Bender-Wu singularities

Riccardo Giachetti and Vincenzo Grecchi

Citation: J. Math. Phys. **57**, 122109 (2016); doi: 10.1063/1.4972290
View online: http://dx.doi.org/10.1063/1.4972290
View Table of Contents: http://aip.scitation.org/toc/jmp/57/12
Published by the American Institute of Physics

JOURNAL OF MATHEMATICAL PHYSICS **57**, 122109 (2016)

Bender-Wu singularities

Riccardo Giachetti[1] and Vincenzo Grecchi[2]
[1]*Dipartimento di Fisica e Astronomia, Università di Firenze, I.N.F.N., 50019 Sesto Fiorentino, Sezione di Firenze, Italy*
[2]*Dipartimento di Matematica, Università di Bologna, GNFM, INdAM, 40126 Bologna, Roma, Italy*

(Received 19 August 2016; accepted 4 December 2016; published online 27 December 2016)

We consider the properties of the family of double well quantum Hamiltonians $H_\hbar = -\hbar^2 (d^2/dx^2) + i(x^3 - x)$, $x \in \mathbb{R}, \hbar > 0$, starting from the resonances of the cubic oscillator $H_\epsilon = -(d^2/dx^2) + x^2 + \epsilon x^3$, $\epsilon > 0$, and studying their analytic continuations obtained by generalized changes of representation. We prove the existence of infinite crossings of the eigenvalues of $H_\hbar$ together with the selection rules of the pairs of eigenvalues taking part in a crossing. This is a semiclassical localization effect. The eigenvalues at the crossings accumulate at a critical energy for some of the Stokes lines. *Published by AIP Publishing.* [http://dx.doi.org/10.1063/1.4972290]

I. INTRODUCTION AND STATEMENT OF THE RESULTS

In this paper we are concerned about some spectral properties of the family of cubic oscillators described by the closed Hamiltonians

$$H_\hbar = -\hbar^2 (d^2/dx^2) + V(x)\,,\ x \in \mathbb{R}\,, \qquad V(x) = i(x^3 - x)\,. \tag{1.1}$$

Their definition domain is

$$\mathcal{D}_H = D(\,d^2/dx^2) \bigcap D(|x|^3) \tag{1.2}$$

and $\hbar$ is a positive parameter. More precisely we prove the existence of an infinite number of crossings of the eigenvalues $E_n(\hbar)$ (or *levels*) and we specify the selection rules for the levels participating in a given crossing. We show that, actually, the crossing is possible for a pair of levels only: indeed for decreasing $\hbar$ a single node is unstable and the localization can be achieved in only one of the two wells. Since the spectrum is simple, the functions $E_n(\hbar)$ are not holomorphic. At the value of $\hbar$ that corresponds to the crossing, there is a branch point called Bender-Wu singularity.[1–3] For real $\hbar > 0$ the Hamiltonians (1.1), and in particular the potential, are PT-symmetric operators.[4–7] As $V(x)$ has two stationary points at $x_\pm = \pm 1/\sqrt{3}$, we will speak of a PT-symmetric double well oscillator. In contrast with the real case, the eigenfunctions $\psi_n(x) = \psi_n(x,\hbar)$ (or *states*) can possibly localize at one of the wells.

Anharmonic oscillators are basic non-solvable models in quantum mechanics. They pose a problem of convergence of the perturbation series analogous to the one encountered in quantum field theory: this is a major reason for which they have been extensively investigated for a long time.[8–14] The divergence of the perturbation series of the levels is related to the existence of the Bender-Wu singularities.

The cubic oscillator is probably the simplest non-solvable model and has been considered by several approaches.[13] We use here the nodal analysis,[15,16] namely, the study of the confinement of the zeros of the entire eigenfunctions $\psi_n(x)$ and of their derivatives. We take advantage of the following techniques: the Loeffel-Martin method for the control of the zeros;[8] the asymptotic behavior of Sibuya;[17] the semiclassical accumulation of the zeros in some of the Stokes lines;[18] the exact semiclassical quantization (see (4.3) and (4.5) in Section IV). Moreover, the results of perturbation theory[13,16] and the unique summability of the perturbation series are relevant too. In such a way we believe that we can draw an exhaustive picture of the level crossing phenomenon. We must mention that the semiclassical theory[1–3] has produced good results for low values of the parameter $\hbar$ up to the crossing value. These have been extended by the exact semiclassical theory[19–21] to larger, although not very large values of $\hbar$. A still different rigorous technique is then found in Refs. 22 and 23.

Of course all these treatments are very useful and complementary to ours, presented in Refs. 24 and 25 in a preliminary and not completely rigorous form. We refer to these papers for details of the crossing processes as a function of the parameter. The paper[24] is just an announcement of the results of Ref. 25 and of the present paper. In Ref. 24 the relationship between the localization of the states and the crossings was mainly emphasized. In Ref. 25 we gave a detailed description of the crossing essentially based on the results of numerical calculations. We assumed the existence of the exact Stokes lines, dependent upon $(E,\hbar)$ for $E \in \sigma(H_\hbar)$ and $\hbar > 0$, as defined in Ref. 19. The zeros of the state should lay on these lines. We called the 'vibrating string' the exact Stokes line which should contain the nodes and the "sound board" the other one which should contain the remaining zeros. Attention was given to the sequence of nodes and antinodes of a state in the vibrating string. We followed, by computational methods, the crossing process for decreasing $\hbar$ and we presented many figures showing, before the crossing, the presence of an antinode near each well. We thus depicted a smooth transition from delocalization to bilocalization. Moreover, a node of the odd state passes into the sound board, disappearing as a node. A dramatic breaking of the vibrating string then occurs at the crossing and each one of the two emerging states localizes in a different well. The proof of the confinement of the zeros is however absent. Here we achieve this result by using the semiclassical limit of the zeros which accumulate in some of the Stokes lines at the limit energies. We recall that a Stokes line is defined for complex values of the energy E, starts from one of the turning points (called $I_\pm$ and I_0), and satisfies at each one of its points $z \in \mathbb{C}$ the condition

$$- p_0^2(E,z)\, dz^2(z) > 0, \qquad p_0^2(E,z) = V(z) - E\,, \tag{1.3}$$

where $dz(z)$ is the tangent vector at z. The choice of the sign is consistently defined in all the complex plane, so that the Stokes lines are given an orientation. We are interested in two particular Stokes lines: the internal one (hereafter called the *oscillatory range* $\rho(E)$) and the exceptional one (called the *escape line* $\eta(E)$).[18] Their union $\rho(E) \bigcup \eta(E)$ is the *union of the classical trajectories* $\tau(E)$. In particular we consider $\eta(\pm E_0)$ and $\rho(\pm E_0)$ which reduces to a point. In Lemma (2.1) we prove the confinement of $\eta(\pm E_0)$. Using the results of Ref. 18 we then prove the confinement of the zeros and in particular of the nodes. A further new result obtained in this paper is the proof that the sequence of crossings is semiclassical.

We recall that analytic families of self-adjoint Hamiltonians[26] cannot develop level crossings with Bender-Wu singularities. The same happens for many other families of PT-symmetric Hamiltonians with a single well potential,[6,16] where the PT action on a state is $PT\,\psi(x) = \bar{\psi}(-x)$. In the following we will use the results established for two more families defined on the same domain (1.2), whose potentials are still cubic but different from $V(x)$. Those families are related to $H_\hbar$ by a generalized change of representation. The first one is an analytic family of type A[26] of single well complex cubic oscillators,[16]

$$\widetilde{H}_\beta = -(d^2/dx^2) + x^2 + i\sqrt{\beta}x^3, \qquad \beta \neq 0, \quad |\arg(\beta)| < \pi\,. \tag{1.4}$$

This will be necessary in order to identify a semiclassical level as the continuation of a perturbative one. All the levels $\widetilde{E}_n(\beta)$ of $\widetilde{H}_\beta$ and the corresponding states $\widetilde{\psi}_n(\beta)$ are perturbative. Their labels n are determined by the number of zeros which are stable at $\beta = 0$ (or *nodes*). The $n+1$ stationary points of $\widetilde{\psi}_n(\beta)$ stable at $\beta = 0$ will be called *antinodes*. Notice that $H_{\beta=0}$ reduces to a harmonic oscillator whose states are concentrated in the interval $[-\sqrt{E}, \sqrt{E}\,]$, namely, about their antinodes. From the physical point of view, the levels $\widetilde{E}_n(\beta)$ at $\arg\beta = -\pi$ are the resonances (in the sense of the Gamow boundary condition at $-\infty$) of a real cubic oscillator with Hamiltonian[13]

$$H_\epsilon = -(d^2/dx^2) + x^2 + \epsilon x^3\,, \qquad \epsilon = \sqrt{|\beta|}.$$

In the paper[16] Martinez and one of us (V.G.) have extended to (1.4) the proof of the absence of crossings.

