Research Article

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Unbounded solutions for differential equations with p-Laplacian and mixed nonlinearities

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Abstract: The existence of unbounded solutions with different asymptotic behavior for a second order nonlinear differential equation with p-Laplacian is considered. The oscillation of all solutions is investigated. Some discrepancies and similarities between equations of Emden–Fowler-type and equations with mixed nonlinearities are pointed out.

Keywords: Second order nonlinear differential equation, unbounded solutions, oscillatory solution, nonoscillatory solution, continuability

MSC 2010: Primary 34C10; secondary 34C15

Dedicated to Professor Ivan Kiguradze on the occasion of his 80th birthday

1 Introduction

Consider the second order nonlinear differential equation

$$(\Phi(x'))' + f(t,x) = 0, \qquad t \in I = [1,\infty), \tag{E}$$

where $\Phi(u) = |u|^\alpha \operatorname{sgn} u$, $\alpha > 0$, and $f : I \times \mathbb{R} \to \mathbb{R}$ is continuous on $I \times \mathbb{R}$, continuously differentiable with respect to t for any fixed u and $f(t,u)u > 0$ for $u \neq 0$, $t \in I$.

Sometimes, the following conditions will be assumed throughout the paper:

$$\frac{f(t,u)}{|u|^\beta} \text{ is nonincreasing with respect to } u \text{ on } (-\infty,0) \text{ and on } (0,\infty) \tag{1.1}$$

for some $\beta > \alpha$ and any $t \in I$,

$$t^\gamma |f(t,u)| \text{ is nonincreasing with respect to } t \text{ on } [T,\infty) \tag{1.2}$$

for some $T \geq 1$ and for any $u \in \mathbb{R}$, and

$$\lim_{u\to 0+} \frac{f(T,u)}{|u|^\beta \operatorname{sgn} u} = L < \infty, \tag{1.3}$$

where

$$\gamma = \frac{\alpha\beta + 2\alpha + 1}{\alpha + 1}.$$

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There is a large variety of nonlinearities f which satisfy assumptions (1.1)–(1.3). For instance, the functions

$$f_1(t,u) = t^{-\gamma}e^{-|tu|}|u|^\beta \operatorname{sgn} u,$$
$$f_2(t,u) = t^{-\gamma}\Big(1 + \frac{1}{1+\log(1+|u|)}\Big)|u|^\beta \operatorname{sgn} u$$

satisfy conditions (1.1)–(1.3). Moreover, the function f_1 is bounded with respect to the second variable, while $\lim_{u\to\infty} f_2(t,u) = \infty$.

A special case of (E) is the equation

$$(\Phi(x'))' + b(t)g(x)|x|^\beta \operatorname{sgn} x = 0, \quad t \in I,\ \alpha < \beta, \tag{S}$$

where b is a positive continuously differentiable function for $t \in I$ and g is a positive and continuous function on $\mathbb{R}$ which is bounded away from zero.

By a solution of (E) we mean a function which is defined on $[t_x, \infty)$, $t_x \ge 1$, and satisfies (E). This definition is not restricted because when (1.2) is valid, any solution of (E) is continuable to infinity, see Theorem 2.1 below.

A solution x of (E) is said to be *nonoscillatory* if $x(t) \neq 0$ for large t and *oscillatory* if it is nontrivial and has arbitrarily large zeros, i.e., if there exists a sequence $\{\tau_n\}$, $\tau_n \to \infty$, with $x(\tau_n) = 0$. Equation (E) is said to be oscillatory if any of its solutions is oscillatory.

Any eventually positive solution of (E) is nondecreasing for large t. Moreover, the class of all eventually positive solutions of (E) can be divided into the following three subclasses, according to their asymptotic growth at infinity:

$$\lim_{t\to\infty} x(t) = c_x, \tag{1.4}$$
$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} \frac{x(t)}{t} = 0, \tag{1.5}$$
$$\lim_{t\to\infty} \frac{x(t)}{t} = c_x, \tag{1.6}$$

where c_x is a positive constant depending on x; see, e.g., [1, Lemma 3.13.8.] or [4, Section 2]. Solutions satisfying (1.4), (1.5) or (1.6) are referred to as *subdominant solutions, intermediate solutions* or *dominant solutions*, respectively; see, e.g., [4, 13, 16].

In [11], see also [12, Theorem 18.5], I. Kiguradze proved that if for some $\varepsilon > 0$ the function $t^{(\beta+3)/2+\varepsilon}b(t)$ is nonincreasing on a certain interval $[t_0, \infty)$, then all solutions of the Emden–Fowler equation

$$x'' + b(t)|x|^\beta \operatorname{sgn} x = 0, \quad \beta > 1, \tag{1.7}$$

are nonoscillatory.

Thus, an interesting question arises: can the above three types of nonoscillatory solutions coexist for (E)? In particular, is it possible that both types of unbounded nonoscillatory solutions, that is, intermediate solutions and dominant solutions, can coexist simultaneously?

For the general Emden–Fowler differential equation

$$(a(t)|x'|^\alpha \operatorname{sgn} x')' + b(t)|x|^\beta \operatorname{sgn} x = 0, \quad 0 < \alpha < \beta, \tag{1.8}$$

where a is a positive continuous function for $t \in I$, the answer has been given in [6, Corollary 2]; see also Lemma 4.5 below.

