

*Proceedings of the Royal Society of Edinburgh*, **146A**, 213–223, 2016  
DOI:10.1017/S030821051500044X

## Characterization of ellipsoids as $K$ -dense sets

**Rolando Magnanini**

Dipartimento di Matematica e Informatica ‘Ulisse Dini’,  
Università degli Studi di Firenze, viale Morgagni 67/A,  
50134 Firenze, Italy ([magnanin@math.unifi.it](mailto:magnanin@math.unifi.it))

**Michele Marini**

Scuola Normale Superiore, Piazza dei Cavalieri 7,  
56126 Pisa, Italy ([michele.marini@sns.it](mailto:michele.marini@sns.it))

(MS received 27 February 2014; accepted 7 January 2015)

Let  $K \subset \mathbb{R}^N$  be any convex body containing the origin. A measurable set  $G \subset \mathbb{R}^N$  with finite and positive Lebesgue measure is said to be  $K$ -dense if, for any fixed  $r > 0$ , the measure of  $G \cap (x + rK)$  is constant when  $x$  varies on the boundary of  $G$  (here,  $x + rK$  denotes a translation of a dilation of  $K$ ). In a previous work, we proved for the case  $N = 2$  that if  $G$  is  $K$ -dense, then both  $G$  and  $K$  must be homothetic to the same ellipse. Here, we completely characterize  $K$ -dense sets in  $\mathbb{R}^N$ : if  $G$  is  $K$ -dense, then both  $G$  and  $K$  must be homothetic to the same ellipsoid. Our proof, which builds upon results obtained in our previous work, relies on an asymptotic formula for the measure of  $G \cap (x + rK)$  for large values of the parameter  $r$  and a classical characterization of ellipsoids due to Petty.

*Keywords:* uniformly dense sets; convex bodies; affine inequalities

2010 *Mathematics subject classification:* Primary 52A10; 52A20; 52A39; 52A40

### 1. Introduction

Let  $K$  be a convex body containing the origin of  $\mathbb{R}^N$  and let  $G$  be a measurable subset of  $\mathbb{R}^N$  with finite and positive Lebesgue measure  $V(G)$ . We say that  $G$  is  $K$ -dense if there is a function  $c: (0, \infty) \rightarrow (0, \infty)$  such that

$$V(G \cap (x + rK)) = c(r) \quad \text{for } x \in \partial G, r > 0. \quad (1.1)$$

Here,  $\partial G$  is the topological boundary of  $G$  and  $x + rK$  denotes the translation by a vector  $x$  of a dilation of  $K$  by a factor  $r > 0$ . For the case when  $K$  is the unit Euclidean ball  $B$  in  $\mathbb{R}^N$  such sets have been studied in [6], in which it is proved that any  $B$ -dense set  $G$  is a ball and some characterizations are also given for  $B$ -dense sets when  $V(G) = \infty$ .

In [6],  $B$ -dense sets were studied in connection with the so-called stationary isothermic (or time-invariant level) surfaces of solution of the heat equation (see also the related paper [5]). More precisely, it was noted that a set  $G$  is  $B$ -dense if and only the solution  $U = U(x, t)$  of the following Cauchy problem:

$$U_t = \Delta U \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad U = \chi_G \quad \text{on } \mathbb{R}^N \times \{0\}, \quad (1.2)$$

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is such that

$$U(x, t) = a(t) \quad \text{for } (x, t) \in \partial G \times (0, \infty),$$

for some function  $a: (0, \infty) \rightarrow (0, 1)$  (here,  $\mathcal{X}_G$  denotes the characteristic function of the set  $G$ ). If that is the case,  $\partial G$  is called a *stationary isothermic* or *time-invariant* level surface for  $U$ .

While in the case of the (linear) Laplace operator  $\Delta$  it is relatively easy to establish this connection, which is made possible by the fact that  $U$  can be obtained as the convolution of  $\mathcal{X}_G$  with the heat kernel, it is still an open problem to confirm such a link between  $K$ -dense sets and time-invariant level surfaces of the solutions of the problem

$$U_t = \Delta_K U \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad U = \mathcal{X}_G \quad \text{on } \mathbb{R}^N \times \{0\}, \quad (1.3)$$

where  $\Delta_K U = \operatorname{div}(\nabla h_K(\nabla U)h_K(\nabla U))$  denotes the *Finsler Laplacian* of  $U$  generated by the support function  $h_K$  of  $K$ . We hope that this paper may shed some light on this issue.

Plane  $K$ -dense sets have been characterized in [1, 4]. They cannot exist unless they are homothetic to  $K$  itself and, if this is the case, they must be ellipses (together with  $K$ ). In this paper, we shall extend that characterization to general dimensions by proving the following result.

