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Size estimates of unknown boundaries with a Robin-type condition

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We deal with the problem of determining an unknown part of the boundary of an electrical conductor that is inaccessible for external observation and where a corrosion process is going on. We obtain estimates of the size of this damaged region from above and below.

Keywords: inverse problems; size estimates; boundary determination

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1. Introduction

We consider an electrical conductor Ω whose boundary is not fully observable, and denote by Γ the portion of $\partial\Omega$ where it is possible to make measurements. The aim of this paper is to extract information on an unknown subset E contained in $\partial\Omega \setminus \Gamma$, where a corrosion process is going on, by performing boundary measurements on Γ. These problems arise in non-destructive testing of materials and modelling phenomena of surface corrosion in metals (see [16, 23]).

Prescribing a current density g supported on Γ such that $g = 0$ on $\partial\Omega \setminus \Gamma$, we induce a potential u solution to the problem

$$\left.\begin{aligned} \Delta u &= 0 \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \gamma u &= g \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.1}$$

where γ denotes the surface impedance in the form

$$\gamma(x) = \gamma_0(x)\chi_\Gamma + k\chi_E \quad \text{for any } x \in \partial\Omega, \tag{1.2}$$

where k is a constant whose value is unknown and $\gamma_0 \equiv 0$ in $\partial\Omega \setminus \Gamma$. The case in which k is replaced by a variable function can be treated similarly with minor adjustments, while on the remaining portion of the boundary $\partial\Omega \setminus E$ the impedance term γ is fully known.

Our goal is to bound the measure of E by comparing the solution u on the boundary with the solution u_0 of the 'unperturbed' problem

$$\left.\begin{aligned} \Delta u_0 &= 0 \quad \text{in } \Omega, \\ \frac{\partial u_0}{\partial \nu} + \gamma_0 u_0 &= g \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.3}$$

where $E = \emptyset$, i.e. is, in principle, completely known. Note that $\partial u_0/\partial\nu$ vanishes outside Γ.

Specifically, using similar arguments to those developed in the context of the inverse inclusion problem (see [5] and the references therein), we deduce information on the size of E by analysing the so-called power gap, defined as

$$W - W_0 = \int_{\partial\Omega} gu \,\mathrm{d}\sigma - \int_{\partial\Omega} gu_0 \,\mathrm{d}\sigma = \int_\Gamma g(u - u_0)\,\mathrm{d}\sigma.$$

Note that the quantities W and W_0 can be computed from the boundary data that we measure and are meaningful from a physical viewpoint as they represent the power required to maintain the boundary current g.

The idea of bounding the size of an unknown object D enclosed in a given domain Ω goes back to Friedman [13]. The key point is to extract as much information as possible from the boundary measurements available. More precisely, the approach we follow is that proposed by Alessandrini and Rosset [3] and Kang *et al*. [15] and subsequently refined by Alessandrini *et al*. [4].

The basic aim is to gain information on the hidden boundary by studying the power gap, which is sensitive to the presence of the defect. In particular, since such a power gap contains information at the accessible boundary, it is possible to extend this to the inaccessible part of the boundary in a quantitative manner and thus obtain information on its size. This procedure follows the lines of similar problems studied in [3, 15] and later developed in [5, 8, 10–12, 19, 20]. The main novelty of this paper relies on the evaluation of a defect located on the boundary. Such a new feature requires an original approach to relate the power gap and the size of the defect. In order to overcome such a difficulty we find it convenient to analyse the problem in an abstract Hilbert setting (see § 3). Due to its general character, this argument can be applied to inverse problems in other practical contexts. The main technical arguments are based on the use of the three-spheres inequality and the doubling inequality at the boundary as unique continuation tools that allow us to extract information on the unknown defect from the interior and the boundary values of the solution. Another issue that arises in dealing with boundary defects concerns the use of quantitative estimates. With the introduction of a suitable norm (see remark 2.3) and quantitative estimates of unique continuation (see proposition 4.2), it is possible to obtain the desired bounds on the size of the corroded part.

The plan of the paper is the following. In § 2 we define our notation and state the main theorem. In § 3 we present an abstract formulation of our problem that will be applied in § 4 to prove our main result.

2. Assumptions and main result

For a given vector $x = (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$, we write $x = (x', x_n)$, where $x' = (x_1, \ldots, x_{n-1})$. Moreover, we denote by $B_r(x)$ and $B'_r(x)$ the open balls of radius r centred at x and x' in $\mathbb{R}^n, \mathbb{R}^{n-1}$, respectively.

Definition 2.1. Let Ω be a bounded domain in $\mathbb{R}^n$. Given k, α with $k \in \mathbb{N}$, $0 < \alpha \leqslant 1$, we say that a portion S of $\partial\Omega$ is of class $C^{k,\alpha}$ with constants r_0, M if, for any $P \in S$, there exists a rigid transformation of coordinates under which we have $P = 0$ and

$$\Omega \cap B_{r_0}(0) = \{x \in B_{r_0}(0) \colon x_n > \psi(x')\},$$

where ψ is a $C^{k,\alpha}$ function on $B'_{r_0}(0)$ satisfying

$$\begin{aligned} \psi(0) &= 0, \\ \nabla\psi(0) &= 0 \quad \text{when } k \geqslant 1, \\ \|\psi\|_{C^{k,\alpha}(B'_{r_0}(0))} &\leqslant M r_0. \end{aligned}$$

When $k = 0$ and $\alpha = 1$, we also say that S is of Lipschitz class with constants r_0 and M.

