LOG-CONCAVE FUNCTIONS

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ABSTRACT. We attempt to provide a description of the geometric theory of log-concave functions. We present the main aspects of this theory: operations between log-concave functions; duality; inequalities including the Prékopa-Leindler inequality and the functional form of Blaschke-Santaló inequality and its converse; functional versions of area measure and mixed volumes; valuations on log-concave functions.

1. INTRODUCTION

A function f is log-concave if it is of the form

$$f = e^{-u}$$

where u is convex. This simple structure might suggest that there can not be anything too deep or interesting behind. Moreover, it is clear that log-concave functions are in one-to-one correspondence with convex functions, for which there exists a satisfactory and consolidated theory. Why to develop yet another theory?

Despite these considerations, which may occur to those who meet these functions for the first time, the theory of log-concave functions is rich, young and promising. There are two main reasons for that. The first comes from probability theory: many important examples of probability measures on $\mathbb{R}^n$, starting with the Gaussian mesure, have a log-concave density. These measures are referred to as log-concave probability measures (and thanks to a celebrated results of Borell they admit an equivalent and more direct characterization, see [14]). They have been attracting more and more interest over the last years. Typical results that have been proved for these measures are: Poincaré (or spectral gap) and log-Sobolev inequalities, concentration phenomena, isoperimetric type inequalities, central limit theorems and so on (see [46] for a survey).

The second motivation comes from convex geometry and gives rise to the geometric theory of log-concave functions, which is the theme of this paper. There is a natural way to embed the set of convex bodies in that of log-concave functions, and there are surprisingly many analogies between the theory of convex bodies and that of log-concave

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functions. The extension of notions and propositions from the context of convex bodies to the more recent theory of log-concave functions is sometimes called *geometrization of analysis*. The seeds of this process were the Prékopa-Leindler inequality (see [43], [49]), recognized as the functional version of the Brunn-Minkowski inequality, and the discovery of a functional form of the Blaschke-Santaló inequality due to Ball (see [8]). A strong impulse to the development of geometrization of analysis was then given by the innovative ideas of Artstein-Avidan, Klartag and Milman who, through a series of papers (see [3], [5], [6] and [46]), widened the perspectives of the study of log-concave functions and transformed this subject into a more structured theory. In the course of this paper, we will see how many authors have then contributed in recent years to enrich this theory with new results, concepts and directions for future developments.

Here we try to provide a picture of the current state of the art in this area. We will start from the beginning. In Section 3, we give a precise definition of the space of log-concave functions we work with, denoted by $\mathcal{L}^n$, and we describe basic properties of these functions. Moreover, we define the operations that are commonly used to add such functions and to multiply them by non negative reals. Once equipped with these operations $\mathcal{L}^n$ is a convex cone of functions, just like the family of convex bodies $\mathcal{K}^n$ with respect to the Minkowski addition and the corresponding multiplication by positive scalars.

Section 4 is entirely devoted to the notion of duality. The most natural way to define the dual of a log-concave function $f = e^{-u}$ is to set

$$f^\circ := e^{-u^*}$$

where u^* is the Fenchel (or Legendre) transform of the convex function u. The effectiveness of this definition will be confirmed by the inequalities reported in the subsequent Section 5. In Section 4, we recall the basic properties of this duality relation and the characterization result due to Artstein-Avidan and Milman, which ensures that the duality mapping which takes f in f° is characterized by two elementary properties only: monotonicity, and idempotence. In the same section, we will also see a different duality relation, due to Artstein-Avidan and Milman as well, which can be applied to the subclass of $\mathcal{L}^n$ formed by geometric log-concave functions.

Inequalities are the salt of the earth, as every analyst knows, and log-concave functions are a very fertile ground by this point of view. In Section 5, we review the two main examples of inequalities in this area: the Prékopa-Leindler inequality and the functional versions of the Blaschke-Santaló inequality together with its converse. Concerning the Prékopa-Leindler inequality, we also explain its connection with the Brunn-Minkowski inequality, and we show how its infinitesimal form leads to a Poincaré inequality due to Brascamp and Lieb. In the same section we also introduce the notion of the difference function of a log-concave function and an inequality which can be interpreted as the functional version of the Rogers-Shephard inequality for the volume of the difference body of a convex body.

The analogy between convex bodies and log-concave functions has its imperfections. Here is a first discrepancy: in convex geometry the important notions of mixed volumes and mixed area measures are originated by the polynomiality of the volume of Minkowski linear combinations of convex bodies. This property fails to be true in the case of log-concave functions, at least if the usual addition (the one introduced in Section 3) is in use. Nevertheless, there have been some attempts to overcome this difficulty. In Section 6, we describe two constructions that lead to the definition of functional versions of area measure and mixed volumes for log-concave functions.

A second aspect in which the geometric theory of log-concave functions, at present, differs from that of convex bodies is given by valuations. The theory of valuations on convex bodies is one of the most active and prolific parts of convex geometry (see for instance Chapter 6 of [56] for an updated survey on this subject). Two milestones in this area are the Hadwiger theorem which characterizes continuous and rigid motion invariant valuations, and McMullen's decomposition theorem for continuous and translation invariant valuations. On the other hand, the corresponding theory of valuations on the space of log-concave functions is still moving the first steps, and it is not clear whether neat characterization results will be achieved in the functional setting as well. The situation is depicted in Section 7.

In the appendix of the paper we collected some of the main notions and results from convex geometry, described in a very synthetic way, for the reader's convenience.

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2. Notations

We work in the n-dimensional Euclidean space $\mathbb{R}^n$, $n \geq 1$, endowed with the usual scalar product (x, y) and norm $\|x\|$. B_n denotes the unit ball of $\mathbb{R}^n$.

If A is a subset of $\mathbb{R}^n$, we denote by I_A its *indicatrix* function, defined in $\mathbb{R}^n$ as follows:

$$I_A(x) = \begin{cases} 0 & \text{if } x \in A, \\ \infty & \text{if } x \notin A. \end{cases}$$

The characteristic function of A will be denoted by χ_A:

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

The Lebesgue measure of a (measurable) set $A \subset \mathbb{R}^n$ will be denoted by $V_n(A)$ (and sometimes called the volume of A) and

$$\int_A f dx$$

stands for the integral of a function f over A, with respect to the Lebesgue measure.

A *convex body* is a compact, convex subset of $\mathbb{R}^n$; the family of convex bodies will be denoted by $\mathcal{K}^n$. Some notions and constructions regarding convex bodies, directly used in this paper, are recalled in the appendix. For an exhaustive presentation of the theory of convex bodies the reader is referred to [56].

3. The space $\mathcal{L}^n$

3.1. **The spaces $\mathcal{C}^n$ and $\mathcal{L}^n$.** In oder to define the space of log-concave functions, which we will be working with, in a precise way, we start by the definition of a specific space of *convex* functions. The typical convex function u that we will consider, is defined on the whole space $\mathbb{R}^n$ and attains, possibly, the value ∞. The domain of u is the set

$$\mathrm{dom}(u) = \{x \in \mathbb{R}^n \, : \, u(x) < \infty\}.$$

By the convexity of u, $\mathrm{dom}(u)$ is a convex set. The function u is *proper* if its domain is not empty.

Definition 3.1. *We set*

$$\mathcal{C}^n = \left\{ u \, : \, \mathbb{R} \to \mathbb{R}^n \cup \{\infty\} \, : \, u \text{ convex and s.t.} \lim_{\|x\| \to \infty} u(x) = \infty \right\}$$

and

$$\mathcal{L}^n = e^{-\mathcal{C}^n} = \left\{ f = e^{-u} \, : \, u \in \mathcal{C}^n \right\}.$$

Clearly in the previous definition we adopt the convention $e^{-\infty} = 0$. $\mathcal{L}^n$ is the space of log-concave functions which we will be working with. Note that the *support* of a function $f = e^{-u} \in \mathcal{L}^n$, i.e. the set

$$\mathrm{sprt}(f) = \{x \in \mathbb{R}^n \, : \, f(x) > 0\}$$

coincides with $\mathrm{dom}(u)$.

Remark 3.2. As an alternative to the previous definition (to avoid the use of convex functions), one could proceed as follows. A function $f : \mathbb{R}^n \to [0, \infty)$ is said log-concave if

$$f((1-t)x_0 + tx_1) \geq f(x_0)^{1-t} f(x_1)^t, \quad \forall \, x_0, \, x_1 \in \mathbb{R}^n, \quad \forall \, t \in [0,1]$$

(with the convention: $0^\alpha = 0$ for every $\alpha \geq 0$). Then $\mathcal{L}^n$ is the set of all log-concave functions f such that

$$\lim_{\|x\| \to \infty} f(x) = 0.$$

There are clearly many examples of functions belonging to $\mathcal{L}^n$. We choose two of them which are particularly meaningful for our purposes.

Example 3.3. Let K be a convex body; then $I_K \in \mathcal{C}^n$. As a consequence the function e^{-I_K}, which is nothing but the characteristic function of K, belongs to $\mathcal{L}^n$.

This simple fact provides a one-to-one correspondence between the family of convex bodies and a subset of log-concave functions. In other words, $\mathcal{K}^n$ can be seen as a subset of $\mathcal{L}^n$. We will see that this embedding is in perfect harmony with the natural algebraic structure of $\mathcal{L}^n$ and $\mathcal{K}^n$.

Example 3.4. Another prototype of log-concave function is the Gaussian function

$$f(x) = e^{-\frac{\|x\|^2}{2}}$$

which clearly belongs to $\mathcal{L}^n$.

Remark 3.5. By convexity and the behavior at infinity, any function $u \in \mathcal{C}^n$ is bounded from below. As a consequence

$$f \in \mathcal{L}^n \;\Rightarrow\; \sup_{\mathbb{R}^n} f < \infty.$$

3.2. **Operations on $\mathcal{L}^n$.** We will now define an addition and a multiplication by non-negative reals on $\mathcal{L}^n$. With these operations $\mathcal{L}^n$ becomes a *cone* (but not a vector space) of functions, just like the family of convex bodies $\mathcal{K}^n$ with respect to the Minkowski addition and dilations, is a cone of sets. The operations that we are going to introduce are widely accepted to be the natural ones for $\mathcal{L}^n$. Their construction is not straightforward; the following stepwise procedure might be of some help for the reader.

