Continuation of the solution to the chemotaxis problem beyond its blow-up

GLOBAL IN TIME SOLUTION TO THE KELLER-SEGEL MODEL OF CHEMOTAXIS.

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Abstract. We consider the Keller-Segel model of chemotaxis in the radial-symmetric two-dimensional case. The blow-up occurs if the size of the initial datum is greater than some threshold. We define the continuation of the solution after its blow-up and provide two ways of regularizing the problem that look quite natural and converge to the solution. Finally, we show that if the size of the initial datum is less than threshold, than all the mass diffuses to the infinity for infinite time whereas, if it is greater than threshold, then all the initial mass concentrates asymptotically in the origin.

Key words. chemotaxis, blow-up.

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1. Introduction.

Blow-up of solutions to the system of partial differential equations modelling chemotaxis has been investigated recently by several authors (see e.g. [1]–[10]and e.g. [11] and the literature quoted there for analogous models related to gravitational collapse). But also in the case of the basic Keller–Segel model [12] some problems have received only partial answer. Referring to that model and confining to a two-dimensional radial-symmetric problem in an infinite domain, we will try to contribute to a better understanding of this mathematical phenomenon.

Just for the sake of completeness, let us recall the biological motivation of the model. A living population moves according to a diffusional mechanism and to the stimulus provided by a chemical substance produced by the population itself. The latter acts in the sense that population tends to migrate where higher concentration of the substance is found. Thus, the basic model consists of two partial differential equations for the concentrations of the chemical substance and of the population respectively.

In all radial-symmetric problems, Keller-Segel system can be reduced to a single equation for a suitable mass function; this equation shows clearly that the phenomenon is the combination of diffusion and nonlinear convection. When the latter prevails, blow-up occurs, whereas when diffusion provides sufficiently strong dissipation, no singularity appears for the concentration.

This balance is dependent on the number of dimensions, so that in one space dimension there is no blow-up and eventually the equation is a Burger's equation. In three dimensions blow-up occurs for any size of initial datum, while in the two-dimensional case either the diffusion term or nonlinear convection can be dominant depending on the size of initial datum.

Let us focus our attention to the two-dimensional radial-symmetric situation. In case of blow-up (which will occur in the center for symmetry reasons) some additional questions arise. If the initial mass exceeds the threshold value, is it possible to characterize the fraction that "collapses"in a delta function in the origin, at the blow-up instant?

Will it be 100% of the initial mass or will it be related to the threshold value as some authors claim ([13], [14])? Is there any way of continuing the solution after

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blow-up? If yes, how will the blow-up develop in time (e.g. can it "melt out ")? How will be the asymptotic behaviour? Can a regularization be performed so that this continuation can be actually computed?

In Section 2 of this paper we deal with classical solutions and we characterize cases of blow-up by a comparison technique using as a barrier the only nontrivial asymptotic solution. In Section 3 we define continuation of the classical solutions beyond blow-up time T_0. This definition, although analogous to the usual definition of generalized solution, is in fact non-trivial because, although allowing finite mass to concentrate in the origin after T_0 (without prescribing its amount as a function of time) identifies the solution uniquely. Moreover, two way of regularizing the problem are provided that look quite natural and they are shown to converge to the solution.

Finally, in Section 4 we analyze in more detail the solution after blow-up and its asymptotic behaviour. We outline a feature distinguishing the diffusion-dominated case (no blow-up) from the convection-dominated case. In the former case all the mass is diffused to infinity, while in the latter the mass concentrated at the origin is changing in time and tends asymptotically to the total initial mass, i.e. the "delta-function" absorbs all of the mass in infinite time.

The case in which the initial mass is exactly equal to the critical value is the only one in which a non-trivial asymptotic solution exists, and is easily predictable to be unstable.

2. Classical solution. The two-dimensional radial-symmetric problem we will consider is the following,

$$ru_t = \left(\nu r u_r - r u p_r\right)_r, \ r > 0, \ t > 0, \tag{2.1}$$

$$\left(r p_r\right)_r = -ru, \ r > 0, \ t > 0, \tag{2.2}$$

with initial condition

$$u(r, 0) = u_0(r), \ r > 0, \tag{2.3}$$

where $u_0(r)$ will be assumed such that

$$\lim_{r\to\infty} u_0(r) r^{2+\alpha} = 0, \tag{2.4}$$

for some $\alpha > 0$, so that the total initial mass

$$\widehat{M} = 2\pi \int_0^\infty r u_0(r) dr \tag{2.5}$$

is finite.

We define the mass function $\mathfrak{M}(r, t)$ as

$$\mathfrak{M}(r, t) \stackrel{def}{=} 2\pi \int_0^r \rho u(\rho, t) d\rho. \tag{2.6}$$

Then (2.1), (2.2) reduces to the single equation

$$\mathfrak{M}_t = \nu r \left[\frac{\mathfrak{M}_r}{r}\right]_r + \frac{\mathfrak{M}\mathfrak{M}_r}{2\pi r}, \ r > 0, \ t > 0. \tag{2.7}$$

Next, define

$$s \equiv \frac{r^2}{2}, \tag{2.8}$$

$$M(s,t) = \mathfrak{M}(\sqrt{2s}, t) = 2\pi \int_0^{\sqrt{2s}} \rho u(\rho, t) d\rho. \tag{2.9}$$

Then, (2.7) becomes

$$M_t = 2\nu s M_{ss} + \frac{1}{2\pi} M M_s, \ s > 0, \ t > 0, \tag{2.10}$$

and (2.3) gives

$$M(s,0) = M_0(s) \equiv 2\pi \int_0^{\sqrt{2s}} \rho u_0(\rho) d\rho, \tag{2.11}$$

so that

$$\lim_{s\to\infty} M_0(s) = \widehat{M}. \tag{2.12}$$

Our first aim is to discuss the solvability of the following

PROBLEM 2.1. *Given a bounded nondecreasing function $M_0(s) \in C^1([0,\infty))$, $M_0(0) = 0$, find $T > 0$ and a bounded function $M(s,t) \in C([0,\infty) \times [0,T)) \cap C^{2,1}((0,\infty) \times (0,T))$ satisfying*

$$\begin{cases} M_t = 2\nu s M_{ss} + \frac{1}{2\pi} M M_s, & s > 0, \ t \in (0,T), \\ M(s,0) = M_0(s), & s > 0, \\ M(0,t) = 0, & t \in (0,T). \end{cases} \tag{2.13}$$

$M(s,t)$ will be called classical solution to the problem and interval $(0, T_0)$, $T_0 = \sup\{T\}$, finite or infinite, will be called maximal interval of existence.

