

$$[u_\delta^{1,\#}](x, y, \cdot) \text{ is 1-periodic,} \qquad \text{on } \Omega \times \Gamma\,; \tag{4.23}$$

eq:InitData

$$u_\delta^\#(x, t) = \Psi(x, t)\,, \qquad \text{on } \partial\Omega \times \boldsymbol{R}\,. \tag{4.24}$$

eq:BoundDat

Since f_δ has a strictly positive derivative on $\boldsymbol{R}$, by the results proved in [**?**, Subsection 4.2], a unique periodic solution $(u_\delta^\#, u_\delta^{1,\#})$ of problem (4.19)– (4.24) does exist. Recalling equation (4.40) in the proof of Theorem 4.14 of [**?**]; i.e.

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_\Omega\int_Y \sigma\left(\nabla u_\delta^\#(x,t) + \nabla_y u_\delta^{1,\#}(x,y,t)\right)(\nabla\phi(x,t) + \nabla_y\Phi(x,y,t))\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad + \int_{t_1}^{t_2}\int_\Omega\int_\Gamma f_\delta([u_\delta^{1,\#}(x,y,t)])[\Phi(x,y,t)]\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad\qquad - \alpha\int_{t_1}^{t_2}\int_\Omega\int_\Gamma [u_\delta^{1,\#}(x,y,t)]\frac{\partial}{\partial t}[\Phi(x,y,t)]\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t = 0\,;
\end{aligned} \tag{4.25}$$

eq:a35

we obtain that $(u_\delta^\#, u_\delta^{1,\#})$ satisfies an energy inequality, easily obtained replacing (ϕ, Φ) in (4.25) with $(u_\delta^\# - \Psi, u_\delta^{1,\#})$, which implies

$$\begin{aligned}
&\int_0^1\int_\Omega\int_Y \frac{\sigma}{2}|\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_0^1\int_\Omega\int_\Gamma f_\delta([u_\delta^{1,\#}])[u_\delta^{1,\#}]\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad = \int_0^1\int_\Omega\int_Y \frac{\sigma}{2}|\nabla\Psi|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\,.
\end{aligned} \tag{4.26}$$

eq:a56

Note that $(u_\delta^\#, u_\delta^{1,\#})$ because of their periodicity, can be used as test functions, even if they have not compact support in $[0,1]$; indeed, it is enough to apply a routine approximation procedure.
From (4.26), working as done in (3.11)–(3.12) of Section 3 and taking into account (2.12) and the fact that

$$\int_0^1 [u_{\delta,t}^{1,\#}][u_\delta^{1,\#}]\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}[u_\delta^{1,\#}]^2\,\mathrm{d}t = 0$$

because of the periodicity, we get

$$\int_0^1\int_\Omega\int_Y \sigma|\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_0^1\int_\Omega\int_\Gamma \lambda_1[u_\delta^{1,\#}]^2\,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma\,, \tag{4.27}$$

eq:a57

where γ is a constant depending on $\lambda_1, \lambda_2, |\Gamma|$ and the H^1-norm of Ψ.
Replacing (ϕ, Φ) in (4.25) with $(u_{\delta,t}^\# - \Psi_t, u_{\delta,t}^{1,\#})$, by (4.27), (2.12) and taking into account the fact that

$$\int_0^1 (\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#})(\nabla u_{\delta,t}^\# + \nabla_y u_{\delta,t}^{1,\#})\,\mathrm{d}t = \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}|\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#}|^2\,\mathrm{d}t = 0$$

because of the periodicity, we get

$$\int_0^1\int_\Omega\int_\Gamma f_\delta([u_\delta^{1,\#}])[u_{\delta,t}^{1,\#}]\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t + \alpha\int_0^1\int_\Omega\int_\Gamma [u_{\delta,t}^{1,\#}]^2\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma\,, \tag{4.28}$$

eq:a58

where, again γ depends on $\lambda_1, \lambda_2, |\Gamma|$ and the H^1-norms of Ψ and Ψ_t.
Inequality (4.28), together with (4.27), gives