Remark 2.2. We have chosen to normalize all norms in such a way that their terms are dimensionally homogeneous and coincide with the standard definition as the dimensional parameter equals 1. For instance, the meaning of the norm appearing in the previous definition is as follows:

$$\|\psi\|_{C^{k,\alpha}(B'_{r_0}(0))} = \sum_{i=0}^{k} r_0^i \|D^i\psi\|_{L^\infty(B'_{r_0}(0))} + r_0^{k+\alpha} |D^k\psi|_{\alpha, B'_{r_0}(0)},$$

where

$$|D^k\psi|_{\alpha, B'_{r_0}(0)} = \sup_{x', y' \in B'_{r_0},\ x' \neq y'} \frac{|D^k\psi(x') - D^k\psi(y')|}{|x' - y'|^\alpha}.$$

Similarly, we shall set

$$\|u\|_{L^2(\Omega)} = r_0^{-n/2} \left(\int_\Omega u^2 \right)^{1/2}, \tag{2.1}$$

$$\|u\|_{H^1(\Omega)} = r_0^{-n/2} \left(\int_\Omega u^2 + r_0^2 \int_\Omega |\nabla u|^2 \right)^{1/2}. \tag{2.2}$$

Let $\langle \cdot, \cdot \rangle_{H^{-1/2}, H^{1/2}}$ denote the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$ based on the L^2 scalar product. Given the open and connected portion Γ of $\partial\Omega$, we introduce the trace space $H^{1/2}_{00}(\Gamma)$ as the interpolation space $[H^1_0(\Gamma), L^2(\Gamma)]_{1/2}$ (see [17, ch. 1]). Let us now consider the following space of distributions:

$$H^{-1/2}(\Gamma) = \{\eta \in H^{-1/2}(\partial\Omega) \mid \langle \eta, \varphi \rangle = 0 \ \forall \varphi \in H^{1/2}_{00}(\partial\Omega \setminus \bar{\Gamma})\}.$$

2.1. Assumptions on the domain Ω

Given constants $r_0, M > 0$, we assume that $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$, and

$$\Omega \text{ is of Lipschitz class with constants } r_0, M. \tag{2.3}$$

Furthermore, given $L > 0$, we assume that

$$|\partial\Omega| \leqslant L r_0^{n-1}. \tag{2.4}$$

In addition, we assume that the portion of the boundary

$$\partial\Omega \setminus \Gamma \text{ is of class } C^{1,1} \text{ with constants } r_0, M. \tag{2.5}$$

2.2. Assumptions on the surface impedance γ

Given an open and connected subset E of $\partial\Gamma \setminus \bar{\Gamma}$ and an open and connected subset Γ_0 of Γ, we assume that

$$\gamma \in L^\infty(\partial\Omega). \tag{2.6}$$

Moreover, for a given constant c_0, $0 < c_0 \leqslant 1$, we have that

$$\gamma(x) \geqslant \frac{c_0}{r_0} > 0 \quad \text{on } \Gamma_0. \tag{2.7}$$

Finally, for a given function $\gamma_0(x) \in L^\infty(\partial\Omega)$ supported on Γ and such that

$$\gamma_0(x) \leqslant c_0^{-1}/r_0, \tag{2.8}$$

we have that

$$\gamma(x) = \gamma_0(x)\chi_\Gamma + k\chi_E, \tag{2.9}$$

where $k > 0$ is an unknown constant such that

$$0 < \bar{k}_0 < kr_0 < \bar{k}_1 \tag{2.10}$$

for given constants $\bar{k}_0$ and $\bar{k}_1$.

Here and in the following we shall set

$$\gamma(x) = \frac{\bar{\gamma}(x)}{r_0}, \tag{2.11}$$

$$\gamma_0(x) = \frac{\bar{\gamma}_0(x)}{r_0}, \tag{2.12}$$

$$k = \frac{\bar{k}}{r_0}. \tag{2.13}$$

2.3. Assumptions on the given data *g*

Given $g_0 > 0$ we assume that

$$\|g\|_{H^{-1/2}(\Gamma)} \leqslant g_0. \tag{2.14}$$

Furthermore, given $F > 0$ we assume that

$$\frac{\|g\|_{H^{-1/2}(\Gamma)}}{\|g\|_{H^{-1}(\Gamma)}} \leqslant F. \tag{2.15}$$

This ratio (called frequency) takes into account the oscillatory character of the boundary data. Other choices of norm are possible and we refer the reader to [5] for a discussion on this topic.

REMARK 2.3. We first observe that the standard norm in $H^1(\Omega)$ and the norm

$$\|u\|_* = r_0^{-n/2}\biggl(r_0^2\int_\Omega |\nabla u|^2\,\mathrm{d}x + r_0\int_{\Gamma_0} u^2\,\mathrm{d}\sigma\biggr)^{1/2},$$

are equivalent.

Indeed, we note that, on the one hand, by the standard trace estimate we have

$$\|u\|_{H^{1/2}(\Gamma_0)} \leqslant C\|u\|_{H^1(\Omega)}, \tag{2.16}$$

where $C > 0$ is a constant depending only on L and M. The above inequality leads to

$$r_0^2\int_\Omega |\nabla u|^2\,\mathrm{d}x + r_0\int_{\Gamma_0} u^2\,\mathrm{d}\sigma \leqslant C\biggl(r_0^2\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x\biggr),$$

where $C > 0$ is a constant depending only on L and M.