First we state the following

PROPOSITION 2.2. *Any solution of Problem 2.1 is such that*

$$0 \le M \le \widehat{M}, \ s > 0, \ t \in (0, T_0), \tag{2.14}$$

$$M_s(s,t) \ge 0, \ s > 0, \ t \in (0, T_0). \tag{2.15}$$

Proof. Straightforward application of maximum principle [15] yields inequalities (2.14). On the other hand $U = M_s$ satisfies

$$U_t = 2\nu s U_{ss} + \left(2\nu + \frac{1}{2\pi}\right) U_s + \frac{1}{2\pi} U^2. \tag{2.16}$$

Moreover, $U(0,t) > 0$ since M attains its minimum on $s = 0$. Since $U(s,0) \ge 0$, maximum principle gives (2.15). □

Next, we prove the following result on monotone dependence upon data.

LEMMA 2.3. *Let M_1, M_2 be classical solutions of (2.13) corresponding to data $M_{10}(s)$ and $M_{20}(s)$ respectively, let T_{01} and T_{02} be their respective maximal interval of existence, and let $\widetilde{T}_0 = \min(T_{01}, T_{02})$. Then, if*

$$M_{10}(s) \le M_{20}(s), \ s > 0, \tag{2.17}$$

then

$$M_1(s,t) \le M_2(s,t),\ \ s>0,\ \ t\in(0,\widetilde{T}_0). \tag{2.18}$$

Proof. Consider

$$V(s,t) = M_2(s,t) - M_1(s,t),\ \ s>0,\ \ t\in(0,\widetilde{T}_0),$$

and note that

$$V_t = 2\nu s V_{ss} + \frac{M_1+M_2}{4\pi} V_s + \frac{V}{s}\left(M_1+M_2\right)_s,\ \ s>0,\ \ t\in(0,\widetilde{T}_0). \tag{2.19}$$

Note that V is nonnegative for $t=0$, $s>0$ because of (2.17) and vanishes on $s=0$. In addition, the coefficient of V in (2.19) is nonnegative because of (2.15). Hence (2.18) follows by maximum principle (consider e.g. approximation V^ε whose data are $V^\varepsilon(s,0) = V(s,0)+\varepsilon$, $V^\varepsilon(0,t)=\varepsilon$ and prove that V^ε can never reach the minimum value $\varepsilon/2$). □

Essentially by the same arguments, we can prove

LEMMA 2.4. *Let M solve Problem 2.1. Then the following implications hold:*

$$M_0''(s) \le 0 \ \ \textit{in } R^+ \Longrightarrow M_{ss}(s,t) \le 0 \ \textit{in } R^+ \times (0,T), \tag{2.20}$$

$$4\pi\nu s M_0''(s) + M_0(s)M_0'(s) \ge 0 \ \textit{in } R^+ \Longrightarrow M_t(s,t) \ge 0 \ \textit{in } R^+ \times (0,T), \tag{2.21}$$

$$4\pi\nu s M_0''(s) + M_0(s)M_0'(s) \le 0 \ \textit{in } R^+ \Longrightarrow M_t(s,t) \le 0 \ \textit{in } R^+ \times (0,T). \tag{2.22}$$

Proof. The proof is based again on maximum principle applied to M_{ss} and to M_t. It is just necessary to approximate M by the solution M_α of approximated equations

$$M_{\alpha t} = 2\nu(s+\alpha)M_{\alpha ss} + \frac{1}{2\pi} M_\alpha M_{\alpha s}$$

so that $M_{\alpha ss}(0,t)$ exists and vanishes on $s=0$. □

Investigating stationary solutions to (2.10) will prove to be an useful tool in the sequel. We prove the following

PROPOSITION 2.5. *Equation (2.10) admits nonconstant stationary solutions $\overline{M}(s)$ of finite mass $\widehat{M}$ only if*

$$\widehat{M} \in (4\pi\nu, 8\pi\nu]. \tag{2.23}$$

If $\widehat{M} = 8\pi\nu$, then $\overline{M}(s)$ has the form

$$\overline{M}(s) = 8\pi\nu \frac{s}{s+\beta}, \tag{2.24}$$

for arbitrary $\beta > 0$.
If $\widehat{M} \in (4\pi\nu, 8\pi\nu)$, then $\overline{M}(s)$ has the form

$$\overline{M}(s) = \frac{\widehat{M} s^\gamma + \delta N_0}{s^\gamma + \delta}, \tag{2.25}$$

where $N_0 = 8\pi\nu - \widehat{M} \in (0, 4\pi\nu)$, $\gamma = \widehat{M}/(4\pi\nu) - 1$ and δ is an arbitrary positive constant.