For later use we show here the relationship between (1.1) and (1.4). We first make a translation of $H_\hbar$ centered at each of the two wells, letting $x = y + x_\pm = y \pm 1/\sqrt{3}$. We then define the two isospectral Hamiltonians

$$H_\hbar^\pm = -\hbar^2 (d^2/dy^2) + i(y^3 \pm \sqrt{3} y^2) \pm E_0, \qquad E_0 = -ic\,, \quad c = \frac{2}{3\sqrt{3}}\,. \tag{1.5}$$

In order to apply the perturbation theory,[16] we make the dilations[27]

$$y = \lambda^\pm(\hbar) z\,, \qquad \lambda^\pm(\hbar) = 3^{-1/8} \exp(\mp i\pi/8)\sqrt{\hbar}.$$

Letting

$$c^\pm = 3^{1/4}\sqrt{\pm i}, \qquad \beta^\pm(\hbar) = 3^{-5/4} \exp(\mp i5\pi/4)\,\hbar, \tag{1.6}$$

we find the isospectrality ($\sim$),

$$\widetilde{H}_{\beta^\pm(\hbar)} \sim (1/\hbar c^\pm)\, H_\hbar^\pm \mp E_0\,, \qquad \widetilde{E}_n(\beta^\pm(\hbar)) = (1/\hbar c^\pm)\, E_n^\pm(\hbar) \mp E_0\,. \tag{1.7}$$

It can be proved rather easily that for small enough values of $\hbar$, the levels $E_n^\pm(\hbar)$ are respectively obtained from $E_n(\hbar \exp(\pm i\pi/4))$ by analytic continuations along paths in the complex plane of $\hbar$ with fixed $|\hbar|$ (see Ref. 13). The functions $E_n^\pm(\hbar)$ are non-real and complex conjugated.

The necessity of level crossings is determined by comparing the behavior of the levels $E_n(\hbar)$ for large $\hbar$ with the behavior of $E_n^\pm(\hbar)$ for small $\hbar$. It is thus very convenient to introduce a parametrization of the Hamiltonians (1.1) more suited for large $\hbar$. Once again in the domain (1.2) we define the new family

$$\widehat{H}_\alpha = -(d^2/dx^2) + i(x^3 + \alpha x), \qquad \alpha \in \mathbb{C} \tag{1.8}$$

with levels $\widehat{E}_n(\alpha)$ and states $\widehat{\psi}_n(\alpha)$. The simple regular dilation

$$x \to \lambda x\,, \qquad \lambda = \sqrt{-\alpha} = \hbar^{-2/5} \tag{1.9}$$

yields the relationship

$$\widehat{E}_n(\alpha(\hbar)) = \hbar^{-6/5} E_n(\hbar)\,, \qquad \alpha(\hbar) = -\hbar^{-4/5} \leq 0\,. \tag{1.10}$$

Hence $\hbar^{-6/5} E_n(\hbar) \to \widehat{E}_n(0)$ as $\hbar \to +\infty$. The eigenvalues $\widehat{E}_n(\alpha)$ are therefore holomorphic in a neighborhood of the origin and on the sector

$$\mathbb{C}_\alpha = \{\alpha \in \mathbb{C},\, \alpha \neq 0, |\arg(\alpha)| < 4\pi/5\}. \tag{1.11}$$

In view of (1.7) we have the isospectrality,

$$\widetilde{H}_\beta \sim \beta^{1/5}\, \widehat{H}_{\alpha(\beta)} - \frac{2}{27\beta}, \quad \widetilde{E}_n(\beta) = \beta^{1/5}\, \widehat{E}_n(\alpha(\beta)) - \frac{2}{27\beta}, \quad \alpha(\beta) = \frac{1}{3\beta^{4/5}}\,. \tag{1.12}$$

These relations are obtained by composing the following analytic translation and dilation:[27]

$$x \to x + i/(3\sqrt{\beta})\,, \qquad x \to \beta^{-1/10}\, x\,. \tag{1.13}$$

Later on we will prove that the eigenvalues $\widehat{E}_n(\alpha)$ are real analytic for $\alpha \in \mathbb{R}$ in a neighborhood of the origin. Through (1.10) and (1.12) the levels $E_n(\hbar)$ are analytic continuations of the perturbative levels $\widetilde{E}_n(\beta)$ and can be extended as many-valued functions to the sector of the $\hbar$ complex plane

$$\mathbb{C}^0 = \{\hbar \in \mathbb{C} \mid \hbar \neq 0, |\arg(\hbar)| < \pi/4\}\,. \tag{1.14}$$

We can now formulate the crossing selection rule in very simple terms. The two positive levels $E_{m^\pm}(\hbar)$, $m^\pm = 2n + (1/2) \pm (1/2)$, defined for sufficiently high $\hbar$, undergo a crossing at a value $\hbar = \hbar_n$ and become the two complex conjugate levels $E_n^\pm(\hbar)$ defined in (1.7) for $\hbar < \hbar_n$. The corresponding states $\psi_n^\pm(\hbar)$ are PT-conjugated. We call $E_n^c > 0$ the limit level at $\hbar = \hbar_n$; ψ_n^c is the corresponding PT-symmetric state.

This process is made possible by the instability of a single node of $\psi_{m^+}(\hbar)$ and by the instability of the PT-symmetry of both the states $\psi_{m^\pm}(\hbar)$. More explicitly, the crossing rule is described in terms of the analytic continuations as follows. The two functions $E_{m^\pm}(\hbar)$ are holomorphic for large $|\hbar|$; they are analytically continued in $\mathbb{C}$ along the positive semi-axis for decreasing $\hbar$ by passing above the singularity at $\hbar = \hbar_n$; they respectively become the two levels $E_n^{\mp}(\hbar)$ for small $\hbar > 0$. Thus the Bender-Wu singularities are square root branch points. We have also proved that the sequence of crossings is semiclassical: as $n \to \infty$, we have the limits $\hbar_n \to 0$ and $E_n^c \to E^c \geq 0$, where E^c is an instability point of the set of the Stokes lines, supposedly unique.

Some more final comments are in order. In the first place we believe that our method for the control of the nodes, based here on the accumulation of the zeros and on the confinement of the Stokes lines, gives clear hints to understand the physical aspects of the problem. Second, we observe that we have extended the method from the case in which the singularities are absent,[16] to a more general context which admits the presence of singularities. Moreover we think that the relation between singularities and double well localization is universal in polynomial oscillators.

We give a brief summary of the content of Secs. II–V, where all the statements will be rigorously proved. In Sec. II we study the levels and the states for small parameter $\hbar$. In particular, we show the stability of the nodes in one of the half-planes $\pm\Re z > 0$. In Sec. III we deal with the behavior of the levels and of the nodes for large values of $\hbar$. We prove the confinement of the nodes, the positivity of the spectrum, and the possible existence of only one imaginary node, which is the one becoming unstable at $\hbar_n$. We also prove the selection rules of the crossings. In Sec. IV we show the semiclassical nature of the problem, we state the exact quantization rules, and we prove the local boundedness of the levels. We then determine the crossing rules and we consider the Riemann surfaces of the levels in a neighborhood of the real axis of $\hbar$. In Sec. V we draw some conclusions and suggest possible extensions to complex values of the parameter $\hbar$.

II. CONFINEMENT OF THE ESCAPE LINE AT $E = E_n^{\pm}(0)$ AND THE ZEROS OF $\psi_n^{\pm}(z, \hbar)$ FOR SMALL $\hbar$

As announced in the Introduction, we prove here the confinement of the nodes for low values of the parameter $\hbar$. For later use it is convenient to introduce the following notations:

$$\mathbb{C}^{\pm} = \{z\,,\,\Re(z) \gtrless 0\}, \qquad \mathbb{C}_{\pm} = \{z\,,\,\Im(z) \gtrless 0\}. \tag{2.1}$$

With the definitions (1.5) of E_0 and c, we prove the following lemma.

Lemma 2.1. The escape lines $\eta(E)$ at the levels $E = \mp E_0$, are in $\mathbb{C}^{\pm}$ respectively. $\rho(\mp E_0)$ are the stationary points $x_{\mp} \in \mathbb{C}^{\mp}$.

