BEST RANK K APPROXIMATION FOR BINARY FORMS

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Abstract. In the tensor space $\mathrm{Sym}^d\mathbb{R}^2$ of binary forms we study the best rank k approximation problem. The critical points of the best rank 1 approximation problem are the eigenvectors and it is known that they span a hyperplane. We prove that the critical points of the best rank k approximation problem lie in the same hyperplane. As a consequence, every binary form may be written as linear combination of its critical rank 1 tensors, which extends the Spectral Theorem from quadratic forms to binary forms of any degree. In the same vein, also the best rank k approximation may be written as a linear combination of the critical rank 1 tensors, which extends the Eckart-Young Theorem from matrices to binary forms.

1. Introduction

The symmetric tensor space Sym^dV, with $V=\mathbb{R}^2$ (resp. $V=\mathbb{C}^2$), contains real (resp. complex) binary forms, which are homogeneous polynomials in two variables. The forms which can be written as v^d, with $v\in V$, correspond to polynomials which are the d-power of a linear form, they have rank one. We denote by $C_d\subset \mathrm{Sym}^dV$ the variety of forms of rank one. The k-secant variety $\sigma_k(C_d)$ is the closure of the set of forms which can be written as $\sum_{i=1}^k \lambda_i v_i^d$ with $\lambda_i\in\mathbb{R}$ (resp. $\lambda_i\in\mathbb{C}$).

We say that a nonzero rank 1 tensor is a critical rank one tensor for $f\in\mathrm{Sym}^dV$ if it is a critical point of the distance function from f to the variety of rank 1 tensors. Critical rank one tensors are important to determine the best rank one approximation of f, in the setting of optimization [5, 7, 14]. Critical rank one tensors may be written as λv^d with $\lambda\in\mathbb{C}$ and $v\cdot v=1$, the last scalar product is the Euclidean scalar product. The corresponding vector $v\in V$ has been called tensor eigenvector, independently by Lim and Qi, [7, 12]. In this paper we concentrate on critical rank one tensors λv^d, which live in Sym^dV (not in V like the eigenvectors), for a better comparison with critical rank k tensors, see Definition 3.10 .

There are exactly d critical rank one tensor (counting with multiplicities) for any f different from $c(x^2+y^2)^{d/2}$ (with d even), while there are infinitely many critical rank one tensors for $f=(x^2+y^2)^{d/2}$ (see Prop. 3.4).

The critical rank one tensors for f are contained in the hyperplane H_f (called the singular space, see [9]), which is orthogonal to the vector $D(f)=yf_x-xf_y$. We review this statement at the beginning of §4.

The main result of this paper is the following extension of the previous statement to critical rank k tensors, for any $k\geq 1$.

Theorem 1.1. *Let $f \in \mathrm{Sym}^d\mathbb{C}^2$.*

i) All critical rank k tensors for f are contained in the hyperplane H_f, for any $k \geq 1$.

ii) Any critical rank k tensor for f may be written as a linear combination of the critical rank 1 tensors for f.

Theorem 1.1 follows after Theorem 4.2 and Proposition 5.1. Note that Theorem 1.1 may applied to the best rank k approximation of f, which turns out to be contained in H_f and may then be written as a linear combination of the critical rank 1 tensors for f. This statement may be seen as a weak extension of the Eckart-Young Theorem to tensors. Indeed, in the case of matrices, the best rank k approximation is exactly the sum of the first k critical rank one tensors, by the Eckart-Young Theorem, see [9]. The polynomial f itself may be written as linear combination of its critical rank 1 tensors, see Corollary 5.2, this statement may be seen as a *spectral decomposition for f*. All these statements may be generalized to the larger class of tensors, not necessarily symmetric, in any dimension, see [4].

In §6 we report about some numerical experiments regarding the number of real critical rank 2 tensors in $\mathrm{Sym}^4\mathbb{R}^2$.