On the other hand, by the argument in [6, example 3.6], we deduce that

$$r_0^2\int_\Omega |\nabla u|^2\,\mathrm{d}x + \int_\Omega u^2\,\mathrm{d}x \leqslant C\biggl(r_0^2\int_\Omega |\nabla u|^2\,\mathrm{d}x + r_0\int_{\Gamma_0} u^2\,\mathrm{d}\sigma\biggr),$$

where $C > 0$ is a constant depending only on L and M.

Again denoting by $\langle\cdot,\cdot\rangle_{H^{-1/2},H^{1/2}}$ the duality pairing between $H^{-1/2}(\partial\Omega)$ and $H^{1/2}(\partial\Omega)$, with a slight abuse of notation, we shall write

$$\langle g, f\rangle_{H^{-1/2},H^{1/2}} = \int_{\partial\Omega} gf\,\mathrm{d}\sigma$$

for any $g \in H^{-1/2}(\partial\Omega)$ and $f \in H^{1/2}(\partial\Omega)$.

REMARK 2.4. By solution to (1.1) we mean a function $u \in H^1(\Omega)$ such that

$$\int_\Omega \nabla u\cdot\nabla v\,\mathrm{d}x + \int_{\partial\Omega}\gamma(x)uv\,\mathrm{d}\sigma = \int_{\partial\Omega} gv\,\mathrm{d}\sigma \quad \forall v \in H^1(\Omega).$$

As a consequence of remark 2.3, we deduce that the existence and uniqueness of the weak solution to problem (1.1) follow from standard theory on the boundary-value problem for the Laplace equation and the sign condition (2.7).

The inverse problem we are addressing is to estimate the size of the corroded part E of the boundary from a knowledge of Cauchy data $\{g, u|_\Gamma\}$. To do this we shall compare u with the solution u_0 of the problem when $E = \emptyset$ and $\gamma \equiv \gamma_0$. Precisely, $u_0 \in H^1(\Omega)$ is such that

$$\int_\Omega \nabla u_0 \cdot \nabla v \, \mathrm{d}x + \int_{\partial\Omega} \gamma_0(x) u_0 v \, \mathrm{d}\sigma = \int_{\partial\Omega} gv \, \mathrm{d}\sigma \quad \forall v \in H^1(\Omega).$$

As earlier, we denote by W and W_0 the power required to maintain the current density g on $\partial\Omega$ when E is and is not present, respectively, namely

$$W = \int_{\partial\Omega} gu \, \mathrm{d}\sigma = \int_\Omega \nabla u \cdot \nabla u \, \mathrm{d}x + \int_{\partial\Omega} \gamma u^2 \, \mathrm{d}\sigma,$$
$$W_0 = \int_{\partial\Omega} gu_0 \, \mathrm{d}\sigma = \int_\Omega \nabla u_0 \cdot \nabla u_0 \, \mathrm{d}x + \int_{\partial\Omega} \gamma_0 u_0^2 \, \mathrm{d}\sigma.$$

From now on we shall refer to the following set of quantities as the *a priori* data: M, L, $\bar{k}_0$, $\bar{k}_1$, c_0, g_0, F.

We can now state the main result we want to prove.

THEOREM 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class $C^{0,1}$. Let $\gamma, \gamma_0 \in L^\infty(\partial\Omega)$ defined as above. Then there exist positive constants $C_1, C_2, p > 1$ depending only on the* a priori *data such that*

$$C_1 {r_0}^{n-1} \frac{W - W_0}{W_0} \leqslant |E| \leqslant C_2 {r_0}^{n-1} \left(\frac{W - W_0}{W_0} \right)^{1/p}. \tag{2.17}$$

3. Abstract formulation

To prove theorem 2.5 we shall make use of techniques developed in the context of the inverse conductivity problem [5]. The difference from other situations is that we want to determine a defect in the external boundary of the specimen, whereas in the other cases the inhomogeneity is fully contained in the domain. To overcome this difficulty we shall rephrase our argument in an abstract way, disconnecting it from the physical context.

We denote by H a Hilbert space and by H' its dual. Let $a_1(\cdot,\cdot)$ and $a_0(\cdot,\cdot)$ be two bilinear symmetric forms on H and let $F \in H'$. By the Lax–Milgram theorem, there exist u_1 and u_0 in H such that

$$a_j(u_j, v) = \langle F, v \rangle \quad \forall v \in H, \ j = 0, 1,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between H and H'.