**THEOREM 1.1.** *Let  $K \subset \mathbb{R}^N$  be a convex body and assume that there is a set  $G \subset \mathbb{R}^N$  of finite positive measure such that (1.1) holds.*

*Then, both  $K$  and  $G$  must be homothetic to the same ellipsoid.*

The case  $N = 2$  was first settled in [1] under some smoothness assumptions ( $\partial K$  of class  $C^2$  and  $\partial G$  of class  $C^4$ ). It should also be noted that the proof in [1] works even if condition (1.1) holds when  $r$  takes a sufficiently small interval  $(0, r_0)$ , since it only uses local information on  $\partial G$ .

In [4], we were able to remove such regularity assumptions. In fact, we showed that in the plane the occurrence of property (1.1) implies that both  $\partial G$  and  $\partial K$  are necessarily of class  $C^\infty$ . Moreover, we gave an alternative proof of the characterization, which is based on some local information on  $\partial G$  derived from (1.1) and classical affine inequalities for convex bodies.

In [4], we also established some facts that hold in general dimensions and will be useful in the remainder of this paper: let  $K \subset \mathbb{R}^N$  be a convex body and assume that property (1.1) holds. Then

- (i)  $G$  is strictly convex,
- (ii)  $\partial G$  is at least of class  $C^{1,1}$ ,
- (iii) if  $K$  is centrally symmetric (i.e.  $-K = K$ ), then  $K = G - G$  up to dilations,  $K$  is strictly convex and  $\partial K$  is at least of class  $C^{1,1}$ ,
- (iv) if  $\partial G$  is differentiable at  $x$ , then

$$V(G \cap (x + rK)) = V_0(x)r^N + o(r^N) \quad \text{as } r \rightarrow 0^+,$$

(v) if  $\partial G$  is of class  $C^2$  in a neighbourhood of  $x$ , then

$$V(G \cap (x + rK)) = V_0(x)r^N + V_1(x)r^{N+1} + o(r^{N+1}) \quad \text{as } r \rightarrow 0^+.$$

The coefficients  $V_0(x)$  and  $V_1(x)$  are explicitly computed;  $G - G$  denotes the *Minkowski sum* of  $G$  and  $-G$ :  $G - G = G + (-G) = \{x - y : x, y \in G\}$ .

It will be useful to understand the mechanism of our proof in [4]. Since (1.1) holds, (ii) and (iv) imply that the function  $V_0$  is constant on  $\partial G$ . By the explicit expression of  $V_0(x)$ , one gets that

$$V(\{y \in K : y \cdot \nu(x) \geq 0\}) = \frac{1}{2}V(K) \quad \text{for every } x \in \partial G, \tag{1.4}$$

where  $\nu(x)$  denotes the exterior unit normal to  $\partial G$  at  $x$ . When  $N = 2$ , due to (i), it is not difficult to show that (1.4) implies that  $K$  is centrally symmetric; indeed, that is also true for  $N \geq 3$ , by a non-trivial result of [8]. Thus, (iii) comes into play and we can infer further regularity ( $C^{2,1}$ ) for  $\partial G$ . Hence, (v) can be used: the function  $V_1$  must also be constant on  $\partial G$ . This condition gives a pointwise constraint on the curvature of  $\partial G$  (see [4, (1.8)]) that, for  $N = 2$ , ensures that  $K = 2G$  up to homotheties and, with the help of Minkowski's inequality for mixed volumes and an inequality involving the *affine surface area* of  $\partial G$ , gives the desired conclusion.

Now, let us look at the case in which  $N \geq 3$ . Of course, (i) and (ii) still hold if  $G$  is  $K$ -dense. Thus, the formula in (iv) still makes sense, and hence, by the aforementioned result [8],  $K$  is centrally symmetric; consequently, (iii) holds too. Therefore, (v) also makes sense and, even now, we can deduce that  $V_1$  must be constant on  $\partial G$ . Unfortunately, the pointwise constraint on the principal curvatures [4, (1.8)] is no longer enough to deduce that  $K = 2G$  and to conclude.

In this paper, we succeed in our aim by changing strategy: we give up the asymptotic expansion for  $r \rightarrow 0^+$  in (v) in favour of an expansion such as

$$V(G \cap (x + rK)) = V(G) + W(x)(r_G - r)^{(N+1)/2} + o((r_G - r)^{(N+1)/2}) \tag{1.5}$$

as  $r \rightarrow r_G^-$ , where

$$r_G = \inf\{r > 0 : G \subseteq x + rK\}, \quad x \in \partial G.$$

Note that, if  $G$  is  $K$ -dense, then  $r_G$  is independent on  $x \in \partial G$ ; since our problem is invariant with respect to dilations of  $K$ , throughout the paper, we shall assume that  $r_G = 1$ .