2. Preliminaries

Let $V = \mathbb{R}^2$ equipped with the Euclidean scalar product. The associated quadratic form has the coordinate expression $x^2 + y^2$, with respect to the orthonormal basis x, y. The scalar product can be extended to a scalar product on the tensor space Sym^dV of binary forms, which is $SO(V)$-invariant. For powers l^d, m^d where $l, m \in V$, we set $\langle l^d, m^d\rangle := \langle l, m\rangle^d$ and by linearity this defines the scalar product on the whole Sym^dV (see Lemma 3.5).

Denote as usual $\|f\| = \sqrt{\langle f, f\rangle}$.

For binary forms which split in the product of linear forms we have the formula

$$\langle l_1 l_2 \cdots l_d, m_1 m_2 \cdots m_d\rangle = \frac{1}{d!}\sum_{\sigma}\langle l_1, m_{\sigma(1)}\rangle\langle l_2, m_{\sigma(2)}\rangle \cdots \langle l_d, m_{\sigma(d)}\rangle \tag{1}$$

The powers l^d are exactly the tensors of rank one in Sym^dV, they make a cone C_d over the rational normal curve.

The sums $l_1^d + \ldots + l_k^d$ are the tensors of rank $\leq k$, and equality holds when the number of summands is minimal. The closure of the set of tensors of rank $\leq k$, both in the Euclidean or in the Zariski topology, is a cone $\sigma_k C_d$, which is the k-secant variety of C_d.

The Euclidean distance function $d(f, g) = \|f - g\|$ is our objective function. The optimization problem we are interested is, given a real f, to minimize $d(f, g)$ with the constraint that $g \in (\sigma_k C_d)_{\mathbb{R}}$. This is equivalent to minimize the square function $d^2(f, g)$, which has the advantage to be algebraic. The number of complex critical points of the square distance function d^2 is called the Euclidean distance degree (EDdegree [3]) of $\sigma_k C_d$ and has been computed for small values of

k, d in the rightmost chart in Table 4.1 of [10]. We do not know a closed formula for these values, although [10, Theorem 3.7] computes them in the case of a general quadratic distance function, not $SO(2)$-invariant.

3. Critical points of the distance function

Let us recall the notion of eigenvector for symmetric tensors (see [7, 12],[9, Theorem 4.4]).

Definition 3.1. *Let $f \in \mathrm{Sym}^d V$. We say that a nonzero rank 1 tensor is a critical rank one tensor for f if it is a critical point of the distance function from f to the variety of rank 1 tensors. It is convenient to write a critical rank one tensor in the form λv^d with $\|v\| = 1$, in this way v is defined up to d-th roots of unity and is called an eigenvector of f with eigenvalue λ.*

Remark 3.2. *Let $d = 2$ and let f be a symmetric matrix. All the critical rank one tensors of f have the form λv^2 where v is a classical eigenvector of norm 1 for the symmetric matrix f, with eigenvalue λ.*

Lemma 3.3. *Given $f \in \mathrm{Sym}^d V$, the point λv^d of rank 1, with $\|v\| = 1$, is a critical rank one tensor for f if and only if $\langle f, v^{d-1}w\rangle = \lambda\langle v, w\rangle$ $\forall w \in V$, which can be written (identifying V with $V^\vee$ according to the Euclidean scalar product) as*

$$f \cdot v^{d-1} = \lambda v,$$

with $\lambda = \langle f, v^d\rangle$.

Proof. The property of critical point is equivalent to $f - \lambda v^d$ being orthogonal to $v^{d-1}w$ $\forall w \in V$, which gives $\langle f, v^{d-1}w\rangle = \langle \lambda v^d, v^{d-1}w\rangle$ $\forall w \in V$. The right-hand side is $\|v\|^{2d-2}\,\lambda\langle v, w\rangle = \lambda\langle v, w\rangle$, as we wanted. Setting $w = v$ we get $\langle f, v^d\rangle = \lambda$. $\square$

On the other hand, eigenvectors correspond to critical points of the function $f(x, y)$ restricted on the circle $S^1 = \{(x, y)|x^2 + y^2 = 1\}$ ([7, 12]). By Lagrange multiplier method, we can compute the eigenvectors of f as the normalized solutions (x, y) of:

$$\operatorname{rank}\begin{bmatrix} f_x & f_y \\ x & y \end{bmatrix} \leq 1 \tag{2}$$

This corresponds with the roots of discriminant polynomial $D(f) = yf_x - xf_y$. D is a well known differential operator which satisfies the Leibniz rule, i.e. $D(fg) = D(f)g + fD(g)$ $\forall f, g \in \mathrm{Sym}^d V$. For any $l = ax + by \in V$ denote $l^\perp = D(l) = -bx + ay$. Note that $\langle l, l^\perp\rangle = 0$.