Proof. We fix $E = -E_0$. The case $E = E_0$ is completely analogous. The oriented exceptional Stokes line starts from the turning point $I_0 = 2/\sqrt{3} \in \mathbb{C}^+$. The two turning points $I_{\pm}$ coincide and we have $\rho(E) = I_+ = I_-$. Using the variable $w = y - ix$, the condition (1.3) for the Stokes field becomes

$$-\,p_0^2(E, w)\,dw^2 = (w^3 + w + E)\,dw^2 > 0. \tag{2.2}$$

For $w = -iI_0 + \delta$, at the first order in δ we have $p_0^2(-iI_0 + \delta)\,\delta^2 \sim 3\delta^3 < 0$. Hence $\delta^3 < 0$ and $\arg\delta = \pm\pi/3$. In the z plane, $z = I_0 + i\delta$ with $\arg(i\delta) = (\pi/2) \pm \pi/3$. The choice $\arg(i\delta) = (\pi/2) + \pi/3 = 5\pi/6$ yields the oriented exceptional Stokes line $\eta(-E_0)$. In the z plane this line is asymptotic to the imaginary axis at $+i\infty$. For large $y > 0$ the behavior of the action integral is

$$S(w) = \int_{-iI_0}^{w} \sqrt{-p_0^2(E, w)}\; dw(w) \;\sim\; S_a(w) = (2/5)w^{5/2} + w^{1/2} + Ew^{-1/2}. \tag{2.3}$$

Thus, if $w = w(y) = y - ix(y)$ and $x(y) \to 0$ as $y \to \infty$, we have

$$S_a(w(y)) = (2/5)\,y^{5/2} + y^{1/2} - i\,y^{3/2}\big((y^2 + 1/2)\,x(y) - \Im E\big)\,. \tag{2.4}$$

$\Im S_a(w(y)) = 0$ implies therefore

$$x(y) \sim c/(y^2 + 1/2) \tag{2.5}$$

and the escape line $\eta(-E_0)$ stays in $\mathbb{C}^+$. On the imaginary axis, the vectors $\pm dw(y)$ are determined by the condition

$$p_0^2(E, y)\,dw^2(y) < 0, \qquad p_0^2(y) = -(y^3 + y) + E. \tag{2.6}$$

The regular field of velocities on the imaginary axis satisfies $dy > 0$. By making explicit the two conditions (2.6),

$$\begin{aligned} \Re(p_0^2(E,y)\,dw^2(y)) &= (y^3+y)(dx^2-dy^2)+2\,c\,dx\,dy\,<0, \\ \Im(p_0^2(E,y)\,dw^2(y)) &= (y^3+y)\,2\,dy\,dx - c\,(dx^2-dy^2)=0, \end{aligned} \tag{2.7}$$

and substituting the equality (2.7) into the inequality, we find

$$(2/c)\big((y^3+y)^2+c^2\big)\,dx\,dy\,<0.$$

From $dy > 0$ we get $dx < 0$. Moreover, since the field of vectors $dw(y)$ is regular on the imaginary axis, the oriented line $\eta(E_0)$ could exit but not come back to $\mathbb{C}^+$: so that it always stays in $\mathbb{C}^+$. □

The following useful result is a consequence of Lemma 2.1 and of the exact semiclassical theory (see Ref. 18, Theorem 1).

Corollary 2.2. Consider the zeros of $\psi_n^\pm(z,\hbar)$ with energy $E = E_n^\pm(\hbar)$ and small $\hbar$. In the limit $\hbar \to 0^+$, $E_n^\pm(\hbar) \to \pm E_0$, all the n nodes go to $x_\pm \in \mathbb{C}^\pm$ and all the other zeros go to $\eta(\pm E_0) \in \mathbb{C}^\mp$. □

We now study the behavior of levels and states in the semiclassical limit.

Lemma 2.3. For any $n \in \mathbb{N}$, there exists $\hbar_n > 0$ such that for $0 < \hbar < \hbar_n$, the levels $E_n^\pm(\hbar)$ are complex conjugate levels and the corresponding states $\psi_n^\pm(\hbar,x)$ are PT-conjugated,

$$\psi_n^-(\hbar) = PT\psi_n^+(\hbar). \tag{2.8}$$

Each one of the entire functions $\psi^\pm(z)$ has n nodes respectively tending to the points $x_\pm \in \mathbb{C}^\pm$ as $\hbar \to 0^+$. Their kernels in $\mathbb{C}$ are P_x-conjugated, namely, $\ker\psi_n^-(z) = P_x \ker\psi_n^+(z)$, where $P_x f(x+iy) = f(-x+iy)$.

Proof. The isospectrality of $H_\hbar^\pm$ and $\widehat{H}_{\beta^\pm(\hbar)}$ has already been established in (1.5)-(1.7). Notice that for positive $\hbar$ the parameters $\beta^\pm(\hbar)$ are not in the complex plane cut along the negative axis, $\mathbb{C}_c = \{z \in \mathbb{C};\ z \neq 0,\ |\arg z| < \pi\}$. This means that the results of Ref. 16 are not sufficient by themselves, but we also need some of the results of Ref. 13. In particular we use the fact that there exists a $b_n > 0$ such that the perturbative level $\widetilde{E}_n(\beta)$ admits analytic continuations in the open disks of radius b_n and centers at $\exp(\pm i\pi)\,b_n$, respectively. The perturbation theory yields that the semiclassical behavior of the levels is

$$E_n^\pm(\hbar) = \pm E_0 + \hbar c^\pm(2n+1) + O(\hbar^2)\,. \tag{2.9}$$

We prove that the corresponding states $\psi_n^\pm(\hbar)$ are $PT-$ conjugated for a suitable choice of the phase factors. Indeed PT is a bounded involution. Therefore, from the relation $H\psi^+ = E^+\psi^+$ we get

$$(PT\,H\,PT)\,(PT\psi^+) = H(PT\psi^+) = \bar{E}^+(PT\psi^+) = E^-(PT\psi^+),$$

which implies (2.8) since the spectrum is simple. Moreover, the set of the zeros of ψ^- is the reflection with respect to the imaginary axis of the set of zeros of ψ^+, or $\ker(\psi^-) = P_x\ker(\psi^+)$. It is relevant to notice that in the perturbation theory of H_β (1.4), all the nodes of $\widetilde{\psi}_n(\beta^\pm(\hbar))$ have a semiclassical limit in $\rho(2n+1) = [-\sqrt{2n+1}, \sqrt{2n+1}\,]$, while the corresponding nodes of the semiclassical functions $\psi_n^\pm(\hbar)$ go to the stationary points $x_\pm$ respectively. □

We next prove that the zeros $\psi_n^\pm(\hbar)$ are stably confined in $\mathbb{C}^\pm$ respectively, so that such zeros coincide with the nodes tending to $x_\pm$ as $\hbar \to 0$. This shows that no crossing can exist between levels of the same set $\{E_n^-(\hbar)\}$ or $\{E_n^+(\hbar)\}$. Since $E_n^\pm(\hbar)$ are complex conjugated for small $\hbar$, the levels $E_n^-(\hbar)$ and $E_n^+(\hbar)$ will cross at $\hbar_n$, where they become real.

Lemma 2.4. Let $E = E_n^\pm(\hbar)$ be the non-real levels at $0 < \hbar < \hbar_n$. The corresponding states $\psi_n^\pm(z)$ are non-vanishing on the imaginary axis.

Proof. Let E be one of the non-real levels and $\psi(z)$ the corresponding state. Let $\phi(y) = \psi(iy)$, $y \in \mathbb{R}$, be the eigenfunction on the imaginary axis. With $w = y - ix$, for a fixed x, we define the Hamiltonian

$$H^r_\hbar(x) = -\hbar^2(d^2/dy^2) - w^3 - w \tag{2.10}$$

having a level $-E$. The corresponding translated state, $\phi_x(y) = \phi(w)$, has the well-known asymptotic behavior for large y,[17]

$$\phi_x(y) = \frac{C\big(1 + O(y^{-1/2})\big)}{\sqrt{p_0(E,w)}} \cos\Big(\frac{S_a(w)}{\hbar} + \theta\Big), \tag{2.11}$$

where $C > 0$, $\theta \in \mathbb{R}$, $p_0(E,w)$ is defined in (2.2) and $S_a(w)$ in (2.3). We have

$$|\phi(y)|^2 = |\phi_0(y)|^2 = O(|y|^{-3/2}) \quad \text{as} \quad y \to +\infty\,. \tag{2.12}$$

We now consider the Loeffel-Martin formula for $\phi(y)$, producing the same result of the law of the imaginary part of the shape resonances,

$$\hbar^2\Im(\bar{\phi}(y)\phi'(y)) = -\Im E \int_y^\infty |\phi(s)|^2 ds, \forall y \in \mathbb{R}. \tag{2.13}$$

Due to the asymptotic behavior (2.12), the integral exists and is finite. Therefore $\psi^\pm_n(z,\hbar)$ is not vanishing for z on the imaginary axis. □

Lemma 2.5. Let $E = E^\pm_n(\hbar)$ and $\psi^\pm_n(z)$ as above. The large zeros $Z^\pm_j$ of $\psi^\pm_n(z)$ are in the half-planes $\mathbb{C}^\mp$ and their nodes are stable in $\mathbb{C}^\pm$ respectively.