The computation of the coefficient  $W(x)$  is carried out in §2 and involves the *support function*  $h_K : \mathbb{S}^{N-1} \rightarrow \mathbb{R}$  of the convex body  $K$  with respect to the origin, and the *shape operators*  $S_G$  and  $S_K$  are of  $G$  and  $K$ , respectively. In fact, it turns out that for  $x \in \partial G$

$$W(x) = -\frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[\frac{1}{2}S_G(u) - \frac{1}{2}S_K(u)]^{1/2}} \quad \text{with } u = \nu(\bar{x}); \tag{1.6}$$

here,  $\{\bar{x}\} = \partial G \cap (x + K)$ ,<sup>1</sup>  $\nu(\bar{x})$  is the exterior unit normal to  $\partial G$  at  $\bar{x}$ , and  $\omega_{N-1}$  denotes the measure of the unit ball of  $\mathbb{R}^{N-1}$ .

<sup>1</sup> It will be made clear in §2 that  $\bar{x}$  is uniquely determined.

Properties (i) and (1.1) imply that the right-hand side of (1.6) must be constant as a function of  $u \in \mathbb{S}^{N-1}$ . A first consequence of this fact is that  $K = 2G$  up to homotheties; a second consequence is that

$$\kappa_G(u) = ch_G(u)^{N+1} \quad \text{for every } u \in \mathbb{S}^{N-1}, \quad (1.7)$$

for some positive constant  $c$ ; here,  $\kappa_G$  denotes the *Gauss curvature* of  $\partial G$  at the (unique) point  $x \in \partial G$  having normal equal to  $u$ .

The identity (1.7) is well known in the theory of convex bodies: in fact, Petty proved in [7] that it characterizes  $G$  as an ellipsoid.

Section 2 contains all the details.

## 2. The proof of theorem 1.1

Let  $G \subset \mathbb{R}^N$  be a  $C^2$  convex body. In a sufficiently small neighbourhood of a point  $x \in \partial G$ , the set  $\partial G$  is the graph of a  $C^2$ -regular convex function over the tangent space to  $\partial G$  at  $x$ ; we denote by  $S_G$  the Hessian of this function (the associated bilinear form is often called the *shape operator*); it is well known that its determinant  $\kappa_G$  is the *Gaussian curvature* of  $\partial G$  at that point. When  $G$  is strictly convex, without any ambiguity we can think of  $S_G$  as a function over the unit sphere, so that, for a given  $u \in \mathbb{S}^{N-1}$ ,  $S_G(u)$  denotes the shape operator at the only point  $x \in \partial G$  with outward unit normal equal to  $u$ .

We know from [4] that if  $G$  is  $K$ -dense, then  $\partial G$  is of class  $C^2$ , but, unfortunately, we cannot assert that  $\partial K$  is of class  $C^2$ , even if we know that  $K = G - G$  (see, for example, [2]). Nevertheless, in [3] it is shown that if  $G$  is strongly convex,<sup>2</sup> then  $K$  has the same regularity as  $G$ ; in particular, the following result holds.

**THEOREM 2.1** (Krantz and Parks [3]). *If  $A$  is a strongly convex body with boundary of class  $C^\infty$  and  $B$  is a convex body with boundary of class  $C^2$ , then the Minkowski sum  $A + B$  has boundary of class  $C^\infty$ .*

*Moreover, the shape operator of  $A + B$  can be expressed by the following formula:*

$$S_{A+B}(u) = [I + S_A(u)^{-1}S_B(u)]^{-1}S_B(u). \quad (2.1)$$

The proof of this theorem can also be repeated step by step in the case in which the  $C^\infty$ -regularity of  $A$  is replaced by its  $C^2$ -regularity: one then gets that the boundary of  $A + B$  is of class  $C^2$  and that (2.1) holds as well.

Thus, our aim is now to show that  $K$ -dense bodies are strongly convex; then, by theorem 2.1, we will gain the necessary regularity of  $K$  that gives a meaning to (2.1) with  $A = G$  and  $B = -G$ .

In order to do this, for  $x \in \partial G$  we shall study the asymptotic behaviour of  $V(G \setminus (x + rK))$  as  $r \rightarrow 1^-$ . As we shall see, if we want to express  $V(G \setminus (x + rK))$  in terms of the shape operator of  $\partial G$  at some point  $\bar{x} \in \partial G$ , it is important to make sure that  $G$  shares only one point with the boundary of  $x + K$ . We observe that this is not always the case: indeed, consider the Releaux triangle as the set  $G$ , and let  $x$  denote one of its vertices; then,  $K = G - G$  is a ball and  $G \cap (x + K)$  is one of the arcs constituting the triangle's boundary; hence,  $G \setminus (x + rK)$  cannot be localized, so to speak, around any point of  $\partial G$ .