We have the following:

Proposition 3.4. *Consider $f(x, y) \in \mathrm{Sym}^d V$:*

- *If v is eigenvector of f then $D(v) = v^\perp$ is a linear factor of $D(f)$.*
- *Assume that $D(f)$ splits as product of distinct linear factors and $v^\perp|D(f)$, then $\frac{v}{\|v\|}$ is an eigenvector of f.*

We postpone the proof after Prop. 3.11.

Now let us differentiate some cases in terms of $D(f)$ (see Theorem 2.7 of [2]):

- if d is odd: $D(f) = 0$ if and only if $f = 0$, in particular $D : \mathrm{Sym}^d V \to \mathrm{Sym}^d V$ is an isomorphism.
- if d is even: $D(f) = 0$ if and only if $f = c(x^2 + y^2)^{d/2}$ for some $c \in \mathbb{R}$. We will show in Lemma 3.7 which are the eigenvectors in this case. The image of $D : \mathrm{Sym}^d V \to \mathrm{Sym}^d V$ is the space orthogonal to $f = (x^2 + y^2)^{d/2}$.

Lemma 3.5. *([6], Section 2) Suppose $f = \sum_{i=0}^{d} \binom{d}{i} a_i x^i y^{d-i}$ and $g = \sum_{i=0}^{d} \binom{d}{i} b_i x^i y^{d-i}$. Then we get:*

$$\langle f, g \rangle := \sum_{i=0}^{d} \binom{d}{i} a_i b_i \tag{3}$$

where $\langle\, ,\, \rangle$ is the scalar product defined in the introduction.

Proof. By linearity we may assume $f = (\alpha x + \beta y)^d$ and $g = (\alpha' x + \beta' y)^d$. The right-hand side of (3) gives

$$\langle f, g \rangle = \sum_{i=0}^{d} \binom{d}{i} (\alpha\alpha')^i (\beta\beta')^{d-i} = (\alpha\alpha' + \beta\beta')^d$$

which agrees with $\langle \alpha x + \beta y, \alpha' x + \beta' y \rangle^d$. $\square$

Lemma 3.6. *Let $f = (x^2 + y^2)^{d/2} \in \mathrm{Sym}^d V$ with d even, and $v = \alpha x + \beta y \in V$, $v \neq 0$, then $\langle v^d, f \rangle = \|v\|^d$.*

Proof. By applying (1) with a grain of salt (e.g. decomposing $x^2 + y^2$ into two conjugates linear factors) we get

$$\langle v^d, f \rangle = \langle (x^2 + y^2), v^2 \rangle^{d/2} = (\alpha^2 + \beta^2)^{d/2} = \|v\|^d .$$

$\square$

Lemma 3.7. *If $f = (x^2 + y^2)^{d/2} \in \mathrm{Sym}^d V$ then, for every nonzero $v \in V$, $\langle f, v^{d-1} w \rangle = \|v\|^{d-2} \langle v, w \rangle$. In particular every vector v of norm 1 is eigenvector of f with eigenvalue 1.*

Proof. As in Lemma 3.6 we get

$$\langle f, v^{d-1} w \rangle = \langle (x^2 + y^2), v^2 \rangle^{d/2-1} \langle (x^2 + y^2), vw \rangle = \|v\|^{d-2} \langle v, w \rangle.$$

The second part follows by putting $w = v$ and equating with Lemma 3.6. We get $\langle f, v^{d-1} w \rangle = \langle v^d, f \rangle \langle v, w \rangle$ just in the case $|v| = 1$.

