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Factorization of Hermite subdivision operators preserving exponentials and polynomials

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Abstract In this paper we focus on Hermite subdivision operators that act on vector valued data interpreting their components as function values and associated consecutive derivatives. We are mainly interested in studying the exponential and polynomial preservation capability of such kind of operators, which can be expressed in terms of a generalization of the spectral condition property in the spaces generated by polynomials and exponential functions. The main tool for our investigation are convolution operators that annihilate the aforementioned spaces, which apparently is a general concept in the study of various types of subdivision operators. Based on these annihilators, we characterize the spectral condition in terms of factorization of the subdivision operator.

Keywords Hermite subdivision · factorization · annihilators · Taylor operator · exponentials

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1 Introduction

Subdivision schemes are iterative procedures based on the repeated application of *subdivision operators*, which can even differ at different levels of iteration, on discrete data. More specifically, subdivision operators act on bi-infinite sequences

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$\boldsymbol{c} : \mathbb{Z} \to \mathbb{R}$ by means of a *finitely* supported *mask* $\boldsymbol{a} : \mathbb{Z} \to \mathbb{R}$ in the convolution–like form $\mathcal{S}_{\boldsymbol{a}}\boldsymbol{c} = \sum_{\beta\in\mathbb{Z}} a(\cdot - 2\beta)\, c(\beta)$. This type of operators has been generalized in various ways, considering multivariate operators, operators with dilation factors other than 2 or subdivision operators acting on vector or matrix data by means of matrix valued masks. There is such a vast amount of literature meanwhile that we do not even attempt to give specific references here.
It has been observed from very early on that preservation of *polynomial data* is an important property of subdivision operators. For example, the preservation of constants, $\mathcal{S}_{\boldsymbol{a}} 1 = 1$, is necessary for the convergence of the subdivision schemes based on iterations of the same operator $\mathcal{S}_{\boldsymbol{a}}$. More generally, the preservation of polynomial spaces, $\mathcal{S}_{\boldsymbol{a}} \Pi_n = \Pi_n$, plays an important role in the investigation of the differentiability of the limit function of subdivision schemes [1] and in the investigation of approximation properties of subdivisions schemes [16]. In addition, there has been interest in also preserving functions other than polynomials, see for example [18], and it is natural from connections e.g. to systems theory that such functions must be exponential, i.e., of the form $e^{\lambda\cdot}$, cf. [22] as well as [7].

In this paper we will consider preservation of such exponentials by *Hermite subdivision operators* which act on vector data but with the particular understanding that these vectors represent function values and consecutive derivatives up to a certain order (see, for example, [5,8–11,13–15,17,19,20]). We will study the preservation capability of such operators by means of a *cancellation operator*, a concept that applies to subdivision schemes in quite some generality. Based on these cancellation operator, we characterize the spectral condition in terms of factorization of the subdivision operator.

Before we get to the main technical content of the paper, we want to illustrate the idea and the concept through a few examples, while all the formal definitions will be postponed to Section 2.
The simplest example deals with the preservation of constants, $\mathcal{S}_{\boldsymbol{a}} 1 = 1$. Note that constant sequences are exactly the kernel of the difference operator Δ, defined as $\Delta \boldsymbol{c} = c(\cdot + 1) - c$; in other words: the (standard) difference operator is the simplest *cancellation operator* or *annihilator* of the constant functions. Now, whenever $\mathcal{S}_{\boldsymbol{a}}$ *preserves* constants, then $\mathcal{S}_{\boldsymbol{c}} = \Delta \mathcal{S}_{\boldsymbol{a}}$ is a subdivision operator that *annihilates* the constants. As it can easily be shown, any such operator can be written as $\mathcal{S}_{\boldsymbol{c}} = \mathcal{S}_{\boldsymbol{b}} \Delta$ for some other finitely supported mask $\boldsymbol{b}$, hence we get the *factorization* $\Delta \mathcal{S}_{\boldsymbol{a}} = \mathcal{S}_{\boldsymbol{b}} \Delta$. Switching to the calculus of *symbols* which associates to a finitely supported sequence $\boldsymbol{a}$ the *Laurent polynomial* $a^*(z) := \sum_{\alpha\in\mathbb{Z}} a(\alpha)\, z^{\alpha}$, the factorization is equivalent to $(z^{-1} - 1)a^*(z) = b^*(z)(z^{-2} - 1)$ or, equivalently, to the famous "zero at π" condition, since $a^*(z) = \left(z^{-1} + 1\right) b^*(z)$ vanishes at $z = e^{-i\pi}$.

For a slightly more elaborate example let us consider $\lambda_i \in \mathbb{C}\backslash\{0\}$ for $i = 1, \cdots, r$ and the corresponding set $\Lambda = \{\lambda_1, \cdots, \lambda_r\}$. Suppose that now the subdivision operator provides preservation of the subspace

$$V_{p,\Lambda} = \operatorname{span} \left\{1, x, \dots, x^p, e^{\lambda_1 x}, e^{-\lambda_1 x}, \dots, e^{\lambda_r x}, e^{-\lambda_r x}\right\}, \tag{1}$$

in the sense that $\mathcal{S}_{\boldsymbol{a}} V^0_{p,\Lambda} \subseteq V^1_{p,\Lambda}$ where $V^j_{p,\Lambda} := \left\{v(2^{-j}\cdot) \,:\, v \in V_{p,\Lambda}\right\}$, see, for example, [2,4,6,21,22]. Note that one really has to consider different spaces here, because the result of the subdivision operator corresponds to a sequence on the finer grid $\mathbb{Z}/2$ due to the upsampling incorporated into the subdivision operator. Again we approach this problem in terms of cancellation, determining an operator $\mathcal{H}_{p,\Lambda}$ such that $\mathcal{H}_{p,\Lambda} V_{p,\Lambda}^0 = \{0\}$ first. Assuming that $\mathcal{H}_{p,\Lambda}$ is a *convolution operator* (or *LTI filter* in the language of signal processing, cf. [12]) with *impulse response* $\boldsymbol{h}$, it is easily seen and well–known that cancellation of the polynomials of degree at most p implies that $(h^*)^{(k)}(1) = 0$, $k = 0, \dots, p$, hence cancellation of the polynomial part of $V_{p,\Lambda}$ implies that $h^*(z) = \left(z^{-1} - 1\right)^{p+1} b_1^*(z)$. Cancellation of an exponential sequence $e^{\lambda\cdot}$, on the other hand, leads to

$$0 = \sum_{j \in \mathbb{Z}} h(\cdot - j) e^{\lambda j} = \sum_{j \in \mathbb{Z}} h(j) e^{\lambda(\cdot - j)} = e^{\lambda \cdot} h^*(e^{-\lambda}),$$

hence, the annihilation of the space spanned by the exponentials implies that

$$h^*(z) = b_2^*(z) \prod_{j=1}^{r} \left(z^{-1} - e^{\lambda_j}\right) \left(z^{-1} - e^{-\lambda_j}\right).$$

Summarizing, the simplest cancellation operator for $V_{p,\Lambda}$ takes the form

$$h_{p,\Lambda}^*(z) = (z^{-1} - 1)^{p+1} \prod_{j=1}^{r} \left(z^{-1} - e^{\lambda_j}\right) \left(z^{-1} - e^{-\lambda_j}\right),$$

and the associated factorization by means of cancellation operators,

$$\mathcal{H}_{p,2^{-1}\Lambda} \mathcal{S}_{\boldsymbol{a}} = \mathcal{S}_{\boldsymbol{b}} \mathcal{H}_{p,\Lambda}, \tag{2}$$

is easily verified to be equivalent to the symbol factorization

$$a^*(z) = b^*(z) \, (z^{-1} + 1)^{p+1} \prod_{j=1}^{r} \left(z^{-1} + e^{\lambda_j/2}\right) \left(z^{-1} + e^{-\lambda_j/2}\right), \tag{3}$$

given in [22]. Note that this also says that the symbol of any cancellation operator for the space $V_{p,\Lambda}$ is a multiple of $h_{p,\Lambda}^*$.

The last example considers *Hermite subdivision schemes* as they were considered so far, for example in [11,17]. In Hermite subdivision, the data are vector valued sequences $\boldsymbol{v} \in \ell^{q+1}(\mathbb{Z})$ with the intuition that the k–th component of such a sequence represents a $(k-1)$–th derivative. Then, as considered for example in [5, 11,17], one defines, for $f \in C^q(\mathbb{R})$, a sequence

$$\boldsymbol{v}_f \; : \; \alpha \mapsto \left[f^{(j)}(\alpha) \; : \; j = 0, \dots, q \right], \qquad \alpha \in \mathbb{Z},$$

and asks when a subdivision operator $\mathcal{S}_{\boldsymbol{C}}$ with *matrix valued* masks $\boldsymbol{C} \in \ell_{00}^{(q+1)\times(q+1)}(\mathbb{Z})$ annihilates all $\boldsymbol{v}_\pi$ for $\pi \in \Pi_q$ which, by the aforementioned machinery, can again be used to describe the *spectral condition*, a "polynomial preservation" rule introduced by Dubuc and Merrien in [11]. Note that it is no mistake or accident that the letter q appears for the maximal order of derivatives and the maximal degree of polynomial cancellation – the space dimension and the order of derivatives are closely tied. It was then shown in [17] that whenever $\mathcal{S}_{\boldsymbol{C}} \boldsymbol{v}_\pi = 0$ for all $\pi \in \Pi_q$, then there exists a finitely supported $\boldsymbol{B} \in \ell_{00}^{(q+1)\times(q+1)}(\mathbb{Z})$ such that

$$C^*(z) = B^*(z)\, T_q^*(z^2),$$

where

$$T_q^*(z) = \begin{bmatrix} z^{-1}-1 & -1 & -\frac{1}{2} & \dots & -\frac{1}{q!} \\ 0 & z^{-1}-1 & -1 & \dots & -\frac{1}{(q-1)!} \\ \vdots & & \ddots & & \vdots \\ 0 & & & z^{-1}-1 & -1 \\ 0 & \dots & & 0 & z^{-1}-1 \end{bmatrix}$$

is the symbol of the operator $\mathcal{T}_q$ defined as

$$\left(\mathcal{T}_q v_f\right)_k(\alpha) := f^{(k)}(\alpha+1) - \sum_{j=0}^{q-k} \frac{f^{(k+j)}(\alpha)}{j!} \tag{4}$$

for $f \in C^{q+1}$ and $k = 0, \cdots, q$. Since $\mathcal{T}_q$ measures the difference between a function and its Taylor polynomial approximation at the neighboring point, it is called the *(complete) Taylor operator* of order q. That $\mathcal{T}_q$ annihilates all $\boldsymbol{v}_\pi$, $\pi \in \Pi_q$, is immediate from (4).

