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Spectrum of heavy and light mesons from a unified covariant treatment of hyperfine splitting.

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We present a completely covariant description of a two-quark system interacting by the Cornell potential with a Breit term describing the hyperfine splitting. Using an appropriate numerical procedure to calculate the Breit correction, we find heavy meson masses in excellent agreement with experimental data. We also use our approach to investigate light quarks. Taking average values of the running coupling constant, we show that covariance properties and hyperfine splitting are sufficient to reproduce the light mesons spectrum with good accuracy . The fundamental role of a coherent relativistic formulation for quarkonia models is therefore evident and our unified treatment for any mass of the constituent quarks proves to be a valuable and effective scheme to study potential models for mesons.

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1. Introduction

The main motivation of this work is to investigate the relevance of the relativistic properties on the structure of the energy levels of an interacting two quark systems, both for heavy and light mesons [1]. In particular we will focus on the spin dependent interaction, responsible of the hyperfine structure. In non-relativistic approaches (here and in the following we refer to [1] for detailed references) this is usually modeled by the Breit-Fermi potential with a δ-function centered at the origin and in many cases it yields difficulties in reproducing the spectra: for instance, to get a better description of the small distance behavior, a smearing of the δ-function has been proposed; this point, however, has not been settled [2]. A great attention is also devoted to open flavor quark systems. Indeed, as observed in [3], in this case the relativistic properties actually turn out to be more relevant. In summary, then, we will present: (i) a relativistic wave equation for two fermions with arbitrary masses, (ii) having the correct one particle Dirac equation limit when the mass of one component tends to infinity, (iii) having the correct two-body Schrödinger equation limit in the non-relativistic regime, (iv) including a vector interaction with a Coulomb like time component, (v) including a Breit term responsible for the hyperfine splitting, (vi) containing, finally, also a confining scalar interaction linearly growing at infinity. Many of the existing relativistic or "relativized" models are connected with field theory along the lines of the Bethe-Salpeter equation and the spectra of the resulting equations are not of straightforward computation. Few of them deal with a consistent relativistic quantum mechanical description. We could mention the full spinor treatment in [4], where, however, the confinement is essentially obtained by a cutoff of the wave function at a fixed interparticle separation and the Breit interaction is differently treated for light and heavy mesons by introducing an *ad hoc* contact interaction. Also the papers in [5] formulate a relativistic model with a two-body Dirac equation derived from constraint dynamics: the interaction is first introduced by a relativistic extension of the Adler-Piran potential and then improved by the addition of a time-like confining vector potential, yielding good results. Our formulation originates from a wave equation for two relativistic fermions with arbitrary masses obtained from two Dirac operators coupled by the interaction [6]. The construction is thus different from the previous ones and yields somewhat different results: it is canonical and involves the relative time as a cyclic variable, thus avoiding the difficulties of the relative energy excitations. The Dirac operators entering the wave equation take completely into account the relativistic kinematics: for instance the spin-orbit couplings for each component fermion are directly implied and a safe perturbation treatment of the spin-spin interaction without the smearing puzzle is made possible. In [6] this allowed a calculation of the Positronium hyperfine splitting, finding an agreement better than up to the fourth power of the fine structure constant with the results obtained by QED semi-classical expansions. For the Cornell potential, finally, the correct form of the interactions is obtained according to their tensorial nature: the vector Coulomb-like term is minimally coupled to the energy; the linear scalar term is coupled to the mass. Indeed we recall that only a scalar growing potential is confining, while an unbounded vector interaction is not [7].

2. The two fermion wave equation

The derivation of the wave equation can be found in [6], which we refer to for details. Starting with the variables $x^{\mu}_{(i)}$, $p^{\mu}_{(i)}$ of the two particle with masses m_i, $i=1,2$, we call r_a, q_a, $(a=1,2,3)$, the Wigner vectors of spin one given by the spatial parts of relative coordinates and momenta boosted to the frame with vanishing total spatial momentum and we put $r=(r_a r_a)^{1/2}$ (sum over repeated indexes). We denote by $\gamma_{(i)}$ the gamma matrices acting in the spinor space of the i-th fermion of mass $m_{(i)}$, $M=m_{(1)}+m_{(2)}$ and $\rho=\left|m_{(1)}-m_{(2)}\right|/M$. The vector minimal coupling gives rise to the term $E+b/r$, while the scalar interaction produces $\frac{1}{2}(M+\sigma r)$. Letting the Breit term be