Let u and v be in $\mathcal{C}^n$; their *infimal convolution*, denoted by $u \Box v$, is defined as follows

$$(u \Box v)(x) = \inf_{y \in \mathbb{R}^n} \{u(y) + v(x-y)\}.$$

This operation is thoroughly studied in convex analysis (see for instance the monograph [52] by Rockafellar, to which we will refer for its properties). As a first fact, we have that $u \Box v \in \mathcal{C}^n$, i.e. this is an internal operation of $\mathcal{C}^n$ (see, for instance, [25, Prop. 2.6]). The infimal convolution has the following nice geometric interpretation (which can be easily verified): $u \Box v$ is the function whose epigraph is the vector sum of the epigraphs of u and v:

$$\text{epi}(u \Box v) = \{x + y \,:\, x \in \text{epi}(u),\, y \in \text{epi}(v)\} = \text{epi}(u) + \text{epi}(v),$$

where, for $w \in \mathcal{C}^n$

$$\text{epi}(w) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \,:\, y \geq w(x)\}.$$

Naturally associated to $\Box$ there is a multiplication by positive reals: for $u \in \mathcal{C}^n$ and $\alpha > 0$ we set

$$(\alpha \times u)(x) = \alpha\, u\left(\frac{x}{\alpha}\right).$$

This definition can be extended to the case $\alpha = 0$ by setting

$$0 \times u = I_{\{0\}};$$

the reason being that $I_{\{0\}}$ acts as the identity element: $I_{\{0\}}\Box u = u$ for every $u \in \mathcal{C}^n$. Note that

$$u\Box u = 2 \times u \quad \forall\, u \in \mathcal{C}^n,$$

as it follows easily from the convexity of u.

We are now ready to define the corresponding operations on $\mathcal{L}^n$.

Definition 3.6. *Let* $f = e^{-u}, g = e^{-v} \in \mathcal{L}^n$ *and let* $\alpha, \beta \geq 0$. *We define the function* $\alpha \cdot f \oplus \beta \cdot g$ *as follows*

$$(\alpha \cdot f \oplus \beta \cdot g) = e^{-(\alpha \times u \Box \beta \times v)}.$$

According to the previous definitions, when $\alpha, \beta > 0$ we have that[1]

$$(\alpha \cdot f \oplus \beta \cdot g)(x) = \sup_{y \in \mathbb{R}^n} f\left(\frac{x-y}{\alpha}\right)^{\alpha} g\left(\frac{y}{\beta}\right)^{\beta}. \tag{1}$$

Example 3.7. As an instructive and remarkable example, let us see how these operations act on characteristic functions of convex bodies. Let $K, L \in \mathcal{K}^n$, and $\alpha, \beta \geq 0$. The *Minkowski linear combination* of K and L with coefficients α and β is

$$\alpha K + \beta L = \{\alpha x + \beta y \,:\, x \in K,\, y \in L\}.$$

Te reader may check, as a simple exercise, the following identity

$$\alpha \cdot \chi_K \oplus \beta \cdot \chi_L = \chi_{\alpha K + \beta L}.$$

As $\mathcal{C}^n$ is closed with respect to $\Box$ and $\times$ (see [25, Prop. 2.6]), we have the following result.

Proposition 3.8. *Let* $f, g \in \mathcal{L}^n$ *and* $\alpha, \beta \geq 0$. *Then* $\alpha \cdot f \oplus \beta \cdot g \in \mathcal{L}^n$.

3.3. **The volume functional.** In the parallelism between convex geometry and the theory of log-concave functions it is important to find the corresponding notion of the volume of a convex body, in the functional setting. The natural candidate is the $L^1(\mathbb{R}^n)$-norm. Given $f \in \mathcal{L}^n$ we set

$$I(f) := \int_{\mathbb{R}^n} f(x)dx.$$

To prove that this integral is always finite we exploit the following lemma (see Lemma 2.5 in [25]).

Lemma 3.9. *Let* $u \in \mathcal{C}^n$; *then there exists* $a > 0$ *and* $b \in \mathbb{R}$ *such that*

$$u(x) \geq a\|x\| + b \quad \forall\, x \in \mathbb{R}^n.$$

[1] For this reason the sum defined here is sometimes referred to as the *Asplund product*, see for instance [3].

As a consequence, if $f = e^{-u} \in \mathcal{L}^n$, we have that

$$f(x) \le Ce^{-a\|x\|} \quad \forall\, x \in \mathbb{R}^n$$

for some $a > 0$ and $C > 0$. This implies that

$$I(f) < \infty \quad \forall\, f \in \mathcal{L}^n,$$

i.e.

$$\mathcal{L}^n \subset L^1(\mathbb{R}^n).$$

We will refer to the quantity $I(f)$ as the integral or the *volume* functional, evaluated at f. Note that if K is a convex body and $f = \chi_K$, then

$$I(f) = I(\chi_K) = \int_K dx = V_n(K).$$

3.4. **p-concave and quasi-concave functions.** A one parameter family of sets of functions which includes log-concave functions, is that of p-concave functions, as the parameter p ranges in $\mathbb{R} \cup \{\pm\infty\}$. Roughly speaking a function is p-concave if its p-th power is concave in the usual sense, but the precise definition requires some preparation.

Given $p \in \mathbb{R} \cup \{\pm\infty\}$, $a, b \ge 0$ and $t \in [0,1]$, the p-th mean of a and b, with weights t and $(1-t)$ is

$$M_p(a, b; t) := ((1-t)a^p + tb^p)^{1/p}$$

if $p > 0$. For $p < 0$, we adopt the same definition if $a > 0$ and $b > 0$, while if $ab = 0$ we simply set $M_p(a, b; t) = 0$. For $p = 0$:

$$M_0(a, b; t) := a^{1-t}b^t.$$

Finally, we set

$$M_\infty(a, b; t) := \max\{a, b\}, \quad M_{-\infty}(a, b; t) := \min\{a, b\}.$$

A non-negative function f defined on $\mathbb{R}^n$ is said to be p-concave if

$$f((1-t)x + ty) \ge M_p(f(x), f(y); t) \quad \forall\, x, y \in \mathbb{R}^n,\ \forall\, t \in [0,1].$$

For $p = 0$, we have the condition of log-concavity; for $p = 1$, this clearly gives back the notion of concave functions; for $p = -\infty$, the above conditions identifies the so-called quasi-concave functions, which can be characterized by the convexity of their super-level sets.

In the course of this paper we will see that some of the results that we present for log-concave functions admits a corresponding form for p-concave functions.

4. Duality

The notion of conjugate, or dual, function of a log-concave function that we introduce here (following, for instance, [3]) is based on the well-known relation of duality in the realm of convex functions, provided by the Fenchel, or Legendre, transform, that we briefly recall. Let u be a convex function in $\mathbb{R}^n$; we set

$$u^*(y) = \sup_{x\in\mathbb{R}^n} (x, y) - u(x), \quad \forall\, y \in \mathbb{R}^n.$$

Remark 4.1. Being the supremum of linear functions, u^* is convex. Moreover, unless $u \equiv \infty$, $u^*(y) > -\infty$ for every y. If we require additionally that $u \in \mathcal{C}^n$ (and $u \not\equiv \infty$), then u^* is proper (see [25, Lemma 2.5]). On the other hand, $u \in \mathcal{C}^n$ does not imply, in general, $u^* \in \mathcal{C}^n$. Indeed, for $u = I_{\{0\}}$ we have $u^* \equiv 0$.

Definition 4.2. *For $f = e^{-u} \in \mathcal{L}^n$, we set*

$$f^\circ = e^{-u^*}.$$

A more direct characterization of f° is

$$f^\circ(y) = \inf_{x\in\mathbb{R}^n} \left[\frac{e^{-(x,y)}}{f(x)} \right]$$

(where the involved quotient has to be intended as ∞ when the denominator vanishes). Hence f° is a log-concave function (which does not necessarily belong to $\mathcal{L}^n$).

The idempotence relation (that one would expect)

$$(u^*)^* = u \tag{2}$$

has to be handled with care, as it is not always true in $\mathcal{C}^n$. This depends on the fact that the Fenchel conjugate of a function is always lower semi-continuous (l.s.c., for brevity), while u needs not to have this property. On the other hand, this is the only possible obstacle for (2).

Proposition 4.3. *Let $u \in \mathcal{C}^n$ be l.s.c., then* (2) *holds.*

Corollary 4.4. *Let $f \in \mathcal{L}^n$ be upper semi-continuous (u.s.c.). Then*

$$(f^\circ)^\circ = f. \tag{3}$$

Examples.
1. Let K be a convex body and I_K be its indicatrix function. Then we have

$$(I_K)^*(y) = \sup_{x\in K}(x, y) =: h_K(y) \quad \forall\, y \in \mathbb{R}^n.$$

Here, following the standard notations, we denoted by h_K the *support function* of the convex body K (see the appendix).

2. The Gaussian function is the unique element of $\mathcal{L}^n$ which is self-dual:

$$f = e^{-\frac{\|x\|^2}{2}} \ \Leftrightarrow \ f^\circ \equiv f.$$

Remark 4.5. The Fenchel transform gives another interpretation of the inf-convolution operation and, consequently, of the addition that we have defined on $\mathcal{L}^n$. Indeed, if u and v are in $\mathcal{C}^n$ and $\alpha, \beta \geq 0$, then:

$$(\alpha \times u \,\square\, \beta \times v)^* = \alpha u^* + \beta v^* \ \Rightarrow \ \alpha \times u \,\square\, \beta \times v = (\alpha u^* + \beta v^*)^*,$$

if the function on the left hand-side of the last equalty is l.s.c. (see [25, Prop. 2.1]). Hence, given $f = e^{-u}, g = e^{-v} \in \mathcal{L}^n$ (such that $\alpha \cdot f \oplus \beta \cdot g$ is u.s.c.) we have

$$\alpha \cdot f \oplus \beta \cdot g = e^{-(\alpha u^* + \beta v^*)^*}. \tag{4}$$

In other words, the algebraic structure that we have set on $\mathcal{L}^n$ coincide with the usual addition of functions and multiplication by non-negative reals, applied to the conjugates of the exponents (with sign changed).