LEMMA 3.1. *The following inequalities hold:*

$$J_1 := a_0(u_1 - u_0, u_1 - u_0) - [a_1(u_0, u_0) - a_0(u_0, u_0)] = \langle F, u_1 - u_0 \rangle, \tag{3.1 a}$$
$$J_2 := a_0(u_0 - u_1, u_0 - u_1) - [a_0(u_1, u_1) - a_1(u_1, u_1)] = -\langle F, u_1 - u_0 \rangle, \tag{3.1 b}$$
$$J_3 := a_0(u_1, u_0) - a_1(u_1, u_0) = \langle F, u_1 - u_0 \rangle. \tag{3.1 c}$$

Proof. Let us verify (3.1 *a*).

$$\begin{aligned}a_0(u_1-u_0,u_1-u_0)&-[a_1(u_0,u_0)-a_0(u_0,u_0)]\\&=a_1(u_1,u_1)-2a_1(u_1,u_0)+a_1(u_0,u_0)-a_1(u_0,u_0)+a_0(u_0,u_0)\\&=\langle F,u_1\rangle-2\langle F,u_0\rangle+\langle F,u_0\rangle=\langle F,u_1-u_0\rangle.\end{aligned}$$

Equalities (3.1 *b*) and (3.1 *c*) can be obtained similarly. □

We now define

$$G(u):=a_1(u,u)-a_0(u,u),\quad u\in H.$$

Let us observe that G is a functional depending on the defect. We also define

$$\alpha(u,v):=\tfrac{1}{4}[G(u+v)-G(u-v)],\quad u,v\in H.$$

Trivially, we have

$$a_1(u,v)=a_0(u,v)+\alpha(u,v),\quad u,v\in H.$$

LEMMA 3.2. *If, for every $u\in H$, either $\alpha(u,u)\geqslant 0$ or $\alpha(u,u)\leqslant 0$, then*

$$|\alpha(u,v)|\leqslant|\alpha(u,u)|^{1/2}|\alpha(v,v)|^{1/2}\tag{3.2}$$

for every $u,v\in H$.

Proof. If $\alpha(u,u)=0$ and $\alpha(v,v)=0$, then, assuming $\alpha(w,w)\geqslant 0$ for every $w\in H$, we would have

$$0\leqslant\alpha(u+tv,u+tv)=2t\alpha(u,v)\quad\forall t\in\mathbb{R},$$

which implies $\alpha(u,v)=0$ and (3.2) is proved.

If $\alpha(u,u)\neq 0$ or $\alpha(v,v)\neq 0$, then assuming, for instance, $\alpha(v,v)>0$, we would have

$$0\leqslant\alpha(u+tv,u+tv)=t^2\alpha(v,v)+2t\alpha(u,v)+\alpha(u,u)\quad\forall t\in\mathbb{R},$$

from which

$$(\alpha(u,v))^2-\alpha(u,u)\alpha(v,v)\leqslant 0$$

and (3.2) follows.

If $\alpha(w,w)\leqslant 0$ for every $w\in H$, the thesis follows by similarly applying the above argument to $-\alpha(\cdot,\cdot)$. □

Defining

$$\delta W=\langle F,u_1-u_0\rangle,$$

formula (3.1) can be written as

$$a_1(u_1-u_0,u_1-u_0)-\alpha(u_0,u_0)=\delta W,\tag{3.3 a}$$
$$a_0(u_1-u_0,u_1-u_0)+\alpha(u_1,u_1)=-\delta W,\tag{3.3 b}$$
$$\alpha(u_0,u_1)=\delta W.\tag{3.3 c}$$

We now prove estimates for a and α that will be useful for our purposes.

PROPOSITION 3.3. *Let $\lambda_0, \lambda_1 \in (0,1]$ be given. Assume that a_0 and a_1 satisfy the following conditions:*

$$\lambda_0\|u\|^2 \leqslant a_0(u,u) \leqslant \lambda_0^{-1}\|u\|^2 \quad \forall u \in H, \tag{3.4 a}$$

$$\lambda_1\|u\|^2 \leqslant a_1(u,u) \leqslant \lambda_1^{-1}\|u\|^2 \quad \forall u \in H. \tag{3.4 b}$$

If α satisfies the condition

$$0 \leqslant \alpha(u,u) \leqslant C_0 a_0(u,u) \quad \forall u \in H, \tag{3.5}$$

where C_0 is a positive constant, then

$$|\delta W| \leqslant \alpha(u_0,u_0) \leqslant (1+C_0)|\delta W|. \tag{3.6}$$

Conversely, if α satisfies the condition

$$\alpha(u,u) \leqslant 0 \quad \forall u \in H, \tag{3.7}$$

then

$$C|\delta W| \leqslant -\alpha(u_0,u_0) \leqslant |\delta W|, \tag{3.8}$$

where C is a positive constant depending only on λ_0 and λ_1.

Proof. Let us first consider (3.5). By (3.3 *b*) we have $\delta W \leqslant 0$, and by (3.3 *a*) we have $-\alpha(u_0,u_0) \leqslant \delta W$. Thus,

$$|\delta W| \leqslant \alpha(u_0,u_0). \tag{3.9}$$

Let us now obtain the upper bound for $\alpha(u_0,u_0)$. Using lemma 3.2 we have

$$\begin{aligned}\alpha(u_0,u_0) &\leqslant \alpha(u_0-u_1,u_0-u_1)+\alpha(u_1,u_1)\\ &\qquad + 2|\alpha(u_0-u_1,u_0-u_1)|^{1/2}|\alpha(u_1,u_1)|^{1/2}\\ &\leqslant \alpha(u_0-u_1,u_0-u_1)+\alpha(u_1,u_1)\\ &\qquad + \varepsilon\alpha(u_0-u_1,u_0-u_1)+\frac{\alpha(u_1,u_1)}{\varepsilon}\\ &\leqslant (1+\varepsilon)\left[C_0 a_0(u_0-u_1,u_0-u_1)+\frac{\alpha(u_1,u_1)}{\varepsilon}\right]\\ &\leqslant (1+C_0)|\delta W|,\end{aligned}$$