$\square$

Remark 3.8. *Lemma 3.7 extends the fact that every vector of norm* 1 *is eigenvector of the identity matrix with eigenvalue* 1*. The geometric interpretation of this lemma is that the* 2*-dimensional cone of rank* 1 *degree* d *binary forms cuts any sphere centered in* $(x^2+y^2)^{d/2}$ *in a curve.*

Lemma 3.9. *The normal space at* $l^d \in C_d$ *coincides with* $\left(l^{\perp}\right)^2 \cdot \mathrm{Sym}^{d-2}V$

Proof. The tangent space at l^d is spanned by $l^{d-1}V$ and has dimension 2. The elements in $\left(l^{\perp}\right)^2 \cdot \mathrm{Sym}^{d-2}V$ are orthogonal to the tangent space, moreover the dimension of this space is the expected one $d-1$. □

Definition 3.10. *We say that* $g \in \mathrm{Sym}^d V$ *is a critical rank* k *tensor for* f *if it is a critical point of the distance function* $d(f, \_)$ *restricted on* $\sigma_k C_d$*.*

Proposition 3.11. *Let* $2k \leq d$*. A polynomial* $g = \sum_{i=1}^{k} \mu_i l_i^d \in \sigma_k C_d$ *is a critical rank* k *tensor for* f *if and only if there exist* $h \in \mathrm{Sym}^{d-2k}V$ *such that*

$$f = \sum_{i=1}^{k} \mu_i l_i^d + h \cdot \prod_{i=1}^{k} \left(l_i^{\perp}\right)^2 \tag{4}$$

Proof. By Terracini Lemma, the tangent space of the point $g \in \sigma_k C_d$ is given by the sum of k tangent spaces at $l_i^d = (a_i x + b_i y)^d$. By Lemma 3.9 the normal space of each of these tangent spaces are given by $\left(l_i^{\perp}\right)^2 \cdot \mathrm{Sym}^{d-2}V$. Hence, the normal space to g is given by intersection of the k normal spaces, which is given by polynomials $\prod_{i=1}^{k} \left(l_i^{\perp}\right)^2 \cdot h$ where $h \in \mathrm{Sym}^{d-2k}V$.

Now suppose that g is a critical rank k tensor for f. This means that $f-g$ is in the normal space. Hence, $f-g$ is of the form $\prod_{i=1}^{k} \left(l_i^{\perp}\right)^2 \cdot h$ for some $h \in \mathrm{Sym}^{d-2k}V$.

Conversely, if (4) holds, we need that $f-g$ belongs to the normal space at g which is also true by the construction of the normal space. □

Proof of Prop. 3.4. If v is eigenvector of f then $\langle f, v^d \rangle v^d$ is critical rank 1 tensor for f (by Lemma 3.3). By Prop. 3.11 $f = \langle f, v^d \rangle v^d + h \left(v^{\perp}\right)^2$ where $h \in \mathrm{Sym}^{d-2}V$. Applying the operator D to f we get by Leibniz rule, since $D(v) = v^{\perp}$ and $D(v^{\perp}) = -v$:

$$D(f) = \langle f, v^d \rangle d v^{d-1} v^{\perp} + D(h) \left(v^{\perp}\right)^2 - 2 v v^{\perp} h \Longrightarrow v^{\perp} | D(f)$$

Conversely, since we assume there are d distinct eigenvectors, then we find all the linear factors of $D(f)$. □

This proposition is connected with Theorem 2.5 of [6].

4. The singular space

In [9] it was considered the singular space H_f as the hyperplane orthogonal to $D(f) = y f_x - x f_y$. It follows from Prop. 3.4 that the critical rank 1 tensor for f belong to H_f (since the eigenvectors of f can be computed as the solutions of (2) that coincides with $D(f)$ for binary forms), see [9, Def. 5.3]. It is worth to give a direct proof that the critical rank 1 tensors for f belong to H_f, the hyperplane orthogonal to $D(f)$, based on Prop. 3.11.

Let μl^d be a critical rank 1 tensors for f, then by Prop. 3.11 there exist $h \in \mathrm{Sym}^{d-2}V$ such that $f = \mu l^d + h\left(l^{\perp}\right)^2$.

We have to prove $\langle D(f), l^d\rangle = 0$ which follows immediately from (1) since $l^{\perp}$ divides $D(f)$ by Prop. 3.4.