Proof. For $y \to \infty$, $x(y) \to 0$, we get from (2.11) the asymptotic condition

$$\Im\big(S_a(w(y)) + \hbar\theta\big) \sim -\big(y^{3/2} + 1/2\big)\,x(y) + \Im E + \hbar y^{1/2}\Im\theta \;= 0.$$

If $\Im\theta = 0$, the large zero $Z_j = x(y) + iy$ has an asymptotic behavior with

$$x(y) \sim (\Im E^\pm_n + \hbar y^{1/2}\,\Im\theta)\,/\,(y^2 + 1/2). \tag{2.14}$$

These behaviors (2.14) are imposed by the continuity of the zeros and their impossibility of crossing the imaginary axis established in Corollary 2.2. This proves the stability of the zeros (nodes) in $\mathbb{C}^\mp$ respectively. At the limit of $\hbar \to \hbar^-_n$ the energies $E^\pm_n(\hbar)$ become positive and the large zeros $Z^\pm_j$ become imaginary. □

From Ref. 18, Corollary 2.2, continuity of the nodes, and Lemma 2.4, we have the following.

Corollary 2.6. Let $E = E^\pm_n(\hbar)$ and $\psi^\pm_n(z)$ as above. For small $\hbar$ all the n nodes are contained in a neighborhood of $x_\pm$, $U_\pm \subset \mathbb{C}^\pm$. □

We finally prove the analyticity of the levels $E^\pm_n(\hbar)$.

Proposition 2.7. The two functions $E^\pm_n(\hbar)$ are analytic for $0 < \hbar < \hbar_n$. The two levels $E^\pm_n(\hbar)$ and the two states $\psi^\pm_n(\hbar)$ coincide at the crossing limit $\hbar \to \hbar^-_n$. The limit level E^c_n is positive. The limit state $\psi^c_n(z)$ is P_xT-symmetric and has $2n$ non-imaginary zeros. The large zeros are imaginary.

Proof. According to Lemma 2.3 and Lemma 2.5, for $\hbar < \hbar_n$ the n nodes of the two states $\psi^\pm_n(\hbar)$ are the only zeros in $\mathbb{C}^\pm$, respectively. Since the states $\psi^\pm_n(\hbar)$ are the only ones having n zeros in $\mathbb{C}^\pm$, the functions $E^\pm_n(\hbar)$ are analytic. From the relation (2.8) for $\hbar < \hbar_n$ and the limit $\psi^\pm_n(\hbar) \to \psi^c_n$ when $\hbar \to \hbar^-_n$ we get $\psi^c_n = PT\psi^c_n$. As the states $\psi^\pm_n(\hbar)$ have only n zeros in $\mathbb{C}^\pm$ and at the limit $\hbar \to \hbar^-_n$ these zeros cannot diverge or become imaginary. The limits of the $2n$ non-imaginary zeros of both the states $\psi^\pm(\hbar)$ are all the non-imaginary zeros of ψ^c_n. □

III. ANALYSIS OF LEVELS AND NODES FOR LARGE $\hbar$

We have stated in the Introduction that level crossing comes from looking at the behavior of the levels for small and large $\hbar$. We also described in (1.8)-(1.13) the appropriate scaling for dealing with large values of $\hbar$ or β, corresponding to small values of α. We are now going to prove the confinement in two regions of the zeros of $\widehat{\psi}_n(\alpha)$, for small α. We define "nodes" the zeros confined in one of these regions by identifying them with the nodes of the states $\widetilde{\psi}_n(\beta)$ for large β. We find it useful to introduce the two disjoint sets

$$\begin{aligned} \mathbb{C}_\rho &= \{z = x + iy,\ y < 0, |x| < -\sqrt{3}\, y\} \ \subset\ \mathbb{C}_-, \\ \mathbb{C}_\eta &= \{z = x + iy,\ y > 0, |x| < -\sqrt{3}\, y\} \ \subset\ \mathbb{C}_+. \end{aligned} \tag{3.1}$$

Lemma 3.1. The m nodes of $\widehat{\psi}_m(\alpha)$ for small $|\alpha|$ are confined in $\mathbb{C}_\rho$ and correspond to the nodes of $\widetilde{\psi}_m(\beta)$ in $\mathbb{C}_-$. The remaining zeros of $\widehat{\psi}_m(\alpha)$ are in $\mathbb{C}_\eta$. The function $\widehat{E}_m(\alpha(\beta))$ is real analytic and coincides with $\widetilde{E}_m(\beta)$ by (1.12). The state $\widehat{\psi}_m(\alpha)$ is PT-symmetric and coincides with $\widetilde{\psi}_m(\beta)$.

Proof. Define the translated operator $\widehat{H}_{\alpha=0}$ by $x \to x + iy$,

$$\widehat{H}_{\alpha=0,\ y} = -(d^2/dx^2) + V_y(x), \qquad V_y(x) = y\,(y^2 - 3x^2) + ix\,(x^2 - 3y^2). \tag{3.2}$$

Apply the Loeffel-Martin method[8] to the state $\psi = \widehat{\psi}_m(\alpha = 0)$ with energy $E = \widehat{E}_m(\alpha = 0) > 0$, for $\pm x \geq \sqrt{3}\,|y|$,

$$-\Im\,[\overline{\psi}(x+iy)\partial_x\psi(x+iy)] = \int_x^\infty \Im V_y(s)\,|\psi(s+iy)|^2 ds =$$
$$\int_x^\infty (s^2 - 3y^2)s|\psi(s+iy)|^2 ds = -\int_{-\infty}^x (s^2 - 3y^2)s|\psi(s+iy)|^2 ds \neq 0.$$

For $\alpha = 0$ the nodes are thus rigorously confined in $\mathbb{C}_\rho$. The confinement extends then to $\alpha > 0$. Since the m zeros of $\widehat{\psi}_m(\alpha)$ on $\mathbb{C}_-$ are stable for $\alpha \to +\infty$, they are nodes by definition. By (1.13) the set $\mathbb{C}_-$ is invariant for positive dilations in the limit of infinite β and for $\beta > 0$ the m nodes of $\widetilde{\psi}_m(\beta)$ are in $\mathbb{C}_-$.[16] By (1.12) the level $\widehat{E}_m(\alpha)$ for small $|\alpha|$ connects the perturbative level $\widetilde{E}_m(\beta)$ with the level $E_m(\hbar)$ for large positive β and $\hbar$ respectively. The state $\widehat{\psi}_m(\alpha)$ is defined by the number m of its zeros in $\mathbb{C}_-$ which, by Lemma 3.1, are identified as nodes. By continuity the number of nodes of $\widehat{\psi}_m(\alpha)$ is stable for $0 < |\alpha| < \epsilon$ and the scaling (1.9) is regular and phase preserving, although unbounded. The function $\widehat{E}_m(\alpha)$ is therefore real analytic for all $\alpha > 0$ due to the results of Ref. 16 and to (1.10). This property is stable for small $-\alpha > 0$, up to the first crossing and the function $E_m(\hbar)$ is real analytic by (1.10) and (1.12). Finally the m nodes of $\psi_m(\hbar)$ for large positive $\hbar$ are all its zeros in $\mathbb{C}_\rho$ and in $\mathbb{C}_-$. In an analogous way the state $\widehat{\psi}_m(\alpha(\beta))$ coincides with the PT-symmetric perturbative state $\widetilde{\psi}_m(\beta)$[16] proving the lemma. □

Corollary 3.2. For large $\hbar$ the level $E_m(\hbar)$ exists and is positive. The state $\psi_m(\hbar)$ is PT-symmetric.

Proof. Let $\widehat{\psi}_m(\alpha)$ be the normalized state corresponding to the level $\widehat{E}_m(\alpha)$. The real part of $\widehat{E}_m(\alpha)$ is positive due to the positivity of $-(d^2/dx^2)$,

$$\Re\widehat{E}_m(\alpha) = \Re\langle\widehat{\psi}_m(\alpha), \widehat{H}_\alpha\,\widehat{\psi}_m(\alpha)\rangle = \langle\widehat{\psi}_m(\alpha), -(d^2/dx^2)\,\widehat{\psi}_m(\alpha)\rangle\ >\ 0.$$

The result follows from (1.10). □

We now establish some properties of the PT-symmetric states which will be useful later on. In analogy to (3.2) we consider the translated operator

$$H_{\hbar,y} = -\hbar^2\,(d^2/dx^2) + V(x+iy) \tag{3.3}$$

with V as in (1.1).