$$\frac{\alpha}{2}\int_0^1\int_\Omega\int_\Gamma [u_{\delta,t}^{1,\#}]^2\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \le \gamma\,, \tag{4.29}$$

eq:a59

γ depends on $\lambda_1, \lambda_2, |\Gamma|, L$ and the H^1-norms of Ψ and Ψ_t. Moreover, from (4.26), we obtain

$$\int_0^1\int_\Omega |\nabla u_\delta^\#|^2\,\mathrm{d}x\,\mathrm{d}t \le \gamma\,, \tag{4.30}$$

eq:a60

$$\int_0^1\int_\Omega\int_Y |\nabla_y u_\delta^{1,\#}|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le \gamma\,. \tag{4.31}$$

eq:a61

Indeed,

$$
\begin{aligned}
&\int_0^1\int_\Omega\int_Y |\nabla_y u_\delta^{1,\#}(x,y,t)|^2 \,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t + \int_0^1\int_\Omega |\nabla u_\delta^\#|^2 \,\mathrm{d}x\,\mathrm{d}t \\
\leq&\gamma - 2\int_0^1\int_\Omega\int_Y \nabla_y u_\delta^{1,\#}(x,y,t)\nabla u_\delta^\#(x,t)\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}t \\
=&\gamma - 2\int_0^1\int_\Omega \nabla u_\delta^\# \left(\int_Y \nabla_y u_\delta^{1,\#}(x,y,t)\,\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t \\
\leq&\gamma + 2\int_0^1\int_\Omega |\nabla u_\delta^\#| \left(\int_\Gamma |[u_\delta^{1,\#}(x,y,t)]|\,\mathrm{d}\sigma\right)\mathrm{d}x\,\mathrm{d}t \\
\leq&\gamma + \frac{1}{2|\Gamma|}|\Gamma|\int_0^1\int_\Omega |\nabla u_\delta^\#|^2\,\mathrm{d}x\,\mathrm{d}t + 2|\Gamma|\int_0^1\int_\Omega\int_\Gamma [u_\delta^{1,\#}(x,y,t)]^2\,\mathrm{d}\sigma\,\mathrm{d}x\,\mathrm{d}t \\
\leq&\gamma + \frac{1}{2}\int_0^1\int_\Omega |\nabla u_\delta^\#(x,t)|^2\,\mathrm{d}x\,\mathrm{d}t + \gamma\,.
\end{aligned}
\tag{4.32}
$$

eq:a20

Finally, (4.27) and (4.29)–(4.31) allow us to pass to the limit with respect to δ in the weak formulation of problem (4.19)–(4.24), thus proving that there exists a periodic (in time) solution $(u^\#, u^{1,\#})$ of the homogenized problem

$$
\begin{aligned}
\int_0^T\int_{\Omega\times Y} \sigma(\nabla u^\# + \nabla_y u^{1,\#})\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \int_0^T\int_{\Omega\times Y} \sigma(\nabla u^\# + \nabla_y u^{1,\#})\cdot\nabla_y\Phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \\
\int_0^T\int_\Omega\int_\Gamma \mu[\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t - \alpha\int_0^T\int_\Omega\int_\Gamma [u^{1,\#}]\frac{\partial}{\partial t}[\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t = 0
\end{aligned}
\tag{4.33}
$$