where in the last line we have chosen $\varepsilon = 1/C_0$. Hence, we get

$$\alpha(u_0,u_0) \leqslant (1+C_0)|\delta W|.$$

Let us now consider (3.7). By (3.3 *a*) we get $\delta W \geqslant 0$ and also

$$|\alpha(u_0,u_0)| \leqslant \delta W. \tag{3.10}$$

Let us recover an estimate from below for $|\alpha(u_0,u_0)|$. By (3.3 *c*) we get

$$\begin{aligned}\delta W &= \alpha(u_0,u_1)\\ &\leqslant (-\alpha(u_0,u_0))^{1/2}(-\alpha(u_1,u_1))^{1/2}\\ &\leqslant \frac{\varepsilon}{2}(-\alpha(u_1,u_1))+\frac{1}{2\varepsilon}(-\alpha(u_0,u_0)).\end{aligned} \tag{3.11}$$

Also, by (3.3 b), we have

$$-\alpha(u_1, u_1) = a_0(u_1 - u_0, u_1 - u_0) + \delta W. \tag{3.12}$$

Moreover, by (3.4 a) and (3.4 b), we have

$$a_0(u_1 - u_0, u_1 - u_0) \leqslant \lambda_0^{-1}\lambda_1^{-1} a_1(u_1 - u_0, u_1 - u_0).$$

By the above inequality and (3.12) we obtain

$$-\alpha(u_1, u_1) \leqslant \lambda_0^{-1}\lambda_1^{-1} a_1(u_1 - u_0, u_1 - u_0) + \delta W.$$

Then, substituting (3.11) and using (3.3 a), we have

$$\begin{aligned}
\delta W &\leqslant \frac{\varepsilon}{2}[A a_1(u_1 - u_0, u_1 - u_0) + \delta W] + \frac{1}{2\varepsilon}(-\alpha(u_0, u_0)) \\
&= \frac{\varepsilon}{2}[A(a_1(u_1 - u_0, u_1 - u_0) - \alpha(u_0, u_0)) + A\alpha(u_0, u_0) + \delta W] \\
&\quad + \frac{1}{2\varepsilon}(-\alpha(u_0, u_0)) \\
&= \frac{\varepsilon}{2}(1 + A)\delta W + \left(\frac{1}{2\varepsilon} - A\frac{\varepsilon}{2}\right)(-\alpha(u_0, u_0)),
\end{aligned}$$

where $A := \lambda_0^{-1}\lambda_1^{-1}$. Thus,

$$\left(1 - \frac{\varepsilon}{2}(1 + A)\right)\delta W \leqslant \frac{1 - A\varepsilon^2}{2\varepsilon} \leqslant |\alpha(u_0, u_0)|.$$

If $\varepsilon < \sqrt{A}$, we have

$$\left(\frac{1}{1 - A\varepsilon^2}\right)\left(2\varepsilon\left(1 - \frac{\varepsilon(1 + A)}{2}\right)\right)\delta W \leqslant |\alpha(u_0, u_0)|.$$

Finally, choosing $\varepsilon = 1/(1 + A)$, we get

$$c\delta W \leqslant |\alpha(u_0, u_0)|,$$

where c depends only on λ_0 and λ_1. □

Remark 3.4. In (3.5), condition (3.4) can be weakened by assuming $a_0(\cdot, \cdot)$ and $a_1(\cdot, \cdot)$ are positive semi-definite. Conversely, in (3.7), it is enough to require that $a_0(\cdot, \cdot)$ and $a_1(\cdot, \cdot)$ are positive semi-definite, such that

$$a_0(u, u) \leqslant C_1 a_1(u, u) \quad \forall u \in H,$$

where C_1 is a positive constant.

4. Proof of the main result

We want to make use of estimates obtained in the previous section to prove our bounds on the size of E. To do this we define

$$a_1(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Gamma \gamma_0 uv \, \mathrm{d}\sigma + k \int_E uv \, \mathrm{d}\sigma,$$
$$a_0(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x + \int_\Gamma \gamma_0 uv \, \mathrm{d}\sigma,$$
$$\alpha(u,v) = k \int_E uv \, \mathrm{d}\sigma$$

for $u, v \in H^1(\Omega)$. We immediately obtain

$$\alpha(u,u) = k \int_E u^2 \, \mathrm{d}\sigma \geqslant 0, \qquad a_0(u,u) \leqslant a_1(u,u), \tag{4.1}$$

for every $u \in H^1(\Omega)$.

Lemma 4.1. *There exists a constant $C > 0$ depending only on M and L such that*

$$\int_{\partial\Omega\setminus\Gamma} u^2 \, \mathrm{d}\sigma \leqslant C\left(r_0 \int_\Omega |\nabla u|^2 + \int_{\Gamma_0} u^2 \, \mathrm{d}\sigma \right) \tag{4.2}$$

for every $u \in H^1(\Omega)$.