For the more general equation (E) and its special case (S) necessary and sufficient conditions for the existence of dominant solutions can be easily given; see, e.g., Corollary 2.4 below or [14, Theorem 2.3], [1, Theorem 3.13.12] and [15, Theorem 17.2]. Nevertheless, until now, no general necessary and sufficient conditions for the existence of intermediate solutions of (E) have been known. This is a difficult problem because of the lack of sharp upper and lower bounds for these solutions; also, for (1.8), see, e.g., [1, p. 241], [10, p. 3] and [13, p. 2].

The aim of this paper is to study the existence of intermediate solutions and the possible coexistence of dominant and intermediate ones for (E) and (S). Moreover, some discrepancies and similarities between (1.8) with $a \equiv 1$ and (E) are pointed out. In particular, we show that the oscillation property reads in the same way for (1.8) with $a \equiv 1$ and the special case (S). Our approach is mainly based on certain monotonicity properties of an energy-type function, which can be associated to (E), when (1.1) and (1.2) hold and on some comparison criteria.

Finally, in the last section we show how our results are extended to the more general equation

$$(a(t)\Phi(x'))' + f(t,x) = 0, \tag{G}$$

where

$$\int_1^\infty \frac{dt}{\Phi^*(a(t))} = \infty \tag{1.9}$$

and Φ^* denotes the inverse map to Φ, that is, $\Phi^*(u) = |u|^{1/\alpha} \operatorname{sgn} u$.

Our results here for (E) and (G) extend recent ones in [3, 6–8]. In [3] the asymptotic and oscillatory properties of solutions of (E) have been investigated under a condition opposite to (1.1), that is, $\frac{f(t,u)}{|u|^\beta}$ is nondecreasing with respect to u on $(-\infty, 0)$ and on $(0, \infty)$.

The coexistence problem for nonoscillatory solutions and the existence of intermediate solutions have been investigated for the Emden–Fowler equation (1.8) under assumption (1.9) in [6, 8], while the case when

$$\int_1^\infty \frac{dt}{\Phi^*(a(t))} < \infty$$

has been treated in [7]. In particular, the following result holds, as follows from [8, Theorem 2.1].

Theorem A. *Let*

$$\int_1^\infty t b(t)\,dt < \infty, \qquad \int_1^\infty t^\beta b(t)\,dt = \infty$$

and $F(t) = t^{(\beta+3)/2} b(t)$ *be nonincreasing for* $t \ge T$*. Then* (1.7) *has infinitely many intermediate solutions which are positive increasing on* $[T, \infty)$.

2 Preliminaries

We start with the continuability of solutions of (E).

Theorem 2.1. *We have the following results:*

(i) *If* (1.1) *and* (1.2) *hold, then any solution of* (E) *which is defined on* $[t_x, \tau)$*, where* $t_x \ge T$*, is continuable to infinity, i.e.,* $\tau = \infty$.

(ii) *Every solution of* (S) *is defined on* $[1, \infty)$.

Proof. Assume that there is a solution x defined on $[t_x, \tau)$, $\tau < \infty$, which cannot be extended for $t = \tau$. Then there exists a sequence (t_k) such that $t_k \in [t_x, \tau)$, $\lim_{k\to\infty} t_k = \tau$ and

$$\lim_{k\to\infty} |x'(t_k)| = \infty. \tag{2.1}$$

Define

$$G(t) = |x'(t)|^{\alpha+1} + \frac{\alpha+1}{\alpha} \int_0^{x(t)} f(t,s)\,ds.$$

In view of the fact $f(t,u)u > 0$ for $u \neq 0$ we have

$$G(t) \ge 0 \quad \text{for } t \in [t_x, \tau),$$

and by (2.1) we have $\lim_{k\to\infty} G(t_k) = \infty$. Further, by (1.2) and the fact that f is continuously differentiable with respect to t we have

$$G'(t) = \frac{\alpha+1}{\alpha} \int_0^{x(t)} \frac{\partial}{\partial t} f(t,s)\,ds \le 0.$$

Thus,

$$0 \le G(t) \le G(t_x), \quad t \in [t_x, \tau)$$

which is in contradiction with the fact $\lim_{k\to\infty} G(t_k) = \infty$.

Now consider (S) and assume, by contradiction, that there is a solution x defined on (τ, ∞), $\tau > 1$, which cannot be extended for $t = \tau$. Then there exists a sequence $\{t_k\}$ such that $t_k \in (\tau, \infty)$, $\lim_{k\to\infty} t_k = \tau$ and (2.1) holds. Now consider the function G on (τ, ∞). Proceeding in a way similar to above, we get $\lim_{k\to\infty} G(t_k) = \infty$.

Since b is positive and continuously differentiable, from the Jordan decomposition there exist two non-negative nondecreasing continuously differentiable functions b_i, $i = 1, 2$, such that $b(t) = b_1(t) - b_2(t)$.