4.1. **Characterization of duality.** In the papers [5] and [6], Arstein-Avidan and Milman established several powerful characterizations of duality relations in the class of convex and log-concave functions (as well as in other classes of functions). The space of convex functions in which they work is slightly different from ours. They denote by $Cvx(\mathbb{R}^n)$ the space of functions $u : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, which are convex and l.s.c. One of their results is the following characterizations of the Fenchel conjugate, proved in [6].

Theorem 4.6 (**Artstein-Avidan, Milman**). *Let $\mathcal{T} : Cvx(\mathbb{R}^n) \to Cvx(\mathbb{R}^n)$ be such that:*

(1) *$\mathcal{T}\mathcal{T}u = u$ for every $u \in Cvx(\mathbb{R}^n)$;*
(2) *$u \leq v$ in $\mathbb{R}^n$ implies $\mathcal{T}(u) \geq \mathcal{T}(v)$ in $\mathbb{R}^n$.*

Then $\mathcal{T}$ coincides essentially with the Fenchel conjugate: there exist $C_0 \in \mathbb{R}$, $v_0 \in \mathbb{R}^n$ and an invertible symmetric linear transformation B of $\mathbb{R}^n$ such that for every $u \in Cvx(\mathbb{R}^n)$,

$$\mathcal{T}(u)(y) = u^*(By + v_0) + (x, v_0) + C_0, \quad \forall y \in \mathbb{R}^n.$$

A direct consequence of the previous result, is a characterization of the conjugate that we have introduced before for log-concave functions. Following the notation of [5] and [6] we set

$$LC(\mathbb{R}^n) = \{f = e^{-u} : u \in Cvx(\mathbb{R}^n)\}.$$

Theorem 4.7 (**Artstein-Avidan, Milman**). *Let $\mathcal{T} : LC(\mathbb{R}^n) \to LC(\mathbb{R}^n)$ be such that:*

(1) *$\mathcal{T}\mathcal{T}f = f$ for every $f \in LC(\mathbb{R}^n)$;*
(2) *$f \leq g$ in $\mathbb{R}^n$ implies $\mathcal{T}(f) \geq \mathcal{T}(g)$ in $\mathbb{R}^n$.*

Then there exist $C_0 \in \mathbb{R}$, $v_0 \in \mathbb{R}^n$ and an invertible symmetric linear transformation B of $\mathbb{R}^n$ such that for every $f \in LC(\mathbb{R}^n)$,

$$\mathcal{T}(f)(y) = C_0 e^{-(v_0,x)} f^\circ(Bx + v_0) \quad \forall y \in \mathbb{R}^n.$$

4.2. **Geometric log-concave functions and a related duality transform.** In the paper [7] the authors introduce a special subclass of $Cvx(\mathbb{R}^n)$, called the class of geometric convex functions, and denoted by $Cvx_0(\mathbb{R}^n)$. A function $u \in Cvx(\mathbb{R}^n)$ belongs to $Cvx_0(\mathbb{R}^n)$ if

$$\inf_{\mathbb{R}^n} u = \min_{\mathbb{R}^n} u = u(0) = 0.$$

Correspondingly, they define the class of geometric log-concave functions as follows:

$$LC_g(\mathbb{R}^n) = \{f = e^{-u} \, : \, u \in Cvx_0(\mathbb{R}^n)\}.$$

Note in particular that if $f \in LC_g(\mathbb{R}^n)$, then

$$0 \le f(1) \le 1 = f(0) = \max_{\mathbb{R}^n} f \quad \forall x \in \mathbb{R}^n.$$

For $u \in Cvx_0(\mathbb{R}^n)$ the set

$$u^{-1}(0) = \{x \, : \, u(x) = 0\}$$

is closed (by semicontinuity), convex and it contains the origin, even if not necessarily as an interior point. As an extension of the notion of polar set of a convex body having the origin in its interior (see the appendix), we set

$$(u^{-1}(0))^\circ = \{x \in \mathbb{R}^n \, : \, (x, y) \le 1 \; \forall \, y \in u^{-1}(0)\}.$$

The new duality transform introduced in [7], denoted by $\mathcal{A}$, is defined, for $u \in Cvx_0(\mathbb{R}^n)$, by

$$(\mathcal{A}u)(x) = \begin{cases} \displaystyle\sup_{\{y \,:\, u(y)>0\}} \frac{(x, y) - 1}{u(y)} & \text{if } x \in (u^{-1}(0))^\circ, \\ \\ \infty & \text{otherwise.} \end{cases}$$

Many interesting properties of this transform are proved in [7]; among them, we mention that $\mathcal{A}$ is order reversing and it is an involution, i.e.

$$\mathcal{A}(\mathcal{A}u) = u \quad \forall \, u \in Cvx_0(\mathbb{R}^n). \tag{5}$$

As in the case of Fenchel transform, these features can be used to characterize this operator, together with the Fenchel transform itself.

Theorem 4.8 (**Artstein-Avidan, Milman).** *Let $n \ge 2$ and $\mathcal{T} \, : \, Cvx_0(\mathbb{R}^n) \to Cvx_0(\mathbb{R}^n)$ be a transform which is order reversing and is an involution. Then either*

$$\mathcal{T}u = (u^*) \circ B \quad \forall \, u \in Cvx_0(\mathbb{R}^n),$$

or

$$\mathcal{T}u = C_0(\mathcal{A}u) \circ B \quad \forall \, u \in Cvx_0(\mathbb{R}^n),$$

where B is an invertible linear transformation of $\mathbb{R}^n$, $C_0 \in \mathbb{R}$.

As an application, a corresponding characterization result can be derived for the case of geometric log-concave functions.

5. Inequalities

5.1. **The Prékopa-Leindler inequality.** Let f, g, h be non-negative measurable functions defined in $\mathbb{R}^n$, and let t be a parameter which ranges in $[0,1]$. Assume that the following condition holds:

$$f((1-t)x_0 + tx_1) \geq g(x_0)^{1-t} h(x_1)^t \quad \forall\, x_0,\, x_1 \in \mathbb{R}^n. \tag{6}$$

In other words, f which is evaluated at the convex linear combination of any two points is greater than the geometric mean of g and h at those points. Then the integral of f is greater than the geometric mean of the integrals of g and h:

$$\int_{\mathbb{R}^n} f dx \geq \left(\int_{\mathbb{R}^n} g dx \right)^{1-t} \left(\int_{\mathbb{R}^n} h dx \right)^t. \tag{7}$$

Inequality (7) is the general form of the Prékopa-Leindler inequality; it was proved in [43], [49] and [50].

Though the inequality (7) in itself is rather simple, the condition behind it, i.e. (6), is unusual as it is not a point-wise condition but involves the values of f, g and h at different points. It will become clearer once it is written using the operations that we have introduced for log-concave functions. In fact, our next aim is to discover how Prékopa-Leindler is naturally connected to log-concavity. As a first step in this direction, we observe that, given g and h, one could rewrite inequality (7) replacing f by the smallest function with verifies (6). Namely, let

$$\bar{f}(z) = \sup_{(1-t)x+ty=z} g^{1-t}(x) h^t(y). \tag{8}$$

Then, if $\bar{f}$ is measurable[2], (7) holds for the triple $\bar{f}, g, h$. In view of (1), if $g, h \in \mathcal{L}^n$ then

$$\bar{f} = (1-t) \cdot g \oplus t \cdot g \in \mathcal{L}^n.$$

The second observation concerns equality conditions in (7), in which log-concave functions intervene directly. Note first that if $f = g = h$ (for which we trivially have equality in (7)), then (6) is equivalent to say that these functions are log-concave. Moreover, the converse of this claim is basically true, due to the following result proved by Dubuc (see [28, Theorem 12]). Assume that f, g and h are such that (6) is verified and equality holds in (7); then there exists a log-concave function F, a vector x_0 and constants

[2]In general the measurability of g and h does not imply that of f. See [34] for more information on this point.

$c_1, c_2, \alpha, \beta \geq 0$ such that:

$$\begin{aligned} f(x) &= F(x) \quad \text{a.e. in } \mathbb{R}^n, \\ g(x) &= c_1 F(\alpha x + x_0) \quad \text{a.e. in } \mathbb{R}^n, \\ h(x) &= c_2 F(\beta x + x_0) \quad \text{a.e. in } \mathbb{R}^n. \end{aligned}$$

In view of what we have seen so far, we may rephrase (7) in the realm of log-concave functions in the following way.

Theorem 5.1. *Let $g, h \in \mathcal{L}^n$ and let $t \in [0,1]$. Then*

$$\int_{\mathbb{R}^n} [(1-t)\cdot g \oplus t \cdot h] dx \geq \left(\int_{\mathbb{R}^n} g dx \right)^{1-t} \left(\int_{\mathbb{R}^n} h dx \right)^t, \tag{9}$$

i.e.

$$I((1-t)\cdot g \oplus t \cdot h) \geq I(g)^{t-1} \, I(h)^t.$$

Moreover, equality holds if and only if g coincide with a multiple of h up to a translation and a dilation of the coordinates.

Written in this form, the Prékopa-Leindler inequality is clearly equivalent to the following statement: *the volume functional I is log-concave in the space $\mathcal{L}^n$.* This point of view will be important to derive the infinitesimal form of this inequality. In the sequel we will refer to the Prékopa-Leindler inequality in the form (9) as to (PL).

We note here an important consequence of (PL), which was emphasized and exploited in various ways in [17].

Theorem 5.2. *Let $F = F(x,y)$ be defined in $\mathbb{R}^n \times \mathbb{R}^m$, and assume that F is log-concave. Then the function $f : \mathbb{R}^n \to \mathbb{R}$ defined by*

$$f(x) = \int_{\mathbb{R}^m} F(x,y) dy$$

is log-concave.

The proof is a simple application of (PL).

We conclude this part with some further remarks on the Prékopa-Leindler inequality.

Remark 5.3. One way to look at (PL) is as a reverse form of the Hölder inequality. Indeed, an equivalent formulation of Hölder inequality is the following: if g and h are non-negative measurable functions defined on $\mathbb{R}^n$, and $t \in [0,1]$,

$$\int_{\mathbb{R}^n} g^{1-t} h^t dx \leq \left(\int_{\mathbb{R}^n} g dx \right)^{1-t} \left(\int_{\mathbb{R}^n} h dx \right)^t.$$

Prékopa-Leindler inequality asserts that the previous inequality is reversed if the geometric mean of g and h is replaced by the supremum of their geometric means, in the sense of (8).