Lemma 3.3. A PT-symmetric state $\psi_y(x) = \psi(x + iy)$ of the translated operator $H_{\hbar,y}$ has an even real part and an odd imaginary part. In particular, at the origin it satisfies the conditions

$$\Im\psi_y(0) = \Im\psi(iy) = 0, \qquad \Re\psi'_y(0) = \Re\psi'(iy) = 0. \tag{3.4}$$

Proof. Indeed if $\psi_y(x) = R(x) + iI(x)$ we have

$$PT(R(x) + iI(x)) = R(-x) - iI(-x) = R(x) + iI(x). \qquad \square$$

We previously said that a zero $Z_j(\hbar)$ of $\psi_m(\hbar, z)$ is a node when its continuation $Z_j(\hbar')$ to a parameter $\hbar' > \hbar$, large enough, belongs to $\mathbb{C}_-$. We also recall that an imaginary zero $Z_j(\hbar)$ of the state $\psi_m(\hbar, z)$ stays imaginary for any $\hbar' > \hbar$, because of the P_x-symmetry of its kernel and the simplicity of the spectrum.

Lemma 3.4. We consider the PT-symmetric state $\psi_m(x)$. There is an alternative:

(a) the absence of imaginary nodes of the function $\psi_m(z)$;
(b) the existence of only one imaginary node of the function $\psi_m(z)$. The second case is possible if and only if m is odd.

Proof. For $\hbar$ large enough the Hamiltonian (1.1) admits a positive level $E_m = E_m(\hbar)$ with eigenfunction $\psi_n(z) = \psi_n(z,\hbar)$. We consider the Hamiltonian (2.10) on the imaginary axis, $x = 0$ or $w = y$, and we observe that $H^r_\hbar = H^r_\hbar(x = 0) = -H_\hbar$ is real. The eigenfunction $\phi_m(y) = \psi_m(iy)$ of $H^r_\hbar$ is also real by the conditions (3.4) at the origin. $H^r_\hbar$ and $-H_\hbar$ have the same spectrum and $-E_n = -E_n(\hbar) < 0$ is one of its eigenvalues. Therefore, for large positive y the solution $\phi_m(y)$ is the function $\phi_{x=0}(y)$ given in (2.11). For large $-y > 0$ the solution $\phi_m(y)$ is a real combination of the two fundamental solutions[17] and reads

$$\phi_m(y) = \frac{C'\big(1 + O((-y)^{-1/2})\big)}{\sqrt{p_0(E, y)}} \Big(\exp\big(\frac{S_a(-y)}{\hbar}\big) + a\exp\big(-\frac{S_a(-y)}{\hbar}\big)\Big), \tag{3.5}$$

with $C' > 0$, $a = a_m(\hbar) \in \mathbb{R}$, and $p_0(E, y) = \sqrt{-y^3 - y + E}$.

For $[m/2] = n \in \mathbb{N}$ and $\hbar \geq \hbar_n$ both functions $\psi_m(z)$ have n nodes on each half-plane $\mathbb{C}^+$ and $\mathbb{C}^-$. They are distinguished by the number of imaginary nodes. If we define the complement of the escape line in the imaginary axis,

$$\eta^c(E) = \{z = iy, -\infty < y < y_0\}, \tag{3.6}$$

where $y_0 = -i\,I_0$ and I_0 is the imaginary turning point, then the crossing process for $\hbar \geq \hbar_n$ can be studied by looking at the behaviors $\psi_m(z)$ with energy E_m on the semi-axis $\eta^c(E_m)$. We therefore consider the behavior of the two states $\phi_m(y)$ in an open interval $A \subseteq \eta^c(E_m)$ for large $\hbar$ and we see that a state is concave when it is positive and convex when negative. Since we can assume $\phi_m(y)$ to be positive decreasing for $y \ll y_0$, only two possibilities are allowed:

(a) no zero exists on $\eta^c(E_m)$;
(b) there exists a single zero on $\eta^c(E_m)$.

According to Lemma 3.1, for large positive $\hbar$ an imaginary node of a state $\psi_n(z,\hbar)$ is in $\mathbb{C}_-$. As $y_0 > 0$ when $E > 0$ an imaginary node should lie in the intersection of $\mathbb{C}_-$ with the imaginary axis, contained in $\eta^c(E_m)$. $\square$

Corollary 3.5. Assume $\hbar > \hbar_n$ and let $\psi_m(\hbar)$ be a generic state tending to ψ^c_n for $\hbar \to \hbar^+_n$. Then the non-imaginary zeros of $\psi_m(\hbar)$ are exactly $2n$ and they are stable at the limit $\hbar^+_n$. Since there exists at most one imaginary node, the number m of the nodes of $\psi_m(\hbar)$ is no greater than $m^+ = 2n + 1$.

Proof. By Lemma 3.4 when $\hbar > \hbar_n$ the levels $E_m(\hbar)$ are positive and the states $\psi_m(\hbar)$ are PT-symmetric. The spectrum is simple and the nodes are symmetric with respect to the imaginary axis. Therefore an imaginary zero axis cannot leave the imaginary axis and a non-imaginary zero cannot become purely imaginary. A non-imaginary zero of the state $\psi_m(\hbar)$, however, can go to infinity moving along a path having the imaginary axis as asymptote at infinity.[16,17] Moreover, at a fixed $\hbar > \hbar_n$ the state $\psi_m(\hbar)$ has the behavior (2.11) so that it is non-vanishing for large y and small $|x| \neq 0$. We can therefore conclude that the large zeros are imaginary and the non-imaginary nodes are stable. The non-imaginary zeros of the two states $\psi_m(\hbar)$ and of the limiting state ψ_n^c are $2n$. By Lemma 3.4 there exists at most one zero on the imaginary axis, so that the number of nodes is $0 \leq m \leq 2n+1$. Due to the independence of the two states $\psi_m(\hbar)$ we actually have two different numbers m, one no greater than $2n+1$ and the other no greater than $2n$. □

Collecting all the previous results we can finally prove the following.

Theorem 3.6. *For each $n \in \mathbb{N}$, there exists a crossing parameter $\hbar_n$. Two levels $E_{m^\pm}(\hbar)$, $m^\pm = 2n + (1 \pm 1)/2$, are defined for $\hbar > \hbar_n$ and two levels $E_n^\pm(\hbar_n)$ for $\hbar < \hbar_n$. The two pairs cross at $\hbar_n$. Both the states $\psi_{m^\pm}(z,\hbar)$ have a P_x- symmetric set of $2n$ non-imaginary nodes. Only $\psi_{m^+}(z,\hbar)$ has an imaginary node.*

Proof. Before giving the proof, an observation is in order. Actually we do not prove the uniqueness of this crossing and the possible existence of a next pair of anti-crossing and crossing is left open. It could happen that the two levels $E_{m^\pm}(\hbar)$, for $\hbar_n' > \hbar > \hbar_n$, and the two levels $E_n^\pm(\hbar)$ for $\hbar > \hbar_n'$, cross at $\hbar_n'$. Moreover, $E_{m^\pm}(\hbar)$, for $\hbar > \hbar_n''$, and $E_n^\pm(\hbar_n)$, for $\hbar < \hbar_n''$, cross at $\hbar_n''$. For simplicity, we disregard this possibility.

Let us now proceed with the proof of the theorem. The non-reality of $E_n^\pm(\hbar)$ for small $\hbar$ and positivity of $E_m(\hbar)$ for large $\hbar$ necessarily yield the existence of crossings. Seen from $\hbar \leq \hbar_n$ the crossing occurs when the two levels $E_n^\pm(\hbar)$ become real. Consider the pairs of positive levels, of the kind called $E_m(\hbar)$, obtained by the crossing at $\hbar_n$. Only the pairs of numbers m equal to $m^\pm$ are compatible with the uniqueness of such levels for large $\hbar$. Therefore only the sequence of pairs $\{(E_{2n}(\hbar), E_{2n+1}(\hbar))\}_{n=0,1,\ldots}$ corresponds exactly to the sequence $\{E_m(\hbar)\}_{m=0,1,\ldots}$ we have for large $\hbar$. As a consequence we also have that the non-imaginary zeros of the states $\psi_m(\hbar)$ are the non-imaginary nodes.