<sup>2</sup> That is,  $S_K(u) > 0$  for every  $u \in \mathbb{S}^{N-1}$ , by which we mean that  $S_K(u)$  is positive definite for every  $u \in \mathbb{S}^{N-1}$ .

Note that such a  $G$  is strictly convex, but  $\partial G$  is not differentiable. Likewise, if we consider differentiable bodies that are not strictly convex, we can still provide an example of the same phenomenon: in fact, it is enough to set  $G = B + Q$ , where  $B$  is the unit ball and  $Q$  is the unit square.

The following lemma shows that we can get the desired result if we assume that  $G$  is both differentiable and strictly convex.

LEMMA 2.2. *Let  $G$  be a strictly convex body with differentiable boundary and set  $K = G - G$ . Then, for each  $x \in \partial G$ , the set  $\partial(x + K) \cap G$  consists of only one point,  $\bar{x} \in \partial G$ , characterized by  $\nu_K(\bar{x} - x) = -\nu_G(x)$ .*

*Proof.* Let  $z \in \partial K \cap (G - x)$  and let  $u = \nu_K(z)$ . Clearly,  $z + x \in \partial G$  and, since  $G - x$  is contained in  $K$  and touches  $K$  at  $z$  from inside, then  $\nu_G(z + x) = u$ . Since  $K = G - G$ , we have

$$\begin{aligned} h_G(u) + h_G(-u) &= h_K(u) = \langle z, u \rangle \\ &= \langle z + x, u \rangle + \langle x, -u \rangle \\ &= h_G(u) + \langle x, -u \rangle. \end{aligned}$$

Thus,  $h_G(-u) = \langle x, -u \rangle$ , that is  $\nu_G(x) = -u$ . It is then enough to set  $\bar{x} = z + x$ .

Now, suppose that there exists another point,  $z'$ , such that  $z' \in \partial K \cap (G - x)$  and set  $u' = \nu_K(z')$ ; by the same argument as above, we get that  $\nu_G(x) = -u'$ , and hence  $u = u'$ . Since  $K$  is strictly convex (as  $G$  is strictly convex), we finally find  $z = z'$ .  $\square$

The following lemma is helpful to prove that a  $K$ -dense set is positively curved.

LEMMA 2.3. *Let  $G$  be a strictly convex body with boundary of class  $C^2$  and let  $K = G - G$ . For  $x \in \partial G$  and  $\bar{x} \in \partial G$  such that  $u = \nu_G(x) = -\nu_G(\bar{x})$ , the following hold:*

(i) *if  $\kappa_G(u) = 0$ , then*

$$\liminf_{r \rightarrow 1^-} \frac{V(G \setminus (x + rK))}{(1 - r)^{(N+1)/2}} = +\infty;$$

(ii) *if  $\kappa_G(u) > 0$ , then*

$$\limsup_{r \rightarrow 1^-} \frac{V(G \setminus (x + rK))}{(1 - r)^{(N+1)/2}} \leq \frac{2^{(N+1)/2} \omega_{N-1}}{N + 1} \kappa_G(u) h_K(u)^{(N+1)/2} (1 + \Lambda)^{(N-1)/2},$$

*where  $\Lambda$  is the maximal principal curvature of  $\partial G$  at  $x$ .*

*Proof.* First, note that, by the above lemma, our choice of  $x$  and  $\bar{x}$  ensures that  $\{\bar{x}\} = \partial(x + K) \cap G$ . Without loss of generality, we can always assume that  $\bar{x} = 0$  and that  $u = (0, 0, \dots, -1)$ ; then, in a neighbourhood of  $\bar{x}$ ,  $\partial G$  can be parametrized by

$$y_N = \frac{1}{2} \langle S_G(u)y, y \rangle + o(|y|^2) \quad \text{as } |y| \rightarrow 0, \tag{2.2}$$

where  $y = (y_1, \dots, y_{N-1})$  ranges in the tangent space to  $\partial G$  at  $\bar{x}$ .