$$V_B(r)=\frac{b}{2r}\,\gamma^0_{(1)}\gamma_{(1)_a}\gamma^0_{(2)}\gamma_{(2)_b}\left(\delta_{ab}+\frac{r_a r_b}{r^2}\right)$$

the operator acting on the wave function $\Psi(\vec{r})$ reads

$$\left(\gamma^0_{(1)}\gamma_{(1)_a}-\gamma^0_{(2)}\gamma_{(2)_a}\right)q_a+\frac{1}{2}\left(\gamma^0_{(1)}+\gamma^0_{(2)}\right)\left(M+\sigma r\right)+\frac{1}{2}\left(\gamma^0_{(1)}-\gamma^0_{(2)}\right)M\rho-\left(E+\frac{b}{r}\right)+V_B(r)$$

We proved in [6] that the first perturbation order of the Breit term is numerically obtained by substituting $V_B(r)$ with $\varepsilon V_B(r)$ in the wave operator and taking the first derivative of the eigenvalues with respect to ε in $\varepsilon=0$: this is an elementary application of the spectral correspondence, also known as the Feynman-Hellman theorem. We introduce the dimensionless variables Ω, w, s defined by

$$\sigma=\frac{M^2}{4}\,\Omega^{\frac{3}{2}},\qquad E=\frac{M}{2}\,(2+\Omega w),\qquad r=\frac{2}{M}\,\Omega^{-\frac{1}{2}}\,s,$$

By diagonalizing angular momentum and parity, we obtain a fourth order radial system

$$\begin{pmatrix} du_1(s)/ds \\ du_2(s)/ds \\ du_3(s)/ds \\ du_4(s)/ds \end{pmatrix}+\begin{pmatrix} 0 & A_0(s) & -B_0(s) & 0 \\ A_\varepsilon(s) & 1/s & 0 & B_\varepsilon(s) \\ C_\varepsilon(s) & 0 & 2/s & A_\varepsilon(s) \\ 0 & D_\varepsilon(s) & A_0(s) & 1/s \end{pmatrix}\begin{pmatrix} u_1(s) \\ u_2(s) \\ u_3(s) \\ u_4(s) \end{pmatrix}=0.$$

Introducing

$$h(s)=(2+\Omega w)/\sqrt{\Omega}+b/s\,,\qquad k(s)=(2+\Omega s)/(2\sqrt{\Omega})\,,\qquad J^2=j(j+1)\,,$$

and letting $A_0=A_\varepsilon|_{\varepsilon=0}$, $B_0=B_\varepsilon|_{\varepsilon=0}$, the even parity coefficients are

$$A_\varepsilon(s)=\frac{2\sqrt{J^2}\,\rho}{\sqrt{\Omega}\,(s\,h(s)-2\varepsilon b)}\,,\qquad C_\varepsilon(s)=\frac{h(s)}{2}+\frac{2\varepsilon b}{s}+\frac{2J^2}{2\varepsilon bs-s^2h(s)}+\frac{2s\,k^2(s)}{4\varepsilon b-s\,h(s)}\,,$$

$$B_\varepsilon(s)=\frac{(h^2(s)/2-2\rho^2/\Omega)\,s^2-2\varepsilon^2b^2}{s^2h(s)-2\varepsilon bs}\,,\quad D_\varepsilon(s)=\frac{2J^2}{s^2h(s)}-\frac{4b^2\varepsilon^2-s^2h^2(s)+4s^2k^2(s)}{4\varepsilon bs-2s^2h(s)}\,.$$