It should have become clear by now that there is a common structure behind all these examples. *Preservation* of a subspace that can be written as the kernel of a convolution operator is related to a commuting property provided that the convolution operator factorizes or "divides" *any* annihilator of the subspace. This can be seen as a minimality property with respect to the partial ordering given by divisibility and justifies the following terminology where we identify any function $f \in V$ with the sequence $\boldsymbol{f} = (f(\alpha) \,:\, \alpha \in \mathbb{Z})$.

Definition 1 A linear operator $\mathcal{H} : \ell^m(\mathbb{Z}) \to \ell^m(\mathbb{Z})$ is called a *convolution operator* for a space V if there exists a matrix sequence $\boldsymbol{H} \in \ell^{m\times m}(\mathbb{Z})$, called the *impulse response* of $\mathcal{H}$, such that

$$\mathcal{H}\boldsymbol{f} = \boldsymbol{H} * \boldsymbol{f} = \sum_{\beta\in\mathbb{Z}} H(\cdot - \alpha) f(\alpha), \qquad \boldsymbol{f} \in V.$$

Definition 2 A convolution operator $\mathcal{H} : \ell^m(\mathbb{Z}) \to \ell^m(\mathbb{Z})$ is called a *minimal annihilator* for a space V with respect to

1. *subdivision*, if for any $\boldsymbol{C} \in \ell_{00}^{m\times m}(\mathbb{Z})$ such that $\mathcal{S}_{\boldsymbol{C}} V = 0$ there exists $\boldsymbol{B} \in \ell_{00}^{m\times m}(\mathbb{Z})$ with $\mathcal{S}_{\boldsymbol{C}} = \mathcal{S}_{\boldsymbol{B}} \mathcal{H}$.
2. *convolution*, if for any $\boldsymbol{C} \in \ell_{00}^{m\times m}(\mathbb{Z})$ such that $\boldsymbol{C} * V = \{0\}$ there exists $\boldsymbol{B} \in \ell_{00}^{m\times m}(\mathbb{Z})$ with $\boldsymbol{C} = \boldsymbol{B} * \boldsymbol{H}$,

respectively. If $\mathcal{H}$ satisfies both properties it is simply called a *minimal annihilator*.

The goal of this paper is to use this general concept to understand preservation of exponentials and polynomials by Hermite subdivision operators. In more technical terms, we will derive the analogy of the Taylor operator for the case of preservation of exponentials and prove in Theorem 5 that it is again a minimal annihilator. We will see that even the cancellation operator depends only on the space $V_{p,\Lambda}$ and on the level. We will also see that the existence of the annihilator operator is strongly connected with the factorization of the subdivision operator satisfying specific preservation properties. Such results can certainly be useful for studying convergence, but we think that they also are of independent interest by themselves. Their use for convergence analysis of Hermite subdivision schemes is presently under investigation.

The organization of the paper is as follows. After providing the necessary notation and terminology, the main results on Hermite subdivision schemes and their reproduction capabilities will be derived in Section 3. To better explain the underlying ideas, we will first consider the case of adding a single frequency to the polynomial space and then extend the results and methods to an arbitrary number of frequencies. These descriptions will be in terms of appropriate cancellation operators. Thereafter, in Section 4 we will use these cancellation operators to derive factorization properties which will also verify that the cancellation operators are minimal. Finally, we will illustrate our results with specific examples.

2 Notation and basic facts

We begin by fixing the notation and recalling some known facts about subdivision operators. We denote by $\ell^m(\mathbb{Z})$ and $\ell^{m\times m}(\mathbb{Z})$ the linear spaces of all sequences of m–vectors and $m \times m$ matrices, respectively. Operators acting on that spaces are denoted by capital calligrafic letter. Sequences in $\ell^m(\mathbb{Z})$ and $\ell^{m\times m}(\mathbb{Z})$ will be denoted by boldface lower case and upper case letters, respectively. For example, $\boldsymbol{c} \in \ell^m(\mathbb{Z})$ is $\boldsymbol{c} = (c(\alpha) : \alpha \in \mathbb{Z})$, while $\boldsymbol{A} \in \ell^{m\times m}(\mathbb{Z})$ stands for $\boldsymbol{A} = (A(\alpha) : \alpha \in \mathbb{Z})$, indexing $A \in \mathbb{R}^{m\times m}$ as $A = \left[a_{jk} : j,k = 0,\dots,m-1\right]$. As usual, $\ell_{00}^m(\mathbb{Z})$ and $\ell_{00}^{m\times m}(\mathbb{Z})$ will denote the subspaces of finitely supported sequences, and $\mathbb{N}_0$ denotes the set $\{0,1,2,\dots\}$.
For $\boldsymbol{A} \in \ell_{00}^{m\times m}(\mathbb{Z})$ and $\boldsymbol{c} \in \ell_{00}^m(\mathbb{Z})$ we define the associated *symbols* as the *Laurent polynomials*

$$A^*(z) := \sum_{\alpha\in\mathbb{Z}} A(\alpha)\, z^\alpha, \qquad c^*(z) := \sum_{\alpha\in\mathbb{Z}} c(\alpha)\, z^\alpha, \qquad z \in \mathbb{C}\setminus\{0\}.$$

For $\boldsymbol{A} \in \ell_{00}^{m\times r}(\mathbb{Z})$ and $\boldsymbol{B} \in \ell_{00}^{r\times q}(\mathbb{Z})$ the *convolution* $\boldsymbol{C} = \boldsymbol{A} * \boldsymbol{B}$ in $\ell_{00}^{m\times q}(\mathbb{Z})$ is defined as

$$C(\alpha) := \sum_{\beta\in\mathbb{Z}} A(\beta)B(\alpha-\beta), \qquad \alpha \in \mathbb{Z}.$$

The *subdivision operator* $\mathcal{S}_{\boldsymbol{A}} : \ell^m(\mathbb{Z}) \to \ell^m(\mathbb{Z})$ based on the matrix sequence or *mask* $\boldsymbol{A} \in \ell_{00}^{m\times m}(\mathbb{Z})$ is defined as

$$\mathcal{S}_{\boldsymbol{A}}\boldsymbol{c}(\alpha) = \sum_{\beta\in\mathbb{Z}} A(\alpha-2\beta)\, c(\beta), \qquad \alpha \in \mathbb{Z}, \qquad \boldsymbol{c} \in \ell^m(\mathbb{Z}). \tag{5}$$

Alternatively, using symbol calculus notation, we can also decribe the action of the subdivision operator in the form

$$(\mathcal{S}_{\boldsymbol{A}}\boldsymbol{c})^*(z) = A^*(z)\, c^*\left(z^2\right), \qquad z \in \mathbb{C} \setminus \{0\}, \tag{6}$$

though, strictly speaking, (6) is only valid for $\boldsymbol{c} \in \ell_{00}^m(\mathbb{Z})$.
A *subdivision scheme* consists of the successive application of subdivision operators $\mathcal{S}_{\boldsymbol{A}^{[n]}}$, constructed from a sequence of masks $\left(\boldsymbol{A}^{[n]} \,:\, n \in \mathbb{N}_0\right)$ where $\boldsymbol{A}^{[n]} = \left(A^{[n]}(\alpha) \,:\, \alpha \in \mathbb{Z}\right) \in \ell_{00}^{m\times m}(\mathbb{Z})$ is called the *level n subdivision mask* and is assumed to be of finite support.

For some initial sequence $\boldsymbol{c}^{[0]} \in \ell^m(\mathbb{Z})$ the subdivision scheme iteratively produces sequences

$$\boldsymbol{c}^{[n+1]} := \mathcal{S}_{\boldsymbol{A}^{[n]}}\boldsymbol{c}^{[n]}, \qquad n \in \mathbb{N}_0,$$

where the components of $\boldsymbol{c}^{[n]}$ can be interpreted as values at $2^{-n}\mathbb{Z}$.

3 Hermite subdivision operators and reproduction

As already mentioned, Hermite subdivision operators act on vector valued data $\boldsymbol{c} \in \ell^{d+1}(\mathbb{Z})$, whose k-th component corresponds to a $(k-1)$–th derivative. We are interested in studying the exponential and polynomial preservation capabilities of such kind of operators.

Using the formula for the derivation of composite functions we see that for $g := f(2^{-n}\cdot)$ we have $\frac{d^r}{dx^r} g = 2^{-nr}\frac{d^r f}{dx^r}(2^{-n}\cdot)$, $r = 0, \cdots, d$. Hence,

$$\left[\frac{d^j}{dx^j} f(2^{-n}\cdot) \,:\, j = 0, \ldots, d\right] = D^n \left[f^{(j)}(2^{-n}\cdot) \,:\, j = 0, \ldots, d\right], \tag{7}$$

where

$$D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \frac{1}{2^d} \end{bmatrix}.$$

Therefore, the level-n Hermite subdivision operator is of the following form:

$$D^{n+1}\boldsymbol{c}^{[n+1]} = \sum_{\beta\in\mathbb{Z}} A^{[n]}(\cdot - 2\beta) D^n \boldsymbol{c}^{[n]}(\beta), \quad n \in \mathbb{N}_0. \tag{8}$$

3.1 Single exponential frequency

In the first step of our analysis of reproduction capability of a Hermite subdivision operator of type (8), we add only *a single* pair of exponentials $e^{\pm\lambda x}$ and therefore consider $\Lambda = \{\lambda\}$ and the space

$$V_{p,\Lambda} = \operatorname{span}\left\{1, x, \ldots, x^p, e^{\lambda x}, e^{-\lambda x}\right\}, \qquad \lambda \in \mathbb{C} \setminus \{0\}. \tag{9}$$

To keep notation simple we will denote $V_{p,\Lambda}$ simply by $V_{p,\lambda}$. Also, we will denote by d the integer $p+2\#\Lambda$, that is $d=p+2$. Note that $d+1$ is the dimension of the space $V_{p,\Lambda}$.

To better explain the underlying ideas, we will first carefully investigate this situation and then extend it in a quite straightforward though technically more involved fashion to the general case.

Remark 1 As can be seen in (9), the addition of an exponential frequency λ always means the addition of the pair $e^{\pm\lambda\cdot}$ of functions. On the one hand, this is motivated by the fact that choosing $\lambda = i$ equals reproduction of the trigonometric functions $\sin x$ and $\cos x$. Moreover, our approach to construct the annihilator and the factorization actually strongly depends on the presence of this pair of frequencies. Whether or not similar results will be available for the case where only $e^{\lambda\cdot}$ but not $e^{-\lambda\cdot}$ is considered, we do not know at present. However, our techniques to construct factorizations, used for example in the proof of Lemma 2, strongly rely on this property which is at least an indication that it is not completely pointless.