Proof. By direct computation we have

$$s\overline{M}' + \frac{1}{8\pi\nu}\overline{M}^2 - \overline{M} - \frac{1}{8\pi\nu}\widehat{M}^2 + \widehat{M} = 0,$$

since assumption (2.4) implies $\lim_{s\to\infty} s\overline{M}' = 0$. On the other hand, $\overline{M} < \widehat{M}$ and $\overline{M}' \geq 0$ yield $\widehat{M} > 4\pi\nu$ and, consequently, $\lim_{s\to 0+} s\overline{M}' = 0$. Then (2.24), (2.25) follow at once. Note that in (2.24)

$$\frac{1}{\beta} = \frac{\overline{M}'(0)}{8\pi\nu} = \frac{\overline{u}(0)}{4\nu}, \tag{2.26}$$

and that in (2.25)

$$\overline{M}(0) = N_0, \tag{2.27}$$

whereas (2.24) is the only form of nontrivial stationary solution having $\overline{M}(0) = 0$. □
We also have

PROPOSITION 2.6. *Let*

$$M_\infty(t) = 2\pi \int_0^\infty ru(r,t)dr = \lim_{s\to\infty} M(s,t). \tag{2.28}$$

Then it is

$$M_\infty(t) = M_\infty(0) = \widehat{M}, \quad \forall t \in (0, T_0). \tag{2.29}$$

Proof. Let $Q(s,t)$ solve the linear problem

$$Q_t = 2\nu s Q_{ss},\ s > 0,\ t \in (0,T), \tag{2.30}$$

$$Q(s,0) = M_0(s),\ s > 0,\ Q(0,t) = 0,\ t \in (0,T). \tag{2.31}$$

It is immediately seen that

$$M(s,t) \geq Q(s,t),\ s > 0,\ t \in (0,T). \tag{2.32}$$

But it is also true that

$$\lim_{s\to\infty} Q(s,t) = \lim_{s\to\infty} Q(s,0) = \widehat{M}. \tag{2.33}$$

Indeed, it is easy to get (2.33) writing the problem solved by Q_s (possibly using the approximations as in the proof of Lemma 2.4) and integrating it over any rectangle $R^+ \times (0,t)$). At this point, (2.33) and (2.14) yield (2.29). □

Now, we state and prove the main result of this section.

THEOREM 2.7. *If*

$$\widehat{M} \leq 8\pi\nu, \tag{2.34}$$

then Problem 2.1 is uniquely solvable and $T_0 = \infty$.
If on the contrary

$$\widehat{M} > 8\pi\nu, \tag{2.35}$$

then Problem 2.1 is uniquely solvable , T_0 is finite and

$$\lim_{t\to T_0-} M(s,t) \geq 8\pi\nu,\ \forall s \in R^+. \tag{2.36}$$

Proof. Local existence is rather standard. Uniqueness for a given T is a corollary of Lemma 2.3. We also note that as long as

$$\lim_{\substack{s\to 0+,\\ t\to T-}} M(s,t) = 0$$

(i.e. as long as $u(0,t)$ is finite) the solution can be continued beyond T.

First, we prove the theorem when

$$\widehat{M} < 8\pi\nu. \tag{2.37}$$

Indeed, the stationary solution (2.24) is uniformly and monotonically convergent to $8\pi\nu$ in any closed subset of R^+ when $\beta \to 0$. Hence, for sufficiently small β, $M_0(s) \leq \overline{M}(s)$. Therefore, Lemma 2.3 enables us to claim that

$$M(s,t) \leq \overline{M}(s) \text{ in } R^+ \times (0,T) \tag{2.38}$$

and thus the solution can be continued beyond any $T > 0$.

Next, we prove that the same is true also when

$$\widehat{M} = 8\pi\nu. \tag{2.39}$$

Let

$$\widetilde{u} = \max(8\pi\nu, \max_{[0,1]} M_0'(s)), \tag{2.40}$$

and define

$$v_0(s) = \begin{cases} \widetilde{u}s, & s \in [0, 8\pi\nu/\widetilde{u}], \\ 8\pi\nu, & s > 8\pi\nu/\widetilde{u}. \end{cases} \tag{2.41}$$

If $v(s,t)$ is the solution of Problem 2.1 with initial datum $v_0(s)$ and $(0,\widetilde{T})$ is its maximal interval of existence, Lemma 2.3 ensures that $M(s,t) \leq v(s,t)$ in their common interval of existence and that $T_0 \geq \widetilde{T}$. So, to complete the proof of the first statement of the theorem we only need to prove that $\widetilde{T} = +\infty$.

To this end, let $\varepsilon > 0$ and let P_ε be the solution of the following problem

$$\begin{aligned} P_{\varepsilon t} &= 2\nu\varepsilon P_{\varepsilon ss} + \frac{1}{2\pi} P_\varepsilon P_{\varepsilon s},\ s > 0,\ t \in (\widetilde{T}/2, \widetilde{T}), \\ P_\varepsilon(s, \widetilde{T}/2) &= v(s, \widetilde{T}/2), \quad s > 0, \\ P_\varepsilon(0,t) &= 0,\ t \in (\widetilde{T}/2, \widetilde{T}). \end{aligned}$$

Since $v_{ss} \leq 0$ in $R^+ \times (0,\widetilde{T})$ (recall (2.20) after proper smoothing of $v_0(s)$). we have that

$$v(s,t) \leq P_\varepsilon(s,t) \text{ in } R^+ \times (\widetilde{T}/2, \widetilde{T}). \tag{2.42}$$

Consider the nonlinear wave propagation equation

$$\begin{aligned} P_t &= \frac{1}{2\pi} P P_s, \text{ in } R^+ \times (\widetilde{T}/2, \widetilde{T}), \\ P(s, \widetilde{T}/2) &= v(s, \widetilde{T}/2), \quad s > 0, \\ P_\varepsilon(0, t) &= 0, \ t \in (\widetilde{T}/2, \widetilde{T}). \end{aligned}$$

The problem is solvable, since $v'(s, \widetilde{T}/2) > 0$, and such that

$$P(s, t) < 8\pi\nu, \text{ in } R^+ \times (\widetilde{T}/2, \widetilde{T}). \tag{2.43}$$

Using (2.42) and its limit for $\varepsilon \to 0$ and (2.43) we have that, for any $\bar{t} \in (\widetilde{T}/2, \widetilde{T})$, it is

$$v(s, \bar{t}) < 8\pi\nu, \ s > 0,$$

and taking $\bar{t}$ as a new initial time we are back in the case studied above.