The coefficients for the odd parity system read:

$$A_\varepsilon(s)=\frac{2\sqrt{J^2}\,k(s)}{2\varepsilon b-s\,h(s)},\qquad C_\varepsilon(s)=\frac{h(s)}{2}+\frac{2J^2}{2\varepsilon bs-s^2h(s)}+\frac{2\varepsilon b}{s}+\frac{2s\rho^2}{\Omega(4\varepsilon b-s\,h(s))}\,,$$

$$B_\varepsilon(s)=\frac{4\varepsilon^2b^2-s^2h^2(s)+4s^2k^2(s)}{4\varepsilon bs-2s^2h(s)}\,,\quad D_\varepsilon(s)=-\frac{h(s)}{2}+\frac{2J^2}{s^2h(s)}-\frac{\varepsilon b}{s}+\frac{2\rho^2 s}{\Omega\,(s\,h(s)-2bs)}$$

In the following we use a parameter $\alpha=3/4b$.

State		Exp	Num		Exp	Num
$(1^1s_0)\ 0^+(0^{-+})$	η_b	9390.90±2.8	9390.39	η_c	2978.40±1.2	2978.26
$(1^3s_1)\ 0^-(1^{--})$	Υ	9460.30±.25	9466.10	J/ψ	3096.916±.011	3097.91
$(1^3p_0)\ 0^+(0^{++})$	χ_{b0}	9859.44±.73	9857.41	χ_{c0}	3414.75±.31	3423.88
$(1^3p_1)\ 0^+(1^{++})$	χ_{b1}	9892.78±.57	9886.70	χ_{c1}	3510.66±.07	3502.83
$(1^1p_1)\ 0^-(1^{+-})$	h_b	9898.60±1.4	9895.35	h_c	3525.41±.16	3523.67
$(1^3p_2)\ 0^+(2^{++})$	χ_{b2}	9912.21±.57	9908.14	χ_{c2}	3556.20±.09	3555.84
$(2^1s_0)\ 0^+(0^{-+})$	η_b	-	9971.14	η_c	3637±4	3619.64
$(2^3s_1)\ 0^-(1^{--})$	Υ	10023.26±.0003	10009.04	ψ	3686.09±.04	3692.91
$(1^3d_1)\ 0^-(1^{--})$	Υ	-	10143.84	ψ	3772.92±.35	3808.48
$(1^3d_2)\ 0^-(2^{--})$	Υ_2	10163.70±1.4	10152.69	-	-	3833.62
$(1^1d_2)\ 0^+(2^{-+})$	η_{b2}	-	10154.79	-	-	3839.20
$(1^3d_3)\ 0^-(3^{--})$	Υ_3	-	10160.91	-	-	3855.18
$(2^3p_0)\ 0^+(0^{++})$	χ_{b0}	10232.50±.0009	10232.36	χ_{c0}	-	3898.00
$(2^3p_1)\ 0^+(1^{++})$	χ_{b1}	10255.46±.0005	10256.58	χ_{c1}	-	3961.21
$(2^1p_1)\ 0^-(1^{+-})$	h_b	-	10263.61	h_c	-	3977.71
$(2^3p_2)\ 0^+(2^{++})$	χ_{b2}	10268.65±.0007	10274.26	χ_{c2}	3927±2.6	4003.93
$(3^1s_0)\ 0^+(0^{-+})$	η_b	-	10334.98	η_c	-	4064.21
$(3^3s_1)\ 0^-(1^{--})$	Υ	10355.20±.0005	10364.52	ψ	4039±1	4122.95
$(2^3d_1)\ 0^-(1^{--})$	-	-	-	ψ	4153±3	4200.51
$(3^3p_0)\ 0^+(0^{++})$	χ_{b0}		10534.86	-	-	-
$(3^3p_1)\ 0^+(1^{++})$	χ_{b1}	$<10530\pm.014>_J$	10556.59	-	-	-
$(3^3p_2)\ 0^+(2^{++})$	χ_{b2}		10572.44	-	-	-
$(4^3s_1)\ 0^-(1^{--})$	Υ	10579.40±.0012	10655.34	ψ	4421±4	4479.22
$(5^3s_1)\ 0^-(1^{--})$	Υ	10876±11	10910.35	-	-	-

Table 1: The $b\bar{b}$ and $c\bar{c}$ levels in MeV. The term symbol and the $I^G(JPC)$ numbers are displayed in the first column. We then report the names, the experimental and the numerical values for the bottomonium and charmonium levels. σ=1.111 GeV/fm for both cases, the masses of b and c quarks have been assumed m_b=4725.5 and m_c=1394.5 MeV. The parameter α has been fixed at the average values α=0.3272 and α=0.435 for $b\bar{b}$ and $c\bar{c}$ respectively. Experimental data from [8].