eq:a88

for every test function $\varphi \in \mathcal{C}_c^1\big(\Omega\times(0,T)\big)$ and $\Phi\in\mathcal{C}^1\big(\overline{\Omega}\times[0,T];\mathfrak{C}_\#^1(Y)\big)$ 1-periodic in time (recall the definition of $\mathfrak{C}_\#^1(Y)$ given in Subsection 2.1). It remains to identify μ. To this purpose, we follow the Minty monotone operators method. Let us consider a sequence of 1-periodic in time test functions $\psi_k(x,y,t) = \phi_0^k(x,t) + \phi_1^k(x,y,t) + \lambda\phi_2(x,y,t)$, with $\phi_0^k\in\mathcal{C}^1(\Omega\times\boldsymbol{R})$, $\phi_1^k\in\mathcal{C}^1\big(\overline{\Omega}\times\boldsymbol{R};\mathfrak{C}_\#^1(Y)\big)$, $\phi_2\in\mathcal{C}_c^1\big(\Omega\times\boldsymbol{R};\mathfrak{C}_\#^1(Y)\big)$, with $\phi_0^k(\cdot,t),\phi_1^k(\cdot,y,t)$ vanishing on $\partial\Omega$ for every $t\in\boldsymbol{R}$ and every $y\in Y$, $\phi_0^k\to u^\#$ strongly in $L^2_{loc}\big(\boldsymbol{R};H^1(\Omega)\big)$, $\phi_1^k\to u^{1,\#}$ strongly in $L^2_{loc}\big(\boldsymbol{R};L^2(\Omega;\mathcal{X}_\#^1(Y))\big)$ and $[\phi_{1,t}^k]$

stable in $L^2([0,1];L^2(\Omega\times\Gamma)\big)$ because of inequality (4.28), i.e.

$$\int_0^1\int_\Omega \|\phi_1^k(x,\cdot,t)-u^{1,\#}(x,\cdot,t)\|^2_{H^1(E_i)}\,\mathrm{d}t\,\mathrm{d}x + \int_0^1\int_\Omega \|[\phi_1^k(x,\cdot,t)]-[u^{1,\#}(x,\cdot,t)]\|^2_{L^2(\Gamma)}\,\mathrm{d}t\,\mathrm{d}x\to 0\,,\qquad \text{for } k\to+\infty,\ i=1,2;$$

and

$$\int_0^1\int_\Omega \|[\phi_{1,t}^k(x,\cdot,t)]\|^2_{L^2(\Gamma)}\,\mathrm{d}t\,\mathrm{d}x\le\gamma\,,$$

with γ independent of k. Taking only into account the monotonicity assumption on f and the periodicity in time of ϕ_0^k and ϕ_1^k, we obtain

$$\begin{aligned}
&\int_0^1\int_{\Omega\times Y}\sigma(\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#} - \nabla\phi_0^k - \nabla_y\phi_1^k - \lambda\nabla_y\phi_2)\cdot(\nabla u_\delta^\# - \nabla\phi_0^k)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&+\int_0^1\int_{\Omega\times Y}\sigma(\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#} - \nabla\phi_0^k - \nabla_y\phi_1^k - \lambda\nabla_y\phi_2)\cdot(\nabla_y u_\delta^{1,\#} - \nabla_y\phi_1^k - \lambda\nabla_y\phi_2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\quad+\alpha\int_0^1\int_\Omega\int_{\Omega\times\Gamma}\frac{\partial}{\partial t}\Big([u_\delta^{1,\#}]-[\phi_1^k+\lambda\phi_2^k]\Big)\Big([u_\delta^{1,\#}]-[\phi_1^k+\lambda\phi_2^k]\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\quad+\int_0^1\int_{\Omega\times\Gamma}\Big(f_\delta([u_\delta^{1,\#}])-f_\delta\big([\phi_1^k+\lambda\phi_2^k]\big)\Big)\Big([u_\delta^{1,\#}]-[\phi_1^k+\lambda\phi_2^k]\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t\\
&\qquad=\int_0^1\int_{\Omega\times Y}\sigma|\nabla u_\delta^\# + \nabla_y u_\delta^{1,\#} - \nabla\phi_0^k - \nabla_y\phi_1^k - \lambda\nabla_y\phi_2|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t\\
&\quad+\alpha\int_0^1\int_\Omega\int_{\Omega\times\Gamma}\frac{\partial}{\partial t}\Big([u_\delta^{1,\#}]-[\phi_1^k+\lambda\phi_2^k]\Big)\Big([u_\delta^{1,\#}]-[\phi_1^k+\lambda\phi_2^k]\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t\\
&+\int_0^1\int_{\Omega\times\Gamma}\Big(f_\delta([u_\delta^{1,\#}])-f_\delta\big([\phi_1^k+\lambda\phi_2^k]\big)\Big)\Big([u_\delta^{1,\#}]-[\phi_1^k+\lambda\phi_2^k]\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t\ge 0\,.
\end{aligned}\tag{4.34}$$