To conclude the proof of the theorem we have to consider case in which (2.35) holds.

Define

$$G(t) = \int_0^\infty r^3 u(r, t) dr, \ t \in (0, T). \tag{2.44}$$

Multiply equations (2.1) and (2.2) by r^3 and integrate over R^+. After integration by parts we find

$$G_t = \frac{\widehat{M}}{2\pi}(8\pi\nu - \widehat{M}), \ t \in (0, T_0), \tag{2.45}$$

where (2.29) has been taken into account.

Since $G(t) \geq 0$ in $(0, T_0)$ it is clear that

$$T_0 \leq \frac{2\pi G(0)}{\widehat{M}(8\pi\nu - \widehat{M})}, \tag{2.46}$$

thus excluding global existence of classical solution when (2.35) holds.

To prove (2.36), assume that for a given $s_0 > 0$, there exists $\delta > 0$ such that

$$M(s_0, T_0) = \lim_{t \to T\hat{}_0 -} M(s_0, t) = 8\pi\nu - 2\delta. \tag{2.47}$$

Therefore, for some $\sigma > 0$

$$M(s_0, t) < 8\pi\nu - \delta, \ t \in (T_0 - \sigma, T_0),$$

and hence (recall $M_s \geq 0$)

$$M(s, T_0 - \sigma) \leq 8\pi\nu - \delta, \ s \in [0, s_0]. \tag{2.48}$$

But this would imply the possibility of defining $\overline{M}(s)$ as in (2.24) with sufficiently small $\beta > 0$ so that

$$M(s, T_0 - \sigma) \leq \overline{M}(s), \ s \in R^+. \tag{2.49}$$

By Lemma 2.3

$$M(s,T_0) \le \overline{M}(s) \tag{2.50}$$

and thus we find that contradicting (2.36) will imply that $(0,T_0)$ is not the maximal interval of existence of classical solution. □

Remark 2.8. *Note that (2.36) only gives a lower bound for the exact value of "blow-up mass". To find the exact value of $M(0+,T_0-)$ is still an open problem in our knowledge. Partial answer is given by Herrero and Velazquez [4], [5] where an example is provided (for a slightly different version of the Keller-Segel model) where the blow-up mass is exactly $8\pi\nu$. This shows in any case that estimate (2.36) is sharp.*

3. Global in time solutions. We state

Problem 3.1. *Given a bounded nondecreasing function $M_0(s) \in C^1([0,\infty))$, $M_0(0) = 0$, such that $\widehat{M} > 8\pi\nu$, let T_0 be the maximum interval of existence of the classical solution. We look for a bounded nondecreasing function M defined on $R^+ \times (0,\infty)$ and such that $\forall T > 0$*

$$\begin{cases} M_t = 2\nu s M_{ss} + \frac{1}{2\pi} M M_s, & s > 0,\ t \in (0,T), \\ M(s,0) = M_0(s), & s > 0, \\ M(0,t) = \begin{cases} 0, & t < T_0, \\ \ge 0, & t > T_0, \end{cases} & t \in (0,T). \end{cases} \tag{3.1}$$

$M(s,t)$ will be called global-in-time solution to our problem.

Remark 3.2. *The global solution is obtained by "glueing" a smooth classical solution and a singular (at $r = 0$) solution at the blow-up instant $t = T_0$.*

As a preliminary, we note that classical results on parabolic equations yield the following

Proposition 3.3. *For any global-in-time solution we have $M \in C^\infty(R^+ \times (0,T))$.*

Let us prove that blow-up singularity never "melts" in a sense that

Lemma 3.4. *For any global-in-time solution we have*

$$M(s,t) \ge 8\pi\nu \text{ for all } s > 0,\ t \ge T_0. \tag{3.2}$$

Proof. Using Proposition 3.3 we find

$$M_t = 2\nu s M_{ss} + \frac{1}{2\pi} M M_s,\ s > 0,\ t > T_0, \tag{3.3}$$

$$M(s,T_0) \ge 8\pi\nu,\ s > 0, \tag{3.4}$$

$$M(0,t) \ge 0,\ t > T_0. \tag{3.5}$$

Applying Lemma 2.3 we obtain that

$$M(s,t) \ge \overline{M}(s) \text{ in } R^+ \times (T_0,\infty) \tag{3.6}$$

where $\overline{M}(s)$ is any stationary solution defined by (2.24). Since the stationary solutions are uniformly and monotonically convergent to $8\pi\nu$ in any closed subset of R^+ when $\beta \to 0$, (3.6) yields the estimate (3.2). □

Next, we prove uniqueness of the global-in-time solutions.

THEOREM 3.5. *Let M_1 and M_2 be continuations of the same classical solution beyond T_0. Then $M_1 = M_2$ in $R^+ \times (T_0, T)$.*

Proof. Integrating the equation

$$2\nu(sM_s)_s = M_t - \frac{1}{4\pi}\left[M^2 - 8\pi\nu M\right]_s$$

in s and t, we find for any global-in-time solution

$$sM_s \in L^\infty(0,\infty; L^1(0,T)) \text{ for all } T > 0. \tag{3.7}$$

Define

$$N(s,t) = M_1(s,t) - M_2(s,t). \tag{3.8}$$

Because of Proposition 3.3

$$N_t = 2\nu(sN_s)_s + \frac{1}{4\pi}\left(N(M_1 + M_2 - 8\pi\nu)\right)_s \tag{3.9}$$

is satisfied in $R^+ \times (T_0, T)$.