We give some details on the numerical method we have used. The origin and infinity are the only singular points of the boundary value problem. No further singularities arise from the matrix of the coefficients. The solution was obtained by a double shooting method and the spectral relation comes from the vanishing of the 4×4 determinant obtained by imposing the continuity of the four components at a crossing point [6]. In order to improve the accuracy of the approximate solutions at zero and infinity, we have used Padé techniques to sum the asymptotic series. The integration precision has always been kept very high and tested against the stability of the spectral values.

3. The numerical results

We will now make some comments on the results for the meson masses obtained from our model and presented in the Tables 1-3. First of all we recall that the coefficient of the Coulomb-like part of the potential is related to the QCD running coupling constant (*rcc*) α_S [8]. In our calculations we have used average values of the *rcc* for the different families of mesons, verifying *a posteriori* that the ratios of the assumed values are in agreement with those obtained from the well known α_S curve: this is possible because in each family the spread of the masses we have

State	Exp	Num	State	Exp	Num
$(1^1s_0)\ 0^+(0^{-+})$	-	818.12	$(1^1s_0)\ 0(0^-)\ B_s^0$	5366.8±.2	5387.41
$(1^3s_1)\ 0^-(1^{--})\ \phi$	1019.46±.02	1019.44	$(1^3s_1)\ 0(1^-)\ B_s^*$	5415±2.1	5434.34
$(1^3p_0)\ 0^+(0^{++})$	-	1206.44	$(1^3p_0)\ 0(0^+)$	-	5711.71
$(1^3p_1)\ 0^+(1^{++})\ f_1(1420)$	1426.4±.9	1412.84	$(1^3p_1)\ 0(1^+)$	-	5753.89
$(1^1p_1)\ 0^-(1^{+-})$	-	1458.59	$(1^1p_1)\ 0(1^+)\ B_{s1}(5830)$	5829.4±.7	5817.80
$(1^3p_2)\ 0^+(2^{++})\ f'_2(1525)$	1525±5	1525.60	$(1^3p_2)\ 0(2^+)\ B_{s2}(5840)$	5839.7±.6	5829.33
$(2^1s_0)\ 0^+(0^{-+})$	-	1554.68	$(1^1s_0)\ 0(0^-)\ D_s$	1968.49±.32	1961.24
$(2^3s_1)\ 0^-(1^{--})\ \phi$	1680±20	1698.41	$(1^3s_1)\ 0(1^-)\ D_s^*$	2112.3±.50	2101.78
$?^?(1^{--})$ X(1750)	1753.5±3.8		$(1^3p_0)\ 0(0^+)\ D_{s0}(2317)$	2317.8±.6	2339.94
$(1^3d_1)\ 0^-(1^{--})$	-	1776.53	$(1^3p_1)\ 0(1^+)\ D_{s1}(2460)$	2459.6±.6	2466.15
$(1^3d_2)\ 0^-(2^{--})$	-	1838.72	$(1^1p_1)\ 0(1^+)\ D_{s1}(2536)$	2535.12±.13	2535.82
$(2^3p_0)\ 0^+(0^{++})$	-	1841.12	$(1^3p_2)\ 0(2^+)\ D_{s2}^*(2573)$	2571.9±.8	2574.92
$(1^1d_2)\ 0^+(2^{-+})$	-	1851.44	$(2^1s_0)\ 0(0^-)\ D_s(2632)$	2632.6±1.6	2613.98
$(1^3d_3)\ 0^-(3^{--})\ \phi_3(1850)$	1854±7	1880.85	$(2^3s_1)\ 0(1^-)\ D_{sJ}^*(2710)$	2709±9	2716.67
$(2^3p_1)\ 0^+(1^{++})$	-	1988.38	$(1^3d_1)\ 0(1^-)$	-	2821.30
$(2^1p_1)\ 0^-(1^{+-})$	-	2021.97	$(1^3d_2)\ 0(2^-)$	-	2857.08
$(2^3p_2)\ 0^+(2^{++})\ f_2(2010)$	2011±70	2073.15	$(1^1d_2)\ 0(2^-)$	-	2881.48
$(3^1s_0)\ 0^+(0^{-+})$	-	2099.15	$(2^3p_0)\ 0(0^+)\ D_{sJ}(2860)$	2856.6±6.5	2885.44
$(3^3s_1)\ 0^-(1^{--})\ \phi$	2175±15	2217.57	$(1^3d_3)\ 0(3^-)\ D_{sJ}(2860)$	2862±7	2900.14
-	-	-	$(2^3p_1)\ 0(1^+)$	-	2983.53
$(1^1s_0)\quad 0(0^-)\ B_c^\pm$	6277±.006	6277	$(2^1p_1)\ 0(1^+)\ D_{sJ}(3040)$	3044±38	3029.01
-	-	-	$(2^3p_2)\ 0(2^+)$	-	3062.61