For any function $f \in C^d(\mathbb{R})$ and any integer $n \in \mathbb{N}_0$ we consider the vector sequence $\boldsymbol{v}_{f,n} \in \ell^{d+1}(\mathbb{Z})$, defined, for $\alpha \in \mathbb{Z}$, as

$$v_{f,n}(\alpha) := \begin{bmatrix} f(2^{-n}\alpha) \\ 2^{-n}f'(2^{-n}\alpha) \\ \vdots \\ 2^{-nd}f^{(d)}(2^{-n}\alpha) \end{bmatrix}.$$

The following definition is consistent with [17, Definition 1] in the case $\Lambda = \emptyset$, though formulated in a slightly different way.

Definition 3 A subdivision operator $\mathcal{S}_{\boldsymbol{A}^{[n]}}$ based on a mask $\boldsymbol{A}^{[n]} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ satisfies the $V_{p,\lambda}$-*spectral condition* if

$$\mathcal{S}_{\boldsymbol{A}^{[n]}}\boldsymbol{v}_{f,n} = \boldsymbol{v}_{f,n+1}, \qquad f \in V_{p,\lambda}.$$

Remark 2 In the previous definition, we somehow abused the term "spectral". Indeed the functions are not really eigenfunctions (the exponentials are not dilation invariant), but we wanted to keep some consistency with the existing literature. And in fact, for the polynomial members in $V_{p,\Lambda}$ this is the *spectral condition* due to Dubuc and Merrien [11].

Since we plan to extend difference operators and Taylor operators, we next recall their formal definitions.

Definition 4 The (complete) Taylor operator $\mathcal{T}_p$ of order p, acting on $\ell^{p+1}(\mathbb{Z})$, is defined as

$$\mathcal{T}_p := \begin{bmatrix} \Delta & -1 & \cdots & -\frac{1}{(p-1)!} & -\frac{1}{p!} \\ 0 & \Delta & \ddots & \vdots & \vdots \\ \vdots & & \ddots & -1 & \vdots \\ 0 & \dots & & \Delta & -1 \\ 0 & \dots & & 0 & \Delta \end{bmatrix}, \tag{10}$$

where Δ is the forward difference operator.

The Taylor operator $\mathcal{T}_p$ is a cancellation operator for the space Π_p. Its symbol $T_p^*(z) \in \mathbb{R}^{(p+1)\times(p+1)}$ takes the form

$$T_p^*(z) := \begin{bmatrix} (z^{-1}-1) & -1 & \cdots & -\frac{1}{(p-1)!} & -\frac{1}{p!} \\ 0 & (z^{-1}-1) & \ddots & \vdots & \vdots \\ \vdots & & \ddots & -1 & \vdots \\ 0 & \dots & & (z^{-1}-1) & -1 \\ 0 & \dots & & 0 & (z^{-1}-1) \end{bmatrix}. \tag{11}$$

Definition 5 A *level-n cancellation operator* $\mathcal{H} : \ell^{d+1}(\mathbb{Z}) \to \ell^{d+1}(\mathbb{Z})$ for a linear function space $V \subset C^d(\mathbb{R})$ is a convolution operator such that

$$\mathcal{H}\boldsymbol{v}_{f,n} = \sum_{\alpha\in\mathbb{Z}} H(\cdot-\alpha)v_{f,n}(\alpha) = 0, \qquad f \in V. \tag{12}$$

Lemma 1 *An operator $\mathcal{H}$ is a level-n cancellation operator for the space $V_{p,\lambda}$ if and only if it satisfies*

$$H^*(z) = \begin{bmatrix} T_p^*(z) & * \\ 0 & * \end{bmatrix}, \quad H^*(z) \in \mathbb{R}^{(p+3)\times(p+3)} \tag{13}$$

and

$$H^*\left(e^{\mp 2^{-n}\lambda}\right) D^n \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix} = 0. \tag{14}$$

Proof To annihilate polynomials of degree p, condition (12) has to be satisfied for the vector sequences

$$\boldsymbol{v}_{(\cdot)^j,n} = D^n \left[(2^{-n}\cdot)^j, j(2^{-n}\cdot)^{j-1}, \dots, j!, \underbrace{0,\, 0, \cdots,\, 0}_{d-j} \right]^T, \qquad j = 0, \dots, p,$$

and since these sequences are exactly annihilated by the complete Taylor operator as shown in [17], any decomposition of the form (13) annihilates polynomials of degree at most p, and any annihilator must have this form. To describe cancellation of exponentials, we first observe that

$$\boldsymbol{v}_{e^{\pm\lambda\cdot},n} = e^{\pm\lambda 2^{-n}\cdot} D^n \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix},$$

so that (12) becomes

$$
\begin{aligned}
0 &= \sum_{\alpha\in\mathbb{Z}} H(\cdot-\alpha)\, e^{\pm 2^{-n}\lambda\alpha} D^n \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix} \\
&= \sum_{\alpha\in\mathbb{Z}} H(\alpha)\, e^{\pm 2^{-n}\lambda(\cdot-\alpha)} D^n \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix} = e^{\pm 2^{-n}\lambda\cdot} H^*(e^{\mp 2^{-n}\lambda})\, D^n \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix},
\end{aligned}
$$

which yields (14). □

By $\mathcal{H}_{p,\lambda}^{[n]}$ we denote from now on a level-n cancellation operator for the function space spanned by $V_{p,\lambda}$.

Remark 3 If we are able to find, for given p and λ, an operator $\mathcal{H}_{p,\lambda}$ that satisfies (13) and

$$
H_{p,\lambda}^*\left(e^{\mp\lambda}\right) \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix} = 0, \tag{15}
$$

then we automatically obtain the level-n cancellation operators $\mathcal{H}_{p,\lambda}^{[n]}$ for any $n\in\mathbb{N}_0$ by setting

$$
\mathcal{H}_{p,\lambda}^{[n]} := \mathcal{H}_{p,2^{-n}\lambda}.
$$

In fact, this follows from the simple observation that the identity

$$
H_{p,2^{-n}\lambda}^*\left(e^{\mp 2^{-n}\lambda}\right) D^n \begin{bmatrix} 1 \\ \pm\lambda \\ \vdots \\ (\pm\lambda)^d \end{bmatrix} = H_{p,2^{-n}\lambda}^*\left(e^{\mp 2^{-n}\lambda}\right) \begin{bmatrix} 1 \\ \pm\frac{\lambda}{2^n} \\ \vdots \\ (\pm\frac{\lambda}{2^n})^d \end{bmatrix} = 0,
$$

is equivalent to (15), as can be verified by just replacing λ with $2^{-n}\lambda$.

The rest of the section is dedicated to the construction of a specific cancellation operator $\mathcal{H}_{p,\lambda}$, which will eventually turns out to be minimal. To that end, we write its symbol in the form

$$
H_{p,\lambda}^*(z) = \begin{bmatrix} T_p^*(z) & Q^*(z) \\ 0 & R^*(z) \end{bmatrix}, \quad Q^*(z)\in\mathbb{R}^{(p+1)\times 2},\ R^*(z)\in\mathbb{R}^{2\times 2} \tag{16}
$$

and determine the remaining parts of $H_{p,\lambda}^*(z)$, namely the Laurent polynomial matrices $Q^*(z)$ and $R^*(z)$. We begin by explicitly computing the last two elements of the first line $(H_{p,\lambda}^*)_{0,:}(z)$, where the ":" is to be understood in the sense of Matlab notation.

Lemma 2 *The condition*

$$(H^*_{p,\lambda})_{0,:}\left(e^{\mp\lambda}\right)\begin{bmatrix}1\\ \pm\lambda\\ \vdots\\ (\pm\lambda)^d\end{bmatrix}=0\,, \tag{17}$$

can be fulfilled by setting

$$(H_{p,\lambda})_{0,d-1}=h_{0,d-1}=\frac{\lambda^{1-d}}{2}\begin{cases} e^{-\lambda}-e^{\lambda}+2\displaystyle\sum_{2j+1\le d-2}\frac{\lambda^{2j+1}}{(2j+1)!}, & d\in 2\mathbb{Z},\\ -\left(e^{-\lambda}+e^{\lambda}-2\displaystyle\sum_{2j\le d-2}\frac{\lambda^{2j}}{(2j)!}\right), & d\in 2\mathbb{Z}+1,\end{cases} \tag{18}$$

and

$$(H_{p,\lambda})_{0,d}=h_{0,d}=\frac{\lambda^{-d}}{2}\begin{cases} -\left(e^{-\lambda}+e^{\lambda}+2\displaystyle\sum_{2j\le d-2}\frac{\lambda^{2j}}{(2j)!}\right), & d\in 2\mathbb{Z},\\ e^{-\lambda}-e^{\lambda}+2\displaystyle\sum_{2j+1\le d-2}\frac{\lambda^{2j+1}}{(2j+1)!}, & d\in 2\mathbb{Z}+1,\end{cases} \tag{19}$$

Proof Substituting (11) into (16), the identity (17) can be written as

$$\begin{aligned}0&=(e^{\pm\lambda}-1)-\sum_{k=1}^{d-2}\frac{(\pm\lambda)^k}{k!}+(\pm\lambda)^{d-1}h_{0,d-1}+(\pm\lambda)^d h_{0,d}\\ &=e^{\pm\lambda}-t_{d-2}\left[e^{\pm\lambda\cdot}\right](1)+(\pm\lambda)^{d-1}h_{0,d-1}+(\pm\lambda)^d h_{0,d},\end{aligned}$$

where $t_k[f]=\displaystyle\sum_{j=0}^{k}\frac{f^{(j)}(0)}{j!}(\cdot)^j$, denotes the Taylor polynomial of f of order k expanded at 0. Adding and subtracting the above conditions we get

$$\begin{aligned}0=&\left(e^{\lambda}\pm e^{-\lambda}\right)-t_{d-2}\left[e^{\lambda\cdot}\pm e^{-\lambda\cdot}\right](1)\\ &+\left(\lambda^{d-1}\pm(-\lambda)^{d-1}\right)h_{0,d-1}+\left(\lambda^d\pm(-\lambda)^d\right)h_{0,d}.\end{aligned}$$