Moreover,

$$N(s, T_0) = 0 \text{ in } R^+. \tag{3.10}$$

Multiply (3.9) by $f_\gamma\left(N(s,t)\right)$, where $f_\gamma(N)$ are smooth approximations of $sgn(N)$:

$$f_\gamma(N) = \frac{N}{\sqrt{N^2+\gamma}}, \quad \gamma > 0. \tag{3.11}$$

Then, for any $\delta > 0$, integrate on $(\delta, +\infty)$. After integration by parts the following inequality is obtained

$$\frac{d}{dt}\int_\delta^\infty \frac{sN^2}{\sqrt{N^2+\gamma}}ds \le 2\nu\left[-s^2 f_\gamma N_s\right]|_{s=\delta} + 2\nu\int_\delta^\infty f_\gamma N ds - \tag{3.12}$$

$$-\frac{1}{4\pi}\left[N f_\gamma\left(M_1+M_2-8\pi\nu\right)\right]|_{s=\delta} - \frac{1}{4\pi}\int_0^\infty N\left(M_1+M_2-8\pi\nu\right)\frac{\partial f_\gamma}{\partial s} s ds -$$

$$-\frac{1}{4\pi}\int_0^\infty N\left(M_1+M_2-8\pi\nu\right) f_\gamma ds.$$

Now, integrate in t, take the limit $\gamma \to 0$ let δ tend to zero and apply (3.7) to obtain

$$\int_0^\infty s|N(s,t)|ds \le -\frac{1}{4\pi}\int_0^\infty |N|\left(M_1+M_2-16\pi\nu\right)ds,\ t \in (T_0,T), \tag{3.13}$$

and making use of Lemma 3.3 completes the proof. □

To prove existence, we will use a monotonicity argument based on two different types of regularization.

(i) First, we regularize the Keller-Segel model as we did in [16], [17], i.e. by assuming that the coefficient of chemotactic response vanishes when concentration of cells exceeds a threshold value (maximum packing). The regularized problem takes the form

$$ru_t^\varepsilon = \left(\nu r u_r^\varepsilon - H\left(\frac{1}{\varepsilon} - u^\varepsilon\right) r u^\varepsilon p_r^\varepsilon\right)_r, \; r > 0, \; t > 0, \tag{3.14}$$

$$(rp_r^\varepsilon)_r = -ru^\varepsilon, \; r > 0, \; t > 0, \tag{3.15}$$

$$u^\varepsilon(r,0) = u_0(r), \; r > 0. \tag{3.16}$$

In (3.11) H is the Heaviside graph.

(ii) Alternatively, regularization will be based on the assumption that the coefficient of chemotactic response vanishes for $r < \varepsilon$, so that the problem takes the form

$$ru_t^\varepsilon = \left(\nu r u_r^\varepsilon - H(r - \varepsilon) r u^\varepsilon p_r^\varepsilon\right)_r, \; r > 0, \; t > 0, \tag{3.17}$$

$$(rp_r^\varepsilon)_r = -ru^\varepsilon, \; r > 0, \; t > 0, \tag{3.18}$$

$$u^\varepsilon(r,0) = u_0(r), \; r > 0. \tag{3.19}$$

In both cases (i) and (ii), by sending ε to zero we will obtain the classical solution for $T < T_0$ and its continuation beyond T_0.

We start with approach (i). Through the same steps used to obtain (2.10), we get for $\varepsilon < \varepsilon_0 = 2\pi / \sup M_0'$

$$M_t^\varepsilon = 2\nu s M_{ss}^\varepsilon + \frac{1}{2\pi} H\left(\frac{2\pi}{\varepsilon} - M_s^\varepsilon\right) M^\varepsilon M_s^\varepsilon, \; s > 0, \; t > 0, \tag{3.20}$$

$$M^\varepsilon(s,0) = M_0(s), \; s > 0. \tag{3.21}$$

Thanks to the cutting effect on the nonlinear term, existence of a unique global solution (3.17), (3.18) such that

$$M^\varepsilon(0,t) = 0, \; t > 0. \tag{3.22}$$

The following comparison result is immediately found

Lemma 3.6. *The solution of (3.20)–(3.22) is such that*

$$M^\varepsilon(s,t) \geq Q(s,t), \; s > 0, \; t > 0, \tag{3.23}$$

where Q solves (2.30), (2.31) for an arbitrary $T > 0$.

Next, we prove

Lemma 3.7. *Assume*

$$M_0'' \leq 0, \; s > 0. \tag{3.24}$$

Then

$$M_{ss}^\varepsilon \leq 0, \; a.e. \; s > 0, \; t > 0. \tag{3.25}$$

Moreover, for two different initial data M_{10}, M_{20} *satisfying (3.24) it is*

$$M_{10}(s) \geq M_{20}(s) \Longrightarrow M_1^\varepsilon(s,t) \geq M_2^\varepsilon(s,t). \tag{3.26}$$

Proof. We apply the argument of Lemma 2.4 in the domain where $M_s^\varepsilon < 2\pi/\varepsilon$.
To prove (3.26), note that when (3.25) holds then

$$H\left(\frac{2\pi}{\varepsilon} - M_s^\varepsilon\right) = H\left(\frac{2\pi s}{\varepsilon} - M^\varepsilon\right). \tag{3.27}$$

Then , we approximate H in L^2 by smooth monotonic functions H_n and we are reduced to prove (3.26) for the corresponding smooth solutions $\widetilde{M_1^\varepsilon}$, $\widetilde{M_2^\varepsilon}$. But this follows from the maximum principle applied to the difference $\widetilde{N} = \widetilde{M_1^\varepsilon} - \widetilde{M_2^\varepsilon}$, $\widetilde{N}(s,0) \geq 0$ that satisfies

$$\widetilde{N}_t = 2\nu\nu\widetilde{N}_{ss} + \frac{1}{2\pi}H_n^{(1)}M_1^\varepsilon\widetilde{N}_s + \frac{1}{2\pi}H_n^{(1)}M_{2s}^\varepsilon\widetilde{N} + \frac{1}{2\pi}H_n(v)\widetilde{N}M_2^\varepsilon M_{2s}^\varepsilon,\ s>0,\ t>0, \tag{3.28}$$

where

$$H_n^{(1)} = H_n\left(\frac{2\pi s}{\varepsilon} - M_1^\varepsilon\right) \text{ and } v \in \left(M_1^\varepsilon - \frac{2\pi s}{\varepsilon}, M_2^\varepsilon - \frac{2\pi s}{\varepsilon}\right).$$