From [2, Theorem 2] the function G satisfies

$$G(s) \le \exp\Big(\int_s^t \frac{b_2(r)}{b(r)}\,dr\Big) G(t), \quad \tau \le s < t < \infty.$$

Thus G is bounded on $[\tau, t]$ which is a contradiction to the fact that $\lim_{k\to\infty} G(t_k) = \infty$. □

Define

$$K_\lambda = \int_1^\infty f(s, \lambda s)\,ds.$$

The following existence result for dominant solutions holds. It is a minor extension of similar results, see, e.g., [14, Theorem 2.3], [1, Theorem 3.13.12] and [15, Theorem 17.2], stated for equations with different nonlinearities.

Proposition 2.2. *Assume* (1.1) *and let* $0 < \alpha < \beta$. *If* $K_\lambda < \infty$ *for some* $\lambda > 0$, *then for any* $\ell, \lambda < \Phi^*(\ell)$, *equation* (E) *has a nonoscillatory solution* x *such that*

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} x'(t) = \ell. \tag{2.2}$$

Proof. For fixed $\ell > \Phi(\lambda)$ choose h, $h > \lambda$, such that

$$\Phi(\lambda) < \ell < \Phi(h) \tag{2.3}$$

and let t_0 be large such that

$$m = \ell + \Big(\frac{h}{\lambda}\Big)^\beta \int_{t_0}^\infty f(s, \lambda s)\,ds < \Phi(h). \tag{2.4}$$

Since $\ell < m$, let $c > 0$ be such that

$$\Phi^*(\ell) t_0 < c < \Phi^*(m) t_0. \tag{2.5}$$

In the Fréchet space $C[t_0, \infty)$ of continuous functions defined in $[t_0, \infty)$, endowed with the topology of uniform convergence on the compact subsets of $[t_0, \infty)$, consider the operator $\mathcal{T}$ given by

$$\mathcal{T}(u)(t) = c + \int_{t_0}^t \Phi^*\Big(\ell + \int_s^\infty f(r, u(r))\,dr\Big)\,ds,$$

where the function u belongs to the set Ω given by

$$\Omega = \{u \in C[t_0, \infty) : \lambda t \le u(t) \le ht \text{ for } t \ge t_0\}.$$

From (1.1) we have

$$f(r,u(r)) = \frac{f(r,u(r))}{u^\beta(r)}u^\beta(r) \le \frac{f(r,\lambda r)}{\lambda^\beta r^\beta}u^\beta(r) \le \Big(\frac{h}{\lambda}\Big)^\beta f(r,\lambda r) \tag{2.6}$$

for any $u \in \Omega$. Since $K_\lambda < \infty$, the operator $\mathcal{T}$ is well defined. Moreover, from (2.4) and (2.6) we have

$$\mathcal{T}(u)(t) \le c + \int_{t_0}^{t} \Phi^*\Big(\ell + \Big(\frac{h}{\lambda}\Big)^\beta \int_{t_0}^{\infty} f(r,\lambda r)\,dr\Big)\,ds = c + \Phi^*(m)(t-t_0),$$

that is, in view of (2.5), $\mathcal{T}(u)(t) \le \Phi^*(m)t$. Since $\Phi^*(m) < h$, we get $T(u)(t) \le ht$ for $t \ge t_0$. Similarly, from (2.3) and (2.5) we obtain

$$\mathcal{T}(u)(t) \ge c + \int_{t_0}^{t} \Phi^*(\ell)\,ds = c + \Phi^*(\ell)(t-t_0) \ge \Phi^*(\ell)t \ge \lambda t.$$

Hence, the operator $\mathcal{T}$ maps Ω into itself.

Let us show that $\mathcal{T}(\Omega)$ is relatively compact, i.e., $\mathcal{T}(\Omega)$ consists of functions which are equibounded and equicontinuous on every compact interval of $[t_0,\infty)$. Because $\mathcal{T}(\Omega) \subset \Omega$, the equiboundedness follows. Moreover, from (2.6), we have

$$0 \le \frac{d}{dt}\mathcal{T}(u)(t) \le \Phi^*\Big(\ell + \Big(\frac{h}{\lambda}\Big)^\beta \int_{t}^{\infty} f(r,\lambda r)\,dr\Big)$$

for any $u \in \Omega$, which yields the equicontinuity of the elements in $\mathcal{T}(\Omega)$. Now, we prove the continuity of $\mathcal{T}$ in Ω. Let $\{u_n\}$, $n \in \mathbb{N}$, be a sequence in Ω which uniformly converges on every compact interval of $[t_0,\infty)$ to $\bar{u} \in \Omega$. Because $\mathcal{T}(\Omega)$ is relatively compact, the sequence $\{\mathcal{T}(u_n)\}$ admits a subsequence $\{\mathcal{T}(u_{n_j})\}$ converging, in the topology of $C[t_0,\infty)$, to $\bar{x}_u \in \overline{\mathcal{T}(\Omega)}$. Since

$$\int_{t}^{\infty} f(r,u_{n_j}(r))\,dr \le \Big(\frac{h}{\lambda}\Big)^\beta \int_{t}^{\infty} f(r,\lambda r)\,dr,$$

according to (2.6), by the Lebesgue dominated convergence theorem the sequence $\{\mathcal{T}(u_{n_j})(t)\}$ pointwise converges to $\mathcal{T}(\bar{u})(t)$. In view of the uniqueness of the limit, $\mathcal{T}(\bar{u}) = \bar{x}_u$ is the only cluster point of the compact sequence $\{\mathcal{T}(u_n)\}$, that is, the continuity of $\mathcal{T}$ in the topology of $C[t_0,\infty)$. Hence, by the Tychonov fixed point theorem the operator $\mathcal{T}$ has a fixed point x which, clearly, is a solution of (E) satisfying (2.2). ☐

Lemma 2.3. *Assume* (1.1) *and let* $\lambda > 0$ *be such that* $K_\lambda = \infty$. *Then equation* (E) *does not have a nonoscillatory solution* x *such that*

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} x'(t) = \ell \tag{2.7}$$

for any ℓ, $0 < \ell < \lambda$.