If d is even, this implies that

$$\begin{aligned}h_{0,d-1}&=\frac{e^{-\lambda}-e^{\lambda}-t_{d-2}\left[e^{-\lambda\cdot}-e^{\lambda\cdot}\right](1)}{2\lambda^{d-1}},\\ h_{0,d}&=-\frac{e^{-\lambda}+e^{\lambda}-t_{d-2}\left[e^{-\lambda\cdot}+e^{\lambda\cdot}\right](1)}{2\lambda^{d}},\end{aligned}$$

while for odd d we get

$$\begin{aligned}h_{0,d-1}&=-\frac{e^{-\lambda}+e^{\lambda}-t_{d-2}\left[e^{-\lambda\cdot}+e^{\lambda\cdot}\right](1)}{2\lambda^{d-1}},\\ h_{0,d}&=\frac{e^{-\lambda}-e^{\lambda}-t_{d-2}\left[e^{-\lambda\cdot}-e^{\lambda\cdot}\right](1)}{2\lambda^{d}}.\end{aligned}$$

Since

$$\frac{d^k}{dx^k}\left(e^{\lambda x} \pm e^{-\lambda x}\right) = \lambda^k \left(e^{\lambda x} \pm (-1)^k e^{-\lambda x}\right),$$

we have that

$$t_{d-2}\left[e^{\lambda\cdot} - e^{-\lambda\cdot}\right](1) = 2\lambda + \frac{2}{3}\lambda^3 + \cdots = 2 \sum_{2j+1\le d-2} \frac{\lambda^{2j+1}}{(2j+1)!},$$

and

$$t_{d-2}\left[e^{\lambda\cdot} + e^{-\lambda\cdot}\right](1) = 2 + \lambda^2 + \cdots = 2 \sum_{2j\le d-2} \frac{\lambda^{2j}}{(2j)!}.$$

Substituting these identities readily gives (18) and (19). □

Taking into account the structure of $H^*_{p,\lambda}(z)$, we can now easily give also the last two entries of the other lines.

Corollary 1 *For $k = 0, \dots, d-2$, we have that*

$$h_{k,d-1} = \frac{\lambda^{1-d+k}}{2} \begin{cases} e^{-\lambda} - e^{\lambda} + 2 \displaystyle\sum_{2j+1\le d-2-k} \frac{\lambda^{2j+1}}{(2j+1)!}, & d-k \in 2\mathbb{Z}, \\ -\left(e^{-\lambda} + e^{\lambda} - 2 \displaystyle\sum_{2j\le d-2-k} \frac{\lambda^{2j}}{(2j)!}\right), & d-k \in 2\mathbb{Z}+1, \end{cases} \tag{20}$$

$$h_{k,d} = \frac{\lambda^{-d+k}}{2} \begin{cases} -\left(e^{-\lambda} + e^{\lambda} - 2 \displaystyle\sum_{2j\le d-2-k} \frac{\lambda^{2j}}{(2j)!}\right), & d-k \in 2\mathbb{Z}, \\ e^{-\lambda} - e^{\lambda} + 2 \displaystyle\sum_{2j+1\le d-2-k} \frac{\lambda^{2j+1}}{(2j+1)!}, & d-k \in 2\mathbb{Z}+1, \end{cases} \tag{21}$$

in particular, $h_{k-1,d-1} = h_{k,d}$.

To complete the construction of $H^*_{p,\lambda}(z)$, we have to define the lower right block $R^*(z)$ as

$$R^*(z) = \begin{bmatrix} z^{-1} - \dfrac{e^{\lambda} + e^{-\lambda}}{2} & \dfrac{e^{-\lambda} - e^{\lambda}}{2\lambda} \\ \lambda \dfrac{e^{-\lambda} - e^{\lambda}}{2} & z^{-1} - \dfrac{e^{\lambda} + e^{-\lambda}}{2} \end{bmatrix} \tag{22}$$

$$= L_{p,\lambda}\, \Delta^*_{\pm\lambda}(z)\, L^{-1}_{p,\lambda}, \tag{23}$$

with

$$L_{p,\lambda} := \begin{bmatrix} \lambda^{p+1} & (-\lambda)^{p+1} \\ \lambda^{p+2} & (-\lambda)^{p+2} \end{bmatrix}, \quad \Delta^*_{\pm\lambda}(z) := \begin{bmatrix} z^{-1} - e^{\lambda} & 0 \\ 0 & z^{-1} - e^{-\lambda} \end{bmatrix}$$

for which the validity of (14) is easily verified by direct computations.

Example 1 As an example, we provide the explicit structures of $\mathcal{H}_{0,\lambda}$, $\mathcal{H}_{1,\lambda}$ for the spaces $V_{0,\lambda} = \text{span}\left\{1, e^{-\lambda x}, e^{\lambda x}\right\}$ and $V_{1,\lambda} = \text{span}\left\{1, x, e^{-\lambda x}, e^{\lambda x}\right\}$. They are

$$H^*_{0,\lambda}(z) = \begin{bmatrix} z^{-1}-1 & \dfrac{e^{-\lambda}-e^{\lambda}}{2\lambda} & -\dfrac{e^{-\lambda}+e^{\lambda}-2}{2\lambda^2} \\ 0 & z^{-1}-\dfrac{e^{-\lambda}+e^{\lambda}}{2} & \dfrac{e^{-\lambda}-e^{\lambda}}{2\lambda} \\ 0 & \lambda\dfrac{e^{-\lambda}-e^{\lambda}}{2} & z^{-1}-\dfrac{e^{-\lambda}+e^{\lambda}}{2} \end{bmatrix}, \tag{24}$$

and

$$H^*_{1,\lambda}(z) = \begin{bmatrix} z^{-1}-1 & -1 & \dfrac{2-e^{-\lambda}-e^{\lambda}}{2\lambda^2} & \dfrac{2\lambda+e^{-\lambda}-e^{\lambda}}{2\lambda^3} \\ 0 & z^{-1}-1 & \dfrac{e^{-\lambda}-e^{\lambda}}{2\lambda} & -\dfrac{e^{-\lambda}+e^{\lambda}-2}{2\lambda^2} \\ 0 & 0 & z^{-1}-\dfrac{e^{-\lambda}+e^{\lambda}}{2} & \dfrac{e^{-\lambda}-e^{\lambda}}{2\lambda} \\ 0 & 0 & \lambda\dfrac{e^{-\lambda}-e^{\lambda}}{2} & z^{-1}-\dfrac{e^{-\lambda}+e^{\lambda}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} z^{-1}-1 \;\; -1 \;\; \dfrac{2-e^{-\lambda}-e^{\lambda}}{2\lambda^2} \;\; \dfrac{2\lambda+e^{-\lambda}-e^{\lambda}}{2\lambda^3} \\ 0 \\ 0 \qquad H^*_{0,\lambda}(z) \\ 0 \end{bmatrix}. \tag{25}$$

Of course, the above construction of $\mathcal{H}_{p,\lambda}$ is only one of many possibilities to construct a cancellation operator for $V_{p,\lambda}$. However, our construction is well–chosen in the sense that it includes the Taylor operator as action on the polynomials and that it in fact extends the Taylor operator.

Theorem 1 *Let $d = p + 2$. Then,*

$$\lim_{\lambda \to 0} \mathcal{H}_{p,\lambda} = \mathcal{T}_d. \tag{26}$$

Proof It follows immediately from (22) that

$$R^*(z) \to \begin{bmatrix} z^{-1}-1 & -1 \\ 0 & z^{-1}-1 \end{bmatrix},$$

as $\lambda \to 0$, hence it suffices to show that $h_{k,d-1} \to -\frac{1}{(d-1-k)!}$ and $h_{k,d} \to -\frac{1}{(d-k)!}$ as $\lambda \to 0$. Suppose that $d-k$ is even in which case we get

$$h_{k,d-1} = \frac{e^{-\lambda} - e^{\lambda} - t_{d-2-k}\left[e^{-\lambda\cdot} - e^{\lambda\cdot}\right](1)}{2\lambda^{d-1-k}} = \frac{1}{2\lambda^{d-1-k}} \sum_{j=d-1-k}^{\infty} \left((-1)^j - 1\right) \frac{\lambda^j}{j!}$$
$$= -\frac{1}{(d-1-k)!} + \lambda^2 \sum_{j=d+1-k} \frac{(-1)^j + 1}{2j!} \lambda^{j-(d+1-k)},$$

which converges as desired when $\lambda \to 0$. The arguments for $h_{k,d}$ and the case of odd $d-k$ are identical. □

3.2 Several exponential frequencies

Having understood the case of a single frequency λ, it is not hard any more to extend the construction to arbitrary sets of frequencies. To that end, let $\Lambda = \{\lambda_1, \dots, \lambda_r\} \subset \mathbb{C} \setminus \{0\}$ consist of r different frequencies, and let us construct a cancellation operator $\mathcal{H}_{p,\Lambda}$ for the space

$$V_{p,\Lambda} := \text{span}\left\{1, \dots, x^p, e^{\pm\lambda_1\cdot}, \dots, e^{\pm\lambda_r\cdot}\right\}, \quad \lambda_j \in \mathbb{C} \setminus \{0\}.$$

In this setting d is the integer number $p + 2\#\Lambda$, that is $d = p + 2r$. As in the previous subsection, $d+1$ is the dimension of the space $V_{p,\Lambda}$. The conditions for cancellation extend in a straightforward way.

Lemma 3 *The block operator* $\mathcal{H} := \begin{bmatrix} \mathcal{T}_p & \mathcal{Q} \\ 0 & \mathcal{R} \end{bmatrix}$ *with symbol*

$$H^*(z) = \begin{bmatrix} T_p^*(z) & Q^*(z) \\ 0 & R^*(z) \end{bmatrix}, \quad Q^*(z) \in \mathbb{R}^{(p+1)\times 2r},\ R^*(z) \in \mathbb{R}^{2r\times 2r}, \tag{27}$$

annihilates $V_{p,\Lambda}$ *if and only if*

$$H^*(z)\left(e^{\mp\lambda_j}\right)\begin{bmatrix} 1 \\ \pm\lambda_j \\ \vdots \\ (\pm\lambda_j)^d \end{bmatrix} = 0, \qquad j = 1, \dots, r. \tag{28}$$

Proof Since the Taylor part of $\mathcal{H}$ annihilates the polynomials, we only need to perform the computations used to derive (14) for any λ_j to show that cancellation of the exponential polynomials is equivalent to (28). □

The construction of $\mathcal{H}_{p,\Lambda}$ for given p and Λ now follows the same lines as before, namely by determining the matrix symbols $Q^*(z)$ and $R^*(z)$ in (27).