□

Now we consider the following nonlinear wave propagation problem

$$P_t = \frac{1}{2\pi}PP_s,\ \text{ in } R^+ \times R^+, \tag{3.29}$$

$$P(s,0) = M_0(s),\ s>0. \tag{3.30}$$

Note that $P(s,t)$ is the limit of the solutions to the following family of problems for Burgers equations

$$P_{nt} = \frac{1}{n^2}P_{nss} + \frac{1}{2\pi}P_nP_{ns} \text{ in } R^+ \times R^+, \tag{3.31}$$

$$P_n(s,0) = M_0(s),\ s>0, \tag{3.32}$$

$$P_n(0,t) = 0,\ t>0. \tag{3.33}$$

We can prove the following comparison lemma

Lemma 3.8. *If (3.24) holds, then*

$$M^\varepsilon(s,t) \leq P(s,t),\ s>0,\ t>0. \tag{3.34}$$

Proof. We note that M^ε is the limit of M_n^ε solving

$$M_{nt}^\varepsilon = 2\nu\left(s + \frac{1}{n^2}\right)M_{nss}^\varepsilon + \frac{1}{2\pi}H\left(\frac{2\pi}{\varepsilon} - M_{ns}^\varepsilon\right)M_n^\varepsilon M_{ns}^\varepsilon,\ \text{ in } R^+ \times R^+, \tag{3.35}$$

$$M_n^\varepsilon(s,0) = M_0(s),\ s>0, \tag{3.36}$$

$$M_n^\varepsilon(0,t) = 0,\ t>0. \tag{3.37}$$

Recalling (3.21), we note that $M_n^\varepsilon \leq P_n$ by maximum principle and hence (3.24) is obtained letting $n \to \infty$. □

We also have

$$(sM_s^\varepsilon)_s = M_s^\varepsilon + \frac{1}{2\nu}\left(M_t^\varepsilon - \frac{1}{2\pi} H M^\varepsilon M_s^\varepsilon\right),$$

and hence, for any $T > 0$

$$\int_0^T sM_s^\varepsilon dt \le \widehat{M}\left(1 + \frac{1}{4\pi\nu}\right)T + \frac{s\widehat{M}}{2\nu} \tag{3.38}$$

and condition (i) in Problem 3.1 is fulfilled uniformly with respect to $\varepsilon > 0$.

At this point we are in position of proving

THEOREM 3.9. *Assume (3.24) holds. Then the limit of the solutions of (3.20)–(3.22) as ε tends to zero coincides with the classical solution in $(0, T_0)$ and provides its unique continuation beyond T_0.*

Proof. Because of (3.27) H is nonincreasing both with respect to ε and to M so that

$$\varepsilon_1 < \varepsilon_2 \Longrightarrow M^{\varepsilon_1} \le M^{\varepsilon_2} \text{ in } R^+ \times R^+. \tag{3.39}$$

¿¿From (3.39) we have that a function $\mu(x,t)$ exists such that

$$\mu(s,t) = \lim_{\varepsilon \to 0} M^\varepsilon(s,t),\ s > 0,\ t > 0. \tag{3.40}$$

It is easy to check that μ fulfills conditions in the statement of Problem 3.1 and is smooth in $R^+ \times R^+$. Next, compare M^ε with the (classical) solution $M(s,t)$ of Problem 2.1 in its maximal interval of existence. Noting that $H \le 1$, we have $M^\varepsilon(s,t) \le M(s,t)$, $s > 0$, $t \in (0, T_0)$ and passing to the limit

$$\mu(s,t) \le M(s,t),\ s > 0,\ t \in (0, T_0). \tag{3.41}$$

But this means that $\mu(0,t) = 0$ in $(0, T_0)$ and hence (μ) solves Problem 2.1 and its maximal interval of existence contains $(0, t_0)$, since classical solution is unique

$$\mu(s,t) = M(s,t),\ s > 0,\ t \in (0, T_0). \tag{3.42}$$

The use of the uniqueness beyond blow-up (Theorem 3.5) completes the proof. □

Now we consider the regularization (ii) with the aim of avoiding assumption (3.21). In this case we denote by M^ε the solution of

$$M_t^\varepsilon = 2\nu s M_{ss}^\varepsilon + \frac{1}{2\pi} H\left(s - \frac{\varepsilon^2}{2}\right) M^\varepsilon M_s^\varepsilon,\ s > 0,\ t > 0, \tag{3.43}$$

$$M^\varepsilon(s,0) = M_0(s),\ s > 0, \tag{3.44}$$

$$M^\varepsilon(0,t) = 0,\ t > 0. \tag{3.45}$$

We proceed as in proving Lemmas 3.5 and 3.7 and obtain the uniform estimates

LEMMA 3.10. *The approximating solutions are bounded from below by the diffusion-dominated problem and from above by the nonlinear wave propagation problem:*

$$Q(s,t) \le M^\varepsilon(s,t) \le P(s,t),\ s > 0,\ t > 0. \tag{3.46}$$

Moreover M^ε depends monotonically on the initial datum.

Proof. No major changes are needed in proofs of Lemmas 3.5 and 3.7. Remark that in the present case we do need assumption (3.24) since "cutting" in (3.44) does not depend on M^ε. □

THEOREM 3.11. *The limit of the solutions to the approximating problems (3.43)–(3.45) is the global-in-time solution to the problem 3.1.*

Proof. The proof is a slight modification of that of Theorem 3.8 and is based on the use of the monotonic dependence of the solutions on $\varepsilon > 0$. □

Finally, let us generalize the statements of Lemma 2.4 to the global-in-time solutions.