Table 2: The $s\bar{s}$, $b\bar{c}$, $b\bar{s}$, $c\bar{s}$ levels in MeV. m_s=134.27MeV, m_b and m_c as in Table 1. σ=1.111 GeV/fm for $b\bar{c}$ and $b\bar{s}$, σ=1.227 GeV/fm for $c\bar{s}$, σ=1.34 GeV/fm for $s\bar{s}$. The values of α are α=0.3591, 0.3975, 0.5344, 0.6075 for $b\bar{c}$, $b\bar{s}$, $c\bar{s}$, $s\bar{s}$ respectively. Experimental data from [8].

analyzed is sufficiently small, with the irreducible exception of the pion. The comparisons are shown in Table 4, where we give also some explicit evaluations of the corrections due to the Breit term. Further improvements of the potential are an important issue which should be developed at a more phenomenological level of the investigation: for instance, a fine tuning of the *rcc*, modeled according to the α_S curve, should produce better results. Flavor independence can be expected for heavy quarks; this is indeed what we find by doing separate fits for $b\bar{b}$, $b\bar{s}$ and $c\bar{c}$, getting the same string tension σ within the computation precision. The same value of σ is taken for the unique measured $b\bar{c}$ state. In the spectra the levels are found to be grouped into doublets of s states and quadruplets of $p, d, \ldots$ states: this feature is commonly shared by all potential models. The results are generally in very good agreement with experimental data below the thresholds of B and D mesons [8] for $b\bar{b}$ and $c\bar{c}$ respectively. Above these thresholds the calculated energies exceed the experimental ones and a potential with a softened asymptotic behavior that takes into account the pair production could help in reproducing the data of higher levels. In any case, the regularity of the pattern is maintained. For $c\bar{c}$, for instance, as the resonance X(3782) has the two possible assignments $J^{PC} = 1^{++}$ and 2^{-+} [8], the model could indicate a χ_{c1} classification. For $b\bar{b}$, where there are no unclassified physical states, we make some prediction for yet unobserved levels. We also point out the good estimate of the recently discovered $\chi_b(3P)$ resonance [8], staying

State	Exp	Num
$(1^1\mathrm{s}_0)\ 0^+(0^{-+})\ \pi^\pm$	139.57018±.00035	616.45$^{(*)}$
$(1^3\mathrm{s}_1)\ 1^+(1^{--})\ \rho(770)$	775.49±.39	826.14
$(1^3\mathrm{p}_0)\ 1^-(0^{++})\ \mathrm{a}_0(980)$	980.±20	970.34
$(1^3\mathrm{p}_1)\ 1^-(1^{++})\ \mathrm{a}_1(1260)$	1230.±40	1204.66
$(1^1\mathrm{p}_1)\ 1^+(1^{+-})\ \mathrm{b}_1(1235)$	1229.5±3.2	1274.76
$(1^3\mathrm{p}_2)\ 1^-(2^{++})\ \mathrm{a}_2(1320)$	1318.3±.6	1325.40
$(2^1\mathrm{s}_0)\ 1^-(0^{-+})\ \pi(1300)$	1300±100	1337.36
$(2^3\mathrm{s}_1)\ 1^+(1^{--})\ \rho(1450)$	1465±25	1497.63
$(1^3\mathrm{d}_1)\ 1^+(1^{--})\ \rho(1570)$	1570$^{(**)}$	1565.42
$(3^1\mathrm{s}_0)\ 1^-(0^{-+})\ \pi(1800)$	1812±12	1882.30
$(3^3\mathrm{s}_1)\ 1^+(1^{--})\ \rho(1900)$	1900$^{(**)}$	2016.35