Proof. By contradiction, let x be a nonoscillatory solution x of (E) satisfying (2.7). Since $\ell < \lambda$, for a fixed $\varepsilon > 0$ with $\varepsilon < \min\{\ell, \lambda - \ell\}$ there exists $\tau \ge 1$ such that

$$(\ell - \varepsilon)t \le x(t) \le (\ell + \varepsilon)t \le \lambda t$$

on $[\tau,\infty)$. Integrating (E), for $t \ge \tau$ we get

$$\begin{aligned} -\Phi(\ell) + \Phi(x'(t)) &= \int_{t}^{\infty} \frac{f(s,x(s))}{x^\beta(s)} x^\beta(s)\,ds \\ &\ge \lambda^{-\beta} \int_{t}^{\infty} f(s,\lambda s)\Big(\frac{x(s)}{s}\Big)^\beta\,ds \end{aligned}$$

or

$$-\Phi(\ell) + \Phi(x'(t)) \ge \Big(\frac{\ell - \varepsilon}{\lambda}\Big)^\beta \int_{t}^{\infty} f(s,\lambda s)\,ds,$$

which contradicts $K_\lambda = \infty$. ☐

From Lemma 2.3 and Proposition 2.2, we get the following corollary.

Corollary 2.4. *Let* $0 < \alpha < \beta$ *and assume* (1.1). *Equation* (E) *has a solution* x *which satisfies* (2.7) *for some* $\ell > 0$ *if and only if there exists* $\overline{\lambda}$ *such that* $K_{\overline{\lambda}} < \infty$.

3 Intermediate solutions

Our main result is the following.

Theorem 3.1. *Assume* (1.1)–(1.3) *and* $K_\lambda = \infty$ *for any* $\lambda > 0$. *Then equation* (E) *has infinitely many nonoscillatory solutions* x *defined on* $[T, \infty)$ *such that*

$$\lim_{t\to\infty} x(t) = \infty, \qquad \lim_{t\to\infty} x'(t) = 0.$$

For the proof the following lemmas are useful.

Lemma 3.2. *Assume* (1.1). *For* $u \in \mathbb{R}$ *and* $t \in I$ *we have*

$$f(t,u)u \le (\beta + 1)\int_0^u f(t,s)\,ds.$$

Proof. Clearly, if $u = 0$, the assertion holds. For $u > 0$ by (1.1) we have

$$\begin{aligned} uf(t,u) - (\beta+1)\int_0^u f(t,s)\,ds &= uf(t,u) - (\beta+1)\int_0^u \frac{f(t,s)}{s^\beta} s^\beta\,ds \\ &\le uf(t,u) - (\beta+1)\frac{f(t,u)}{u^\beta}\int_0^u s^\beta\,ds = 0. \end{aligned}$$

Similarly, the assertion follows for $u < 0$. □

Lemma 3.3. *Assume* (1.1) *and* (1.2). *Then for any solution* x *of* (E) *which is defined on* $[t_x, \infty)$, $t_x \ge T$, *the function*

$$E_x(t) = (tx'(t) - x(t))\Phi(x'(t)) + \frac{\alpha+1}{\alpha} t \int_0^{x(t)} f(t,s)\,ds \tag{3.1}$$

is nonincreasing on $[t_x, \infty)$.

Proof. Rewriting E_x as

$$E_x(t) = t(|x'(t)|^\alpha)^{\frac{\alpha+1}{\alpha}} - x(t)|x'(t)|^\alpha \operatorname{sgn} x'(t) + \frac{\alpha+1}{\alpha} t \int_0^{x(t)} f(t,s)\,ds,$$

we have

$$\begin{aligned} E_x'(t) = (|x'(t)|^\alpha)^{\frac{\alpha+1}{\alpha}} &+ \frac{\alpha+1}{\alpha} t(|x'(t)|^\alpha)^{1/\alpha} \operatorname{sgn} x'(t)(-f(t,x(t))) - |x'(t)|^{\alpha+1} \\ &- x(t)(-f(t,x(t))) + \frac{\alpha+1}{\alpha}\Big(\int_0^{x(t)} f(t,s)\,ds + tf(t,x(t))x'(t) + t\int_0^{x(t)} \frac{d}{dt} f(t,s)\,ds\Big) \end{aligned}$$

or

$$E_x'(t) = x(t)f(t,x(t)) + \frac{\alpha+1}{\alpha}\int_0^{x(t)} f(t,s)\,ds + \frac{\alpha+1}{\alpha} t \int_0^{x(t)} \frac{d}{dt}(f(t,s)t^\gamma t^{-\gamma})\,ds.$$