(i) Set  $\varepsilon = 1 - r$ . Let  $\varepsilon_n$  be an infinitesimal sequence of positive numbers such that

$$\liminf_{r \rightarrow 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{(N+1)/2}} = \lim_{n \rightarrow \infty} \frac{V(G_n)}{\varepsilon_n^{(N+1)/2}},$$

where  $G_n := G \setminus (x + (1 - \varepsilon_n)K)$ ; then (2.2) suggests that, by possibly extracting a subsequence from  $\varepsilon_n$ , we can fit in  $G_n$  the set  $E_n$  bounded by the paraboloid

$$y_N = \frac{1}{2} \langle S_G(u)y, y \rangle + \frac{1}{n} |y|^2$$

and the hyperplane  $\varepsilon_n h_K(u)u + u^\perp$  supporting the set  $x + (1 - \varepsilon_n)K$  at the point whose outer unit normal coincides with  $u$ . In our coordinates,

$$E_n = \left\{ (y, y_N) : \frac{1}{2} \langle S_G(u)y, y \rangle + \frac{1}{n} |y|^2 < y_N < \varepsilon_n h_K(u) \right\}$$

and  $E_n \subseteq G_n$ .

Thus, by Fubini's theorem and some calculations, we get

$$\begin{aligned} V(G_n) &\geq V(E_n) = \int_0^{\varepsilon_n h_K(u)} \mathcal{H}^{N-1} \left( \left\{ y : \left\langle \left[ \frac{S_G(u)}{2} + \frac{1}{n} I \right] y, y \right\rangle \leq t \right\} \right) dt \\ &= \frac{\omega_{N-1}}{\det[S_G(u)/2 + I/n]^{1/2}} \int_0^{\varepsilon_n h_K(u)} t^{(N-1)/2} dt \\ &= \frac{2\omega_{N-1} \varepsilon_n^{(N+1)/2} h_K(u)^{(N+1)/2}}{(N+1) \det[S_G(u)/2 + I/n]^{1/2}}. \end{aligned}$$

Therefore,

$$\begin{aligned} \liminf_{\varepsilon \rightarrow 0^+} \frac{V(G \setminus (x + (1 - \varepsilon)K))}{\varepsilon^{(N+1)/2}} &= \lim_{n \rightarrow \infty} \varepsilon_n^{-(N+1)/2} V(G_n) \\ &\geq \lim_{n \rightarrow \infty} \frac{2\omega_{N-1} h_K(u)^{(N+1)/2}}{(N+1) \sqrt{\det[S_G(u)/2 + I/n]}} \\ &= +\infty, \end{aligned}$$

since  $\det S_G(u) = \kappa_G(u) = 0$ .

(ii) We obtain the desired inequality by observing that the domain  $G \setminus (x + (1 - \varepsilon)K)$  can be contained in the region  $F_{\varepsilon, \delta}$  bounded by two paraboloids: one outside  $G$  and tangential to  $\partial G$  at  $\bar{x}$ , the other tangent to the boundary of  $x + (1 - \varepsilon)K$  from inside. In order to show this, we assume as before that  $\bar{x} = 0$  and  $u = -e_N$  and, moreover, that  $S_G(u) = I$  (this can be done since the affine transformation  $S_G(u)$  is invertible, as  $\det S_G(u) = \kappa_G(u) > 0$ ): the desired formula will then be obtained by multiplying the right-hand side of (2.3) by the factor  $\kappa_G(u)$ .

We proceed to construct  $F_{\varepsilon, \delta}$ . We choose any number  $\lambda > 0$  such that  $\lambda I > S_G(-u)$ , i.e. such that  $\lambda > \Lambda$ . Since  $\kappa_G(u) > 0$ , theorem 2.1 implies that  $\partial K$  is twice differentiable at  $\bar{x} - x$ ; moreover, (2.1) turns into

$$S_K(u) < \frac{\lambda}{1 + \lambda} I;$$

hence,

$$S_{(1-\varepsilon)K}(u) < \frac{\lambda}{(1+\lambda)(1-\varepsilon)}I.$$

For  $\varepsilon > 0$  sufficiently small, we define  $F_{\varepsilon,\delta}$  as

$$F_{\varepsilon,\delta} = \left\{ (y, y_N) : \frac{1}{2}\delta|y|^2 \leq y_N \leq \varepsilon h_K(u) + \frac{\lambda}{2(1+\lambda)(1-\varepsilon)}|y - \varepsilon x_*|^2 \right\},$$

where  $\delta$  is chosen in the interval  $(\lambda/(1+\lambda)(1-\varepsilon), 1)$  and  $x_*$  is the projection of  $x$  on the tangent space to  $\partial G$  at  $\bar{x}$ ; in this way,

$$G \setminus (x + (1-\varepsilon)K) \subset F_{\varepsilon,\delta}.$$

Indeed, (2.2) guarantees that the above inclusion holds, at least inside a small neighbourhood of  $\bar{x}$ ; however, by lemma 2.2, we know that  $G \setminus (x + (1-\varepsilon)K)$  is contained in a ball  $B_r$  around  $\bar{x}$  whose radius  $r = r(\varepsilon)$  tends to 0 as  $\varepsilon \rightarrow 0$ .