At the crossing, the imaginary node of the state $\psi_{m^+}(\hbar_n)$ coincides with the lowest imaginary zero of $\psi_{m^-}(\hbar_n)$. The crossing between the levels $E_{m^\pm}(\hbar)$ is possible due to the stability of the $2n$ non-imaginary nodes of both the entire functions $\psi_{m^\pm}(z,\hbar)$ and due to the instability of the imaginary node of the functions $\psi_{m^+}(z,\hbar)$. Because of the P_xT-symmetry, both the states $\psi_{m^\pm}(z,\hbar)$ have n nodes in both the half-planes. If we continue the state $\psi_{m^+}(z,\hbar)$ along a path in the $\hbar$ complex plane, coming from and returning to a $\hbar > 0$ large enough and turning around $\hbar_n$, we eventually get the state with m^- zeros in the half-plane $\mathbb{C}_-$ of z, without the imaginary one. Finally, by continuing to $\hbar < \hbar_n$ the n nodes of both the states $\psi_{m^\pm}(z,\hbar)$ in $\mathbb{C}^+$ ($\mathbb{C}^-$) we obtain the nodes of $\psi_n^+(z,\hbar)$ ($\psi_n^-(z,\hbar)$). □

Remarks 3.7. (i) For large $\hbar$ all the zeros in the upper half-plane are imaginary. This statement strengthens the confinement of the zeros of $\psi_m(z,\hbar)$ for large $\hbar$ obtained above. Hence all the non-imaginary zeros are nodes, and all the nodes are in the lower half-pane for large $\hbar$.

(ii) We have seen that the states $\widetilde{\psi}_n = \widetilde{\psi}_n(0)$ of $\widetilde{H}_\beta$ at fixed $\beta = 0$ have definite parity: $P\widetilde{\psi}_n = (-1)^n\widetilde{\psi}_n$. This means that $|\widetilde{\psi}_n|^2$ is P-symmetric, and the expectation value of the parity is $\langle\widetilde{\psi}_n, P\widetilde{\psi}_n\rangle = (-1)^n$. We recall that the state at the crossing, $\psi_n^c = \psi_n^\pm(\hbar_n)$, has a vanishing average value of the parity, $\langle\psi_n^c, P\psi_n^c\rangle = 0$, so that it is totally P-asymmetric in the sense that ψ_n^c is orthogonal to $P\psi_n^c$.[25]

IV. BOUNDEDNESS OF THE LEVELS, QUANTIZATION, AND SELECTION RULES

In this section we examine further properties of the levels in connection with the quantization and the selection rules. The levels are always semiclassical and are given by some semiclassical quantization rules excluding the divergence of the level. The semiclassical nature of the problem is made clear using the dilations. By a regular scaling $x \to \lambda x$, $\lambda = 1/\sqrt{\delta} > 1$ we get the equivalent operator

$$\check{H}_k(\delta) = -k^2\,(d^2/dx^2) + i(x^3 - \delta x) \sim \delta^{3/2} H_\hbar\,, \qquad k = \hbar\,\delta^{5/4}, \tag{4.1}$$

where the new parameter k vanishes, for any fixed $\hbar$, as $\delta \to 0$. In this new representation the energy is $\check{E}_m(k,\delta) = \delta^{3/2} E_m(\hbar)$. Fixing $\delta = 0$ and setting $k = \hbar$ we reproduce the well-known semiclassical operator[5]

$$\check{H}_\hbar(0) = -\hbar^2\,(d^2/dx^2) + i x^3. \tag{4.2}$$

$\check{H}_\hbar(0)$ is related by a scaling to the operator $\widehat{H}_{\alpha=0}$ studied above, its spectrum is positive and the Stokes lines have the trivial dependence $\tau(E) = E^{1/3}\tau(1)$ upon $E > 0$, so that there are no critical energies. Thus we can consider a large enough scaling factor λ in order to replace $\hbar$ by the parameter k as small as we want.

Bounded levels $E_n^{\pm}(\hbar)$, $E_m(\hbar)$ are obtained by two different quantizations. We recall that the nodes of a state ψ are confined in $\mathbb{C}^{\pm}$ according to whether its energy satisfies the condition $E \in \mathbb{C}^{\mp}$. Therefore, for $\hbar < \hbar_n$ both levels $E_n^{\pm}(\hbar)$ satisfy the unique conditions on the imaginary part and on the nodes of the states have no crossing and are analytic. At $\hbar = \hbar_n$ the levels cross and become positive. We have seen that there are two continuations of $E_n^{\pm}(\hbar)$ from $\hbar < \hbar_n$ to $\hbar > \hbar_n$ and that the continuations of the corresponding states have n nodes, each one in the half-planes $\mathbb{C}^{\pm}$. There exist two regular regions $\Omega^{\pm} \subset \mathbb{C}^{\pm}$ large enough, whose boundaries $\gamma^{\pm} = \partial\Omega^{\pm}$ satisfy $P_x\gamma^{+} = \gamma^{-}$ such that the exact quantization conditions are

$$J^{\pm}(E,\hbar) = \frac{\hbar}{2i\pi}\oint_{\gamma^{\pm}} \frac{\psi'(z)}{\psi(z)}dz + \frac{\hbar}{2} = \hbar\Big(n + \frac{1}{2}\Big), \tag{4.3}$$

with $E = E_n^{\pm}(\hbar) \in \mathbb{C}^{\mp}$ and $\psi(z) = \psi_n^{\pm}(z,\hbar)$. For small $\hbar$ and bounded $n\hbar$, from (4.3) we get the semiclassical quantization

$$J^{\pm}(E,\hbar) = \frac{1}{2i\pi}\oint_{\gamma^{\pm}} \sqrt{V(z) - E}\,dz + O(\hbar^2) = \hbar\Big(n + \frac{1}{2}\Big), \tag{4.4}$$

where $\gamma^{\pm}$ squeeze along both the edges of $\rho(E)$. At the critical value $\hbar = \hbar_n$ the two quantizations (4.3) yield equal solutions E_n^c, ψ_n^c. When $\hbar > \hbar_n$, both (4.3) admit the two solutions $E_m(\hbar)$, $\psi_m(\hbar)$, $[m/2] = n$, selected by condition $E_{2n+1}(\hbar) > E_{2n}(\hbar)$ compatible with the order of the levels $\widehat{E}_{2n+1}(0) > \widehat{E}_{2n}(0)$. Thus we have the boundedness and the continuity of the functions $E_n^{\pm}(\hbar)$, $\hbar \le \hbar_n$, becoming $E_{m^{\pm}}(\hbar)$ for $\hbar \ge \hbar_n$ and both the functions $E_{m^{\pm}}(\hbar)$ are analytic in the hypothesis of maximal analyticity.

We now consider the semiclassical regularity of the levels $E_m(\hbar)$ for large $1/\hbar$ and $m = 2n$ or $m = 2n+1$. We expect to find positive semiclassical levels with $E > E^c = 0{,}352\,268$,[21] by the semiclassical quantization we consider here. If $\Omega_m \subset \mathbb{C}_-$ is large enough in order to contain all the m nodes and $\Gamma_m = \partial\Omega_m$, for a fixed $\hbar$, we have the exact quantization rules

$$J_2(E,\hbar) = \frac{\hbar}{2i\pi}\oint_{\Gamma_m} \frac{\psi'(z)}{\psi(z)}dz + \frac{\hbar}{2} = \hbar\Big(m + \frac{1}{2}\Big), \tag{4.5}$$

where $E = E_m(\hbar)$, $\psi(z) = \psi_m(\hbar,z)$. For large m, small $\hbar$, $m\hbar$ bounded, we have

$$J_2(E,\hbar) = \frac{1}{2i\pi}\oint_{\Gamma_m} \sqrt{V(z) - E}\,dz + O(\hbar^2) = \hbar\Big(m + \frac{1}{2}\Big)\,. \tag{4.6}$$

We expect the bounded limits $2n\hbar_n \to J_2^c$ and $E_n^c \to E^c$ as $n \to \infty$. The peculiarity of E^c is the instability of $\tau(E)$ and the connection of $\rho(E)$ and $\eta(E)$ at this point. The quantization conditions (4.4) and (4.6) are compatible with the existence of $\lim_n E_n^c = E^c > 0$ for $n \to \infty$, $J_2^c = J_2(E^c,0) = 2J(E^c,0)$ with $\Omega = \Omega^{+} \bigcup \Omega^{-}$. The localization of the nodes near $\rho(E)$ for small $\hbar$ and the localization of the other zeros near $\eta(E)$ implies this property of $\tau(E)$ at E^c.

We can now establish the local boundedness of the levels in the real axis.

Lemma 4.1. Each of the four continuous functions $E_n^{\pm}(\hbar)$ for $\hbar < \hbar_n$ and $E_{m^{\pm}}(\hbar)$ for $\hbar > \hbar_n$ is locally bounded.