For $k = 0, \dots, p$ and $j = 1, \dots, r$, (28) is equivalent to the conditions

$$\begin{aligned} 0 &= (\pm\lambda_j)^k \left(e^{\pm\lambda_j} - 1 - \sum_{\ell=1}^{p-k} \frac{(\pm\lambda_j)^\ell}{\ell!} \right) + \sum_{m=p+1}^{p+2r} (\pm\lambda_j)^m h_{k,m} \\ &= (\pm\lambda_j)^k \left(e^{\pm\lambda_j} - t_{p-k} \left[e^{\pm\lambda_j \cdot} \right](1) \right) \\ &\quad + \sum_{\ell=1}^{r} \left((\pm\lambda_j)^{p+2\ell-1} h_{k,p+2\ell-1} + (\pm\lambda_j)^{p+2\ell} h_{k,p+2\ell} \right). \end{aligned}$$

Again, we add and subtract to obtain

$$\begin{aligned} 0 &= (\pm\lambda_j)^k \left(e^{\lambda_j} \pm (-1)^k e^{-\lambda_j} \right) - t_{p-k} \left[e^{\lambda_j \cdot} \pm (-1)^k e^{-\lambda_j \cdot} \right](1) \\ &+ \sum_{\ell=1}^{r} \left(\left(\lambda_j^{p+2\ell-1} \pm (-\lambda_j)^{p+2\ell-1} \right) h_{k,p+2\ell-1} + \left(\lambda_j^{p+2\ell} \pm (-\lambda_j)^{p+2\ell} \right) h_{k,p+2\ell} \right). \end{aligned}$$

This again decomposes depending on the parity of p and k. Supposing, for example, that p and k are even, we get for $j = 1, \dots, r$

$$\sum_{\ell=0}^{r-1} \lambda_j^{2\ell} h_{k,p+2\ell+1} = -\frac{\left(e^{\lambda_j} - e^{-\lambda_j} \right) - t_{p-k} \left[e^{\lambda_j \cdot} - e^{-\lambda_j \cdot} \right](1)}{2\lambda_j^{p+1-k}}, \tag{29}$$

$$\sum_{\ell=0}^{r-1} \lambda_j^{2\ell} h_{k,p+2\ell+2} = -\frac{\left(e^{\lambda_j} + e^{-\lambda_j} \right) - t_{p-k} \left[e^{\lambda_j \cdot} + e^{-\lambda_j \cdot} \right](1)}{2\lambda_j^{p+2-k}}, \tag{30}$$

and since the polynomials $1, x^2, \dots, x^{2r-2}$ form a Chebychev system on $\mathbb{R}_+$, this system of equations has a unique solution in terms of $h_{k,\ell}$.

Defining the vectors

$$\begin{aligned} w_+ &:= \left[-\frac{e^{\lambda_j} + e^{-\lambda_j} - t_{p-k} \left[e^{\lambda_j \cdot} + e^{-\lambda_j \cdot} \right](1)}{2\lambda_j^{p+2-k}} \; : \; j = 1, \dots, r \right], \\ w_- &:= \left[-\frac{e^{\lambda_j} - e^{-\lambda_j} - t_{p-k} \left[e^{\lambda_j \cdot} - e^{-\lambda_j \cdot} \right](1)}{2\lambda_j^{p+1-k}} \; : \; j = 1, \dots, r \right], \end{aligned}$$

and the *Vandermonde matrices*

$$L_\Lambda = \left[\lambda_j^{2\ell} \; : \; \begin{matrix} j = 1, \dots, r \\ \ell = 0, \dots, r-1 \end{matrix} \right] \in \mathbb{R}^{r \times r},$$

we can therefore write down the construction of the cancellation operator explicitly.

Lemma 4 *The condition (28) can be satisfied by setting, for $k = 0, \dots, p$,*

$$\begin{aligned} \left[h_{k,p+2\ell+1} \; : \; \ell = 0, \dots, r-1 \right] &= \begin{cases} L_\Lambda^{-1} w_-, & p - k \in 2\mathbb{N}, \\ L_\Lambda^{-1} w_+, & p - k \in 2\mathbb{N} + 1, \end{cases} \\ \left[h_{k,p+2\ell+2} \; : \; \ell = 0, \dots, r-1 \right] &= \begin{cases} L_\Lambda^{-1} w_+, & p - k \in 2\mathbb{N}, \\ L_\Lambda^{-1} w_-, & p - k \in 2\mathbb{N} + 1. \end{cases} \end{aligned}$$

The completion of $\mathcal{H}_{p,\Lambda}$ by means of $\mathcal{R}$ is now a straightforward extension of (22), namely

$$R^*(z) = L_{p,\Lambda}\, \Delta^*_\Lambda(z)\, L^{-1}_{p,\Lambda}, \tag{31}$$

where

$$L_{p,\Lambda} := \begin{bmatrix} \lambda_1^{p+1} & (-\lambda_1)^{p+1} & \dots & \lambda_r^{p+1} & (-\lambda_r)^{p+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^d & (-\lambda_1)^d & \dots & \lambda_r^d & (-\lambda_r)^d \end{bmatrix} \in \mathbb{R}^{2r\times 2r}, \tag{32}$$

and

$$\Delta^*_\Lambda(z) := \operatorname{diag}\left[\Delta^*_{\pm\lambda_j}(z) \;:\; j = 1,\dots,r\right] = \begin{bmatrix} z^{-1} - e^{\lambda_1} & & & & \\ & z^{-1} - e^{-\lambda_1} & & & \\ & & \ddots & & \\ & & & z^{-1} - e^{\lambda_r} & \\ & & & & z^{-1} - e^{-\lambda_r} \end{bmatrix}. \tag{33}$$

With

$$M_\Lambda := \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \\ \lambda_1 & -\lambda_1 & \dots & \lambda_r & -\lambda_r \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{2r-1} & (-\lambda_1)^{2r-1} & \dots & \lambda_r^{2r-1} & (-\lambda_r)^{2r-1} \end{bmatrix} =: L_{p,\Lambda}\, D_\Lambda^{-p-1},$$

where

$$D_\Lambda := \begin{bmatrix} \lambda_1 & & & & \\ & -\lambda_1 & & & \\ & & \ddots & & \\ & & & \lambda_r & \\ & & & & -\lambda_r \end{bmatrix},$$

(31) can be rewritten as

$$R^*(z) = M_\Lambda D_\Lambda^{p+1} \Delta^*_\Lambda(z) D_\Lambda^{-p-1} M_\Lambda^{-1} = M_\Lambda \Delta^*_\Lambda(z) M_\Lambda^{-1}, \tag{34}$$

and since M_Λ is the transpose of a Vandermonde matrix for interpolation at the distinct nodes $\pm\Lambda$, it is nonsingular as well as $L_{p,\Lambda}$.

Finally, we again describe what happens if certain frequencies tend to zero.

Theorem 2 *For any $j \in \{1,\dots,r\}$ we have*

$$\lim_{\lambda_j \to 0} \mathcal{H}_{p,\Lambda} = \mathcal{H}_{p+2,\Lambda\setminus\{\lambda_j\}}. \tag{35}$$

Applying Theorem 2 r times then immediately yields the counterpart of Theorem 1.

Corollary 2 *Let $d = p + 2\#\Lambda$. Then,*

$$\lim_{\Lambda\to\{0,\cdots,0\}} \mathcal{H}_{p,\Lambda} = \mathcal{T}_d. \tag{36}$$

Proof (of Theorem 2) We first observe that, by the same computations as in the proof of Theorem 1, the right hand side of (29) becomes, for $k \le p$, k, p even,

$$
\begin{aligned}
&-\frac{\left(e^{\lambda_j} - e^{-\lambda_j}\right) - t_{p-k}\left[e^{\lambda_j \cdot} - e^{-\lambda_j \cdot}\right](1)}{2\lambda_j^{p+1-k}} \\
&= -\frac{1}{2\lambda_j^{p+1-k}} \sum_{\ell=p-k+1}^{\infty} \left(1-(-1)^\ell\right) \frac{\lambda_j^\ell}{\ell!} \\
&= -\frac{1}{(p-k+1)!} + \lambda_j^2 \sum_{\ell=p-k+3}^{\infty} \left(1-(-1)^\ell\right) \frac{\lambda_j^{\ell-2}}{\ell!},
\end{aligned}
$$

which yields together with (29) that

$$
\begin{aligned}
&h_{k,p+1} - \frac{-1}{(p-k+1)!} \\
&= \lambda_j^2 \left(-\sum_{\ell=0}^{r-2} \lambda_j^{2\ell} h_{k,p+2\ell+3} + \sum_{\ell=p-k+3}^{\infty} \left(1-(-1)^\ell\right) \frac{\lambda_j^{\ell-2}}{\ell!} \right).
\end{aligned} \tag{37}
$$

The same argument applied to (30) or the respective identities for the remaining combinations of parities shows that

$$
\lim_{\lambda_j \to 0} h_{k,p+\ell} = -\frac{1}{(p+\ell-k)!}, \qquad k = 0, \dots, p, \ \ell = 1, 2. \tag{38}
$$

To consider the limit of $\mathcal{R}$ we first note that, since this expression is symmetric in Λ, we can assume that $\lambda_1 \to 0$. Since

$$
\begin{aligned}
M_\Lambda(I - e_2 e_1^T) &= \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 2\lambda_1 & -\lambda_1 & \lambda_2 & \dots & -\lambda_r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 2\lambda_1^{2r-2} & (-\lambda_1)^{2r-2} & \lambda_2^{2r-2} & \dots & (-\lambda_r)^{2r-2} \\ 2\lambda_1^{2r-1} & (-\lambda_1)^{2r-1} & \lambda_2^{2r-1} & \dots & (-\lambda_r)^{2r-1} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & -\lambda_1 & \lambda_2 & \dots & -\lambda_r \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{2r-3} & (-\lambda_1)^{2r-2} & \lambda_2^{2r-2} & \dots & (-\lambda_r)^{2r-2} \\ \lambda_1^{2r-2} & (-\lambda_1)^{2r-1} & \lambda_2^{2r-1} & \dots & (-\lambda_r)^{2r-1} \end{bmatrix} \begin{bmatrix} 2\lambda_1 & \\ & I \end{bmatrix} \\
&=: A(\lambda_1) \begin{bmatrix} 2\lambda_1 & \\ & I \end{bmatrix},
\end{aligned}
$$

where, with the abbreviation $\Lambda' = \Lambda \setminus \{\lambda_1\}$,

$$A(0) = \left[\begin{array}{cc|cccc} 0 & 1 & \lambda_2^{-2} & \dots & (-\lambda_r)^{-2} \\ 1 & 0 & \lambda_2^{-1} & \dots & (-\lambda_r)^{-1} \\ \hline 0 & 0 & 1 & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \lambda_2^{2r-3} & \dots & (-\lambda_r)^{2r-3} \end{array}\right] \left[\begin{array}{cc|ccc} 1 & & & & \\ & 1 & & & \\ \hline & & \lambda_2^2 & & \\ & & & \ddots & \\ & & & & (-\lambda_r)^2 \end{array}\right]$$
$$=: \begin{bmatrix} J & B_{\Lambda'} \\ 0 & M_{\Lambda'} \end{bmatrix} \begin{bmatrix} I & \\ & D_{\Lambda'}^2 \end{bmatrix},$$

hence

$$\det A(0) = \det J \, (\det D_{\Lambda'})^2 \, \det M_{\Lambda'} = -\left(\prod_{j=2}^{r} \lambda_j\right)^4 \det M_{\Lambda'} \neq 0,$$

so that $A(\lambda_1)$ is nonsingular for sufficiently small λ_1 with

$$\lim_{\lambda_j \to 0} A^{-1}(\lambda_1) = A(0)^{-1} = \begin{bmatrix} I & \\ & D_{\Lambda'}^{-2} \end{bmatrix} \begin{bmatrix} J & -JB_{\Lambda'}M_{\Lambda'}^{-1} \\ 0 & M_{\Lambda'}^{-1} \end{bmatrix}$$