Remark 5.4. A more general form of (PL) is the Borell-Brascamp-Lieb inequality (see for instance Section 10 of [34]). This inequality asserts that if f, g, h are non-negative measurable functions defined on $\mathbb{R}^n$ such that for some $p \geq -\frac{1}{n}$ and $t \in [0,1]$

$$f((1-t)x + ty) \geq M_p(g(x), h(y); t) \quad \forall\, x, y \in \mathbb{R}^n,$$

then

$$\int_{\mathbb{R}^n} f dx \geq M_{\frac{p}{mp+1}} \left(\int_{\mathbb{R}^n} g dx, \int_{\mathbb{R}^n} h dx; t \right).$$

Here we have used the definition of p-mean introduced in subsection 3.4.

In the same way as (PL) has a special meaning for log-concave functions, Borell-Brascamp-Lieb inequality is suited to p-concave functions.

Remark 5.5. Prékopa-Leindler inequality can also be seen as a special case of a very general class of inequalities proved by Barthe in [9]. One way (even if limiting) of looking at Barthe's inequalities is as a multifunctional version of (PL). Barthe's inequalities are in turn the reverse form of Brascamp-Lieb inequalities, which have as a simple special case the Hölder inequality. A neat presentation of these inequalities can be found in [34], Section 15.

5.2. **Proof of the Prékopa-Leindler inequality.** For completeness we supply a proof of the Prékopa-Leindler inequality in its formulation (9), i.e. restricted to log-concave functions (omitting the characterization of equality conditions).

As preliminary steps, note that if one of the functions g and h is identically zero then the inequality is trivial. Hence we assume that $g \not\equiv 0$ and $h \not\equiv 0$. Moreover, as it is easy to check, it is not restrictive to assume

$$\sup_{\mathbb{R}^n} g = \sup_{\mathbb{R}^n} h = 1 \tag{10}$$

(see also Remark 3.5).

The rest of the proof proceeds by induction on the dimension n. For simplicity we will set

$$f = (1-t) \cdot g \oplus t \cdot h$$

throughout. For convenience of notations we will in general denote by x, y and z the variable of f, g and h, respectively.

The case $n = 1$. Fix $s \in [0,1]$; by the definition of the operations $\cdot$ and $\oplus$, we have the following set inclusion

$$\{x \,:\, f(x) \geq s\} \supset (1-t)\{y \,:\, g(y) \geq s\} + t\{z \,:\, h(z) \geq s\}.$$

As f, g and h are log-concave, their super-level sets are intervals, and, by the behavior of these functions at infinity, they are bounded. Note that if if I and J are bounded interval of the real line we have

$$V_1(I + J) = V_1(I) + V_1(J)$$

(which is the one-dimensional version of the Brunn-Minkowski inequality, in the case of "convex sets"). Hence

$$V_1(\{x\,:\,f(x)\geq s\})\geq(1-t)V_1(\{y\,:\,g(y)\geq s\})+tV_1(\{z\,:\,h(z)\geq s\}).$$

Now we integrate between 0 and 1 and use the layer cake principle

$$\int_{\mathbb{R}} f\,dx\geq(1-t)\int_{\mathbb{R}} g\,dy+t\int_{\mathbb{R}} h\,dz\geq\left(\int_{\mathbb{R}} g\,dy\right)^{1-t}\left(\int_{\mathbb{R}} h\,dz)\right)^{t}$$

where we have used the arithmetic-geometric mean inequality. This concludes the proof in dimensional one. Note that in this case one obtains (under the assumption (10)) a stronger inequality, namely the integral of f is greater than the arithmetic mean of those of g and h.

The case $n\geq 1$. Assume that the inequality is true up to dimension $(n-1)$. Fix $\bar{y}_n$ and $\bar{z}_n$ in $\mathbb{R}$, and let $\bar{x}_n=(1-t)\bar{y}_n+t\bar{z}_n$. Moreover let $\bar{f},\bar{g},\bar{h}\,:\,\mathbb{R}^{n-1}\to\mathbb{R}$ be defined by

$$\bar{f}(x_1,\dots,x_{n-1})=f(x_1,\dots,x_{n-1},\bar{x}_n),\ \bar{g}(y_1,\dots,y_{n-1})=g(y_1,\dots,y_{n-1},\bar{y}_n),$$
$$\bar{h}(z_1,\dots,z_{n-1})=h(z_1,\dots,z_{n-1},\bar{z}_n).$$

As x_n is the convex linear combination of y_n and z_n, and as $f=(1-t)\cdot g\oplus t\cdot h$, we have that $\bar{f}$, $\bar{g}$ and $\bar{h}$ verify the assumption of (PL), so that, by induction,

$$\int_{\mathbb{R}^{n-1}}\bar{f}dx\geq\left(\int_{\mathbb{R}^{n-1}}\bar{g}dy\right)^{1-t}\left(\int_{\mathbb{R}^{n-1}}\bar{h}dz\right)^{t}. \tag{11}$$

Next define $F,G,H\,:\,\mathbb{R}\to\mathbb{R}$ as

$$F(x)=\int_{\mathbb{R}^{n-1}}f(x_1,\dots,x_{n-1},x)dx_1\dots dx_{n-1},$$
$$G(y)=\int_{\mathbb{R}^{n-1}}g(y_1,\dots,y_{n-1},y)dy_1\dots dy_{n-1},$$
$$H(z)=\int_{\mathbb{R}^{n-1}}h(z_1,\dots,z_{n-1},z)dz_1\dots dz_{n-1}.$$

By Theorem 16 these are log-concave functions; moreover (11) is exactly condition (6) for them. Hence, by induction,

$$\int_{\mathbb{R}}Fdx\geq\left(\int_{\mathbb{R}}Gdy\right)^{1-t}\left(\int_{\mathbb{R}}Hdz\right)^{t},$$

and this is nothing but the required inequality for f,g,h.

□

5.3. **Prékopa-Leindler and Brunn-Minkowski inequality.** One way to understand the importance of Prékopa-Leindler inequality is to set it in relation to the Brunn-Minkowski inequality, one of the most important results in convex geometry.

Theorem 5.6 (Brunn-Minkowski inequality). *Let K and L be convex bodies and $t \in [0,1]$. Then*

$$[V_n((1-t)K + tL)]^{1/n} \geq (1-t)[V_n(K)]^{1/n} + t[V_n(L)]^{1/n}. \tag{12}$$

In case both K and L have non-empty interior, equality holds if and only if they are homothetic, i.e. they coincide up to a translation and a rotation.

The article [34] by Gardner contains an exhaustive survey on this result. Here we only mention that Brunn-Minkowski inequality ((BM) for brevity) is a special case of the family of Aleksandrov-Fenchel inequalities (see [56]), and that a simple argument leads in few lines from this inequality to the isoperimetric inequalitiy (restricted to convex bodies):

$$V_n(K) \leq c \left[\mathcal{H}^{n-1}(\partial K)\right]^{n/(n-1)} \quad \forall\, K \in \mathcal{K}^n \,:\, \mathrm{int}(K) \neq \emptyset. \tag{13}$$

Here c is a dimensional constant and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. Moreover equality holds if and only if K is a ball. The argument to deduce (13) from (12) is rather known and can be found, for instance, in [34].

In what follows we show that (BM) can be easily proved through (PL).

Proof of the Brunn-Minkowsi inequality. Let K, L and t be as in Theorem 5.6. Let

$$g = \chi_K, \quad h = \chi_L, \quad f = (1-t)\cdot g \oplus t \cdot h.$$

As we saw in example 3.7,

$$f = \chi_{(1-t)K+tL}.$$

By (PL) we get

$$V_n((1-t)K + tL) \geq V_n(K)^{1-t}\, V_n(L)^t. \tag{14}$$

This is usually referred to as the multplicative form of the Brunn-Minkowski inequality. From that, by exploiting the homogeneity of volume, (BM) in its standard form can be deduced as follows. Given K, L and t as above, assume that the volumes of K and L are strictly positive (the general case can be obtained by approximation). Let

$$\bar{K} = \frac{1}{V_n(K)^{1/n}}\, K, \quad \bar{L} = \frac{1}{V_n(L)^{1/n}}\, L,$$

so that

$$V_n(\bar{K}) = V_m(\bar{L}) = 1.$$

We set also

$$\bar{t} = \frac{V_n(L)^{1/n}}{(1-t)V_n(K)^{1/n} + tV_n(L)^{1/n}}\, t.$$

Applying (14) to $\bar{K}$, $\bar{L}$ and $\bar{t}$ leads to

$$1 \le V_n((1-\bar{t})\bar{K} + \bar{t}\bar{L}) = V_n\left(\frac{1}{(1-t)V_n(K)^{1/n} + tV_n(L)^{1/n}}\,(1-t)K + tL\right).$$

□

5.4. **The infinitesimal form of (PL).** Both Prékopa-Leindler and Brunn-Minkowski inequalities are concavity inequalities. More precisely, (BM) asserts that the volume functional to the power $1/n$ is concave on the family of convex bodies $\mathcal{K}^n$, while, according to (PL), the logarithm of the integral functional I is concave on $\mathcal{L}^n$. The concavity of a functional F can be expressed by the usual inequality:

$$F((1-t)x_0 + tx_1) \ge (1-t)F(x_0) + tF(x_1) \quad \forall\, x_0, x_1;\ \forall\, t \in [0,1],$$

or by its infinitesimal version

$$D^2F(x) \le 0 \quad \forall\, x, \tag{15}$$

where $D^2F(x)$ denotes the *second variation* of F at x (if it exists, and whatever its meaning can be). The infinitesimal form of the Brunn-Minkowski inequality has been investigated in [24], where it is shown that (15) provides a class of Poincaré type inequalities on the unit sphere of $\mathbb{R}^n$. Here we will show that correspondingly, the infinitesimal form of (PL) is equivalent to a class of (known) inequalities, also of Poincaré type, on $\mathbb{R}^n$, with respect to log-concave probability measures. These inequalities have been proved by Brascamp and Lieb in [17].