eq:monotoni

Taking the function $(u_\delta^{\#} - \phi_0^k, u_\delta^{1,\#} - \phi_1^k - \lambda\phi_2)$ as a test function (φ, Φ) in the weak formulation of problem (4.19)–(4.24), inequality (4.34) can be rewritten as

$$
\begin{aligned}
&- \int_0^1 \int_{\Omega\times Y} \sigma(\nabla\phi_0^k + \nabla_y\phi_1^k + \lambda\nabla_y\phi_2)\cdot(\nabla u_\delta^{\#} - \nabla\phi_0^k)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_0^1 \int_{\Omega\times Y} \sigma(\nabla\phi_0^k + \nabla_y\phi_1^k + \lambda\nabla_y\phi_2)\cdot(\nabla_y u_\delta^{1,\#} - \nabla_y\phi_1^k - \lambda\nabla_y\phi_2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad - \alpha\int_0^1 \int_\Omega \int_{\Omega\times\Gamma} \frac{\partial}{\partial t}[\phi_1^k + \lambda\phi_2]\Big([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad\quad - \int_0^1 \int_{\Omega\times\Gamma} f_\delta\big([\phi_1^k + \lambda\phi_2]\big)\Big([u_\delta^{1,\#}] - [\phi_1^k + \lambda\phi_2]\Big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \ge 0\,. \quad (4.35)
\end{aligned}
$$

eq:monotoni

Hence, passing to the limit as $\delta \to 0$ and using (4.28), it follows

$$
\begin{aligned}
&- \int_0^1 \int_{\Omega\times Y} \sigma(\nabla\phi_0^k + \nabla_y\phi_1^k + \lambda\nabla_y\phi_2)\cdot(\nabla u^{\#} - \nabla\phi_0^k)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\quad - \int_0^1 \int_{\Omega\times Y} \sigma(\nabla\phi_0^k + \nabla_y\phi_1^k + \lambda\nabla_y\phi_2)\cdot(\nabla_y u^{1,\#} - \nabla_y\phi_1^k - \lambda\nabla_y\phi_2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \\
&\qquad - \alpha\int_0^1 \int_\Omega \int_{\Omega\times\Gamma} \frac{\partial}{\partial t}[\phi_1^k + \lambda\phi_2]\big([u^{1,\#}] - [\phi_1^k + \lambda\phi_2]\big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \\
&\qquad\quad - \int_0^1 \int_{\Omega\times\Gamma} f\big([\phi_1^k + \lambda\phi_2]\big)\big([u^{1,\#}] - [\phi_1^k + \lambda\phi_2]\big)\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t \ge 0\,. \quad (4.36)
\end{aligned}
$$

eq:monotoni

Now, letting $k \to +\infty$, we obtain

$$
\begin{aligned}
&\int_0^1 \int_\Omega \int_Y \sigma(\nabla u^{\#} + \nabla_y u^{1,\#} + \lambda\nabla_y\phi_2)\cdot\lambda\nabla_y\phi_2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \\
&\qquad \alpha\int_0^1 \int_\Omega \int_\Gamma \frac{\partial}{\partial t}[u^{1,\#} + \lambda\phi_2]\lambda[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t \\
&\qquad\quad + \int_0^1 \int_\Omega \int_\Gamma f\big([u^{1,\#} + \lambda\phi_2]\big)\lambda[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t \ge 0\,. \quad (4.37)
\end{aligned}
$$

eq:monotoni

Taking into account (4.33) with $\varphi \equiv 0$ and $\Phi = \phi_2$, it follows

$$\lambda^2 \int_0^1 \int_\Omega \int_Y \sigma \nabla_y \phi_2 \cdot \nabla_y \phi_2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \alpha\lambda^2 \int_0^1 \int_\Omega \int_\Gamma \frac{\partial}{\partial t}[\phi_2][\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t$$
$$- \lambda \int_0^1 \int_\Omega \int_\Gamma \mu[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t + \lambda \int_0^1 \int_\Omega \int_\Gamma f\left([u^{1,\#} + \lambda\phi_2]\right)[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t \geq 0\,. \tag{4.38}$$