By using the rescaling  $(y, y_N) = (\sqrt{\varepsilon}\xi, \varepsilon\xi_N)$ , we obtain that

$$V(F_{\varepsilon,\delta}) = \varepsilon^{(N+1)/2}V(F'_{\varepsilon,\delta}),$$

where

$$F'_{\varepsilon,\delta} = \left\{ (\xi, \xi_N) : \frac{1}{2}\delta|\xi|^2 \leq \xi_N \leq h_K(u) + \frac{\lambda}{2(1+\lambda)(1-\varepsilon)}|\xi - \sqrt{\varepsilon}x_*|^2 \right\},$$

and it is easy to show that  $V(F'_{\varepsilon,\delta}) \rightarrow V(F'_{0,\delta})$ . By a straightforward computation of  $V(F'_{0,\delta})$ , we get that

$$\limsup_{\varepsilon \rightarrow 0} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \leq \frac{\omega_{N-1}}{N+1} \times \frac{2^{(N+1)/2}h_K(u)^{(N+1)/2}}{(\delta - \lambda/(1+\lambda))^{(N-1)/2}},$$

and minimizing the right-hand side of this formula for  $\lambda/(1+\lambda) < \delta < 1$  and  $\lambda > \Lambda$  then gives

$$\limsup_{\varepsilon \rightarrow 0} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \leq \frac{\omega_{N-1}}{N+1} \times 2^{(N+1)/2}h_K(u)^{(N+1)/2}(1+\Lambda)^{(N-1)/2}. \tag{2.3}$$

□

**COROLLARY 2.4.** *If  $G$  is  $K$ -dense, then  $\partial K$  is of class  $C^2$  and every point of  $\partial G$  is a point of strong convexity. The latter conditions and the fact that  $K = G - G$  allow us to write, due to theorem 2.1,*

$$S_K(u) = [I + S_G(u)^{-1}S_G(-u)]^{-1}S_G(-u). \tag{2.4}$$

*Proof.* Since  $G$  is  $K$ -dense, the limits in lemma 2.3(i) and (ii) do not depend on the particular point  $x \in \partial G$ ; in other words, they must be constant functions on  $\partial G$ . Since  $G$  is a convex body and  $\partial G$  is of class  $C^2$ ,  $\kappa_G$  is not identically zero; hence, the limit in lemma 2.3(ii) is a finite constant. As a consequence, (i) of the same lemma implies that  $\kappa_G > 0$  (and hence  $S_G > 0$ ) on  $\partial G$ . Formula (2.4) is then a straightforward consequence of theorem 2.1. □

THEOREM 2.5. *Let  $G$  be a strongly convex body with boundary of class  $C^2$ , and set  $K = G - G$ . Choose  $x, u$  and  $\bar{x}$  as in lemma 2.3; then*

$$\lim_{r \rightarrow 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{(N+1)/2}} = \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1) \det[\frac{1}{2}S_G(u) - \frac{1}{2}S_K(u)]^{1/2}}.$$

*Proof.* Again we set  $\varepsilon = 1 - r$ . We begin by showing that

$$\limsup_{\varepsilon \rightarrow 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \leq \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1) \det[\frac{1}{2}S_G(u) - \frac{1}{2}S_K(u)]^{1/2}}. \quad (2.5)$$

As in the proof of lemma 2.3, without loss of generality, we can set  $u = -e_N$  and  $\bar{x} = 0$ .

We recall that  $0 = \bar{x} \in \partial(x + K)$ . Thus,  $-x \in \partial K$  and  $-(1-\varepsilon)x \in \partial((1-\varepsilon)K)$ ; namely,  $\varepsilon x \in \partial(x + (1-\varepsilon)K)$ , and  $u$  is the unit normal to  $\partial(x + (1-\varepsilon)K)$  at that point; also, by a scaling argument, we know that

$$S_{x+(1-\varepsilon)K}(u) = \frac{S_K(u)}{1-\varepsilon}.$$

Note that (2.4) implies that  $S_G(u) > S_K(u)$ ; hence, we can choose  $\bar{n} \in \mathbb{N}$  such that

$$\frac{S_G(u) - S_K(u)}{4} > \frac{I}{\bar{n}}. \quad (2.6)$$

In order to get an estimate from above for  $V(G \setminus (x + (1-\varepsilon)K))$  we construct a set  $C_{\varepsilon,n}$  containing  $G \setminus (x + (1-\varepsilon)K)$ . In fact, for  $n > \bar{n}$  we set