Proof. Let $E(\hbar)$ be one of the levels $E_n^{\pm}(\hbar)$ for $\hbar < \hbar_n$ with one of its continuations $E_{m^{\pm}}(\hbar)$ for $\hbar > \hbar_n$ and let $\psi(z)$ be the corresponding state. Assume that the lemma is false and that there is a divergence of $E(\hbar)$ at $\hbar^c \gg \hbar_n$. We rescale the Hamiltonian $H_\hbar$ by $x \to |E(\hbar)|^{1/3} x$. Upon dividing by $|E(\hbar)|$ we get the operator

$$-k^2 (d^2/dz^2) + i\, z^3 - i\, |E|^{-2/3} z - E/|E|\,, \qquad k = |E|^{-5/6}\hbar.$$

As $\hbar \to \hbar^c$, so that $k \to 0$, and neglecting the linear term in z, the semiclassical quantization reads

$$\frac{1}{2\pi i}\oint_{\Gamma_m} \sqrt{iz^3 - E/|E|}\; dz = k\left(m + \frac{1}{2}\right) + O(k^2), \tag{4.7}$$

where Γ_m is the boundary of a region $\Omega_m \subset \mathbb{C}_-$ containing the m nodes of $\psi(\hbar^c)$. For $k \to 0$, (4.7) could be satisfied only if $E/|E| \to 0$, obviously absurd. □

On the sector $\mathbb{C}^0$ (1.14) of the complex plane of $\hbar$, the functions $E_{m^{\pm}}(\hbar)$ are analytic with Riemann sheets $\mathbb{C}^0_{m^{\pm}}$ having a square-root-type singularity and a cut $\gamma_n = (0, \hbar_n\,]$ on the real axis. We assume the inequality

$$E_{m^+}(\hbar) > E_{m^-}(\hbar), \qquad \hbar > \hbar_n\,, \tag{4.8}$$

the only one compatible with the order of the levels $\widehat{E}_{2n+1}(0) > \widehat{E}_{2n}(0)$. We prove the following:

Theorem 4.2. *At the edges of γ_n the behaviors of the positive analytic functions $E_{m^{\pm}}(\hbar)$ are given by*

$$E_{m^-}(\hbar \pm i0^+) = E_n^{\pm}(\hbar)\,, \qquad E_{m^+}(\hbar \pm i0^+) = E_n^{\mp}(\hbar), \tag{4.9}$$

where $E_n^{\pm}(\hbar) \to \pm E_0$ as $\hbar \to 0$.

Proof. In the hypothesis of uniqueness of the crossing (see the proof of Theorem 3.6) for $\hbar > \hbar_n$, we admit the inequality $E_{2n+1}(\hbar) > E_{2n}(\hbar)$, the only one compatible with the order of the levels $\widehat{E}_{2n+1}(0) > \widehat{E}_{2n}(0)$. Since both the functions $E_{m^{\pm}}(\hbar)$ have a square root singularity at $\hbar_n$ and $E_{m^+}(\hbar_n + \epsilon) - E_{m^-}(\hbar_n + \epsilon) = O(\sqrt{\epsilon}) > 0$ for small positive ϵ, then $\pm\Im\left[\, E_{m^+}(\hbar_n + \exp(\pm i\pi)\epsilon) - E_{m^-}(\hbar_n + \exp(\pm i\pi)\epsilon)\right] < 0$ and $\mp\Im E_n^{\pm}(h) > 0$. We necessarily have

$$E_{m^+}\big(\hbar_n + \exp(\pm i\pi)\epsilon\big) = E_n^{\mp}(\hbar_n - \epsilon), \quad E_{m^-}\big(\hbar_n + \exp(\pm i\pi)\epsilon\big) = E_n^{\pm}(\hbar_n - \epsilon)$$

and the result extends to any $\epsilon < \hbar_n$. □

Remarks 4.3. (*i*) We can look at the crossing process following a path which starts from $\hbar = 0^+$, encircles the singularity $\hbar_n$, and comes back to $\hbar = 0^+$. At the beginning of the path the state $\psi_n^-(z, 0^+)$ is mainly localized around x_+ and at the end turns into $\psi_n^-(z, 0^-)$, mainly localized around x_-. We can also look at a path beginning at a large $\hbar$, going around $\hbar_n$, and returning to the initial $\hbar$. If the initial state $\psi_{m^+}(\hbar)$ is odd, it eventually becomes the even state $\psi_{m^-}(\hbar)$ and the imaginary node in the lower half-plane is changed into the lowest zero on the positive imaginary axis.

(*ii*) It is possible that the Riemann sheet $\mathbb{C}^0_0$ of the fundamental level has only the square root branch point $\hbar_0$ with the cut $\gamma_0 = [0, \hbar_0]$ on the real axis.[21] From Theorem 4.2 the discontinuity on the cut γ_0 is

$$E_0(\hbar + i0^+) - E_0(\hbar - i0^+) = 2\, i\, \Im E_0^{\pm}(\hbar)\,.$$

The function $E_0(\hbar)$, analytic for large $|\hbar|$, if continued to small $|\hbar|$ while keeping $\arg\hbar = \pm\pi/4$, coincides by definition with $E_0^{\pm}(\hbar)$. The absence of complex singularities is compatible with the identities on the edges of the cut γ_0,

$$E_0(\hbar \pm i0^+) = E_0^{\pm}(\hbar \pm i0^+).$$

We finally prove the theorem.

Theorem 4.4. *There exists an instability point, $E^c \geq 0$, of $\rho(E)$. For $n \to \infty$, we have the limits $E_n^c \to E^c$, $2n\hbar_n \to J_2^c$, where $J_2^c = J_2(E^c, 0)$ as in (4.6).*

Proof. Since the eigenvalue problem of $H_\hbar$ is semiclassical, the existence of the infinite crossings is possible if there exists a critical point $E^c \geq 0$ of $\rho(E)$, which we assume to be unique. The existence of a critical point is due to the P_x-symmetry of $\rho\big(E_m(\hbar)\big)$, for $\hbar > 0$ and $E_m(\hbar) > 0$ large, together with the symmetry breaking for small $\hbar$. Actually, at the limit $\hbar \to 0^+$ we have at $E_n^\pm \to \pm E_0$ and $\rho(\pm E_0)$ reduces to the points $x_\pm$, respectively. The P_x-symmetry breaking of $\rho(E)$ at $E^c \geq 0$ is possible only if $I_0(E^c) \in \rho(E^c)$, so that $\rho(E)$ is a line touching the turning points $I_\pm$ with $\Re I_\pm \neq 0$ and containing the point I_0. Thus the symmetry breaking of $\rho(E)$ implies its redefinition as one half of it containing only a pair of turning points, (I_0, I_+) or (I_-, I_0). Our eigenvalue problem is always semiclassical and the change of the semiclassical regime is related to the instability of the nodes used for the semiclassical quantization. Since it is possible to change representation by the scaling (4.1), it is always possible to have a sequence of parameters $k_n \to 0$ together with a sequence δ_n. If a subsequence $\delta_{n(j)}$ vanishes as $j \to \infty$, this is incompatible with the absence of crossings of the levels $\widehat{E}_m(\alpha)$ at $\alpha = 0$. Thus, we have the inequality of the sequence $\delta_n > \epsilon > 0$ for an $\epsilon > 0$ and $n > n_\epsilon$. This means that the original sequence also vanishes, namely, $\hbar_n \to 0$. Moreover, the sequence E_n^c has the limit $E_n^c \to E^c > 0$ as $n \to \infty$ because in the semiclassical limit there is the instability of the nodes at $E = E^c$ only. However, due to the semiclassical quantization rules, we also have a bounded limit of $2n\hbar_n \to J_2^c > 0$ (4.6), where $J_2^c = J_2(E^c, 0)$, as $n \to \infty$. □

V. CONCLUSIONS

We have proved the existence of a crossing for any pair of levels $E_{m^\pm}(\hbar)$, with $m^\pm = 2n + (1 \pm 1)/2$, producing the pair of levels $E_n^\pm(\hbar)$ for smaller $\hbar$. In this PT-symmetric model we see the competition of two different effects: the conservation of the symmetry and the semiclassical localization of the states. The semiclassical localization prevails for small $\hbar$. This semiclassical transition is impossible in families of self-adjoint Hamiltonians. Thus, these PT-symmetric models can be used to describe the appearance of the classical world in non-isolated systems.

In order to understand the physical meaning of the classical trajectories $\tau(E)$, we consider $\hbar$ at the border of the sector $\mathbb{C}^0$. At $\arg \hbar = \pi/4$ we can factorize the imaginary unit and consider the real cubic oscillator described by formal Hamiltonian

$$H_r(\hbar) = p^2 + V^r(x), \qquad p = -i\hbar\,(d/dx), \qquad V^r(x) = x^3 - x, \qquad \hbar > 0 \tag{5.1}$$

for an energy value $E \in A_0 = (-c, c)$, with $c = 2/(3\sqrt{3})$. We get the classical Hamiltonian $H_r(p, x)$ by substituting in $H_r(\hbar)$ the classical momentum p with the operator $-i\hbar\,(d/dx)$. The union $\tau(E)$ of the classical trajectories at energy $E \in A_0$ consists of the oscillation range $\rho(E) = [\, I_-(E), I_+(E)\,]$ and the escape line

$$\eta = \eta(E) = (-\infty, I_0(E)\,], \qquad I_0(E) < -1/\sqrt{3} < I_-(E) < I_+(E).$$

The real potential makes clear the meaning of our definitions of the oscillatory range $\rho(E)$ and of the escape line $\eta(E)$ previously used with complex potential. Notice that $\tau(E)$ is unstable at $E = c$, where $\rho(E)$ touches $\eta(E)$.