Since
$$t\int_0^{x(t)} \frac{d}{dt}(f(t,s)t^{\gamma}t^{-\gamma})\,ds = t\int_0^{x(t)} \frac{d}{dt}(f(t,s)t^{\gamma})t^{-\gamma}\,ds - \gamma\int_0^{x(t)} f(t,s)\,ds,$$
in view of (1.2) and Lemma 3.2 we get
$$\begin{aligned} E_x'(t) &\le x(t)f(t,x(t)) + \frac{\alpha+1}{\alpha}(1-\gamma)\int_0^{x(t)} f(t,s)\,ds \\ &= x(t)f(t,x(t)) - (\beta+1)\int_0^{x(t)} f(t,s)\,ds \le 0, \end{aligned}$$
and the assertion follows. □

Lemma 3.4. *Assume* (1.1) *and* (1.2). *If x is a subdominant solution or an oscillatory solution of* (E) *defined on* $[t_x,\infty)$, $t_x \ge T$, *then we have*
$$\lim_{t\to\infty} E_x(t) \ge 0.$$

Proof. Let x be an oscillatory solution of (E) such that x' vanishes at some $t^* > t_x$, that is, $x'(t^*) = 0$. Thus,
$$E_x(t^*) = \frac{\alpha+1}{\alpha}t^*\int_0^{x(t^*)} f(t^*,s)\,ds.$$
Since $f(t^*,s)s \ge 0$, we obtain $E_x(t^*) \ge 0$ and the assertion follows from Lemma 3.3.

Now, let x be a subdominant solution of (E). Since
$$\lim_{t\to\infty} x(t)\Phi(x'(t)) = 0 \quad\text{and}\quad tx'(t)\Phi(x'(t)) = t|x'(t)|^{\alpha+1},$$
the assertion follows again. □

Proof of Theorem 3.1. Since any solution of (E) is continuable to infinity, see Lemma 2.1, it is sufficient to show that (E) has solutions y for which $E_y(\bar t) < 0$ at some $\bar t \ge T \ge 1$. Consider the solution y of (E) satisfying the initial conditions
$$y(T) = Y,\quad y'(T) = d,$$
where Y and d are positive constants. In view of (1.1) and (1.3), for any $u > 0$ and $T \ge 1$ we have
$$\int_0^Y f(T,s)\,ds = \int_0^Y \frac{f(T,s)}{s^\beta}s^\beta\,ds \le \int_0^Y Ls^\beta\,ds = \frac{L}{\beta+1}Y^{\beta+1}.$$
Thus, from (3.1) we get
$$E_y(T) \le Td^{\alpha+1} - Yd^\alpha + kTY^{\beta+1},$$
where
$$k = \frac{(\alpha+1)}{\alpha(\beta+1)}L.$$
Define
$$g(d,Y) = Td^{\alpha+1} - Yd^\alpha + kTY^{\beta+1}$$
and
$$\psi(d) = (mT)^{-1/\beta}d^{\alpha/\beta},$$
where
$$m > k = \frac{(\alpha+1)}{\alpha(\beta+1)}L. \tag{3.2}$$

A standard calculation yields

$$g(d,\psi(d)) = Td^{\alpha+1} - \varepsilon(mT)^{-1/\beta}d^{(\alpha+\alpha\beta)/\beta},$$

where, in view of (3.2),

$$\varepsilon = 1 - \frac{k}{m} > 0.$$

Thus we obtain

$$g(d,\psi(d)) = Td^{\alpha+1}\Big(1 - \frac{\varepsilon}{T(mT)^{1/\beta}}d^{(\alpha-\beta)/\beta}\Big)$$

and so for any d satisfying

$$0 < d < \Big(\frac{\varepsilon^\beta}{mT^{1+\beta}}\Big)^{1/(\beta-\alpha)}$$

the function $g(d,\psi(d))$ is negative. Hence, the function $E_y(T)$ is negative too. From Lemma 3.3 we get $E_y(t) < 0$ for $t \geq T$ and so, using Lemma 3.4, we obtain that y is neither an oscillatory solution nor a subdominant solution. From (3.1) we obtain

$$E_y(t) \geq (t|y'(t)|^{\alpha+1} - y(t))\Phi(y'(t))$$

for $t \geq T$, and so $y(t) > 0$, $y'(t) > 0$ on $[T,\infty)$. Since $K_\lambda = \infty$ for any $\lambda > 0$, by Lemma 2.3 equation (E) does not have dominant solutions. Hence, y is an intermediate solution of (E). Clearly, if the point $(\overline{y}_2,\overline{y}_1)$ belongs to the graph of the function g and

$$0 < \overline{y}_2 < \Big(\frac{\varepsilon^\beta}{mT^{1+\beta}}\Big)^{1/(\beta-\alpha)},$$

then any solution $\overline{y}$ of (E), starting at $\overline{y}(T) = \overline{y}_1$, $\overline{y}'(T) = \overline{y}_2$, satisfies $E_{\overline{y}}(T) < 0$, thus it is an intermediate solution of (E). □

4 Coexistence of unbounded solutions

Theorem 3.1 requires that $K_\lambda = \infty$ for any $\lambda > 0$, which means that, in virtue of Corollary 2.4, equation (E) does not have dominant solutions.