Theorem 5.7 (Brascamp-Lieb). *Let $f = e^{-u} \in \mathcal{L}^n$ and assume that $u \in C^2(\mathbb{R}^n)$ and $D^2u(x) > 0$ for every $x \in \mathbb{R}^n$. Then for every $\phi \in C^1(\mathbb{R}^n)$ such that*

$$\int_{\mathbb{R}^n} \phi f dx = 0,$$

the following inequality holds:

$$\int_{\mathbb{R}^n} \phi^2 f dx \le \int_{\mathbb{R}^n} ((D^2u)^{-1}\nabla\phi, \nabla\phi) f dx. \tag{16}$$

Remark 5.8. When

$$u(x) = \frac{\|x\|^2}{2},$$

i.e. f is the Gaussian function, (16) becomes the usual Poincaré inequality in Gauss space:

$$\int_{\mathbb{R}^n} \phi^2 d\gamma_n(x) \le \int_{\mathbb{R}^n} \|\nabla\phi\|^2 d\gamma_n(x) \tag{17}$$

for every $\phi \in C^1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} \phi d\gamma_n(x) = 0, \tag{18}$$

where γ_n is the standard Gaussian probability measure. Note that (17) is sharp, indeed it becomes an equality when ϕ is a linear function. In general, the left-hand side of (16) is a weighted $L^2(\mathbb{R}^n, \mu)$-norm of $\nabla\phi$ (squared), where μ is the measure with density f. Note however, that (16) admits extremal functions (i.e. for which equality holds) for every choice of f; this will be clear from the proof that we present in the sequel.

Proof of Theorem 5.7. We will consider a special type of log-concave functions. Let $u \in C^2(\mathbb{R}^n) \cap \mathcal{C}^n$ be such that

$$cI_n \leq D^2 u(x) \quad \forall\, x \in \mathbb{R}^n, \tag{19}$$

where I_n is the $n \times n$ identity matrix and $c > 0$. We denote by $\mathcal{C}_s^n$ the space formed by these functions and set

$$\mathcal{L}_s^n := e^{-\mathcal{C}_s^n} \subset \mathcal{L}^n.$$

We set

$$(\mathcal{C}_s^n)^* = \{u^* \,:\, u \in \mathcal{C}_s^n\}.$$

By standard facts from convex analysis (see for instance [52]), if $u \in \mathcal{C}_s^n$ then $u^* \in C^2(\mathbb{R}^n)$; moreover ∇u is a diffeomorphism between $\mathbb{R}^n$ and itself and

$$\nabla u^* = (\nabla u)^{-1}; \tag{20}$$

$$u^*(y) = ((\nabla u)^{-1}(y), y) - u((\nabla u)^{-1} y) \quad \forall\, y \in \mathbb{R}^n; \tag{21}$$

$$D^2 u^*(y) = (D^2 u((\nabla u)^{-1}(y)))^{-1} \quad \forall\, y \in \mathbb{R}^n. \tag{22}$$

Let $f = e^{-u} \in \mathcal{L}_s^n$; the functional I is defined by

$$I(f) = \int_{\mathbb{R}^n} f(x)dx.$$

By the change of variable $y = \nabla u(x)$ and by the previous relations we get

$$I(f) = \int_{\mathbb{R}^n} e^{u^*(y) - (y, \nabla u^*(y))} \det(D^2 u^*(y)) dy.$$

In other words, $I(e^{-u})$ can be expressed as an integral functional depending on u^*. Given $v \in (\mathcal{C}_s^n)^*$ set

$$J(v) = \int_{\mathbb{R}^n} e^{v(y) - (y, \nabla v(y))} \det(D^2 v(y)) dy.$$

By Remark 4.5, Prékopa-Leindler inequality in its form (9), restricted to $\mathcal{L}_s^n$, is equivalent to say that

$$J \,:\, (\mathcal{C}_s^n)^* \to \mathbb{R} \quad \text{is } \textit{log-concave} \tag{23}$$

where now log-concavity is with respect to the usual addition of functions in $(\mathcal{C}_s^n)^*$. The previous relation is the key step of the proof. We will now determine the second variation of $\ln(J)$ at $v \in (C^n)^*$. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ (i.e. $\psi \in C^\infty(\mathbb{R}^n)$ and it has compact support). There exists $\epsilon > 0$ such that

$$v_s = v + s\psi \quad \text{is convex for every } s \in [-\epsilon, \epsilon].$$

Set

$$g(s) = J(v_s).$$

Then $\ln(g(s))$ is concave in $[-\epsilon, \epsilon]$, so that

$$g(0)g''(0) - g'^2(0) \le 0. \tag{24}$$

After computing $g'(0)$ and $g''(0)$ and returning to the variable x, inequality (24) will turn out to be nothing but the Poincaré inequality of Brascamp and Lieb.

For simplicity, from now on we will restrict ourselves to the one-dimensional case, but the same computation can be done for general dimension (at the price of some additional technical difficulties, consisting in suitable integration by parts formulas), as shown in [24] for the case of the Brunn-Minkowski inequality.

So now v and ψ are functions of one real variable; we denote by v', v'', ψ', ψ'' their first and second derivatives, respectively. The function $g(s)$ takes the form

$$g(s) = \int_{\mathbb{R}} e^{v_s(y) - y v_s'(y)} v_s''(y) dy.$$

Then

$$g'(s) = \int_{\mathbb{R}} e^{v_s(y) - y v_s'(y)} [(\psi(y) - y\psi'(y)) v_s''(y) + \psi''(y)] dy.$$

Note that

$$\int_{\mathbb{R}} e^{v_s(y) - y v_s'(y)} \psi''(y) dy = \int_{\mathbb{R}} y \psi'(y) v_s''(y) dy$$

after an integration by parts (no boundary term appears as ψ has bounded support). Then

$$g'(s) = \int_{\mathbb{R}} e^{v_s(y) - y v_s'(y)} \psi(y) v_s''(y) dy.$$

Differentiating again (this time at $s = 0$) we get

$$\begin{aligned} g''(0) &= \int_{\mathbb{R}} e^{v(y) - y v'(y)} \psi(y) [(\psi(y) - y\psi'(y)) v''(y) + \psi''(y)] dy \\ &= \int_{\mathbb{R}} e^{v(y) - y v'(y)} [\psi(y) v''(y) - (\psi'(y))^2] dy, \end{aligned}$$

where we have integrated by parts again in the second equality. Now set

$$\phi(x) = \psi(u'(x)).$$

Note that $\phi \in C_c^\infty(\mathbb{R})$; moreover, any $\phi \in C_c^\infty(\mathbb{R})$ can be written in the previous form for a suitable ψ. We have:

$$g'(0) = \int_{\mathbb{R}} \phi(x) f(x) dx,$$

and

$$g''(0) = \int_{\mathbb{R}} \phi^2(x) f(x) - \int_{\mathbb{R}} \frac{(\phi'(x))^2}{u''(x)} dx. \tag{25}$$

Hence (18) is equivalent to $g'(0) = 0$. If we now replace (25) in (24) we obtain the desired inequality.

□

Remark 5.9. There are several other examples of the argument used to derive "differential" inequalities (i.e. involving the gradient, or derivatives in general) like Poincaré, Sobolev and log-Sobolev inequalities, starting from Prékopa-Leindler or Brunn-Minkowski inequality; see for instance: [11], [12], and the more recent paper [13].

5.5. **Functional Blaschke-Santaló inequality and its converse.** One of the most fascinating open problems in convex geometry is the Mahler conjecture, concerning the optimal lower bound for the so-called *volume product* of a convex body. If $K \in \mathcal{K}^n$ and the origin is an interior point of K, the polar body (with respect to 0) of K is the set

$$K^\circ = \{x \in \mathbb{R}^n \, : \, (x, y) \leq 1 \, \forall \, y \in K\}.$$

K° is also a convex body. More generally, if K has non-empty interior and z is an interior point of K the polar body of K with respect to z is

$$K^z := (K - z)^\circ.$$

It can be proved that there exists an interior point of K, the *Santaló point*, for which $V_n(K^z)$ is minimum (see [56]).

Roughly speaking, the polar body of a large set is small and vice versa; this suggests to consider the following quantity:

$$\mathcal{P}(K) = V_n(K) V_n(K^z),$$

where z is the Santaló point of K, called the volume product of K. $\mathcal{P}$ is invariant under affine transformations of $\mathbb{R}^n$ and in particular it does not change if K is dilated (or shrunk). It is relatively easy to see that it admits a maximum and a minimum as K ranges in $\mathcal{K}^n$. Then it becomes interesting to find such extremal values and the corresponding extremizers.

The Blashcke-Santaló inequality asserts that

$$\mathcal{P}(K) \leq \mathcal{P}(B_n) \quad \forall K \in \mathcal{K}^n,$$

(we recall that B_n is the unit ball) and equality holds if and only if K is an ellipsoid (see for instance [56]). On the other hand, the problem of finding the minimum of $\mathcal{P}$ is still open, in dimension $n \geq 3$. The Mahler conjecture asserts that

$$\mathcal{P}(K) \geq \mathcal{P}(\Delta) \quad \forall \, K \in \mathcal{K}^n$$

where Δ is a simplex. Correspondingly, in the case of symmetric convex bodies it is conjectured that

$$\mathcal{P}(K) \geq \mathcal{P}(Q) \quad \forall K \in \mathcal{K}^n, \text{ symmetric}$$

where Q is a cube. The validity of these conjectures has been established in the plane by Mahler himself, and, in higher dimension, for some special classes of convex bodies; among them we mention zonoids and unconditional convex bodies. Anyway it would be impossible to give even a synthetic account of all the contributions and results that appeared in the last decades in this area. A recent and updated account can be found in [55]. We mention, as this result has a specific counterpart for log-concave functions, that the best known lower bound for the volume product of symmetric convex bodies (asymptotically optimal with respect n as n tends to ∞), has been established by Bourgain and Milman (see [16]):

$$\mathcal{P}(K) \geq c^n \, \mathcal{P}(Q), \quad \forall \, K \in \mathcal{K}^n, \text{ symmetric,} \tag{26}$$

where c is a constant independent of n. For a recent improvement of the constant c as well as for different proofs of (26), we again refer the reader to [55] (see in particular Section 8).