Proof. By a standard trace inequality (see [1, ch. 7]) we get

$$r_0 \int_{\partial\Omega\setminus\Gamma} u^2 \, \mathrm{d}\sigma \leqslant C\left(\int_\Omega |u|^2 \, \mathrm{d}x + r_0^2 \int_\Omega |\nabla u|^2 \, \mathrm{d}x \right). \tag{4.3}$$

Moreover, by the equivalence between the norm in $H^1(\Omega)$ and $\|\cdot\|_*$ introduced in remark 2.3, the thesis follows. □

The main tools of unique continuation needed in the proof of our main result are contained in [21, lemma 4.5, theorem 4.6 and corollary 4.7], and for a detailed proof we refer the reader to [21]. However, for the reader's convenience and to make this paper as self-contained as possible, we sketch the proof of our main ingredient of unique continuation below.

Proposition 4.2 (A_p property on the boundary). *Let u_0 be a solution to problem (1.3). Then there exist constants $p > 1$, $A > 0$, $\bar{r} > 0$ depending only on the* a priori *data such that for every $x_0 \in \Gamma_{1,2\bar{r}}$ the following holds:*

$$\left(\frac{1}{|\Delta_r(x_0)|} \int_{\Delta_r(x_0)} |u_0|^2 \right) \left(\frac{1}{|\Delta_r(x_0)|} \int_{\Delta_r(x_0)} |u_0|^{-2/(p-1)} \right)^{p-1} \leqslant A, \tag{4.4}$$

where $\Gamma_{1,2\bar{r}} = \{x \in \partial\Omega \colon \mathrm{dist}(x, \Gamma_1) < 2\bar{r}\}$, $\Gamma_1 = \partial\Omega\setminus\bar{\Gamma}$ and $\Delta_r(x_0) = \Gamma_{1,2\bar{r}} \cap B_r(x_0)$ with $0 < r < \bar{r}$.

Proof. We recall that, as our main tool of unique continuation, the following so-called *surface* doubling inequality was achieved in [21]: there exists a constant

$K_1 > 0$ depending only on the *a priori* data, such that, for any $x_0 \in \Gamma_{1,\bar{r}/2}$ and for every $r \in (0, \bar{r})$,

$$\int_{\Delta_{2r}(x_0)} u_0^2 \leqslant K_1 \int_{\Delta_r(x_0)} u_0^2 \tag{4.5}$$

holds. The proof of the latter has two main ingredients. The first is the well-known stability estimate for the Cauchy problem (see, for example, [22]):

$$\begin{aligned}\int_{B_{r/2}(x_0)\cap\Omega} |u_0|^2 \leqslant Cr\biggl(\int_{\Delta_r(x_0)} u_0^2 + r^2 \int_{\Delta_r(x_0)} |\nabla_t u_0|^2\biggr)^{1-\delta} \\ \times \biggl(\int_{\Delta_r(x_0)} u_0^2 + r^2 \int_{\Delta_r(x_0)} |\nabla_t u_0|^2 + \int_{B_r(x_0)\cap\Omega} |\nabla u_0|^2\biggr)^{\delta},\end{aligned} \tag{4.6}$$

where ∇_t denotes the tangential gradient on $\Delta_r(x_0)$ (more precisely, we have $\nabla_t u_0 = \nabla u_0 - (\nabla u_0 \cdot \nu))$, and $C > 0$, $0 < \delta < 1$ are constants depending only on the *a priori* data.

The second main ingredient is the following *volume* doubling inequality (see [21, lemma 4.5]):

$$\int_{B_{\beta r}(x_0)\cap\Omega} |u_0|^2 \leqslant C\beta^K \int_{B_r(x_0)\cap\Omega} |u_0|^2 \tag{4.7}$$

for every r, β such that $\beta > 1$ and $0 < \beta r < 2\bar{r}$, where C and K are positive constants depending only on the *a priori* data. The inequality (4.7) was achieved in [21] by combining the techniques, introduced in [2], that apply to homogeneous Neumann boundary conditions, with a suitable change of variable that fits the problem under the assumption required in [2].

The control on the vanishing rate of the solution on the boundary provided by inequality (4.5) allowed Sincich [21, corollary 4.7] to obtain the following reverse Hölder inequality:

$$\biggl(r^{-2} \int_{\Delta_r(x_0)} u_0^2\biggr)^{1/4} \leqslant \biggl(Cr^{-2} \int_{\Delta_r(x_0)} u_0^2\biggr)^{1/2}, \tag{4.8}$$

which in turn, combined with the powerful theory of Muckenhoupt weights (see [9]), leads to the desired integrability property for $|u_0|^{-1}$ in (4.4). □

Proof of theorem 2.5. By lemma 4.1, there exists a positive constant C_1, depending on M, L, c_0, such that

$$0 \leqslant \alpha(u, u) \leqslant C_1 a_0(u, u) \quad \forall u \in H^1(\Omega).$$

By the above inequality and by proposition 3.3, we have

$$|\delta W| \leqslant k \int_E u_0^2 \, \mathrm{d}\sigma \leqslant (1 + C_1)|\delta W|, \tag{4.9}$$

where $\delta W = \int_{\partial\Omega} g u_0$. The leftmost inequality and standard bounds on the Neumann problem solution lead to the following inequality:

$$|\delta W| \leqslant |E| k \|u_0\|^2_{L^\infty(E)}.$$

Moreover, by a uniform boundedness-type estimate (see [14, ch. 8]) we have that

$$|\delta W| \leqslant C\bar{k}r_0^{-1}|E|\|u_0\|^2_{H^1(\Omega)},$$

where C depends on the *a priori* data only. By remark 2.3 we also have that

$$|\delta W| \leqslant C\bar{k}|E|r_0^{-n-1}\left(r_0\int_{\Gamma_0}u_0^2 + r_0^2\int_{\Omega}|\nabla u_0|^2\right).$$