When $K_{\lambda_0} < \infty$ for some $\lambda_0 > 0$, the following question arises: can equation (E) have simultaneously both types of nonoscillatory unbounded solutions? The following example shows that the answer can be positive.

Example 4.1. Consider the equation

$$x'' + b(t)e^{-|x|}|x|^\beta \operatorname{sgn} x = 0, \quad t \in [1,\infty),\ \beta > 1, \tag{4.1}$$

where

$$b(t) = \frac{1}{4}t^{-(\beta+3)/2}e^{\sqrt{t}}.$$

For (4.1) we have

$$K_\lambda = \frac{\lambda^\beta}{4}\int_1^\infty t^{(\beta-3)/2}e^{\sqrt{t}}e^{-\lambda t}\,dt$$

and so $K_\lambda < \infty$ for any $\lambda > 0$. Hence, for any $\ell > 0$, equation (4.1) has dominant solutions which satisfy (2.2). Moreover, (1.2) does not hold for equation (4.1). Nevertheless, (4.1) has also intermediate solutions because $x(t) = \sqrt{t}$ is its solution. Thus, (4.1) has simultaneously both types of nonoscillatory unbounded solutions.

However, the Emden–Fowler equation

$$x'' + b(t)|x|^\beta \operatorname{sgn} x = 0, \quad t \in I,\ \beta > 1. \tag{4.2}$$

with the same function b is oscillatory, see, e.g., [12, Theorem 1.5.1], and so (4.2) does not have nonoscillatory solutions.

As it is proved in [6, Corollary 1] for the Emden–Fowler equation (1.8), dominant solutions cannot coexist with intermediate ones.

In this section, we show that this property continues to hold for a special cases of (E), in which the nonlinearity is, roughly speaking, close to $|u|^\beta \operatorname{sgn} u$.

Recall equation (S), which was

$$(\Phi(x'))' + b(t)g(x)|x|^\beta \operatorname{sgn} x = 0, \quad t \in I,\ \alpha < \beta, \tag{S}$$

where b is a positive continuously differentiable function for $t \in I$ and g is a positive and continuous function on $\mathbb{R}$ such that

$$g(u) \operatorname{sgn} u \text{ is nonincreasing on } (-\infty, 0) \cup (0, \infty) \tag{4.3}$$

and

$$\lim_{u\to\infty} g(u) = M > 0. \tag{4.4}$$

Clearly, if (4.3) is valid, then the function $f(t, u) = b(t)g(u)|u|^\beta \operatorname{sgn} u$ satisfies (1.1) and (1.3). Observe that when (4.3) and (4.4) are satisfied, for any $u \geq 0$ we have

$$0 < M \leq g(u) \leq g(0). \tag{4.5}$$

Observe that by Theorem 2.1 all solutions of (S) are defined on $[1, \infty)$. The effect of (1.2) to the coexistence problem gives the following results.

Corollary 4.2. *Let $0 < \alpha < \beta$ and assume (4.3) and (4.4). If $F(t) = t^\gamma b(t)$ is nonincreasing bounded away from zero, i.e.,*

$$t^\gamma b(t) \geq H > 0, \quad t \in I,$$

then (S) has infinitely many intermediate solutions and does not have dominant solutions.

Proof. We have $1 + \beta - \gamma > 0$ and so

$$\begin{aligned} K_\lambda &= \int_1^\infty b(s)g(\lambda s)s^\beta\, ds = \int_1^\infty b(s)g(\lambda s)s^\beta s^\gamma s^{-\gamma}\, ds \\ &\geq HM \int_1^\infty s^{-\gamma+\beta}\, ds \geq HM \int_1^\infty s^{-1}\, ds = \infty \end{aligned}$$

for any $\lambda > 0$. Now the assertion follows from Theorem 3.1 and Lemma 2.3. □

Theorem 4.3. *Let $0 < \alpha < \beta$ and assume (4.3) and (4.4). If there exists $\lambda > 0$ such that*

$$\int_1^\infty b(s)g(\lambda s)s^\beta\, ds < \infty, \tag{4.6}$$

then (S) has dominant solutions and does not have intermediate solutions.

For proving Theorem 4.3 the following known results are needed. The first one concerns the change of integration for double integrals; see [5, Lemma 2].