LEMMA 3.12. *Let M be the global-in-time solution to the Problem 3.1. Then the following implications hold:*

$$M_0''(s) \leq 0 \quad \textit{in } R^+ \Longrightarrow M_{ss}(s,t) \leq 0 \textit{ in } R^+ \times (0,\infty), \tag{3.47}$$

$$4\pi\nu s M_0''(s) + M_0(s)M_0'(s) \geq 0 \textit{ in } R^+ \Longrightarrow M_t(s,t) \geq 0 \textit{ in } R^+ \times (0,\infty). \tag{3.48}$$

Proof. Similarly to the proof of Lemma 2.4, we obtain the implications (3.47), (3.48) for the approximating problem (3.43)–(3.45). Sending the regularization parameter to 0 we complete the proof. □

4. Fine structure of the solution after its blow-up. We begin with a consideration of the long-time asymptotic behaviour of the global-in-time solution. First, let us consider the case of the subcritical initial datum:

LEMMA 4.1. *If*

$$\widehat{M} < 8\pi\nu, \tag{4.1}$$

then

$$\lim_{t\to\infty} M(s,t) = 0 \textit{ for all } s \geq 0. \tag{4.2}$$

Proof. We define an upper barrier as a solution $M_+(s,t)$ to the Problem 2.1 with the following initial datum

$$M_+(s,0) = \begin{cases} 8\pi\nu\frac{s}{s+\beta_1}, & s \leq s_0 = \frac{\widehat{M}\beta}{8\pi\nu - \beta_1\widehat{M}}; \\ \widehat{M}, & s > s_0. \end{cases} \tag{4.3}$$

Here, $\beta_1 > 0$ is such that

$$M_0(s) \leq M_+(s,0) \text{ for all } s \geq 0. \tag{4.4}$$

Applying Lemma 2.4 we find

$$M_{+t}(s,t) \leq 0 \text{ for all } s \geq 0 \text{ and } t > 0, \tag{4.5}$$

and, thus, the function M_+ converges in time to the respective stationary solution of the Problem 2.1. Then Proposition 2.5 together with condition $M_+(0) = 0$ yields

$$\lim_{t\to\infty} M_+(s,t) = 0 \text{ for all } s \geq 0. \tag{4.6}$$

Using the inequality (4.4) and Lemma 2.3 we obtain that $M(s,t) \leq M_+(s,t)$ and complete the proof of the statement (4.2). □

Next, we consider the regularized problem (3.17)–(3.19) with total mass $\widehat{M} > 8\pi\nu$.

PROPOSITION 4.2. *If*

$$\widehat{M} > 8\pi\nu, \tag{4.7}$$

then

$$\lim_{t\to\infty} M^\varepsilon(s,t) = M^\varepsilon_\infty(s), \tag{4.8}$$

where M^ε is a solution to the regularized problem (3.17)–(3.19) and $M^\varepsilon_\infty(s)$ is the stationary solution of the problem (3.17)–(3.19)i.e.:

$$\begin{cases} M^\varepsilon_\infty(s) = \sqrt{\widehat{M}^2 - 8\pi\nu\widehat{M}}\left(\frac{2s}{\varepsilon^2}\right),\ 0 \le s \le \frac{\varepsilon^2}{2} \\ \frac{\widehat{M}-M^\varepsilon_\infty(s)}{\widehat{M}+M^\varepsilon_\infty(s)-8\pi\nu} = \frac{\widehat{M}-\sqrt{\widehat{M}^2-8\pi\nu\widehat{M}}}{\widehat{M}+\sqrt{\widehat{M}^2-8\pi\nu\widehat{M}}-8\pi\nu}\left(\frac{2s}{\varepsilon^2}\right)^{1-\widehat{M}/(4\pi\nu)},\ \frac{\varepsilon^2}{2} \le s \end{cases} \tag{4.9}$$

Proof. We start the proof with one more comparison of the solution M^ε with a lower-barrier solution $M_-(s,t)$ to the problem (3.17)–(3.19). We define the initial-value of the lower barrier solution as the following stationary subsolution to the (3.17)–(3.19) corresponding to the total mass $\widehat{M}_- = \widehat{M} - \delta$:

$$\begin{cases} M_-(s,0) = \sqrt{\widehat{M}_- - 8\pi\nu\widehat{M}_- s},\ s \le s_0 \\ \frac{\widehat{M}_- - M_-(s,0)}{\widehat{M}_- + M_-(s,0)-8\pi\nu} = \frac{\widehat{M}_- -\sqrt{\widehat{M}_-^2-8\pi\nu\widehat{M}_-}}{\widehat{M}_- +\sqrt{\widehat{M}_-^2-8\pi\nu\widehat{M}_-}-8\pi\nu}\left(\frac{s}{s_0}\right)^{1-\widehat{M}_-/(4\pi\nu)},\ s_0 \le s \end{cases} \tag{4.10}$$

Since

$$2\nu s M_{-ss} + \frac{1}{2\pi}H\left(s - \frac{\varepsilon^2}{2}\right)M_- M_{-s} \ge 0$$

for any $s_0 > \varepsilon^2/2$, applying Lemma 3.12 we obtain that $\frac{\partial}{\partial t}M_- \ge 0$ and, hence, $M_-(s,t)$ converges in time to the following stationary solution $M^\infty_-(s)$

$$\begin{cases} M^\infty_-(s) = \sqrt{\widehat{M}_-^2 - 8\pi\nu\widehat{M}_-}\left(\frac{2s}{\varepsilon^2}\right),\ 0 \le s \le \frac{\varepsilon^2}{2} \\ \frac{\widehat{M}_- - M^\infty}{\widehat{M}_- + M^\infty -8\pi\nu} = \frac{\widehat{M}_- -\sqrt{\widehat{M}_-^2-8\pi\nu\widehat{M}_-}}{\widehat{M}_- +\sqrt{\widehat{M}_-^2-8\pi\nu\widehat{M}_-}-8\pi\nu}\left(\frac{2s}{\varepsilon^2}\right)^{1-\widehat{M}_-/(4\pi\nu)},\ \frac{\varepsilon^2}{2} \le s \end{cases}$$