eq:monotoni

Assuming firstly that $\lambda > 0$ and then $\lambda < 0$, dividing by λ the previous equation and then letting $\lambda \to 0$, we obtain

$$\int_0^1 \int_\Omega \int_\Gamma \mu[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t = \int_0^1 \int_\Omega \int_\Gamma f\left([u^{1,\#}]\right)[\phi_2]\,\mathrm{d}x\,\mathrm{d}\sigma(y)\,\mathrm{d}t\,,$$

which gives

$$\mu = f\left([u^{1,\#}]\right)\,, \tag{4.39}$$

eq:identifi

so that (4.33) becomes exactly the weak formulation of problem (4.13)–(4.18). Therefore the thesis is achieved. □

r:rem10

Remark 4.12. Note that (4.29) is uniform with respect to δ. Moreover, we can obtain also estimates for $\nabla u^{\#}_{\delta,t}$ and $\nabla_y u^{1,\#}_{\delta,t}$ uniform in δ. Indeed, differentiating formally with respect to t problem (4.19)–(4.24), multiplying equation (4.19) (differentiated with respect to t) by $\left((u^{\#}_{\delta,t} - \Psi_t)\widehat{\upsilon}^\tau, u^{1,\#}_{\delta,t}\widehat{\upsilon}^\tau\right)$, where $\tau \in (0, 1/4)$ and $\widehat{\upsilon}^\tau : [0, +\infty) \to \boldsymbol{R}$ is a function such that $0 \leq \widehat{\upsilon}^\tau \leq 1$, $\widehat{\upsilon}^\tau(t) = 1$, for $t \geq 2\tau$, $\widehat{\upsilon}^\tau(t) = 0$, for $0 \leq t \leq \tau$, and finally integrating by parts, we obtain

$$\int_{2\tau}^1 \int_\Omega \int_Y |\sigma \nabla u^{\#}_{\delta,t} + \sigma \nabla_y u^{1,\#}_{\delta,t}|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t + \alpha \sup_{t\in(2\tau,1)} \int_\Omega \int_\Gamma [u^{1,\#}_{\delta,t}]^2 \,\mathrm{d}x\,\mathrm{d}\sigma$$
$$\leq \int_0^1 \int_\Omega \int_Y |\sigma \nabla u^{\#}_{\delta,t} + \sigma \nabla_y u^{1,\#}_{\delta,t}|^2 \widehat{\upsilon}^\tau(t)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$+ \alpha \sup_{t\in(0,1)} \int_\Omega \int_\Gamma [u^{1,\#}_{\delta,t}]^2 \widehat{\upsilon}^\tau(t)\,\mathrm{d}x\,\mathrm{d}\sigma \leq \gamma(\tau) \tag{4.40}$$

eq:a81

where we used assumption (2.8), (2.13) and (4.29). Now, proceeding as in the proof of (4.30) and (4.31), we obtain

$$\int_{2\tau}^{1}\int_{\Omega} |\nabla u^{\#}_{\delta,t}|^2 \,\mathrm{d}x\,\mathrm{d}t \le \gamma\,, \tag{4.41}$$ eq:a60bis

$$\int_{2\tau}^{1}\int_{\Omega}\int_{Y} |\nabla_y u^{1,\#}_{\delta,t}|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \le \gamma\,. \tag{4.42}$$ eq:a61bis

Therefore, passing to the limit for $\delta \to 0^+$, in (4.29), (4.41) and (4.42), we obtain that the same estimates hold for $(u^{\#}, u^{1,\#})$.
This implies that $(u^{\#}, u^{1,\#})$ belongs to $\mathcal{C}^0([0,1]; H^1(\Omega)) \times \mathcal{C}^0([0,1]; L^2(\Omega; \mathcal{X}^1_{\#}(Y))$. □

It remains to prove that any solution (u, u^1) of the homogenized problem converges to $(u^{\#}, u^{1,\#})$ as $t \to \infty$. This will be stated in the next theorem.