Lemma 4.4. *Let $\alpha < \beta$ and let b be a continuous positive function on $[1, \infty)$. If*

$$\int_1^\infty b(t)t^\beta\, dt < \infty,$$

then

$$\int_1^\infty \Big(\int_t^\infty b(s)\, ds\Big)^{1/\alpha} dt < \infty.$$

Lemma 4.5. *Consider equation* (S) *with* $g(x) \equiv 1$.
(i) *This equation has dominant solutions if and only if*

$$\int_1^\infty b(s)s^\beta\,ds < \infty.$$

(ii) *This equation has subdominant solutions if and only if*

$$\int_1^\infty \Big(\int_t^\infty b(s)\,ds\Big)^{1/\alpha}\,dt < \infty. \tag{4.7}$$

(iii) *This equation does not have simultaneously subdominant, intermediate and dominant solutions.*
(iv) *This equation is oscillatory if and only if*

$$\int_1^\infty \Big(\int_t^\infty b(s)\,ds\Big)^{1/\alpha}\,dt = \infty. \tag{4.8}$$

Proof. Claims (i) and (ii) follow from [9]; see also [12, Sections 18, 19] and [15, Theorems 17.1, 17.2]. For claim (iii) see [6, Corollary 2]. Finally, claim (iv) follows for $\alpha = 1$ from [12, Theorem 1.5.1], and for $\alpha > 0$ from [15, Theorem 6.6]. □

Proof of Theorem 4.3. In view of (4.6) and Proposition 2.2, equation (S) has a nonoscillatory solution z such that

$$\lim_{t\to\infty} z(t) = \infty, \quad \lim_{t\to\infty} z'(t) = \ell_z, \quad 0 < \ell_z < \infty.$$

Now, by contradiction, let x be a solution of (S) such that

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} x'(t) = 0.$$

Without loss of generality, suppose for $t \ge t_0 \ge 1$ that

$$z(t) > x(t) > 0.$$

Consider the following Emden–Fowler equations ($t \ge t_0$):

$$(|v'(t)|^\alpha \operatorname{sgn} v'(t))' + b(t)g(x(t))|v(t)|^\beta \operatorname{sgn} v(t) = 0, \tag{4.9}$$
$$(|w'(t)|^\alpha \operatorname{sgn} w'(t))' + b(t)g(z(t))|w(t)|^\beta \operatorname{sgn} w(t) = 0. \tag{4.10}$$

Obviously, x is a solution of (4.9) and z is a solution of (4.10). From Lemma 4.5 (i) we have

$$\int_{t_0}^\infty b(t)g(z(t))t^\beta\,dt < \infty,$$

and, in view of (4.5),

$$\int_{t_0}^\infty b(t)g(x(t))t^\beta\,dt < \infty.$$

Thus, from Lemma 4.5 (i) equation (4.9) has also dominant solutions. Moreover, from Lemma 4.4 we get

$$\int_{t_0}^\infty \Big(\int_t^\infty b(s)g(x(s))\,ds\Big)^{1/\alpha}\,dt < \infty.$$

Hence, by using again Lemma 4.5 (ii), equation (4.9) has also subdominant solutions, which is a contradiction to Lemma 4.5 (iii). □

5 Oscillation

In the previous section, we have illustrated some similarities between (S) and the corresponding Emden–Fowler equation, which concern the asymptotic behavior of unbounded nonoscillatory solutions. Now, we show that also the oscillation property reads in the same way for (S) and the Emden–Fowler equation

$$(\Phi(x'))' + b(t)|x|^{\beta} \operatorname{sgn} x = 0, \quad t \in I,\ \beta > \alpha. \tag{5.1}$$

Theorem 5.1. *Let* $0 < \alpha < \beta$ *and assume* (4.3) *and* (4.4). *The following three statements are equivalent:*
(i) *Equation* (5.1) *is oscillatory.*
(ii) *Equation* (S) *is oscillatory.*
(iii) *Condition* (4.8) *holds.*

Proof. In virtue of Lemma 4.5, we have (i) ⇔ (iii). Let us prove (iii) ⇒ (ii). By contradiction, assume that (S) has a nonoscillatory solution x such that

$$x(t) > 0, \quad x'(t) > 0 \quad \text{for } t \ge \tau \ge 1.$$

First, let x be an intermediate solution. Integrating (S) on (t, ∞), $t \ge \tau$, and using (4.5), we have

$$\Phi(x'(t)) = \int_t^{\infty} b(s)g(x(s))x^{\beta}(s)\,ds \ge M\int_t^{\infty} b(s)x^{\beta}(s)\,ds \ge Mx^{\beta}(t)\int_t^{\infty} b(s)\,ds$$

or

$$\frac{x'(t)}{x^{\beta/\alpha}(t)} \ge \Big(M\int_t^{\infty} b(s)\,ds\Big)^{1/\alpha}.$$

Thus, integrating again on (τ, t) and setting $\mu = (\beta - \alpha)/\alpha$, we obtain

$$\Big(\frac{1}{x(\tau)}\Big)^{\mu} - \Big(\frac{1}{x(t)}\Big)^{\mu} \ge \mu M^{1/\alpha}\int_{\tau}^{t}\Big(\int_s^{\infty} b(r)\,dr\Big)^{1/\alpha} ds,$$

which is a contradiction to (4.8).

Now, let x be a dominant solution of (S). From Corollary 2.4, there exists $\lambda > 0$ such that $K_\lambda < \infty$. Thus

$$\int_{\tau}^{\infty} b(t)g(\lambda s)s^{\beta}\,ds < \infty.$$

Using (4.5), we obtain

$$\int_{\tau}^{\infty} b(t)s^{\beta}\,ds < \infty.$$

Hence, Lemma 4.4 gives a contradiction.