Table 3: The $u\bar{d}$ levels in MeV. σ=1.34 GeV/fm, α=0.656, m_d=6.1 MeV, m_u=2.94 MeV. $^{(*)}$Since the curve of the *rcc* α_S has a steep increase for low masses, the value for $\pi^\pm$ in the table is obviously affected by a very large error. The experimental value is reproduced by $\alpha = 0.99$. To a smaller extent the argument holds also for $\rho(770)$. $^{(**)}$Meson Summary Table, [8].

just below the B production threshold. The much lighter mass of the s quark highly enhances the relativistic character of the $s\bar{s}$ composite system: the Breit corrections acquires a more fundamental role, giving rise to large hyperfine splittings. Due to these reasons the string tension σ has not been given the same value of the previous systems but has been considered a fitting parameter, finding a value larger than in $b\bar{b}$. Again the model suggests a classification for the unassigned states $\mathrm{f}_1(1420)$, X(1750), $\phi_3(1850)$ and $\phi(2170)$. Moreover, although we cannot have a complete phenomenological confidence in the numerical results, the model could also indicate a term $1^3\mathrm{d}_1$ for X(1750).

The analysis becomes more interesting when studying the open flavor mesons. As observed in the introduction, in these cases the relativistic properties can be expected to be much more relevant. This could be the case of some states whose nature has been debated since a long time, as, in particular, $\mathrm{D}_{\mathrm{s}0}(2317)$ and $\mathrm{D}_{\mathrm{s}1}(2460)$. Due to the fact that they are very narrow and their observation occurred through isospin violating decays, theoretical analyses have been produced to explain these mesons as tetra-quark structures or DK molecules (see refs. in [1]). Proposals investigating the possible $c\bar{s}$ nature of these states have also been done (see refs. in [1]) and more recently the data from their radiative decays have proved to support this interpretation: the mesons $\mathrm{D}_{\mathrm{s}0}(2317)$ and $\mathrm{D}_{\mathrm{s}1}(2460)$ would then be the states completing the p-wave multiplet. This point of view is now widely accepted [9] and our model is in full agreement with it. A similar agreement is also met with $\mathrm{D}^*_{\mathrm{sJ}}(2710)$ corresponding to the first radial excitation of $\mathrm{D}^*_{\mathrm{s}}$. The model then suggests a classification of $\mathrm{D}_{\mathrm{s}}(2632)$ as $2^1\mathrm{s}_0$ and indicates the state $\mathrm{D}_{\mathrm{sJ}}(3040)$ as one of the two states with $J^P = 1^+$, $n = 2$ [9]. Debated, again, is the interpretation of $\mathrm{D}_{\mathrm{sJ}}(2860)$ (see refs. in [1]): however, as it decays into two pseudo-scalars, its quantum numbers can be $J^P = 0^+, 1^-, 2^+, \cdots$. In [9] a $J^P = 3^-$ assignment is suggested: since our model finds the d-wave masses for the $c\bar{s}$ mesons considerably lower than many quark models, we find a very good agreement with such prediction.