Moreover, by the lower bound in (2.3) we deduce that

$$\begin{aligned}|\delta W| &\leqslant C\bar{k}|E|r_0^{1-n}\max\{c_0^{-1},1\}\left(\int_{\Gamma_0}\gamma_0 u_0^2 + \int_{\Omega}|\nabla u_0|^2\right)\\ &\leqslant C\bar{k}|E|r_0^{1-n}\max\{c_0^{-1},1\}\left(\int_{\partial\Omega}\gamma u_0^2 + \int_{\Omega}|\nabla u_0|^2\right).\end{aligned}$$

Finally, by the weak formulation for u_0 (see remark 2.4) we have that

$$|\delta W| \leqslant C\bar{k}|E|r_0^{1-n}\max\{c_0^{-1},1\}\left(\int_{\partial\Omega} g u_0\,\mathrm{d}\sigma\right).$$

Let us consider now the upper bound for E. First, we have to cover properly the unknown part of the boundary (we refer the reader to [8] for a similar construction). Let r be such that

$$r = \frac{1}{4}\min\left\{\frac{r_0}{8\sqrt{n}}, \frac{r_0}{2\sqrt{n}M}\right\} \tag{4.10}$$

and define

$$\Gamma_1^r = \{x \in \Omega\colon \operatorname{dist}(x, \Gamma_1) < r\}, \tag{4.11}$$

where $\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}$.

Let $\{Q_j\}_{j=1}^J$ be a family of closed mutually internally disjoint cubes of size $2r$ such that

$$\Gamma_1^r \cap Q_j \neq \emptyset, \quad j = 1, \dots, J, \tag{4.12}$$

$$\Gamma_1^r \subset \bigcup_{j=1}^J Q_j. \tag{4.13}$$

Let $x_j \in \Gamma_1^r \cap Q_j$, $j = 1, \dots, J$. We have that

$$\Gamma_1^r \subset \bigcup_{j=1}^J B_{4\sqrt{n}r}(x_j).$$

Indeed, for $x \in \Gamma_1^r$, there exists $\bar{x} \in \Gamma_1$ such that $\operatorname{dist}(x, \bar{x}) < 2\sqrt{n}r$. Let j be such that $\bar{x} \in Q_j$. Since $|\bar{x} - x_j| \leqslant 2\sqrt{n}r$, we have

$$|x - x_j| \leqslant |x - \bar{x}| + |\bar{x} - x_j| \leqslant 4\sqrt{n}r,$$

which implies $x \in B_{4\sqrt{n}r}(x_j)$. Making use of the construction argument in [8, proposition 5.2] we can infer that there exists a constant $C > 0$ depending only on M

and L such that

$$\bigcup_{j=1}^{J} Q_j \subset \{x \in \mathbb{R}^n \colon \operatorname{dist}(x, \Gamma_1) \leqslant 4\sqrt{n}r\} \quad \text{and} \quad J \leqslant C, \tag{4.14}$$

where $C > 0$ is a constant depending only on M and L. By the Hölder inequality, (3.6) and (4.3) we have

$$\begin{aligned} |E| &= \int_E |u_0|^{-2/p} |u_0|^{2/p} \\ &\leqslant \left(\int_E |u_0|^{-2/(p-1)} \right)^{(p-1)/p} \left(\int_E |u_0|^2 \right)^{1/p} \\ &\leqslant \left(\int_{\Gamma_1} |u_0|^{-2/(p-1)} \right)^{(p-1)/p} ((1 + C_0)|\delta W|)^{1/p}, \end{aligned} \tag{4.15}$$

where C depends only on M, L and c_0. Now,

$$\begin{aligned} \int_{\Gamma_1} |u_0|^{-2/(p-1)} &\leqslant \int_{\Gamma_1 \cap (\bigcup_{j=1}^{J} B_{4\sqrt{n}r}(x_j))} |u_0|^{-2/(p-1)} \\ &\leqslant \sum_{j=1}^{J} \int_{\Delta_j} |u_0|^{-2/(p-1)} \\ &\leqslant \sum_{j=1}^{J} \frac{Lr_0^{n-1}}{|\Delta_j|} \int_{\Delta_j} |u_0|^{-2/(p-1)}, \end{aligned} \tag{4.16}$$

where $\Delta_j = B_{4\sqrt{n}r}(x_j) \cap \Gamma_1$. By proposition 4.2, we have that

$$\frac{1}{|\Delta_j|} \int_{\Delta_j} |u_0|^{-2/(p-1)} \leqslant \left(\frac{A}{(1/|\Delta_j|) \int_{\Delta_j} |u_0|^2} \right)^{1/(p-1)}, \tag{4.17}$$

where A is a constant depending only on M, L, c_0 and F. Let us assume that the index $\bar{j}, 1 \leqslant \bar{j} \leqslant J$, is such that

$$\frac{1}{|\Delta_{\bar{j}}|} \int_{\Delta_{\bar{j}}} |u_0|^2 = \min_{1 \leqslant j \leqslant J} \frac{1}{|\Delta_j|} \int_{\Delta_j} |u_0|^2. \tag{4.18}$$