Within the framework that we have been describing so far, where results from convex geometry are systematically transferred to the space of log-concave functions, it is natural to expect a functional counterpart of the volume product of convex bodies, and related upper and lower bounds. Given a log-concave function $f = e^{-u} \in \mathcal{L}^n$, we have seen that we can define

$$f^\circ = e^{-u^*}$$

where u^* is the Fenchel conjugate of u (see Section 4). Hence we are led to introduce the following quantity

$$\mathcal{P}(f) := \int_{\mathbb{R}^n} f dx \int_{\mathbb{R}^n} f^\circ dx = I(f)\, I(f^\circ)$$

as a counterpart of the volume product of a convex body. On the other hand, as suggested by the case of convex bodies, it could be important to introduce also a parameter $z \in \mathbb{R}^n$, as the center of polarity. Hence, given $f \in \mathcal{L}^n$ and $z \in \mathbb{R}^n$, we set

$$f_z(x) = f(x - z) \quad \forall \, x \in \mathbb{R}^n,$$

and more generally we consider

$$\mathcal{P}(f_z) := \int_{\mathbb{R}^n} f_z dx \int_{\mathbb{R}^n} (f_z)^\circ dx.$$

The functional Blaschke-Santaló inequality, i.e. an optimal upper bound for $\mathcal{P}(f_z)$, was established in [3] where the authors prove that for every $f \in \mathcal{L}^n$ (with positive integral), if we set

$$z_0 = \frac{1}{I(f)} \int_{\mathbb{R}^n} x f(x) dx$$

then

$$\mathcal{P}(f_{z_0}) \le (2\pi)^n \tag{27}$$

and equality holds if and only if f is (up to a translation of the coordinate system) a Gaussian function, i.e. is of the form

$$f(x) = e^{-(Ax,x)}$$

where A is a positive definite matrix. In the special case of even functions, for which we have $z_0 = 0$, this result was achieved by Ball in [8]. A different proof (which in particular does not exploit its geometric counterpart) of the result by Artstein, Klartag and Milman was given by Lehec in [41]. We also mention that an interesting extension of (27) was given in [30] (see also [42]).

In a similar way, the reverse Blaschke-Santaló inequality (26) have been extended to the functional case. In [39] the authors proved that there exists an absolute constant $c > 0$ (i.e. c does not depend on the dimension n) such that

$$\mathcal{P}(f_0) \ge c^n$$

for every $f \in \mathcal{L}^n$ even. This result has been improved in various ways in the papers [32] and [33]. We also mention that in [31] a sharp lower bound for the functional $\mathcal{P}(f)$ have been given for *unconditional* log-concave functions f (i.e. even with respect to each coordinate). This corresponds to the solution of the Mahler conjecture in the case of unconditional convex bodies.

5.6. **Functional Rogers-Shephard inequality.** Given a convex body K in $\mathbb{R}^n$, its *difference body* DK is defined by

$$DK = K + (-K) = \{x + y \,:\, x \in K,\, -y \in K\}.$$

DK is a centrally symmetric convex body, and, in a sense, any measurement of how far is K from DK could serve as a measure of asymmetry of K. The discrepancy between K and DK can be identified via the volume ratio:

$$\frac{V_n(K)}{V_n(DK)}.$$

If we apply the Brunn-Minkowski inequality to K and $-K$ we immediately get

$$V(DK) \ge 2^n V_n(K).$$

The celebrated Rogers-Shephard inequality (see [51]) provides a corresponding upper bound:

$$V_n(DK) \le \binom{2n}{n} V_n(K). \tag{28}$$

Equality holds in the previous inequality if and only if K is a simplex.

It is natural to wonder whether these facts may find any correspondence for log-concave functions. This question was studied in [23]. The first step is to define a notion of difference function of a log-concave function. Let $f \in \mathcal{L}^n$; we first set

$$\bar{f}(x) = f(-x) \quad \forall\, x \in \mathbb{R}^n$$

(clearly $\bar{f} \in \mathcal{L}^n$). Then we define

$$\Delta f = \frac{1}{2} \cdot f \oplus \frac{1}{2} \cdot \bar{f}.$$

In more explicit terms:

$$\Delta f(x) = \sup \left\{ \sqrt{f(y) f(-z)} \,:\, x = \frac{y+z}{2} \right\}$$

(in fact Δf corresponds to the difference body rescaled by the factor $\frac{1}{2}$).

To get a lower bound for the integral of the difference function we may use the Prékopa-Leindler inequality and obtain:

$$I(\Delta f) = \int_{\mathbb{R}^n} \Delta f dx \geq \int_{\mathbb{R}^n} f dx = I(f).$$

In [23] the following inequality was proved:

$$\int_{\mathbb{R}^n} \Delta f dx \leq 2^n \int_{\mathbb{R}^n} f dx \quad \forall\, f \in \mathcal{L}^n. \tag{29}$$

The previous inequality is sharp. One extremizer is the function f defined by

$$f(x) = f(x_1, \dots, x_n) = \begin{cases} e^{-\sum_{i=1}^n x_i} & \text{if } x_i \geq 0 \text{ for every } i = 1 \dots, n, \\ 0 & \text{otherwise.} \end{cases}$$

All other extremizers can be obtained by the previous function by an affine change of variable and the multiplication by a positive constant (see [23]).

The results of [23] have been recently extended and complemented in the papers [1] and [2], where the authors obtain considerable new developments. To describe an example of their results, given f and g in $\mathcal{L}^n$ one may consider

$$\Delta(f, g) = \frac{1}{2} \cdot f \oplus \frac{1}{2} \cdot \bar{g}.$$

In the above mentioned papers, among other results the authors establish optimal upper bounds for the integral of $\Delta(f, g)$, which in the case $f = g$ returns the inequality (29).

5.7. **The functional affine isoperimetric inequality.** We conclude this section by mentioning yet another inequality for log-concave functions. As we recalled in the introduction, among the main results that can be proved for log-concave probability measures there are log-Sobolev type inequalities (we refer the reader to [40] for this type of inequalities). In the paper [4] the authors prove a reverse form of the standard log-Sobolev inequality (in the case of the Lebesgue measure). The proof of this inequality is based on an important geometric inequality in convex geometry; the *affine isoperimetric inequality*, involving the affine surface area. We refer the reader to [56] for this notion.

The research started in [4] is continued in the papers [19], [20], [21] and [22]. In particular, in these papers several possible functional extensions of the notion of affine surface area are proposed, along with functional versions of the affine isoperimetric inequality.

6. AREA MEASURES AND MIXED VOLUMES

6.1. **The first variation of the total mass functional.** Given two convex bodies K and L, for $\epsilon > 0$ consider the following perturbation of K: $K_\epsilon := K + \epsilon L$. The volume of K_ϵ, as a function of ϵ, is a polynomial and hence admits right derivative at $\epsilon = 0$:

$$\lim_{\epsilon\to 0^+} \frac{V_n(K+\epsilon L) - V_n(K)}{\epsilon} =: V(\underbrace{K,\dots,K}_{(n-1)\text{-times}}, L) = V(K,\dots,K,L). \tag{30}$$

Here we used the standard notations for *mixed volumes* of convex bodies (see the appendix). The mixed volumes $V(K,\dots,K,L)$, when K is fixed and L ranges in $\mathcal{K}^n$, can be computed using the *area measure* of K. Indeed, there exists a unique non-negative Radon measure on $\mathbb{S}^{n-1}$, called the area measure of K and denoted by $S_{n-1}(K,\cdot)$, such that

$$V(K,\dots,K,L) = \frac{1}{n}\int_{\mathbb{S}^{n-1}} h_L(x)dS_{n-1}(K,x) \quad \forall\, L \in \mathcal{K}^n. \tag{31}$$

According to (30) we may say that $V(K,\dots,K,L)$ is the directional derivative of the volume functional at K along the direction L. Moreover, as support function behaves linearly with respect to Minkowski addition (see the appendix), (31) tells us that the first variation of the volume at K is precisely the area measure of K. Note also that if we choose L to be the unit ball B_n of $\mathbb{R}^n$, then we have (under the assumption that K has non-empty interior) that the derivative in (30) is the perimeter of K:

$$V(K,\dots,K,B_n) = \lim_{\epsilon\to 0^+} \frac{V_n(K+\epsilon L) - V_n(K)}{\epsilon} = \mathcal{H}^{n-1}(\partial K)$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure.

One could try to follow a similar path to define a notion of area measure of a log-concave function f, replacing the volume functional by the integral of $f \in \mathcal{L}^n$

$$I(f) = \int_{\mathbb{R}^n} f(x)dx.$$

Then the idea is to compute the first variation of I and deduce as a consequence a surrogate of the area measure. More precisely, in view of (30) and (31), the problem of computing the following limit arises:

$$\delta I(f,g) := \lim_{\epsilon\to 0^+} \frac{I(f\oplus\epsilon\cdot g)-I(f)}{\epsilon} \tag{32}$$

where $f,g\in\mathcal{L}^n$. Here a first striking difference between the geometric and the functional setting appears. While the volume of the linear combination of convex bodies is always polynomial in the coefficients, this is not the case for functions. Indeed (see for instance [25]) there are examples in which $\delta I(f,g)=\infty$.

The idea to compute the limit (32) appeared for the first time in the papers [39], [53] and [54], for a specific choice of the function f (the density of the Gaussian measure), in order to define a notion of mean width (one of the intrinsic volumes) of log-concave functions. The computation of the same limit for general f and g was then considered in [25].

Even if the limit (32) exists (finite of infinite) under the sole assumption $I(f)>0$ (see [25] and [39]), explicit formulas for it (e.g. similar to (31)) have been found only under quite restrictive assumptions. To give an example of such formulas we rephrase Theorem 4.5 in [25]. This result needs some preparation. First of all we denote by $C^2_+(\mathbb{R}^n)$ the set of functions u from $C^2(\mathbb{R}^n)$ such that $D^2u>0$ in $\mathbb{R}^n$. Next we define

$$\mathcal{C}^n_s = \left\{u\in\mathcal{C}^n \,:\, u<\infty \text{ in } \mathbb{R}^n, u\in C^2_+(\mathbb{R}^n),\ \lim_{|x|\to\infty}\frac{u(x)}{|x|}=+\infty\right\}$$

and

$$\mathcal{L}^n_s = e^{-\mathcal{C}^n_s} = \{e^{-u}\,:\, u\in\mathcal{C}^n_s\}\subset\mathcal{L}^n.$$

Given $f=e^{-u}$ and $g=e^{-v}\in\mathcal{L}^n_s$, we say that g is an *admissible perturbation of* f if there exists a constant $c>0$ such that

$$u^*-cv^* \quad \text{is convex in } \mathbb{R}^n.$$

This condition can be viewed as the fact the convexity of u^* *controls* that of v^*.