We have finally considered the light $u\bar{d}$ looking for a fit of their masses again with a constant

Ratios of α_{num}	Ratios of α_S	State	$\Delta_B(b\bar{b})$	$\Delta_B(c\bar{c})$	$\Delta_B(s\bar{s})$	$\Delta_B(u\bar{d})$
$\frac{\alpha_{b\bar{b}}}{\alpha_{c\bar{c}}} = 0.752$	$\frac{\alpha_S(\chi_{b1,1p})}{\alpha_S(\chi_{c0,1p})} = 0.754$	$(1^1\text{s}_0)\ 0^+(0^{-+})$	92.31	155.22	296.81	$600.12^{(*)}$
$\frac{\alpha_{b\bar{b}}}{\alpha_{b\bar{c}}} = 0.911$	$\frac{\alpha_S(\chi_{b1,1p})}{\alpha_S(B_c^{\pm})} = 0.914$	$(1^3\text{s}_1)\ 0^-(1^{--})$	18.09	38.80	94.37	106.21
$\frac{\alpha_{b\bar{c}}}{\alpha_{b\bar{s}}} = 0.903$	$\frac{\alpha_S(B_c^{\pm})}{\alpha_S(B_s^{*})} = 0.955$	$(1^3\text{p}_0)\ 0^+(0^{++})$	44.30	117.41	297.14	334.57
$\frac{\alpha_{b\bar{c}}}{\alpha_{c\bar{s}}} = 0.672$	$\frac{\alpha_S(B_c^{\pm})}{\alpha_S(D_c^{*\pm})} = 0.686$	$(1^3\text{p}_1)\ 0^+(1^{++})$	19.98	52.14	127.83	142.63
$\frac{\alpha_{c\bar{c}}}{\alpha_{s\bar{s}}} = 0.716$	$\frac{\alpha_S(\chi_{c0,1p})}{\alpha_S(f_{1,1p})} = 0.714$	$(1^3\text{p}_2)\ 0^+(2^{++})$	7.51	21.10	55.93	63.72
$\frac{\alpha_{s\bar{s}}}{\alpha_{u\bar{d}}} = 0.926$	$\frac{\alpha_S(f_{1,1p})}{\alpha_S(a_{1,1p})} = 0.933$	$(1^3\text{d}_1)\ 0^-(1^{--})$	17.49	49.32	123.85	139.59

Table 4: *Left*: the behavior of α_{num} *vs.* α_S for average values $\Lambda_S = 0.221, 0.296, 0.349$ GeV for $n_f = 5,4,3$. *Right*: the Breit correction Δ_B in MeV for some levels of $b\bar{b}$, $c\bar{c}$, $s\bar{s}$. $^{(*)}$The Breit correction for $\pi^{\pm}$ has been calculated using the value $\alpha = 0.99$ that reproduces the physical mass.

α. The results are in acceptable agreement also for the very light $\rho(770)$, but obviously not for $\pi^{\pm}$, for which the use of a higher α cannot be avoided, due to the steepness of the α_S curve for very low masses. The results are not very sensitive to the mass ratio ρ that we fix at the physical value 0.35, the string tension appears to be the same found for $s\bar{s}$ and the u and d masses turn out to be close to current algebra masses, as opposed to constituent masses, usually much higher in potential models. As for some of the $c\bar{s}$ mesons previously mentioned, also the nature of the state $\text{a}_0(980)$ has been widely debated. The results coming from our model agree with the arguments exposed in [10], explaining that this state is likely to be identified with the lowest $^3\text{p}_0$. Finally, as previously observed, in order to reproduce the pion mass the appropriate α must be chosen, whose value is specified in the caption of Table 3: we could thus give an estimate of α_S at m_π, largely below the domain of applicability of the renormalization group analysis. Indeed, from Table 4 we see that the ratios of α_S with respect to the corresponding α have an average equal to 0.55. Assuming that the scaling parameter remains in this range we would get $\alpha_S(m_{\pi^{\pm}}) \simeq 0.545$.

To conclude, we can summarize our results as follows: we have presented a fully covariant formulation of the two fermion problem, which: (i) realizes a conceptual and effective improvement for dealing with meson spectra, (ii) makes the model consistent, (iii) simplifies the calculations and (iv) allows a more sound extension to light mesons. We have given a unified way of treating the hyperfine interaction without the use of eigenfunctions, eliminating the ambiguity connected with the spread at the origin: this is sufficient to explain with good accuracy even the spectrum of light and heavy-light mesons. We have found a close relationship with the QCD α_S curve and we have brought some arguments in the debate of the nature of some controversial states. Since our treatment also provides a reasonable way of calculating the wave functions, work is now in progress to investigate transition and decay amplitudes.

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