By combining (4.15), (4.16) and (4.18), we have that

$$|E| \leqslant \left(JLr_0^{n-1} \left(\frac{A}{(1/|\Delta_{\bar{j}}|) \int_{\Delta_{\bar{j}}} |u_0|^2} \right)^{1/(p-1)} \right)^{(p-1)/p} ((1 + C_0)|\delta W|)^{1/p}. \tag{4.19}$$

By the *a priori* bound $|\partial\Omega| \leqslant Lr_0^{n-1}$, we easily get that

$$\frac{1}{|\Delta_{\bar{j}}|} \int_{\Delta_{\bar{j}}} |u_0|^2 \geqslant \frac{1}{Lr_0^{n-1}} \int_{\Delta_{\bar{j}}} |u_0|^2. \tag{4.20}$$

By (4.5) and by a standard trace inequality we can infer that

$$\int_{\Delta_{\bar{j}}(x_j)} |u_0|^2 \geqslant Cr_0^{-1} \int_{B_{2\sqrt{n}r}(x_j) \cap \Omega} |u_0|^2, \tag{4.21}$$

where $C > 0$ is a constant depending only on $\bar{k}_0$, $\bar{k}_1$, M, L and F. Let $\bar{x} \in B_{2\sqrt{n}r}(x_j) \cap \Omega$ be such that $B_{\sqrt{n}r/4}(\bar{x}) \subset B_{2\sqrt{n}r}(x_j) \cap \Omega$. Hence, we get

$$\int_{\Delta_{\bar{j}}(x_j)} |u_0|^2 \geqslant C r_0^{-1} \int_{B_{\sqrt{n}r/4}(\bar{x})} |u_0|^2. \tag{4.22}$$

Now, using the arguments developed in [18, proposition 3.1] (see also [7, lemma 5.3]), relying on a standard propagation of smallness, we get

$$\int_{B_{\sqrt{n}r/4}(\bar{x})} |u_0|^2 \geqslant C \int_{\Omega} |u_0|^2, \tag{4.23}$$

where $C > 0$ is a constant depending only on M, L, $\bar{k}_0$, $\bar{k}_1$ and F. Hence, combining (4.20), (4.22) and (4.23), it easily follows that

$$\frac{1}{|\Delta_{\bar{j}}|} \int_{\Delta_{\bar{j}}} |u_0|^2 \geqslant C r_0^{-n} \int_{\Omega} |u_0|^2, \tag{4.24}$$

where $C > 0$ is a constant depending only on M, L, $\bar{k}_0$, $\bar{k}_1$ and F. By the estimate (4.22) and the Caccioppoli inequality we get that

$$\int_{\Delta_{\bar{j}}} |u_0|^2 \geqslant C r_0 \int_{B_{\sqrt{n}r/8}(\bar{x})} |\nabla u_0|^2, \tag{4.25}$$

where $C > 0$ is a constant depending only on M, L, $\bar{k}_0$, $\bar{k}_1$ and F. Repeating the propagation of smallness techniques described in [18, proposition 3.1], this time for the gradient, we get that

$$\frac{1}{|\Delta_{\bar{j}}|} \int_{\Delta_{\bar{j}}} |u_0|^2 \geqslant C r_0^{2-n} \int_{\Omega} |\nabla u_0|^2, \tag{4.26}$$

where $C > 0$ is a constant depending only on M, L, $\bar{k}_0$, $\bar{k}_1$ and F. We can then infer that

$$\frac{1}{|\Delta_{\bar{j}}|} \int_{\Delta_{\bar{j}}} |u_0|^2 \geqslant C \left(r_0^{-n} \int_{\Omega} |u_0|^2 + r_0^{2-n} \int_{\Omega} |\nabla u_0|^2 \right), \tag{4.27}$$

where $C > 0$ is a constant depending only on M, L, $\bar{k}_0$, $\bar{k}_1$ and F. By the equivalence between the standard $H^1(\Omega)$ norm and the norm introduced in remark 2.3 we find that

$$\frac{1}{|\Delta_{\bar{j}}|} \int_{\Delta_{\bar{j}}} |u_0|^2 \geqslant C r_0^{-n} \left(r_0 \int_{\Gamma_0} |u_0|^2 + r_0^2 \int_{\Omega} |\nabla u_0|^2 \right), \tag{4.28}$$

where $C > 0$ is a constant depending only on M, L, $\bar{k}_0$, $\bar{k}_1$ and F. Now, by the *a priori* bound $\gamma_0(x) \leqslant c_0^{-1}/r_0$ on Γ, we get that

$$\begin{aligned} \frac{1}{|\Delta_{\bar{j}}|} \int_{\Delta_{\bar{j}}} |u_0|^2 &\geqslant C r_0^{2-n} \min\{1, c_0\} \left(\int_{\Gamma_0} \gamma_0 |u_0|^2 + \int_{\Omega} |\nabla u_0|^2 \right) \\ &\geqslant C r_0^{2-n} \min\{1, c_0\} \int_{\partial\Omega} g u_0. \end{aligned} \tag{4.29}$$

Combining (4.14), (4.19) and (4.29) and recalling that $\int_{\partial\Omega} gu_0 = W_0$, we obtain that

$$|E| \leqslant Cr_0^{n-1}\left(\frac{W - W_0}{W_0}\right)^{1/p}, \tag{4.30}$$

where $C > 0$ is a constant depending only on M, L, $\bar{k}_0$, $\bar{k}_1$, F and c_0. □

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