With

$$M_\Lambda = A(\lambda_1) \begin{bmatrix} 2\lambda_1 & & \\ & 1 & \\ & & I \end{bmatrix} \begin{bmatrix} 1 & & \\ 1 & 1 & \\ & & I \end{bmatrix} = A(\lambda_1) \begin{bmatrix} 2\lambda_1 & & \\ 1 & 1 & \\ & & I \end{bmatrix},$$

it finally follows from (34) that

$$\begin{aligned}
&\lim_{\lambda_1 \to 0} R^*(z) \\
&= \lim_{\lambda_1 \to 0} A(\lambda_1) \begin{bmatrix} 2\lambda_1 & & \\ 1 & 1 & \\ & & I \end{bmatrix} \begin{bmatrix} z^{-1} - e^{\lambda_1} & & \\ & z^{-1} - e^{-\lambda_1} & \\ & & \Delta_{\Lambda'}^*(z) \end{bmatrix} \begin{bmatrix} \frac{1}{2}\lambda_1^{-1} & & \\ -\frac{1}{2}\lambda_1^{-1} & 1 & \\ & & I \end{bmatrix} A^{-1}(\lambda_1) \\
&= \lim_{\lambda_1 \to 0} A(\lambda_1) \begin{bmatrix} z^{-1} - e^{\lambda_1} & 0 & \\ -\frac{e^{\lambda_1} - e^{-\lambda_1}}{2\lambda_1} & z^{-1} - e^{-\lambda_1} & \\ & & \Delta_{\Lambda'}^*(z) \end{bmatrix} A^{-1}(\lambda_1) \\
&= A(0) \begin{bmatrix} z^{-1} - 1 & 0 & \\ -1 & z^{-1} - 1 & \\ & & \Delta_{\Lambda'}^*(z) \end{bmatrix} A^{-1}(0) \\
&= \begin{bmatrix} J & B_{\Lambda'} \\ 0 & M_{\Lambda'} \end{bmatrix} \begin{bmatrix} z^{-1} - 1 & 0 & \\ -1 & z^{-1} - 1 & \\ & & \Delta_{\Lambda'}^*(z) \end{bmatrix} \begin{bmatrix} J & -JB_{\Lambda'}M_{\Lambda'}^{-1} \\ 0 & M_{\Lambda'}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} J \begin{bmatrix} z^{-1} - 1 & \\ -1 & z^{-1} - 1 \end{bmatrix} J & * \\ 0 & M_{\Lambda'} \Delta_{\Lambda'}^*(z) M_{\Lambda'}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} z^{-1} - 1 & -1 & \\ & z^{-1} - 1 & * \\ 0 & & M_{\Lambda'} \Delta_{\Lambda'}^*(z) M_{\Lambda'}^{-1} \end{bmatrix}.
\end{aligned}$$

The upper right block of this matrix has the form

$$
\begin{aligned}
&-J\begin{bmatrix} z^{-1}-1 & \\ -1 & z^{-1}-1\end{bmatrix} JB_{\Lambda'}M_{\Lambda'}^{-1} + B_{\Lambda'}\Delta^*_{\Lambda'}(z)M_{\Lambda'}^{-1}\\
&= \left(\begin{bmatrix} 1-z^{-1} & 1 \\ & 1-z^{-1}\end{bmatrix} B_{\Lambda'} + B_{\Lambda'}\Delta^*_{\Lambda'}(z)\right) M_{\Lambda'}^{-1}\\
&= \left(\begin{bmatrix} (1-z^{-1})\lambda_2^{-2}+\lambda_2^{-1} & \dots & (1-z^{-1})(-\lambda_r)^{-2}+(-\lambda_r)^{-1} \\ (1-z^{-1})\lambda_2^{-1} & \dots & (z^{-1}-1)(-\lambda_r)^{-1}\end{bmatrix}\right.\\
&\quad \left. + \begin{bmatrix} (z^{-1}-e^{\lambda_2})\lambda_2^{-2} & \dots & (z^{-1}-e^{-\lambda_r})(-\lambda_r)^{-2} \\ (z^{-1}-e^{\lambda_2})\lambda_2^{-1} & \dots & (z^{-1}-e^{-\lambda_r})(-\lambda_r)^{-1}\end{bmatrix}\right) M_{\Lambda'}^{-1}\\
&= \begin{bmatrix} (1-e^{\lambda_2})\lambda_2^{-2}+\lambda_2^{-1} & \dots & (1-e^{-\lambda_r})(-\lambda_r)^{-2}+(-\lambda_r)^{-1} \\ (1-e^{\lambda_2})\lambda_2^{-1} & \dots & (1-e^{-\lambda_r})(-\lambda_r)^{-1}\end{bmatrix} M_{\Lambda'}^{-1}\\
&= -\begin{bmatrix} \frac12+\frac{\lambda_2}{3!}+\frac{\lambda_2^2}{4!}+\cdots & \dots & \frac12-\frac{\lambda_r}{3!}+\frac{\lambda_r^2}{4!}+\cdots \\ 1+\frac{\lambda_2}{2!}+\frac{\lambda_2^2}{3!}+\cdots & \dots & 1-\frac{\lambda_r}{2!}+\frac{\lambda_r^2}{3!}+\cdots\end{bmatrix} M_{\Lambda'}^{-1}\\
&= \sum_{k=0}^{\infty}\begin{bmatrix} -\frac{\lambda_2^k}{(k+2)!} & \cdots & -\frac{(-\lambda_r)^k}{(k+2)!} \\ -\frac{\lambda_2^k}{(k+1)!} & \cdots & -\frac{(-\lambda_r)^k}{(k+1)!}\end{bmatrix} M_{\Lambda'}^{-1}\\
&= \sum_{k=0}^{\infty}\begin{bmatrix} -\frac{1}{(k+2)!} & \\ & -\frac{1}{(k+1)!}\end{bmatrix}\begin{bmatrix} \lambda_2^k & \dots & (-\lambda_r)^k \\ \lambda_2^k & \dots & (-\lambda_r)^k\end{bmatrix} M_{\Lambda'}^{-1}\\
&= \sum_{k=0}^{2r-3}\begin{bmatrix} -\frac{1}{(k+2)!} & \\ & -\frac{1}{(k+1)!}\end{bmatrix}\begin{bmatrix} e_{k+1}^T M_{\Lambda'} \\ e_{k+1}^T M_{\Lambda'}\end{bmatrix} M_{\Lambda'}^{-1} - \sum_{k=2r-2}^{\infty}\begin{bmatrix} \frac{\lambda_2^k}{(k+2)!} & \cdots & \frac{(-\lambda_r)^k}{(k+2)!} \\ \frac{\lambda_2^k}{(k+1)!} & \cdots & \frac{(-\lambda_r)^k}{(k+1)!}\end{bmatrix} M_{\Lambda'}^{-1}\\
&= \begin{bmatrix} -\frac12 & \cdots & -\frac{1}{(2r-1)!} \\ -1 & \dots & -\frac{1}{(2r-2)!}\end{bmatrix} - \sum_{k=2r-2}^{\infty}\begin{bmatrix} \frac{\lambda_2^k}{(k+2)!} & \cdots & \frac{(-\lambda_r)^k}{(k+2)!} \\ \frac{\lambda_2^k}{(k+1)!} & \cdots & \frac{(-\lambda_r)^k}{(k+1)!}\end{bmatrix} M_{\Lambda'}^{-1}.
\end{aligned}
$$

which is of the required form and continuous in the frequencies Λ'. □

4 Factorization

The main result for the use of cancellation operators is related to the factorization of any subdivision operator that satisfies the $V_{p,\Lambda}$-spectral condition in the following sense.

Definition 6 Let $\lambda_i \in \mathbb{C}\backslash\{0\}$ and $\Lambda = \{\lambda_1, \cdots, \lambda_r\}$. A mask $\boldsymbol{A}^{[n]} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ or its associated subdivision operator $\mathcal{S}_{\boldsymbol{A}^{[n]}}$ satisfies the $V_{p,\Lambda}$-spectral condition if

$$\mathcal{S}_{\boldsymbol{A}^{[n]}}\boldsymbol{v}_{f,n} = \boldsymbol{v}_{f,n+1}, \qquad f \in V_{p,\Lambda}.$$

Theorem 3 *If the subdivision operator $\mathcal{S}_{\boldsymbol{A}^{[n]}}$ satisfies the $V_{p,\Lambda}$-spectral condition, then there exists a mask $\boldsymbol{B}^{[n]} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ such that*

$$\mathcal{H}_{p,2^{-(n+1)}\Lambda}\mathcal{S}_{\boldsymbol{A}^{[n]}} = \mathcal{S}_{\boldsymbol{B}^{[n]}}\mathcal{H}_{p,2^{-n}\Lambda}, \tag{39}$$

or, in terms of symbols,

$$H^*_{p,2^{-(n+1)}\Lambda}(z)\,\left(A^{[n]}\right)^*(z) = \left(B^{[n]}\right)^*(z)\,H^*_{p,2^{-n}\Lambda}(z^2). \tag{40}$$

In order to prove this theorem, we first give some results about the factorization of (subdivision and convolution) operators which annihilate the space $V_{p,\Lambda}$.