Lemma 4.1. *Let $l, m \in V$, Then $\langle l^{\perp}, m\rangle + \langle m^{\perp}, l\rangle = 0$.*

Proof. Straightforward. □

Our main result is

Theorem 4.2. *The critical points of the form $\sum_{i=1}^k \mu_i l_i^d$ of the distance function $d(f, -)$ restricted on $\sigma_k C_d$ belong to H_f.*

Proof. Given a decomposition $f = \sum_{i=1}^k \mu_i l_i^d + h \cdot \prod_{i=1}^k \left(l_i^{\perp}\right)^2$, with $h \in \mathrm{Sym}^{d-2k}V$, we compute

$$D(f) = d\sum_{i=1}^k \mu_i l_i^{\perp} l_i^{d-1} - \sum_{i=1}^k 2l_i l_i^{\perp} \prod_{j\neq i}^k \left(l_j^{\perp}\right)^2 h + D(h)\prod_{i=1}^k \left(l_i^{\perp}\right)^2 \tag{5}$$

and we have to prove

$$\langle D(f), \sum_{j=1}^k l_j^d\rangle = 0. \tag{6}$$

We compute separately the contribution of the three summands in (5) to the scalar product with l_j^d.

We have for the first summand

$$\langle \left(\sum_{i=1}^k l_i^{\perp} l_i^{d-1}\right), l_j^d\rangle = \sum_{i=1}^k \langle l_i^{\perp}, l_j\rangle\langle l_i \cdot l_j\rangle^{d-1}$$

Summing over j we get zero by Lemma 4.1.

We have for the second summand

$$\langle \left(\sum_{i=1}^k l_i, l_i^{\perp} \prod_{p\neq i}^k \left(l_p^{\perp}\right)^2 h\right), l_j^d\rangle = \langle \left(l_j l_j^{\perp} \prod_{p\neq j}^k \left(l_p^{\perp}\right)^2 h\right), l_j^d\rangle = 0$$

We have for the third summand

$$\langle \left(D(h)\prod_{i=1}^k \left(l_i^{\perp}\right)^2\right), l_j^d\rangle = 0$$

Summing up, this proves (6) and then the thesis.

□

Example 4.3. *If $f = x^3y + 2y^4$ then there are 6 critical points of the form $l_1^4 + l_2^4$ and x^3y which lies on the tangent line at x^4. It cannot be written as $l_1^4 + l_2^4$ and indeed it has rank 4.*

5. The scheme of eigenvectors for binary forms

Suppose $f \in \mathrm{Sym}^d V$ a symmetric tensor and $\dim V = 2$. We denote by Z the scheme defined by the polynomial $D(f)$, embedded in $\mathbb{P}(\mathrm{Sym}^d V)$ by the d-Veronese embedding in $\mathbb{P}V$ (see [1] for the case of matrices).

Proposition 5.1. $\langle Z\rangle = H_f$.

Proof. (i) If $D(f)$ has d distinct roots then it is known that $\langle Z\rangle \subseteq H_f$, since H_f is the hyperplane orthogonal to $D(f)$ (Theorem 4.2 with $k = 1$). Hence $\langle Z\rangle \subseteq H_f$.

(ii) Now let us suppose that $D(f)$ has multiple roots but $f \neq (x^2 + y^2)^{d/2}$. We show that $\langle Z\rangle \subseteq H_f$ by a limit argument. For every tensor f such that $f \neq 0$ and $f \neq (x^2 + y^2)^{d/2}$ there exists a sequence (f_n) such that $f_n \to f$ and $D(f_n)$ has distinct roots for all n. Then, $H_{f_n} \to H_f$ because the differential operator is continuous. Moreover $H(f_n)$ is a hyperplane for all n. On the other hand, by definition we have that $\langle Z_{f_n}\rangle$ is the spanned of the roots of $D(f_n)$. When f_n goes to the limit we get that $\langle Z_{f_n}\rangle \to \langle Z\rangle$. Hence, $\langle Z\rangle \subseteq H_f$.

(iii) In the case that $f = (x^2 + y^2)^{d/2}$ with d even, then by Lemma 3.7 we know that every unitary vector is an eigenvector and H_f is the ambient space. Hence, $\langle Z\rangle = H_f$.