Theorem 6.1. *Let $f=e^{-u}$, $g=e^{-v}\in\mathcal{L}^n_s$ and assume that g is an admissible perturbation of f. Then $\delta I(f,g)$ exists, is finite and is given by*

$$\delta I(f,g) = \int_{\mathbb{R}^n} v^*(\nabla u(x))\, f(x)\, dx. \tag{33}$$

Using a different point of view, we may consider the measure $\tilde\mu_f$ on $\mathbb{R}^n$, with density f with respect to the Lebesgue measure. Then we define μ_f as the push-forward of $\tilde\mu$ through the gradient map ∇u. At this regard note that, as $f=e^{-u}\in\mathcal{L}^n_s$, ∇u is a diffeomorphism between $\mathbb{R}^n$ and itself. Then (33) is equivalent to

$$\delta I(f,g) = \int_{\mathbb{R}^n} v^*(y)\, d\mu_f(y) = \int_{\mathbb{R}^n} (-\ln(g))^*(y)\, d\mu_f(y). \tag{34}$$

Roughly speaking, as the linear structure on $\mathcal{L}^n$ is the usual addition and multiplication by scalars, transferred to the conjugates of the exponents (with minus sign), (34) says that the measure μ_f is the first variation of the functional I at the function f; for this reason this measure could be interpreted as the area measure of f. Note that this fact can not be considered to be too general: if we change the assumptions on f (i.e. the fact that $f \in \mathcal{L}^n_s$) then the expression of $\delta I(f, g)$ may change significantly (see for instance Theorem 4.6 in [25]) .

It is interesting to note that the measure μ_f was studied also by Cordero-Erausquin and Klartag in [27], with a different perspective.

6.2. **Mixed volumes of log-concave functions.** As we saw in the previous section, if we endow $\mathcal{L}^n$ with the addition defined in Section 3.2, the total mass functional of linear combinations of log-concave functions is in general not a polynomial in the coefficients. This is a clear indication that, within the frame of this linear structure, it is not possible to define mixed volumes of generic log-concave functions. On the other hand, there exists a choice of the operations on $\mathcal{L}^n$ which permits to define mixed volumes. These facts were established mainly in the papers [47] and [48] (see also [10] for related results), and here we briefly describe the main points of this construction.

As we said, we have to abandon for a moment the addition previously defined on $\mathcal{L}^n$ and introduce a new one. Given $f, g \in \mathcal{L}^n$ we set

$$(f \tilde{+} g)(z) = \sup\{\min\{f(x), g(y)\} \,:\, x + y = z\}. \tag{35}$$

This apparently intricate definition has in fact a simple geometric interpretation:

$$\{z \in \mathbb{R}^n \,:\, (f \tilde{+} g)(z) \geq t\} = \{x \in \mathbb{R}^n \,:\, f(x) \geq t\} + \{y \in \mathbb{R}^n \,:\, g(y) \geq t\}$$

for every $t > 0$ such that each of the two sets on the right hand-side is non-empty. In other words, the super-level sets of $f \tilde{+} g$ are the Minkowski addition of the corresponding super-level sets of f and g.

The addition (35) preserves log-concavity (see, for instance, [47]), and then it is an internal operation of $\mathcal{L}^n$ (but it is in fact also natural for quasi-concave functions; see [10], [47]).

A notion of multiplication by non-negative scalars is naturally associated to the previous addition: for $f \in \mathcal{L}^n$ and $\lambda > 0$ we define $\lambda \tilde{\cdot} f$ by

$$(\lambda \tilde{\cdot} f)(x) = f\left(\frac{x}{\lambda}\right).$$

In this new frame, the functional I evaluated at linear combinations of log-concave functions admits a polynomial expansion. More precisely, the following theorem, proved in [47], provides the definition of mixed volumes of log-concave functions.

Theorem 6.2. *There exists a function* $V : (\mathcal{L}^n)^n \to \mathbb{R}$ *such that, for every* $m \in \mathbb{N}$, $f_1, \dots, f_m \in \mathcal{L}^n$ *and* $\lambda_1, \dots, \lambda_m > 0$,

$$I(\lambda_1 \tilde{\cdot} f_1 \tilde{+} \dots \tilde{+} \lambda_m \tilde{\cdot} f_m) = \sum_{i_1, \dots, i_n = 1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(f_{i_1}, \dots, f_{i_n}).$$

In [47] the authors prove several inequalities for mixed volumes of log-concave (and, more generally, quasi-concave) functions, including versions of the Bunn-Minkowski and Alexandrov-Fenchel inequalities.

As in the case of convex bodies, several interesting special cases of mixed volumes can be enucleated. For instance, if we fix $f \in \mathcal{L}^n$ and consider, for $i \in \{0, \dots, n\}$,

$$V_i(f) := V(\underbrace{f, \dots, f}_{i\text{-times}}, \underbrace{I_{B_n}, \dots, I_{B_n}}_{(n-i)\text{-times}}),$$

we have a notion which can be regarded as the i-intrinsic volume of f. These quantities have been studied in [10] and [47].

7. VALUATIONS ON $\mathcal{L}^n$

We start by valuations on convex bodies. A (real-valued) valuation on $\mathcal{K}^n$ is a mapping $\sigma : \mathcal{K}^n \to \mathbb{R}$ such that

$$\sigma(K \cup L) + \sigma(K \cap L) = \sigma(K) + \sigma(L) \quad \forall K, L \in \mathcal{K}^n \text{ s.t. } K \cup L \in \mathcal{K}^n. \tag{36}$$

The previous relation establishes a finite additivity property of σ. A typical example of valuation is the volume (i.e. the Lebesgue measure), which, as a measure, is countably additive and then fulfills (36). Another, simple, example is provided by the Euler characteristic, which is constantly 1 on $\mathcal{K}^n$ and then it obviously verifies (36). Note that both volume and Euler characteristic are also continuous with respect to Hausdorff metric, and invariant under rigid motions of $\mathbb{R}^n$. Surprisingly, there are other examples of this type; namely each intrinsic volume V_i, $i = 0, \dots, n$, (see the appendix for a brief presentation) is a rigid motion invariant and continuous valuation on $\mathcal{K}^n$.

The celebrated Hadwiger theorem (see [35], [36], [37]), asserts that, conversely, every rigid motion invariant and continuous valuation can be written as the linear combination of intrinsic volumes; in particular the vector space of such valuations has finite dimension n and $\{V_0, \dots, V_n\}$ is a basis of this space. If rigid motion invariance is replaced by weaker assumption of translation invariance, still the relevant space of valuations preserves a rather strong algebraic structure. It was proved by McMullen (see [45]) that any translation invariant and continuous valuation σ on $\mathcal{K}^n$ can be written as

$$\sigma = \sum_{i=0}^{n} \sigma_i$$

where σ_i has the same property of σ and it is i-homogeneous with respect to dilations.

The results that we have mentioned are two of the milestones in this area and stimulated a great development of the theory of valuations on convex bodies, which now counts many ramifications. The reader may find an updated survey on this subject in [56, chapter 6].

The richness of this part of convex geometry recently motivated the start of a parallel theory of valuations on spaces of functions. Coherently with the theme of this article, we restrict ourselves to valuations on $\mathcal{L}^n$; the reader may find a survey of the existing literature on this field of research in [18], [26] and [44].

A mapping $\mu : \mathcal{L}^n \to \mathbb{R}$ is called a (real-valued) valuation if

$$\mu(f \vee g) + \mu(f \wedge g) = \mu(f) + \mu(g), \quad \forall\, f, g \in \mathcal{L}^n \text{ s.t. } u \vee g \in \mathcal{L}^n,$$

where "$\vee$" and "$\wedge$" denote the point-wise maximum and minimum, respectively (note that the minimum of two functions in $\mathcal{L}^n$ is still in $\mathcal{L}^n$). In other words, sets are replaced by functions and union and intersection are replaced by maximum and minimum. One reason for this definition is that, when restricted to characteristic functions, it gives back the ordinary notion of valuation on relevant sets.

Having the picture of valuations on $\mathcal{K}^n$ in mind, it becomes interesting to consider valuations μ on $\mathcal{L}^n$ which are:

- invariant with respect to some group G of transformations of $\mathbb{R}^n$:
$$\mu(f \circ T) = \mu(f) \quad \forall\, f \in \mathcal{L}^n, \forall\, T \in G;$$
- continuous with respect to some topology τ in $\mathcal{L}^n$:
$$f_i \to f \quad \text{as } i \to \infty \text{ w.r.t. } \tau \quad \Rightarrow \quad \mu(f_i) \to \mu(f).$$

The investigation in this area is still at the beginning, and satisfactory characterizations of valuations with the previous properties are not known. At this regard we report a result which can be deduced from [18], preceded by some preparatory material.

Let μ be a valuation defined on $\mathcal{L}^n$. For G we chose the group of rigid motions of $\mathbb{R}^n$; hence we assume that μ is rigid motion invariant.

Next we want to define a continuity property for μ. Note that, while in $\mathcal{K}^n$ the choice of the topology induced by the Hausdforff metric is natural and effective, the situation in $\mathcal{L}^n$ is rather different. For a discussion on this topic we refer the reader to [26, section 4.1]. Here we consider the following notion of continuous valuation on $\mathcal{L}^n$. A sequence f_i, $i \in \mathbb{N}$, contained in $\mathcal{L}^n$, is said to converge to $f \in \mathcal{L}^n$ if:

- f_i is increasing with respect to i;
- $f_i \le f$ in $\mathbb{R}^n$ for every i;
- f_i converges to f point-wise in the *relative interior* of the support of f (see the definition in the appendix).