Theorem 4 *If $\boldsymbol{C} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ is a finitely supported mask such that $\mathcal{S}_{\boldsymbol{C}} V_{p,\Lambda} = 0$, then there exists a finitely supported mask $\boldsymbol{B} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ such that $\mathcal{S}_{\boldsymbol{C}} = \mathcal{S}_{\boldsymbol{B}}\, \mathcal{H}_{p,\Lambda}$.*

Proof We first recall from [17] that whenever $\mathcal{S}_{\boldsymbol{c}} \Pi_p = 0$, then there exists $\boldsymbol{B} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ such that

$$\mathcal{S}_{\boldsymbol{C}} = \mathcal{S}_{\boldsymbol{B}} \begin{bmatrix} \mathcal{T}_p & 0 \\ 0 & I \end{bmatrix},$$

and $\boldsymbol{B}$ has a symbol with structure

$$B^*(z) = \left[B_p^*(z), C_{2r}^*(z)\right] := \left[b_0^*(z), \dots, b_p^*(z), c_{p+1}^*(z), \dots, c_d^*(z)\right],$$

where $c_{p+1}^*(z), \dots, c_d^*(z)$ are columns of the original $C^*(z)$. We define the matrix sequence

$$\boldsymbol{W} := [\boldsymbol{v}_{e^{\lambda_1 \cdot}}, \boldsymbol{v}_{e^{-\lambda_1 \cdot}}, \dots, \boldsymbol{v}_{e^{\lambda_r \cdot}}, \boldsymbol{v}_{e^{-\lambda_r \cdot}}] \in \ell^{(d+1)\times 2r}(\mathbb{Z}).$$

By assumption, $\mathcal{S}_{\boldsymbol{C}} \boldsymbol{W} = 0$, and since also $\mathcal{H}_{p,\Lambda} \boldsymbol{W} = \begin{bmatrix} \mathcal{T}_p & \mathcal{Q} \\ 0 & \mathcal{R} \end{bmatrix} \boldsymbol{W} = 0$, we thus get

$$\begin{aligned}
0 = \mathcal{S}_{\boldsymbol{C}} \boldsymbol{W} &= \mathcal{S}_{\boldsymbol{B}} \begin{bmatrix} \mathcal{T}_p & 0 \\ 0 & I \end{bmatrix} \boldsymbol{W} = \mathcal{S}_{\boldsymbol{B}} \left(\begin{bmatrix} \mathcal{T}_p & 0 \\ 0 & I \end{bmatrix} - \mathcal{H}_{p,\Lambda} \right) \boldsymbol{W} \\
&= \mathcal{S}_{\boldsymbol{B}} \begin{bmatrix} 0 & -\mathcal{Q} \\ 0 & I - \mathcal{R} \end{bmatrix} \boldsymbol{W} = \mathcal{S}_{\boldsymbol{B}} \begin{bmatrix} 0 & -\mathcal{Q} \\ 0 & I \end{bmatrix} \boldsymbol{W} = \mathcal{S}_{\boldsymbol{B}} \begin{bmatrix} -\mathcal{Q} L_{p,\Lambda} \\ L_{p,\Lambda} \end{bmatrix} \operatorname{diag}\left(e^{\pm \Lambda \cdot}\right) \\
&= \sum_{\alpha \in \mathbb{Z}} \left(-B_p(\cdot - 2\alpha) Q(\cdot - 2\alpha) + C_{2r}(\cdot - 2\alpha)\right) L_{p,\Lambda} \operatorname{diag}\left(e^{\pm \Lambda \cdot}\right),
\end{aligned}$$

where

$$\operatorname{diag}\left(e^{\pm \Lambda \cdot}\right) := \begin{bmatrix} e^{\lambda_1 \cdot} & 0 & & & 0 \\ 0 & e^{-\lambda_1 \cdot} & & & \\ & & \ddots & & \\ & & & e^{\lambda_r \cdot} & 0 \\ 0 & & & 0 & e^{-\lambda_r \cdot} \end{bmatrix}.$$

This implies that for $\epsilon \in \{0,1\}$ and $j = 1, \dots, r$ we must have

$$0 = e^{\lambda_j \cdot} \sum_{\alpha \in \mathbb{Z}} \left(-B_p(\epsilon + 2\alpha) Q(\epsilon + 2\alpha) + C_{2r}(\epsilon + 2\alpha)\right) L_{p,\Lambda} e_{2j-1} e^{-\lambda_j \alpha}, \tag{41}$$

$$0 = e^{-\lambda_j \cdot} \sum_{\alpha \in \mathbb{Z}} \left(-B_p(\epsilon + 2\alpha) Q(\epsilon + 2\alpha) + C_{2r}(\epsilon + 2\alpha)\right) L_{p,\Lambda} e_{2j} e^{\lambda_j \alpha}, \tag{42}$$

with e_j the standard j-th unit vector in $\mathbb{R}^{d+1}$, from which it follows that

$$\left(-B_p^* Q^* + C_{2r}^*\right) L_{p,\Lambda} e_{2j-1} \left(\pm e^{-\lambda_j/2}\right) = 0,$$

and that

$$\left(-B_p^* Q^* + C_{2r}^*\right) L_{p,\Lambda} e_{2j} \left(\pm e^{\lambda_j/2}\right) = 0.$$

Hence, there exists $b^*_{2j-1}(z)$ and $b^*_{2j}(z)$ such that

$$\left(-B^*_p(z)Q^*(z^2)+C^*_{2r}(z)\right)L_{p,\Lambda}e_{2j-1}=\left(z^{-2}-e^{\lambda_j}\right)b^*_{2j-1}(z),\quad j=1,\dots,r,\tag{43}$$

and

$$\left(-B^*_p(z)Q^*(z^2)+C^*_{2r}(z)\right)L_{p,\Lambda}e_{2j}=\left(z^{-2}-e^{-\lambda_j}\right)b^*_{2j}(z),\quad j=1,\dots,r.\tag{44}$$

Setting $B^*_{2r}(z)=\left[b^*_j(z)\ :\ j=1,\dots,2r\right]$, (43) and (44) can be conveniently combined into

$$\left(-B^*_p(z)Q^*(z^2)+C^*_{2r}(z)\right)L_{p,\Lambda}=B^*_{2r}(z)\,\Delta^*_\Lambda(z^2)$$

which leads to

$$C^*_{2r}(z)=B^*_{2r}(z)\,L^{-1}_{p,\Lambda}L_{p,\Lambda}\Delta^*_\Lambda(z^2)L^{-1}_{p,\Lambda}+B^*_p(z)Q^*(z^2),$$

and consequently

$$\begin{aligned}B^*(z)&=\left[B^*_p(z),C^*_{2r}(z)\right]=\left[B^*_p(z),B^*_{2r}(z)L^{-1}_{p,\Lambda}\,L_{p,\Lambda}\Delta^*_\Lambda(z^2)L^{-1}_{p,\Lambda}+B^*_p(z)Q^*(z^2)\right]\\&=\left[B^*_p(z),B^*_{2r}(z)L^{-1}_{p,\Lambda}\right]\begin{bmatrix}I & Q^*(z^2)\\0 & R^*(z^2)\end{bmatrix}.\end{aligned}$$

This eventually gives

$$\begin{aligned}C^*(z)&=B^*(z)\begin{bmatrix}T^*_p(z^2) & 0\\0 & I\end{bmatrix}\\&=\left[B^*_p(z),B^*_{2r}(z)L^{-1}_{p,\Lambda}\right]\begin{bmatrix}I & Q^*(z^2)\\0 & R^*(z^2)\end{bmatrix}\begin{bmatrix}T^*_p(z^2) & 0\\0 & I\end{bmatrix}\\&=\left[B^*_p(z),B^*_{2r}(z)L^{-1}_{p,\Lambda}\right]\begin{bmatrix}T^*_p(z^2) & Q^*(z^2)\\0 & R^*(z^2)\end{bmatrix}\\&=\left[B^*_p(z),B^*_{2r}(z)L^{-1}_{p,\Lambda}\right]H^*_{p,\Lambda}(z^2),\end{aligned}$$

and completes the proof. □

As a consequence of Theorem 4 and Remark 3 we get the desired result that extends the observations made in the introduction.

Corollary 3 *If $\boldsymbol{C}^{[n]}\in\ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ is such that $\mathcal{S}_{\boldsymbol{C}^{[n]}}\boldsymbol{v}_{f,n}=0$, $f\in V_{p,\Lambda}$, then there exists a finitely supported mask $\boldsymbol{B}^{[n]}\in\ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ such that $\mathcal{S}_{\boldsymbol{C}^{[n]}}=\mathcal{S}_{\boldsymbol{B}^{[n]}}\mathcal{H}_{p,2^{-n}\Lambda}$.*

Using this result, Theorem 3 is now easy to prove.

Proof (of Theorem 3) Set $\mathcal{S}_{\boldsymbol{C}^{[n]}}:=\mathcal{H}_{p,2^{-(n+1)}\Lambda}\,\mathcal{S}_{\boldsymbol{A}^{[n]}}$. Since for $f\in V_{p,\Lambda}$ we have

$$\mathcal{S}_{\boldsymbol{C}^{[n]}}\boldsymbol{v}_{f,n}=\mathcal{H}_{p,2^{-(n+1)}\Lambda}\,\mathcal{S}_{\boldsymbol{A}^{[n]}}\boldsymbol{v}_{f,n}=\mathcal{H}_{p,2^{-(n+1)}\Lambda}\,\boldsymbol{v}_{f,n+1}=0,$$

it follows from Corollary 3 that there exists $\boldsymbol{B}^{[n]}$ such that

$$\mathcal{H}_{p,2^{-(n+1)}\Lambda}\,\mathcal{S}_{\boldsymbol{A}^{[n]}}=\mathcal{S}_{\boldsymbol{B}^{[n]}}\,\mathcal{H}_{p,2^{-n}\Lambda},$$

as claimed. □

A careful inspection of the proof of Theorem 4 shows that the factorization can also be extended to convolution operators.

Theorem 5 *If* $\boldsymbol{C} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ *is such that* $\boldsymbol{C} * V_{p,\Lambda} = 0$*, then there exists a finitely supported mask* $\boldsymbol{B} \in \ell_{00}^{(d+1)\times(d+1)}(\mathbb{Z})$ *such that* $\boldsymbol{C} = \boldsymbol{B} * \boldsymbol{H}_{p,\Lambda}$.