Going back to the Hamiltonians $H_\hbar$ at the limit cases $\arg(\hbar) = \pm i(\pi/4)^-$, we have the critical values of the energy

$$E^c\big(\arg(\hbar) = \pm(\pi/4)^-\big) = \pm ic = \mp E_0 = E_n^\mp(0).$$

Thus we expect the existence of an infinite set of crossings in the complex $\hbar$ plane with complex accumulation points $E^c(\theta)$ of the crossing energies for $|\hbar| \to 0$ along the direction $\arg(\hbar) = \theta \neq 0$. A preliminary discussion on all the crossings in the $\mathbb{C}^0$ complex sector of the $\hbar$ variable is found in Ref. 25. We state the generalized crossing rules in terms of the four limits

$$E_{m^\pm}(\hbar_{(n^-,n^+)} + \epsilon) - E^\pm_{m^\pm}(\hbar_{(n^-,n^+)} - \epsilon) \to 0, \ \text{ as } \ \epsilon \to 0, \ \ m^\pm = n^- + n^+ + (1 \pm 1)/2,$$

at $\hbar_{(n^-,n^+)}$, where $\hbar_{(n,n)} = \hbar_n$. Thus we expect, as a general rule, the instability of one of the nodes of the state $\psi_{m^+}(\hbar)$ and the partition between the two states $\psi^\pm_{n^\pm}(\hbar)$ of the $n^- + n^+$ nodes.

The hypothesis of minimality concerning the singularities gives the following picture. The analytic function $E_m(\hbar)$ for large $|\hbar|$ has a sequence of singularities in $\mathbb{C}^0$, ordered by the increasing values of $\Im\hbar_{j,k}$,

$$\hbar_{m,0},\ \hbar_{m-1,0}\ ,\hbar_{m-1,1},\ \hbar_{m-1,2},\ldots,\hbar_{1,m-2},\ \hbar_{1,m-1},\ \hbar_{0,m-1},\ \hbar_{0,m}. \tag{5.2}$$

It is possible to divide $\mathbb{C}^0$, for small $|\hbar|$, by cuts going from 0 to the branch points (5.2) in a sequence of stripes

$$S_m^-,\ S_0^+,\ S_{m-1}^-,\ S_1^+,\ldots,S_1^-,\ S_{m-1}^+,\ S_0^-,\ S_m^+,$$

in which, for $\hbar\to 0$, the levels $E_m(\hbar)$ have the behavior

$$E_m^-(\hbar),\ E_0^+(\hbar),\ E_{m-1}^-(\hbar),\ E_1^+(\hbar),\ldots,E_1^-(\hbar),\ E_{m-1}^+(\hbar),\ E_0^-(\hbar),\ E_m^+(\hbar)$$

respectively.

ACKNOWLEDGMENTS

It is a pleasure to thank Professor André Martinez for useful discussions and suggestions.

[1] C. M. Bender and T. T. Wu, "Anharmonic oscillator," Phys. Rev. **184**, 1231–1260 (1969).
[2] L. Benassi and V. Grecchi, "Resonances in the Stark effect and strongly asymptotic approximants," J. Phys. B: At. Mol. Phys. **13**, 911 (1980).
[3] G. Alvarez, "Bender-Wu branch points in the cubic oscillator," J. Phys. A: Math. Gen. **27**, 4589–4598 (1995).
[4] V. Buslaev and V. Grecchi, "Equivalence of unstable anharmonic oscillators and double wells," J. Phys. A: Math. Gen. **26**, 5541–5549 (1993).
[5] C. M. Bender and S. Boettcher, "Real spectra in non-Hermitian Hamiltonian having PT symmetry," Phys. Rev. Lett. **80**, 5243 (1998).
[6] K. C. Shin, "On the reality of eigenvalues for a class of PT-Symmetric oscillators," Commun. Math. Phys. **229**(3), 543–564 (2002).
[7] R. Giachetti and V. Grecchi, "PT-symmetric operators and metastable states of the 1D relativistic oscillators," J. Phys. A: Math. Theor. **44**, 09538 (2011).
[8] J. Loeffel, A. Martin, B. Simon, and A. Wightman, "Padé approximants and the anharmonic oscillator," Phys. Lett. B **30**, 656 (1969).
[9] B. Simon, "Coupling constant analyticity for the anharmonic oscillator," Ann. Phys. **58**, 76–136 (1970).
[10] S. Graffi, V. Grecchi, and B. Simon, "Borel summability: Application to the anharmonic oscillator," Phys. Lett. B **32**, 631–634 (1970).
[11] C. M. Bender and T. T. Wu, "Anharmonic oscillator. II. A study of perturbation theory in large order," Phys. Rev. D **7**, 1620 (1973).
[12] B. Simon, "Borel summability of the ground-state energy in spatially cutoff," Phys. Rev. Lett. **25**, 1583 (1970).
[13] E. Caliceti, "Distributional Borel summability of odd anharmonic oscillators," J. Phys. A: Math. Gen. **33**, 3753 (2000); E. Caliceti, S. Graffi, and M. Maioli, "Perturbation theory of odd anharmonic oscillators," Commun. Math. Phys. **75**, 51 (1980).
[14] C. M. Bender and G. V. Dunne, "Large-order perturbation theory for a non-Hermitian PT-symmetric Hamiltonian," J. Math. Phys. **40**, 4616 (1999); C. M. Bender and E. J. Weniger, "Numerical evidence that the perturbation expansion for a non-Hermitian PT -symmetric Hamiltonian is Stieltjes," J. Math. Phys. **42**, 2167–2183 (2001).
[15] P. E. Shanley, "Spectral properties of the scaled quartic anharmonic oscillator," Ann. Phys. **186**, 292–324 (1988); "Nodal properties of the quartic anharmonic oscillator," Ann. Phys. **186**, 325–354 (1988).
[16] V. Grecchi and A. Martinez, "The spectrum of the cubic oscillator," Commun. Math. Phys. **319**, 479–500 (2013); "Padé summability of the cubic oscillator," J. Phys. A: Math. Theor. **42**, 425208 (2009); "The top resonances of the cubic oscillator," J. Phys. A: Math. Theor. **43**, 474027 (2010).
[17] Y. Sibuya, "Global theory of a second order linear ordinary differential equation with a polynomial coefficient," in *North-Holland Mathematics Studies* (North-Holland-American Elservier Publishing Company, Amsterdam, New York, 1975), Chap. 7, p. 18.
[18] S. Giller, "The semiclassical small- limit of loci of roots of subdominant solutions for polynomial potentials," J. Math. Phys. **52**, 063514 (2011).
[19] A. Voros, "Exact quantization condition for anharmonic oscillators (in one dimension)," J. Phys. A **27**, 4653–4661 (1994).
[20] E. Delabaere, H. Dillinger, and F. Pham, "Exact semiclassical expansions for one-dimensional quantum oscillators," J. Math. Phys. **38**(12), 6126–6184 (1997); E. Delabaere and F. Pham, "Unfolding the quartic oscillator," Ann. Phys. **261**, 180–218 (1997).
[21] E. Delabaere and D. T. Trinh, "Spectral analysis of the complex cubic oscillator," J. Phys. A: Math. Gen. **33**, 8771–8796 (2000).
[22] A. Eremenko and A. Gabrielov, "Analytic continuation of eigenvalues of a quartic oscillator," Commun. Math. Phys. **287**, 431–457 (2009).
[23] A. Eremenko and A. Gabrielov, "Singular perturbations of polynomial potentials in the complex domain with applications to PT-symmetric families," Moscow Math. J. **11**, 473–503 (2011).
[24] R. Giachetti and V. Grecchi, "Localization of the states of a PT-symmetric double well," Int. J. Theor. Phys. **54**, 3889 (2015).
[25] R. Giachetti and V. Grecchi, "Level crossings in a PT-symmetric double well," J. Phys. A: Math. Theor. **49**, 105202 (2016).
[26] T. Kato, *Perturbation Theory for Linear Operators* (Springer, New York, 1966).
[27] J. Aguilar and J. M. Combes, "A class of analytic perturbations for one-body Schrdinger Hamiltonians.," Commun. Math. Phys. **22**, 269–279 (1971).