Now, let x be a subdominant solution of (S). Integrating (S) on (t, ∞), $t \ge \tau$, and using (4.5), we have

$$\Phi(x'(t)) = \int_t^{\infty} b(s)g(x(s))x^{\beta}(s)\,ds \ge Mx^{\beta}(\tau)\int_t^{\infty} b(s)\,ds$$

or

$$x'(t) \ge H\Big(\int_t^{\infty} b(s)\,ds\Big)^{1/\alpha},$$

where $H = (Mx^{\beta}(T))^{1/\alpha}$. Integrating again on (τ, t), we obtain

$$x(t) - x(\tau) \ge H\int_{\tau}^{t}\Big(\int_t^{\infty} b(s)\,ds\Big)^{1/\alpha} dt,$$

which is a contradiction to (4.8).

Finally, let us prove (ii) ⇒ (i). By contradiction, assume that (5.1) has a nonoscillatory solution. Hence, from Lemma 4.5 (iv) we have (4.7). Then a contradiction follows by applying the Tychonov fixed point theorem to the operator $\mathcal{T}$ given by

$$\mathcal{T}(u)(t) = \mu - \int_t^\infty \Phi^*\Big(\int_s^\infty b(r)g(u(r))u^\beta(r)\,dr\Big)\,ds$$

in the set $\Omega \subset C[t_0,\infty)$ defined as

$$\Omega = \Big\{u \in C[t_0,\infty) : \frac{1}{2}\mu \le u(t) \le \mu \text{ for } t \ge t_0\Big\},$$

where μ is a positive constant and t_0 is sufficiently large such that

$$\Phi^*(g(0)\mu^\beta)\int_{t_0}^\infty\Big(\int_t^\infty b(s)\,ds\Big)^{1/\alpha} dt \le \frac{1}{2}\mu.$$

Indeed, reasoning in a similar way as in the proof of Proposition 2.2, we obtain the existence of a subdominant solution x of (S) such that $\lim_{t\to\infty} x(t) = \mu$. The details are left to the reader. □

The following example illustrates our results.

Example 5.2. Consider the equations

$$((x')^2 \operatorname{sgn} x')' + b_1(t)g(x)|x|^{7/2} \operatorname{sgn} x = 0, \quad t \in I = [1,\infty), \tag{5.2}$$

and

$$((x')^2 \operatorname{sgn} x')' + b_2(t)g(x)|x|^{7/2} \operatorname{sgn} x = 0, \quad t \in I = [1,\infty), \tag{5.3}$$

where

$$b_1(t) = t^{-9/2}, \quad b_2(t) = t^{-3}$$

and

$$g(u) = 1 + \frac{1}{1+\log(1+|u|)}.$$

We have $\gamma = 4$. Hence, the function $f(t,u) = b_1(t)g(u)|u|^{7/2}\operatorname{sgn} u$ satisfies (1.1)–(1.3). Moreover, for (5.2) we have $K_\lambda = \infty$ for any $\lambda > 0$. Thus, by Theorem 3.1 and Corollary 2.4, equation (5.2) has infinitely many intermediate solutions which are the only unbounded nonoscillatory solutions of (5.2).

For equation (5.3) the function g satisfies (4.3) and (4.4). Moreover, (4.8) holds and so, by Theorem 5.1, equation (5.3) is oscillatory.

6 Extension

In this section, we illustrate how our results can be extended to the general equation (G) when (1.9) is satisfied. Denote

$$A(t) = \int_1^t \Phi^*\Big(\frac{1}{a(s)}\Big)\,ds + c,$$

where c is a fixed positive constant. The change of the independent variable

$$s = A(t) + 1 - c, \quad X(s) = x(t),\ t \in [1,\infty),\ s \in [1,\infty),$$

transforms (G) to

$$\frac{d}{ds}\Big(\Phi\Big(\frac{d}{ds}X(s)\Big)\Big) + \Phi^* l(a(t(s)))f(t(s),X(s)) = 0,$$

where $t(s)$ is the inverse function of $s(t)$. A standard calculation shows that if $f(t, u)$ satisfies conditions (1.1) and (1.3), then the same occurs for

$$\varphi(s, u) = \Phi^*\big(a(t(s))\big)f(t(s), u).$$

Thus, our previous results for (E) can be easily formulated for (G). For instance, for equation (G) Theorem 3.1 reads as follows.

Theorem 6.1. *Let a be continuously differentiable for $t \in I$. Assume that* (1.1) *and* (1.9) *hold and that for some $T \geq 1$ the function*

$$A^\gamma(t)\Phi^*(a(t))|f(t,u)| \quad \text{is nonincreasing with respect to } t \in [T, \infty),\ T \geq 1,$$

for any $u \in \mathbb{R}$. If condition (1.3) *holds and*

$$\int_1^\infty f(s, \lambda A(s))\, ds = \infty$$

for any $\lambda > 0$, then equation (G) *has infinitely many nonoscillatory solutions x defined on $[T, \infty)$ such that*

$$\lim_{t\to\infty} x(t) = \infty, \quad \lim_{t\to\infty} a(t)\Phi(x'(t)) = 0.$$

In a similar way, Theorem 4.3 and Theorem 5.1 can be formulated for

$$(a(t)\Phi(x'))' + b(t)g(x)|x|^\beta \operatorname{sgn} x = 0, \quad t \in I,\ \alpha < \beta.$$

The details are left to the reader.

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