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Fractional and fractal dynamics approach to anomalous diffusion in porous media: application to landslide behavior

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In the past three decades, fractional and fractal calculus (that is, calculus of derivatives and integral of any arbitrary real or complex order) appeared to be an important tool for its applications in many fields of science and engineering. This theory allows to face, analytically and/or numerically, fractional differential equations and fractional partial differential equations. In particular, one of the several applications deals with anomalous diffusion processes. The latter phenomena can be clearly described from the statistical viewpoint. Indeed, in various complex systems, the diffusion processes usually no longer follow Gaussian statistics, and thus Fick's second law fails to describe the related transport behavior. In particular, one observes deviations from the linear time dependence of the mean squared displacement

$$\left\langle x^2\left(t\right)\right\rangle \propto t, \tag{1}$$

which is characteristic of Brownian motion, *i.e.*, a direct consequence of the central limit theorem and the Markovian nature of the underlying stochastic process [1-17]. Instead, anomalous diffusion is found in a wide diversity of systems and its feature is the non-linear growth of the mean squared displacement over time. Especially the power-law pattern, with exponent γ different from 1

$$\left\langle x^2\left(t\right)\right\rangle \propto t^{\gamma}, \tag{2}$$

characterizes many systems [18, 19], but a variety of other rules, such as a logarithmic time dependence, exist [20]. The anomalous diffusion, as expressed in Eq. (2) is connected with the breakdown of the central limit theorem, caused by either broad distributions or long-range correlations, *e.g.*, the extreme statistics and the power law distributions, typical of the self-organized criticality [42, 43]. Instead, anomalous diffusion rests on the validity of the Levy-Gnedenko generalized central limit theorem [21-23]. Particularly, broad spatial jumps or waiting time distributions lead to non-Gaussian distribution and non-Markovian time evolution of the system.

Anomalous diffusion has been known since Richardson's treatise on turbulent diffusion in 1926 [24] and today, the list of system displaying anomalous dynamical behavior is quite extensive. We only report some examples: charge carrier transport in amorphous semiconductors [25], porous systems [26], reptation dynamics in polymeric systems [27, 28], transport on fractal geometries [29], the long-time dynamics of DNA sequences [30].

In this scenario, the fractional calculus is used to generalized the Fokker-Planck linear equation

$$\frac{\partial}{\partial t}P(\mathrm{x},t)\text{=}D\nabla^2 P(\mathrm{x},t), \tag{3}$$

where $P(\mathbf{x},t)$ is the density of probability in the space $\mathbf{x}$=[x_1, x_2, x_3] and time t, while D >0 is the diffusion coefficient. Such processes are characterized by Eq. (1).

An example of Eq. (3) generalization is

$$\tfrac{\partial}{\partial t}P(\mathrm{x},t)\text{=}D\nabla^{\alpha}P^{\beta}(\mathrm{x},t) \quad -\infty<\alpha\leq 2 \quad \beta>-1\ , \tag{4}$$

where the fractional based-derivatives Laplacian $\Sigma(\partial^{\alpha}/\partial x^{\alpha})_i$, ($i$ = 1, 2, 3), of non-linear term $P^{\beta}(\mathbf{x},t)$ is taken into account [31].

Another generalized form is represented by equation

$$\tfrac{\partial^{\delta}}{\partial t^{\delta}}P(\mathrm{x},t)\text{=}D\nabla^{\alpha}P(\mathrm{x},t) \quad \delta>0 \quad \alpha\leq 2\ , \tag{5}$$

that considers also the fractional time-derivative [32]. These fractional-described processes exhibit a power law patters as expressed by Eq. (2).

This general introduction introduces the presented work, whose aim is to develop a theoretical model in order to forecast the triggering and propagation of landslides, using the techniques of fractional calculus. The latter is suitable for modeling the water infiltration ($i.e.$, the pore water pressure diffusion in the soil) and the dynamical processes in the fractal media [33]. Alternatively the fractal representation of temporal and spatial derivative (the fractal order only appears in the denominator of the derivative) is considered and the results are compared to the fractional one.

The prediction of landslides and the discovering of the triggering mechanism, is one of the challenging problems in earth science. Landslides can be triggered by different factors but in most cases the trigger is an intense or long rain that percolates into the soil causing an increasing of the pore water pressure. In literature two type of models exist for attempting to forecast the landslides triggering: statistical or empirical modeling based on rainfall thresholds derived from the analysis of temporal series of daily rain [34] and geotechnical modeling, $i.e.$, slope stability models that take into account water infiltration by rainfall considering classical Richardson equations [35-39]. Regarding the propagation of landslides, the models follow Eulerian ($e.g.$, finite element methods, [40]) or Lagrangian approach ($e.g.$, particle or molecular dynamics methods [41-46]). In a preliminary work [44], the possibility of the integration between fractional-based infiltration modeling and molecular dynamics approach, to model both the triggering and propagation, has been investigated in order to characterize the granular material varying the order of fractional derivative taking into account the equation

$$\frac{\partial^\delta}{\partial t^\delta}\theta(z,t)=D\frac{\partial^2\theta(z,t)}{\partial z^2}, \tag{6}$$

where $\theta(z,t)$ represents the water content depending on time t and soil depth z [47], while the parameter δ, with $0.5 \leq \delta < 1$, represents the fractional derivative order to consider anomalous sub-diffusion [48]; when $\delta = 1$ we have classical derivative, $i.e.$, normal diffusion, and when $\delta > 1$ super-diffusion [32]. To sum up, in [44], a three-dimensional model is developed, the water content is expressed in term of pore pressure (interpreted as a scalar field acting on the particles), whose increasing induces the shear strength reduction. The latter is taking into account by means of Mohr-Coulomb criterion that represents a failure criterion based on limit equilibrium theory [49, 50]. Moreover, the fluctuations depending on positions, in term of pore pressure, are also considered. Concerning the interaction between particles, a Lennard-Jones potential is taking into account and other active forces as gravity, dynamic friction and viscosity are also considered. For the updating of positions, the Verlet algorithm is used [51]. The outcome of simulations are quite satisfactory and, although the model proposed in [44] is still quite schematic, the results encourage the investigations in this direction as this types of modeling can represent a new method to simulate landslides triggered by rainfall. Particularly, the results are consistent with the behavior of real landslides, $e.g.$, it is possible to apply the method of the inverse surface displacement velocity for predicting the failure time (Fukuzono method [52]). An interesting behavior emerges from the dynamic and statistical points of view. In the simulations emerging phenomena such as detachments, fractures and arching are observed. Finally, in the simulated system, a transition of the mean energy increment distribution from Gaussian to power law, varying the value of some parameters ($i.e.$, viscosity coefficient) is observed or, fixed all parameters, the same behavior can be observed in the time, during single simulation, due to the stick and slip phases.

As mentioned, considering that our understanding of the triggering mechanisms is limited and alternative approaches based on interconnected elements are meaningful to reproduce transition from slowly moving mass to catastrophic mass release, we are motivated to investigate mathematical methods, as fractional calculus, for the comprehension of non-linearity of the infiltration phenomena and particle-based approach to achieve a realistic description of the behavior of granular materials.

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