Given this definition, we say that μ is continuous if

$$\lim_{i\to\infty} \mu(f_i) = \mu(f),$$

whenever a sequence f_i converges to f in the way specified above. To be able to characterize μ we need two additional properties: μ is increasing, i.e.

$$f_1 \le f_2 \quad \text{in } \mathbb{R}^n \quad \Rightarrow \quad \mu(f_1) \le \mu(f_2);$$

and μ is simple, i.e.

$$f \equiv 0 \quad \text{a.e. in } \mathbb{R}^n \quad \Rightarrow \quad \mu(f) = 0.$$

Theorem 7.1. *μ is a rigid motion invariant, continuous, increasing and simple valuation on $\mathcal{L}^n$ if and only if there exists a function $F : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\mu(f) = \int_{\mathbb{R}^n} F(f(x))dx, \tag{37}$$

and, moreover, F is continuous, increasing, vanishes at 0 and verifies the following integrability condition:

$$\int_0^1 \frac{(-\ln(t))^{n-1}}{t} F(t)dt < +\infty. \tag{38}$$

The proof is a direct application of the results proved in [18] for valuations on the space of convex functions $\mathcal{C}^n$. Indeed, we set $\bar{\mu} : \mathcal{C}^n \to \mathbb{R}$ defined by

$$\bar{\mu}(u) = \mu(e^{-u}) \quad \forall\, u \in \mathcal{C}^n$$

$\bar{\mu}$ inherits the features of μ. The valuation property follows immediately from the monotonicity of the exponential function. Rigid motion invariance and monotonicity are straightforward (note that $\bar{\mu}$ is decreasing). As for continuity, the reader may check that the convergence that we have introduced in $\mathcal{L}^n$ induces precisely the one defined in [18]. The property of being simple for μ implies that $\bar{\mu}(u) = 0$ for every $u \in \mathcal{C}^n$ such that $u \equiv \infty$ a.e. in $\mathbb{R}^n$. Hence we may apply Theorem 1.3 in [18], and deduce the integral representation (37). The integrability condition (38) follows from (1.5) in [18].

Other type of valuations on $\mathcal{L}^n$ can be generated by taking wighed means of intrinsic volumes of super-level sets. More precisely, let $f \in \mathcal{L}^n$. For every $t > 0$ the set

$$L_f(t) = \text{cl}(\{x \in \mathbb{R}^n : f(x) \ge t\})$$

(where "cl" denotes the closure) is (either empty or) a compact convex set, i.e. a convex body, by the properties of f. Note that, for every $f, g \in \mathcal{L}^n$,

$$L_{f\vee g} = L_f(t) \cap L_g(t), \quad L_{f\wedge g} = L_f(t) \cup L_g(t).$$

Using these relations and the valuation property of intrinsic volumes (see (41)) we easily get, for an arbitrary $i \in \{0, \dots, n\}$,

$$V_i(L_{f\vee g}(t)) + V_i(L_{f\wedge g}(t)) = V_i(L_f(t)) + V_i(L_g(t)).$$

In other words, the map $\mathcal{L}^n \to \mathbb{R}$:

$$f \longrightarrow V_i(L_f(t))$$

is a valuation on $\mathcal{L}^n$. More generally we may multiply this function by a non-negative number depending on t and sum over different values of t (keeping i fixed). The result will be again a valuation. The most general way to do it is to consider a continuous sum, that is an integral. In other words, we may take the application:

$$\mu(f) = \int_0^\infty V_i(L_f(t))d\nu(t) \tag{39}$$

where ν is a Radon measure. These type of valuations have been considered in [18] for convex functions. In particular, it follows from condition (1.11) in [18] that $\mu(f)$ is finite for every $f \in \mathcal{L}^n$ if and only if ν verifies the integrability condition

$$\int_0^1 \frac{(-\ln(t))^i}{t} d\nu(t) < +\infty. \tag{40}$$

Moreover, μ is homogeneous with respect to dilations of $\mathbb{R}^n$. More precisely, given $f \in \mathcal{L}^n$ and $\lambda > 0$, define the function f_λ as

$$f_\lambda(x) = f\left(\frac{x}{\lambda}\right) \quad \forall\, x \in \mathbb{R}^n.$$

Then

$$\mu(f_\lambda) = \lambda^i \mu(f) \quad \forall\, f \in \mathcal{L}^n.$$

By theorem 1.4 in [18] and an argument similar to that used in the proof of theorem 7.1, we obtain the following result.

Theorem 7.2. *A mapping $\mu : \mathcal{L}^n \to \mathbb{R}$ is a rigid motion invariant, continuous, monotone valuation, which is in addition homogeneous of some order α, if and only if $\alpha = i \in \{0, \dots, n\}$, and μ can be written in the form* (39)*, for some measure ν verifying condition* (40).

Remark 7.3. In the case $i = n$ formulas (37) and (39) are the same via the layer cake principle.

It would be very interesting to remove part of the assumptions (e.g. monotonicity or homogeneity) in theorems 7.1 and 7.2 and deduce corresponding characterization results.

APPENDIX A. BASIC NOTIONS OF CONVEX GEOMETRY

This part of the paper contains some notions and constructions of convex geometry that are directly invoked throughout this paper. Our main reference text on the theory of convex bodies is the monograph [56].

A.1. **Convex bodies and their dimension.** We denote by $\mathcal{K}^n$ the class of convex bodies, i.e. compact convex subsets of $\mathbb{R}^n$.

Given a convex body K its dimension is the largest integer $k \in \{0, \ldots, n\}$ such that there exists a k-dimensional hyperplane of $\mathbb{R}^n$ containing K. In particular, if K has non-empty interior then its dimension is n. The *relative interior* of K is the set of points $x \in K$ such that there exists a k-dimensional ball centered at x included in K, where k is the dimension of K. If the dimension of K is n then the relative interior coincides with usual interior.

A.2. **Minkowski addition and Hausdorff metric.** The *Minkowski linear combination* of $K, L \in \mathcal{K}^n$ with coefficients $\alpha, \beta \geq 0$ is

$$\alpha K + \beta L = \{x + y \,:\, x \in K,\, y \in L\}.$$

It is easy to check that this is still a convex body.

A.3. **Support function.** The *support function* of a convex body K is defined as:

$$h_K \,:\, \mathbb{R}^n \to \mathbb{R}, \quad h_K(x) = \sup_{y \in K}(x, y).$$

This is a 1-homogeneous convex function in $\mathbb{R}^n$. Vice versa, to each 1-homogeneous convex function h we may assign a unique convex body K such that $h = h_K$. Support functions and Minkowski additions interact in a very simple way; indeed, for every K and L in $\mathcal{K}^n$ and $\alpha, \beta \geq 0$ we have

$$h_{\alpha K + \beta L} = \alpha h_K + \beta h_L.$$

A.4. **Hausdorff metric.** $\mathcal{K}^n$ can be naturally equipped with a metric: the Hausdorff metric d_H. One way to define d_H is as the $L^\infty(\mathbb{S}^{n-1})$ distance of support functions, restricted to the unit sphere:

$$d_H(K, L) = \|h_K - h_L\|_{L^\infty(\mathbb{S}^{n-1})} = \max\{|h_K(x) - h_L(x)| \,:\, x \in \mathbb{S}^{n-1}\}.$$

Hausdorff metric has many useful properties; in particular we note that $\mathcal{K}^n$ is a locally compact space with respect to d_H.

A.5. **Intrinsic volumes.** An easy way to define intrinsic volumes of convex bodies is through the Steiner formula. Let K be a convex body and let B_n denote the closed unit ball of $\mathbb{R}^n$. For $\epsilon > 0$ the set

$$K + \epsilon B_n = \{x + \epsilon y \,:\, x \in K,\, y \in B\} = \{y \in \mathbb{R}^n \,:\, \mathrm{dist}(x, K) \leq \epsilon\}$$

is called the parallel set of K and denoted by K_ϵ. The Steiner formula asserts that the volume of K_ϵ is a polynomial in ϵ. The coefficients of this polynomial are, up to dimensional constants, the intrinsic volumes $V_0(K), \ldots, V_n(K)$ of K:

$$V_n(K_\epsilon) = \sum_{i=0}^{n} V_i(K)\epsilon^{n-i}\kappa_{n-i}.$$

Here κ_j denotes the j-dimensional volume of the unit ball in $\mathbb{R}^j$, for every $j \in \mathbb{N}$. Among the very basic properties of intrinsic volumes, we mention that: V_0 is constantly 1 for every K; V_n is the volume; V_{n-1} is $(n-1)$-dimensional Hausdorff measure of the boundary (only for those bodies with non-empty interior). Moreover, intrinsic volumes are continuous with respect to Hausdorff metric, rigid motion invariant, monotone, and homogeneous with respect to dilations (V_i is i-homogeneous). Finally, each intrinsic volume is a valuation

$$V_i(K \cup L) + V_i(K \cap L) = V_i(K) + V_i(L) \tag{41}$$

for every K and L in $\mathcal{K}^n$, such that $K \cup L \in \mathcal{K}^n$. Hadwiger's theorem claims that every rigid motion invariant and continuous valuation can be written as the linear combination of intrinsic volumes.

A.6. **Mixed volumes.** The Steiner formula is just an example of the polynomiality of the volume of linear combinations of convex bodies. A more general version of it leads to the notions of mixed volumes. Let $m \in \mathbb{N}$ and $K_1, \dots, K_m$ be convex bodies; given $\lambda_1, \dots, \lambda_m \geq 0$, the volume of the convex body $\lambda_1 K_1 + \cdots + \lambda_m K_m$ is a homogeneous polynomial of degree n in the variables λ_i's, and its coefficients are the mixed volumes of the involved bodies. The following more precise statement is a part of Theorem 5.16 in [56]. There exists a function $V : (\mathcal{K}^n)^n \to \mathbb{R}_+$, the *mixed volume*, such that

$$V_n(\lambda_1 K_1 + \cdots + \lambda_m K_m) = \sum_{i_1,\dots,i_n=1}^{m} \lambda_{i_1} \cdots \lambda_{i_n} V(K_{i_1}, \dots, K_{i_n})$$

for every $K_1, \dots, K_m \in K^n$ and $\lambda_1, \dots, \lambda_m \geq 0$. Hence a mixed volume is a function of n convex bodies. Mixed volumes have a number of interesting properties. In particular they are non-negative, symmetric and continuous; moreover they are linear and monotone with respect to each entry.

A.7. **The polar body.** The *polar* of a convex body K, having the origin as an interior point, is the set

$$K^\circ = \{y \,:\, (x,y) \leq 1 \quad \forall x \in K\}.$$

This is again a convex body, with the origin in its interior, and $(K^\circ)^\circ = K$.

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