t:t6 **Theorem 4.13.** *Let (u, u^1) be the solution of problem (4.5)–(4.9). Then, for $t \to +\infty$, $(u, u^1) \to (u^{\#}, u^{1,\#})$ in the following sense:*

$$\lim_{t\to+\infty} \|\big(u(\cdot,t), u^1(\cdot,\cdot,t)\big) - \big(u^{\#}(\cdot,t), u^{1,\#}(\cdot,\cdot,t)\big)\|_{L^2(\Omega\times Y)} = 0\,; \tag{4.43}$$ eq:decayper

$$\lim_{t\to+\infty} \|\big(\nabla u(\cdot,t), \nabla_y u^1(\cdot,\cdot,t)\big) - \big(\nabla u^{\#}_{\varepsilon}(\cdot,t), \nabla_y u^{1,\#}_{\varepsilon}(\cdot,\cdot,t)\big)\|_{L^2(\Omega\times Y)} = 0\,; \tag{4.44}$$ eq:decayper

$$\lim_{t\to+\infty} \|[u^1](\cdot,\cdot,t) - [u^{1,\#}](\cdot,\cdot,t)\|_{L^2(\Omega\times\Gamma)} = 0\,. \tag{4.45}$$ eq:decayper

Proof. As usual, let $(r, r^1) := (u^{\#}, u^{1,\#}) - (u, u^1)$, so that the pair (r, r^1) satisfies:

$$\int_0^t\int_{\Omega}\int_{Y} \sigma\left(\nabla r + \nabla_y r^1\right)\left(\nabla\phi + \nabla_y\Phi\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t$$
$$+\int_0^t\int_{\Omega}\int_{\Gamma} \frac{f([u^{1,\#}]) - f([u^1])}{[u^{1,\#}] - [u^1]}[r^1][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t + \alpha\int_0^t\int_{\Omega}\int_{\Gamma} [r^1_t][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma\,\mathrm{d}t = 0\,, \qquad \forall t > 0\,, \tag{4.46}$$ eq:a62

where $u = 0$ on $\partial\Omega \times [0,T]$ in the trace sense, u^1 is periodic in Y and has zero mean value in Y for every t. Here ϕ is a any regular function depending on (x,t), with compact support in Ω and Φ is a any function depending on (x,y,t) which jumps across Γ, is zero when $t = T$ and is regular elsewhere. Differentiating (4.46) with respect to t, we get

$$\int_{\Omega}\int_{Y} \sigma\left(\nabla r + \nabla_y r^1\right)\left(\nabla\phi + \nabla_y\Phi\right)\,\mathrm{d}x\,\mathrm{d}y + \int_{\Omega}\int_{\Gamma} \frac{f([u^{1,\#}]) - f([u^1])}{[u^{1,\#}] - [u^1]}[r^1][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma$$
$$+\,\alpha\int_{\Omega}\int_{\Gamma} [r^1_t][\Phi]\,\mathrm{d}x\,\mathrm{d}\sigma = 0\,. \tag{4.47}$$ eq:a63

Replacing (ϕ, Φ) with (r, r^1) in (4.47), we get

$$\int_\Omega \int_Y \sigma |\nabla r + \nabla_y r^1|^2 \,\mathrm{d}x\,\mathrm{d}y + \int_\Omega \int_\Gamma \frac{f([u^{1,\#}]) - f([u^1])}{[u^{1,\#}] - [u^1]} [r^1]^2 \,\mathrm{d}x\,\mathrm{d}\sigma + \alpha \int_\Omega \int_\Gamma [r^1_t][r^1] \,\mathrm{d}x\,\mathrm{d}\sigma = 0 \,. \qquad (4.48)$$