$$C_{\varepsilon,n} = \left\{ (y, y_N) : \left\langle \left( \frac{S_G(u)}{2} - \frac{1}{n}I \right) y, y \right\rangle < y_N < \varepsilon h_K(u) + \left\langle \left[ \frac{S_K(u)}{2(1-\varepsilon)} + \frac{1}{n}I \right] (y - \varepsilon x_*) , (y - \varepsilon x_*) \right\rangle \right\},$$

where  $x_*$  denotes the projection of  $x$  on  $u^\perp$ ;  $C_{\varepsilon,n}$  is the region bounded by two paraboloids, one touching  $\partial G$  at  $\bar{x}$  from below, the other touching  $\partial(x + (1-\varepsilon)K)$  at  $\varepsilon x$  from above and, for  $\varepsilon$  small enough, we have

$$G \setminus (x + (1-\varepsilon)K) \subset C_{\varepsilon,n}.$$

Also, condition (2.6) guarantees that

$$\frac{S_G(u)}{2} - \frac{I}{n} > \frac{S_K(u)}{2(1-\varepsilon)} + \frac{I}{n} > 0$$

for  $\varepsilon$  small enough, thus forcing  $C_{\varepsilon,n}$  to be bounded.

The usual change of variables  $(y, y_N) = (\sqrt{\varepsilon}\xi, \varepsilon\xi_N)$  gives that

$$V(C_{\varepsilon,n}) = \varepsilon^{(N+1)/2} V(C'_{\varepsilon,n}),$$



where

$$C'_{\varepsilon,n} = \left\{ (\xi, \xi_N) : \left\langle \left[ \frac{S_G(u)}{2} - \frac{1}{n}I \right] \xi, \xi \right\rangle < \xi_N < h_K(u) + \left\langle \left[ \frac{S_K(u)}{2(1-\varepsilon)} + \frac{1}{n}I \right] (\xi - \sqrt{\varepsilon}x_*), (\xi - \sqrt{\varepsilon}x_*) \right\rangle \right\}.$$

Since clearly  $V(C'_{\varepsilon,n}) \rightarrow V(C'_{0,n})$  as  $\varepsilon \rightarrow 0$ , a straightforward computation gives

$$\begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} &\leq V(C'_{0,n}) \\ &= \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[S_G(u)/2 - S_K(u)/2 - 2I/n]^{1/2}}. \end{aligned} \tag{2.7}$$

Since (2.7) holds for all  $n$  large enough, (2.5) follows at once by taking the limit for  $n \rightarrow \infty$ .

The converse inequality,

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \geq \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[\frac{1}{2}S_G(u) - \frac{1}{2}S_K(u)]^{1/2}},$$

is proved by using the same strategy used for (2.5): we choose  $\bar{n}$  such that

$$S_G(u) > \frac{I}{\bar{n}}$$

and then we construct, for  $n > \bar{n}$  and  $\varepsilon$  small, a set  $D_{\varepsilon,n} \subseteq G \setminus (x + (1-\varepsilon)K)$ :

$$D_{\varepsilon,n} = \left\{ (y, y_N) : \left\langle \left( \frac{S_G(u)}{2} + \frac{1}{n}I \right) y, y \right\rangle < y_N < \varepsilon h_K(u) + \left\langle \left[ \frac{S_K(u)}{2(1-\varepsilon)} - \frac{1}{n}I \right] (y - \varepsilon x_*), (y - \varepsilon x_*) \right\rangle \right\}.$$

As before, the usual rescaling gives

$$\liminf_{\varepsilon \rightarrow 0^+} \frac{V(G \setminus (x + (1-\varepsilon)K))}{\varepsilon^{(N+1)/2}} \geq \frac{2\omega_{N-1}h_K(u)^{(N+1)/2}}{(N+1)\det[S_G(u)/2 - S_K(u)/2 + 2I/n]^{1/2}}.$$

Again, we conclude by taking the limit for  $n \rightarrow \infty$ . □

**COROLLARY 2.6.** *Let  $G$  be a  $K$ -dense body. Then (1.5) holds with the coefficient  $W(x)$  given by (1.6). Moreover, the function defined by*

$$\frac{h_K(u)^{(N+1)/2}}{\det[S_G(u) - S_K(u)]^{1/2}}, \quad u \in \mathbb{S}^{N-1}, \tag{2.8}$$

*is constant.*

*Proof.* Corollary 2.4 ensures that  $G$  satisfies the assumptions of theorem 2.5. Since  $V(G \cap (x + rK)) = V(G) - V(G \setminus (x + rK))$ , the following equality holds for the

function given in (1.6):

$$W(x) = - \lim_{r \rightarrow 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{(N+1)/2}} = \lim_{r \rightarrow 1^-} \frac{V(G \cap (x + rK)) - V(G)}{(1-r)^{(N+1)/2}}. \quad (2.9)$$

Hence, (1.5) holds. Observe that  $W|_{\partial G}$  has to be constant, by (2.9) and the  $K$ -density assumption. Finally, since  $G$  is strictly convex, the last assertion follows from the surjectivity of the Gauss map.  $\square$

Now, we shall show that if  $G$  is  $K$ -dense, then  $G$  and  $K$  must be equal up to homotheties.