Proof The proof follows exactly the lines of the one of Theorem 3 except that (41) and (42) become

$$0 = e^{\lambda_j \cdot} \sum_{\alpha\in\mathbb{Z}} \left(-B_p(\alpha)Q(\alpha) + C_{2r}(\alpha)\right) L_{p,\Lambda} e_{2j-1} e^{-\lambda_j \alpha}, \qquad j = 1, \dots, r,$$
$$0 = e^{-\lambda_j \cdot} \sum_{\alpha\in\mathbb{Z}} \left(-B_p(\alpha)Q(\alpha) + C_{2r}(\alpha)\right) L_{p,\Lambda} e_{2j} e^{\lambda_j \alpha}, \qquad j = 1, \dots, r,$$

that is,

$$\left(-B_p^*(z)Q^*(z) + C_{2r}^*(z)\right) L_{p,\Lambda} e_{2j-1} = \left(z^{-1} - e^{\lambda_j}\right) b_{2j-1}^*(z), \qquad j = 1, \dots, r,$$
$$\left(-B_p^*(z)Q^*(z) + C_{2r}^*(z)\right) L_{p,\Lambda} e_{2j} = \left(z^{-1} - e^{-\lambda_j}\right) b_{2j}^*(z), \qquad j = 1, \dots, r.$$

From there on the arguments can be repeated literally to yield that

$$C^*(z) = B^*(z)\, H_{p,\Lambda}^*(z). \tag{45}$$

Finally, observe that in the same way the argument from [17] can be modified to give the initial factorization by means of the Taylor operator. □

Since $\mathcal{H}_{p,\Lambda}$ is a convolution operator itself and since (45) can be reformulated as the fact that for *any* $\boldsymbol{C}$ that annihilates $V_{p,\Lambda}$, the Laurent polynomial $\det C^*(z)$ must be divisible by $\det H_{p,\Lambda}^*(z)$, this operator is a particular annihilator of $V_{p,\Lambda}$. More explicitly, based on Definition 2, the following corollaries can be derived.

Corollary 4 *The operator* $\mathcal{H}_{p,\Lambda}$ *is a minimal annihilator for* $V_{p,\Lambda}$.

Corollary 5 *The Taylor operator* $\mathcal{T}_p$ *is a minimal annihilator for* $V_{p,\emptyset}$.

5 Examples

To illustrate the results of the preceding sections, we construct two Hermite subdivision operators which reproduce, by construction, polynomials and exponentials from the spaces

$$V_{0,\lambda} = \operatorname{span}\left\{1, e^{-\lambda x}, e^{\lambda x}\right\}, \qquad V_{1,\lambda} = \operatorname{span}\left\{1, x, e^{-\lambda x}, e^{\lambda x}\right\},$$

and explicitly verify for these cases the factorization property via the annihilators in (24) and (25).

For the first one, we start with a function $f \in C^2(\mathbb{R})$ and define the initial sequence of vector data $\boldsymbol{p}^{[0]} = \left(p(\alpha) := [f(\alpha), f'(\alpha), f''(\alpha)]^T \;:\; \alpha \in \mathbb{Z}\right)$ from which we recursively obtain a vector-valued sequence $\boldsymbol{p}^{[n]}$, $n \in \mathbb{N}$, in the following fashion:

having computed $\boldsymbol{p}^{[n]}$, we compute, for $\alpha \in \mathbb{Z}$, a Taylor interpolant $g_\alpha \in V_{0,\lambda}$ by requesting

$$\left[g_\alpha^{(k)}\left(2^{-n}\alpha\right) : k = 0,1,2\right] = p^{[n]}(\alpha).$$

Then we set, for $\alpha \in \mathbb{Z}$,

$$\begin{aligned} p^{[n+1]}(2\alpha) &:= p^{[n]}(\alpha), \\ p^{[n+1]}(2\alpha+1) &:= \left[\frac{g_\alpha^{(k)}\left(2^{-n-1}(2\alpha+1)\right) + g_{\alpha+1}^{(k)}\left(2^{-n-1}(2\alpha+1)\right)}{2} : k = 0,1,2\right], \end{aligned} \tag{46}$$

which gives an interpolatory Hermite subdivision scheme, whose symbol at the n-th iteration is

$$(A^{[n]})^*(z) = \frac{1}{16z}\begin{bmatrix} 8\,(z+1)^2 & \frac{8}{\lambda_n}\left(z^2-1\right)\sinh\frac{\lambda_n}{2} & \frac{8}{\lambda_n^2}\left(1+z^2\right)\left(\cosh\frac{\lambda_n}{2}-1\right) \\ 0 & 4(1+z^2)\cosh\frac{\lambda_n}{2} + 8\,z & \frac{4}{\lambda_n}\left(z^2-1\right)\,\sinh\frac{\lambda_n}{2} \\ 0 & 2\lambda_n\,\left(z^2-1\right)\sinh\frac{\lambda_n}{2} & 2(1+z^2)\,\cosh\frac{\lambda_n}{2} + 4\,z \end{bmatrix}, \tag{47}$$

where we use the abbreviation $\lambda_n := 2^{-n}\lambda$. Observe that the determinant of $(A^{[n]})^*(z)$, $n \in \mathbb{N}_0$, factorizes into

$$\det(A^{[n]})^*(z) = \frac{(z+1)^2 e^{-\lambda_n}\left(e^{\frac{\lambda_n}{2}}+z\right)^2\left(ze^{\frac{\lambda_n}{2}}+1\right)^2}{64z^3}.$$

By construction, this operator satisfies the $V_{0,\Lambda}$-spectral condition and according to Theorem 3 it is possible to find a subdivision operator $\mathcal{S}_{\boldsymbol{B}^{[n]}}$ such that the factorization (39) with the annihilator (24) holds true. At the n-th iteration, its symbol is given by:

$$\left(B^{[n]}\right)^*(z) = \frac{1}{16}\begin{bmatrix} 8(1+z) & -\frac{8}{\lambda_n}\sinh\frac{\lambda_n}{2} & \frac{8}{\lambda_n^2}\left(\cosh\frac{\lambda_n}{2}-1\right) \\ 0 & 4\cosh\frac{\lambda_n}{2} + 4\,z & -\frac{4}{\lambda_n}\sinh\frac{\lambda_n}{2} \\ 0 & -2\lambda_n\,\sinh\frac{\lambda_n}{2} & 2\cosh\frac{\lambda_n}{2} + 2\,z \end{bmatrix}. \tag{48}$$

To construct the second example, we define the initial sequence of vector data $\boldsymbol{p}^{[0]} = \left([f(\alpha), f'(\alpha), f''(\alpha), f'''(\alpha)]^T \ : \ \alpha \in \mathbb{Z}\right)$ and apply the same construction as above, just in $V_{1,\Lambda}$. The symbol at level n can be computed explicitly as

$$32z\left(A^{[n]}\right)^*(z) = \begin{bmatrix} 16\,(1+z)^2 & 8\left(z^2-1\right) & \frac{16}{\lambda_n^2}\left(1+z^2\right)\left(\cosh\frac{\lambda_n}{2}-1\right) & -\frac{8}{\lambda_n^3}\left(z^2-1\right)\left(\lambda_n - 2\sinh\frac{\lambda_n}{2}\right) \\ 0 & 8\,(z+1)^2 & \frac{8}{\lambda_n}\left(z^2-1\right)\sinh\frac{\lambda_n}{2} & \frac{8}{\lambda_n^2}\left(1+z^2\right)\left(\cosh\frac{\lambda_n}{2}-1\right) \\ 0 & 0 & 4(1+z^2)\cosh\frac{\lambda_n}{2} + 8\,z & \frac{4}{\lambda_n}\left(z^2-1\right)\,\sinh\frac{\lambda_n}{2} \\ 0 & 0 & 2\lambda_n\,\left(z^2-1\right)\sinh\frac{\lambda_n}{2} & 2(1+z^2)\,\cosh\frac{\lambda_n}{2} + 4\,z \end{bmatrix} \tag{49}$$

The determinant of $\left(A^{[n]}\right)^*(z)$ factorizes into

$$\det\left(A^{[n]}\right)^*(z) = \frac{(z+1)^4 e^{-\lambda_n}\left(e^{\frac{\lambda_n}{2}}+z\right)^2\left(ze^{\frac{\lambda_n}{2}}+1\right)^2}{1024\,z^4}.$$

This operator satisfies the $V_{1,\Lambda}$-spectral condition and therefore admits the factorization (39) with the annihilator (25) and

$$\left(B^{[n]}\right)^*(z) = \frac{1}{32}\begin{bmatrix} 16\,(1+z) & -8 & \frac{16}{\lambda_n^2}\left(\cosh\frac{\lambda_n}{2}-1\right) & \frac{8}{\lambda_n^3}\left(\lambda_n - 2\sinh\frac{\lambda_n}{2}\right) \\ 0 & 8(1+z) & -\frac{8}{\lambda_n}\sinh\frac{\lambda_n}{2} & \frac{8}{\lambda_n^2}\left(\cosh\frac{\lambda_n}{2}-1\right) \\ 0 & 0 & 4\cosh\frac{\lambda_n}{2}+4\,z & -\frac{4}{\lambda_n}\sinh\frac{\lambda_n}{2} \\ 0 & 0 & -2\lambda_n\sinh\frac{\lambda_n}{2} & 2\cosh\frac{\lambda_n}{2}+2\,z \end{bmatrix}. \quad (50)$$

We conclude this section by observing that, as n tends to infinity, the symbols $\left(A^{[n]}\right)^*(z)$ in (47) and (49) tend to

$$\frac{1}{16z}\begin{bmatrix} 8\,(z+1)^2 & 4\,\left(z^2-1\right) & \left(z^2+1\right) \\ 0 & 4\,(z+1)^2 & 2\,\left(z^2-1\right) \\ 0 & 0 & 2\,(z+1)^2 \end{bmatrix},$$

and

$$\frac{1}{96z}\begin{bmatrix} 48\,(z+1)^2 & 24\,\left(z^2-1\right) & 6\,(z^2+1) & \left(z^2-1\right) \\ 0 & 24\,(z+1)^2 & 12\,\left(z^2-1\right) & 3\,(z^2+1) \\ 0 & 0 & 12\,(z+1)^2 & 6\,\left(z^2-1\right) \\ 0 & 0 & 0 & 6\,(z+1)^2 \end{bmatrix},$$

respectively. These are the symbols of Hermite subdivision operators satisfying a spectral condition. In particular, they reproduce polynomials up to the degree 2 and 3, respectively.

6 Conclusions

In this paper we studied how the so–called *spectral condition* for Hermite subdivision operators extends to spaces generated by polynomials and exponential functions. The main tool are annihilator operators that depend only on the space $V_{p,\Lambda}$ and on the subdivision level. We also showed that the factorization of the subdivision operator satisfying these specific preservation properties is strongly connected with such annihilator operators. Though these results are interesting by themselves, the fact that all the well-known standard proofs of convergence of subdivision schemes rely on factorization and contractivity suggests that they will also be useful to characterize convergence of Hermite subdivision schemes, even with exponential polynomial reproduction. This issue is presently under investigation.

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