eq:a64

As in Section 3, equation (4.48) implies that the function $t \mapsto \alpha \int_\Omega \int_\Gamma [r^1(x,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x$ is a positive, decreasing function of t, hence it tends to a limit value $\overline{r}^1 \geq 0$ as $t \to +\infty$. The value $\overline{r}^1$ must be zero otherwise $\alpha \int_\Omega \int_\Gamma [r^1]^2 \,\mathrm{d}\sigma\,\mathrm{d}x \geq \frac{\overline{r}^1}{2}$ for $t \geq \overline{t}$, for a suitable $\overline{t} > 0$. On the other hand, fixed $t > 0$ and setting $\Gamma_{\overline{r}^1}(t) := \left\{(x,y) \in \Omega \times \Gamma : [r^1]^2(x,y,t) \leq \frac{\overline{r}^1}{4\alpha |\Gamma|\,|\Omega|}\right\}$, reasoning as in the proof of Theorem 3.3, it follows that

$$\alpha \int_\Omega \int_{\Gamma \setminus \Gamma_{\overline{r}^1}(t)} [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x \geq \frac{\overline{r}^1}{4} \,.$$

However, on $\Gamma \setminus \Gamma^{\overline{r}^1}$, $g(x,y,t) := \frac{f([u^{1,\#}]) - f([u^1])}{[u^{1,\#}] - [u^1]} \geq \chi > 0$, where χ is a suitable positive constant depending only on $\overline{r}^1, \alpha, |\Gamma|$ (this last result follows from the assumptions (2.9)–(2.11)). Hence, using (4.48), we get

$$\frac{d}{dt}\left(\frac{\alpha}{2} \int_\Omega \int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x\right) \leq -\int_\Omega \int_{\Gamma \setminus \Gamma_{\overline{r}^1}(t)} g(x,y,t) [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x$$

$$\leq -\chi \int_\Omega \int_{\Gamma \setminus \Gamma_{\overline{r}^1}(t)} [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x \leq -\frac{\overline{r}_1}{4\alpha} \chi < 0 \,. \qquad (4.49)$$

eq:a65

Inequality (4.49) clearly contradicts the asymptotic convergence for $t \to +\infty$ of $\alpha \int_\Omega \int_\Gamma [r^1]^2(x,y,t) \,\mathrm{d}\sigma\,\mathrm{d}x$, hence

$$\lim_{t \to +\infty} \alpha \int_\Omega \int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x = 0 \,, \qquad (4.50)$$

eq:a66

which is exactly (4.45). Integrating (4.48) in $[t, \infty)$ and taking into account (4.50), we get

$$\int_t^{+\infty} \int_\Omega \int_Y \sigma |\nabla r + \nabla_y r^1|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \leq \frac{\alpha}{2} \int_\Omega \int_\Gamma [r^1(x,y,t)]^2 \,\mathrm{d}\sigma\,\mathrm{d}x \,, \qquad (4.51)$$

eq:a67

which implies

$$\lim_{t \to +\infty} \int_t^{+\infty} \int_Y \int_\Omega \sigma |\nabla r + \nabla_y r^1|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t = 0 \,. \qquad (4.52)$$

eq:a68

This last condition guarantees that for every positive η there exists a $\widehat{t}(\eta) > 0$, such that

$$\int_{\widehat{t}}^{+\infty} \int_{\Omega} \int_{Y} \sigma |\nabla r + \nabla_y r^1|^2 \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}t \leq \eta \,,$$

which in turn implies that, for every $n \in \boldsymbol{N}$, there exists a $t_n \in (\widehat{t} + n, \widehat{t} + (n+1))$, such that

$$\int_{\Omega} \int_{Y} \sigma |\nabla r(x, t_n) + \nabla_y r^1(x, y, t_n)|^2 \,\mathrm{d}x \,\mathrm{d}y \leq \eta \,. \tag{4.53}$$ eq:a69

Hence, replacing (ϕ, Φ) with (r_t, r_t^1) in (4.47), we get

$$\int_{\Omega} \int_{Y} \sigma (\nabla r^0 + \nabla_y r^1)(\nabla r_t + \nabla_y r^1) \,\mathrm{d}x \,\mathrm{d}y + \int_{\Omega} \int_{\Gamma} g(x, y, t)[r^1][r_t^1] \,\mathrm{d}\sigma \,\mathrm{d}x$$
$$+ \alpha \int_{\Omega} \int_{\Gamma} [r_t^1(x, y, t)]^2 \,\mathrm{d}\sigma \,\mathrm{d}x = 0 \,, \tag{4.54}$$ eq:a70

and

$$\int_{\Omega} \int_{Y} \sigma (\nabla r + \nabla_y r^1)(\nabla r_t + \nabla_y r_t^1) \,\mathrm{d}x \,\mathrm{d}y \leq \int_{\Omega} \int_{\Gamma} \frac{g^2(x, y, t)}{2\alpha} [r^1(x, y, t)]^2 \,\mathrm{d}\sigma \,\mathrm{d}x \,. \tag{4.55}$$ eq:a71