**PROPOSITION 2.7.** *Let  $G$  be a  $K$ -dense body. Then  $\kappa_G(u) = \kappa_G(-u)$ .*

*Proof.* Let  $u \in \mathbb{S}^{N-1}$  and let  $L = L_u$  be a linear map of  $\mathbb{R}^N$  in itself, which leaves unchanged the unit vector  $u$  and whose restriction to  $u^\perp$  equals  $S_G(u)^{-1/2}$ .

Firstly, note that, as an easy consequence of (1.1), the set  $LG$  is  $LK$ -dense, so that corollary 2.6 holds for this set; in particular, (2.8) implies

$$\begin{aligned} h_{LK}(-u)^{(N+1)/2} \{\det[S_{LG}(-u) - S_{LK}(-u)]\}^{-1/2} \\ = h_{LK}(u)^{(N+1)/2} \{\det[S_{LG}(u) - S_{LK}(u)]\}^{-1/2}. \end{aligned} \quad (2.10)$$

Secondly, we know that  $K$  is centrally symmetric, and so must be  $LK$ ; then,  $S_{LK}(u) = S_{LK}(-u)$  and  $h_{LK}(u) = h_{LK}(-u)$ . Hence, by (2.10),

$$\det[S_{LG}(-u) - S_{LK}(u)] = \det[S_{LG}(u) - S_{LK}(u)]. \quad (2.11)$$

As we shall see, this condition, together with (2.1), is enough to prove that

$$\det[S_{LG}(u)] = \det[S_{LG}(-u)].$$

Indeed, by plugging (2.1) into (2.11) we get

$$\begin{aligned} \det(S_{LG}(-u) - [I + S_{LG}(u)^{-1}S_{LG}(-u)]^{-1}S_{LG}(-u)) \\ = \det(S_{LG}(u) - [I + S_{LG}(u)^{-1}S_{LG}(-u)]^{-1}S_{LG}(-u)); \end{aligned} \quad (2.12)$$

furthermore, our choice of the affine transformation  $L$  ensures that

$$S_{LG}(u) = I$$

and

$$S_{LG}(-u) = S_G(u)^{-1/2}S_G(-u)S_G(u)^{-1/2}. \quad (2.13)$$

Equation (2.12) then turns into

$$\det(S_{LG}(-u) - [I + S_{LG}(-u)]^{-1}S_{LG}(-u)) = \det(I - [I + S_{LG}(-u)]^{-1}S_{LG}(-u)); \quad (2.14)$$

by multiplying both sides of (2.14) by  $\det[I + S_{LG}(-u)]$  and using Binet's identity, we get

$$\det[S_{LG}(-u)^2] = 1.$$

Hence, (2.13) yields  $\det[S_G(u)] = \det[S_G(-u)]$ , i.e.  $\kappa_G(u) = \kappa_G(-u)$ .  $\square$

COROLLARY 2.8. *Let  $G$  be  $K$ -dense. Then  $G$  is symmetric and  $K = 2G$ .*

*Proof.* The two bodies  $G - G$  and  $2G$  have the same Gaussian curvature as a function on  $\mathbb{S}^{N-1}$ ; thus, they only differ by a translation.  $\square$

The following theorem and Petty's characterization of ellipsoids [7] complete the proof of theorem 1.1.

THEOREM 2.9. *Let  $G$  be a  $K$ -dense set. Then, for every  $x \in \partial G$  it holds that*

$$\lim_{r \rightarrow 1^-} \frac{V(G \setminus (x + rK))}{(1-r)^{(N+1)/2}} = \frac{2^N \omega_{N-1} h_K(u)^{(N+1)/2}}{(N+1) \det[S_G(u)]^{1/2}} \quad \text{with } u = \nu(\bar{x})$$

and  $\{\bar{x}\} = \partial G \cap (x + K)$ .

*In particular, there exists a positive constant  $c$ , depending only on  $N$ , such that*

$$\kappa_G(u) = ch_G(u)^{N+1} \quad \text{for every } u \in \mathbb{S}^{N-1}.$$

*Therefore,  $G$  must be an ellipsoid.*

## Acknowledgements

The authors are deeply indebted to the anonymous referee for the careful revision and comments that greatly improved the paper.

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