We prove now that $\dim\langle Z\rangle = \dim H_f$ for (i) and (ii). Since $\mathcal{I}_{Z,\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-d)$,

$$\mathrm{codim}\langle Z\rangle = h^0(\mathcal{I}_{Z,\mathbb{P}^1}(d)) = h^0(\mathcal{O}_{\mathbb{P}^1}(-d + d)) = h^0(\mathcal{O}_{\mathbb{P}^1}) = 1$$

which coincides with the codimension of H_f.

□

As a consequence we obtain the following corollary, which may be seen as a *Spectral Decomposition* of any binary form f.

Corollary 5.2. *Any binary form $f \in \mathrm{Sym}^d V$ with $\dim V = 2$ can be written as a linear combination of the critical rank one tensors for f.*

The previous statement holds even in the special case d even and $f = (x^2 + y^2)^{d/2}$, since from [11, Theorem 9.5] there exists $c_d \in \mathbb{R}$ such that the following decomposition holds $\forall \phi \in \mathbb{R}$

$$(x^2 + y^2)^{d/2} = c_d \sum_{k=0}^{d/2} \left[\cos(\frac{2k\pi}{d+2} + \phi)x + \sin(\frac{2k\pi}{d+2} + \phi)y\right]^d$$

In this decomposition the summands on the right-hand side correspond to $(d+2)/2$ consecutive vertices of a regular $(d+2)$-gon.

In the $d = 2$ case, the Spectral Theorem asserts any binary quadratic form $f \in \mathrm{Sym}^2\mathbb{R}^2$ can be written as sum of its rank one critical tensors. This statement fails for $d \geq 3$, as it can be checked already on the examples $f = x^d + y^d$ for $d \geq 3$, where only two among the d rank one critical tensors are used, namely x^d and y^d, and the coefficients of the remaining $d - 2$ rank one critical tensors in the Spectral Decomposition of f are zero.

6. Real critical rank 2 tensors for binary quartics

We recall the following result by M. Maccioni.

Theorem 6.1. *(Maccioni,* [8, Theorem 1]*) Let f be a binary form.*

$$\#\ \textit{real roots of}\, f\ \leq \#\ \textit{real critical rank 1 tensors for}\ f$$

The inequality is sharp, moreover it is the only constraint between the number of real roots and the number of real critical rank 1 *tensors, beyond parity mod* 2*.*

As a consequence, as it was first proved in [2], hyperbolic binary forms (i.e. with only real roots) have all real critical rank 1 tensors.

We attempted to extend Theorem 6.1 to rank 2 critical tensors. Our description is not yet complete and we report about some numerical experiments in the space $\mathrm{Sym}^4\mathbb{R}^2$. From these experiments it seems that the constraints about the number of real rank 2 critical tensors are weaker than for rank 1 critical tensors.

For quartic binary forms the computation of the critical rank 2 tensors is easier since the dual variety of the secant variety $\sigma_2(C_4)$ is given by quartics which are squares, which make a smooth variety.

The number of complex critical rank 2 tensors for a general binary form of degree d was guessed in [10] to be $3/2d^2 - 9d/2 + 1$. For $d = 4$ this number is 7, which can be confirmed by a symbolic computation on a rational random binary quartic.

In conclusion, for a general binary quartic there are 4 complex critical rank 1 tensors and 7 complex rank 2 critical tensors.

The following table reports some computation done for the case of binary quartic forms, by testing several different quartics. The appearance of "yes" in the last column means that we have found an example of a binary quartic with the prescribed number of distinct and simple real roots, real rank 1 critical tensors and real critical rank 2 tensors. Note that we have not found any quartic with the maximum number of seven real rank 2 critical tensors, we wonder if they exist.

#real roots	#real critical rank 1 tensors	#real critical rank 2 tensors	
0	2	3	yes
2	2	3	yes
0	2	5	yes
2	2	5	yes
0	4	3	yes
2	4	3	yes
4	4	3	yes
0	4	5	yes
2	4	5	yes
4	4	5	?
*	*	7	?

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