Referee's report on the paper

Asymptotics of decreasing solutions of coupled p-Laplacian systems in the framework of regular variation
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The subject of this article is the study of the nonlinear differential system of Emden-Fowler type

$$\begin{aligned}(p(t)\Phi_\alpha(x'))' &= \varphi(t)\Phi_\lambda(y)\\ (q(t)\Phi_\beta(y'))' &= \psi(t)\Phi_\mu(x)\end{aligned}$$

where $\Phi_\zeta(u) = |u|^{\zeta-1}u$, $\alpha, \beta, \lambda, \mu$ are positive constants satisfying subhomogenous condition $\alpha\beta > \lambda\mu$ and p, q, φ, ψ are positive continuous functions defined on $[a, \infty)$.

First, the authors study existence of strongly decreasing solutions i.e. positive decreasing solutions (x, y) satisfying

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = \lim_{t\to\infty} p(t)\Phi_\alpha(x'(t)) = \lim_{t\to\infty} q(t)\Phi_\beta(y'(t)) = 0.$$

In the subsequent section, assuming that p, q, φ, ψ are regularly varying, authors showed that all strongly decreasing solutions are regularly varying and obtained an exact asymptotic formula for such solutions.

This is nice and very interesting paper in which Karamata's theory of regularly varying functions is used to obtain new results on the existence and precise asymptotic behavior of positive solutions. The paper is well organized. Results are mathematically correct and are significant improvement of the existing results both in terms of the existence as well as of the asymptotic behavior of solutions. I think that this work, if published, will certainly attract the attention of many researches working in this area of differential equations. The arguments used in the proofs are clear, precise and easy to follow. Moreover, method of the proof is not at all trivial and represents an additional original contribution of this paper.

Rating of the paper: **Excellent**
Reviewer's conclusions: **I recommend its publication in the Annali di Matematica Pura ed Applicata.**

The manuscript contains some minor typographical flaws and misprints that should be corrected before publication. The list of these flaws now follows:

- $P7_{12}$, $P7_{13}$: At the right side of equations should be $\frac{1}{q^{\frac{1}{\beta}}(t)}$ and $\frac{1}{p^{\frac{1}{\alpha}}(t)}$
- $P10^4$: $\nu\alpha - \alpha + \gamma \le \nu\gamma$ should be $\nu\alpha - \alpha + \gamma \le \nu\alpha$

- $P17_3$: Here $\nu^{[1]} < 0$ implies (20) $\ldots\ \omega^{[1]} < 0$ implies (21)

- $P19^8$: $L_1^{\frac{1}{\alpha\beta}} \ \to L_1^{\frac{1}{\alpha\beta\Lambda}}$

- $P19^{13}$: $\dfrac{1}{s^\delta}\, L_q(s) \to \dfrac{1}{s^\delta L_q(s)}$

- $P19_9$: I suggest to add after "applying the just obtained estimates for x, y" the following : "and using the identities $\lambda\omega+\sigma+1 = (\nu-1)\alpha+\gamma = \nu^{[1]}$ and $\mu\nu+\rho+1 = (\omega-1)\beta+\delta = \omega^{[1]}$" .

  This two identities are not trivial and should be mentioned, especially because they are used later as well, as in (50) and (51) for example.

- $P19_7$: $-y^{[2]}(t) \ \to -y^{[1]}(t)$

- $P21^2$: $zw\Big(\dfrac{-x}{x} \ \to zw\Big(\dfrac{-x'}{x}$

- $P21^{12}$ and $P21^{14}$: $L_q^{\frac{1}{\beta p_2 q_3}} \ \to L_q^{-\frac{1}{\beta p_2 q_3}}$

- $P24^2$: Using above mentioned identities I believe it is sufficient to write that

$$\lambda\,\omega = \lambda + \frac{\lambda}{\beta}(\omega^{[1]} - \delta) = \ldots$$