Moreover, integrating (4.55) in $[t_n, t^*]$, with $t^* \in [t_n, t_n + 2]$, we have

$$\sup_{t \in [t_n, t_n+2]} \left(\int_{\Omega} \int_{Y} \frac{\sigma}{2} |\nabla r(x, t) + \nabla_y r^1(x, y, t)|^2 \,\mathrm{d}x \,\mathrm{d}y \right)$$
$$\leq \frac{\eta}{2} + \frac{2L^2}{2\alpha^2} \sup_{t \in [t_n, +\infty)} \left(\alpha \int_{\Omega} \int_{\Gamma} [r^1(x, y, t)]^2 \,\mathrm{d}\sigma \,\mathrm{d}x \right) \,, \qquad \forall n \in \boldsymbol{N} \,; \tag{4.56}$$ eq:a72

i.e.,

$$\sup_{t \in [\widehat{t}+1, +\infty)} \left(\int_{\Omega} \int_{Y} \frac{\sigma}{2} |\nabla r(x, t) + \nabla_y r^1(x, y, t)|^2 \,\mathrm{d}x \,\mathrm{d}y \right)$$
$$\leq \frac{\eta}{2} + \frac{L^2}{\alpha^2} \sup_{t \in [\widehat{t}, +\infty)} \left(\alpha \int_{\Omega} \int_{\Gamma} [r^1(x, y, t)]^2 \,\mathrm{d}\sigma \right) . \tag{4.57}$$ eq:a73

Because of (4.50) the integral in the right-hand side of (4.57) can be made smaller than $\frac{\eta}{2} \left(\frac{L^2}{\alpha^2} \right)^{-1}$, provided $\widehat{t}$ is chosen sufficiently large in dependence of η. This means that

$$\sup_{t \in [\widehat{t}+1, +\infty)} \left(\int_{\Omega} \int_{Y} \frac{\sigma}{2} |\nabla r(x, t) + \nabla_y r^1(x, y, t)|^2 \,\mathrm{d}x \,\mathrm{d}y \right) \leq \eta \,. \tag{4.58}$$ eq:a74

Inequality (4.58) implies

$$\lim_{t\to+\infty}\int\limits_{\Omega}\int\limits_{Y}\sigma|\nabla r(x,t)+\nabla_y r^1(x,y,t)|^2\,\mathrm{d}x\,\mathrm{d}y=0\,. \tag{4.59}$$

eq:a75

Now, working as done in (4.32), we get

$$\lim_{t\to+\infty}\int\limits_{\Omega}|\nabla r(x,t)|^2\,\mathrm{d}x\,\mathrm{d}y=0\,;\qquad\text{and}\qquad\lim_{t\to+\infty}\int\limits_{\Omega}\int\limits_{Y}|\nabla_y r^1(x,y,t)|^2\,\mathrm{d}x\,\mathrm{d}y=0\,,$$

which gives (4.44). Finally, the previous results together with (4.50) and Poincare's inequality yield

$$\lim_{t\to+\infty}\int\limits_{\Omega}|r(x,t)|^2\,\mathrm{d}x=0\,;\qquad\text{and}\qquad\lim_{t\to+\infty}\int\limits_{\Omega}\int\limits_{Y}|r^1(x,y,t)|^2\,\mathrm{d}x=0\,,$$

which gives (4.43) and concludes the proof. □

r:rem11 *Remark* 4.14. Observe that this asymptotic convergence results implies uniqueness of the periodic solution $(u^{\#},u^{1,\#})$ (in the class of functions specified above). □

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