Since $M^\varepsilon_s(s,t) > 0$ for any $t > 0$, for any $\delta > 0$ we find the value of $s_0 > 0$ such that $M_\varepsilon(s,1) \ge M_-(s,0)$ and comparing the functions $M_\varepsilon(s,t+1)$ and $M_-(s,t)$ we deduce that

$$\underline{\lim}_{t\to\infty} M^\varepsilon(s,t) \ge M^\infty_-(s),\ 0 \le s. \tag{4.11}$$

Next we apply the comparison of $M^\varepsilon(s,t)$ with an upper barrier $M^+(s,t)$.We define the initial value $M^+(s,0)$ as the following stationary supersolution to the (3.17)–(3.19) corresponding to the total mass $\widehat{M}_+ = \widehat{M} + \delta$:

$$\begin{cases} M_+(s,0) = \sqrt{\widehat{M}_+ - 8\pi\nu\widehat{M}_+ s},\ s \le s_1 \\ \frac{\widehat{M}_- - M_+(s,0)}{\widehat{M}_- + M_+(s,0)-8\pi\nu} = \frac{\widehat{M}_+ -\sqrt{\widehat{M}_+^2-8\pi\nu\widehat{M}_+}}{\widehat{M}_+ +\sqrt{\widehat{M}_+^2-8\pi\nu\widehat{M}_+}-8\pi\nu}\left(\frac{s}{s\,1}\right)^{1-\widehat{M}_+/(4\pi\nu)},\ s_1 \le s \end{cases} \tag{4.12}$$

Since

$$2\nu s M_{+ss} + \frac{1}{2\pi} H\left(s - \frac{\varepsilon^2}{2}\right) M_+ M_{+s} \le 0$$

for any $s_0 < \varepsilon^2/2$, applying Lemma we obtain that $\frac{\partial}{\partial t} M_+ \le 0$ and, hence, $M_+(s,t)$ converges in time to the following stationary solution $M_+^\infty(s)$

$$M_+^\infty(s) = \sqrt{\widehat{M}_+^2 - 8\pi\nu \widehat{M}_+} \left(\frac{2s}{\varepsilon^2}\right), \ 0 \le s \le \frac{\varepsilon^2}{2};$$

$$\frac{\widehat{M}_+ - M^\infty}{\widehat{M}_+ + M^\infty - 8\pi\nu} = \frac{\widehat{M}_+ - \sqrt{\widehat{M}_+^2 - 8\pi\nu \widehat{M}_+}}{\widehat{M}_+ + \sqrt{\widehat{M}_+^2 - 8\pi\nu \widehat{M}_+} - 8\pi\nu} \left(\frac{2s}{\varepsilon^2}\right)^{1-\widehat{M}_+/(4\pi\nu)}, \ \frac{\varepsilon^2}{2} \le s.$$

For any $\delta > 0$, we find sufficiently small $s_1 > 0$ such that $M_\varepsilon(s,0) \le M_+(s,0)$ and, thus comparing solutions $M_\varepsilon(s,t)$ and $M_+(s,t)$ we obtain that $M_\varepsilon(s,t) \le M_+(s,t)$ and thus

$$\overline{\lim_{t\to\infty}} M_\varepsilon(s,t) \le M_+^\infty(s), \ 0 \le s. \tag{4.13}$$

Taking the limit $\delta \to 0$ in the estimates (**??**), (4.13) completes the proof. □

LEMMA 4.3. *If*

$$\widehat{M} > 8\pi\nu,$$

then

$$\lim_{t\to\infty} M(s,t) = \widehat{M}, \quad \forall s \ge 0. \tag{4.14}$$

Proof. The proof is based on the comparison

$$M(s,t) \ge M^\varepsilon(s,t), \ s \ge 0, \ t \ge 0, \ \varepsilon > 0. \tag{4.15}$$

Taking the limits $t \to \infty$, $\varepsilon \to 0$ in the estimate (4.15) we obtain that

$$\underline{\lim}_{t\to\infty} M(s,t) \ge \widehat{M}. \tag{4.16}$$

Using the estimate $M(s,t) \le \widehat{M}$ for all $s > 0$, $t > 0$, we complete the proof. □

Next, we prove the main result of this Section

THEOREM 4.4. *The mass $M(0+,t)$ concentrated in the origin changes in the interval $\left[8\pi\nu, \widehat{M}\right)$ for all times after the blow-up instant and in the infinite time all of the mass concentrates in the origin:*

$$\lim_{t\to\infty} M(0+,t) = \widehat{M}. \tag{4.17}$$

Proof. The estimate

$$M(0+,t) \ge 8\pi\nu, \ t \ge T_0, \tag{4.18}$$

follows immediately from Lemma 3.4. Applying the maximum principle to the equation (2.1) we obtain that $M_s > 0$ for all $s > 0$, $t > 0$ and, hence,

$$M(0+,t) < \widehat{M}, \ t \ge 0. \tag{4.19}$$

To prove the convergence (4.17) we use one more time the comparison of the global-in-time solution $M(s,t)$ with an appropriate lower-barrier solution $M_-(s,t)$. We define $M_-(s,t)$ as the global-in-time solution to the problem (3.1) initially equal to the stationary solution of the same problem corresponding to higher diffusivity and lower total mass:

$$M_-(s,0) = \left(\widehat{M} - \delta\right) \frac{s}{s+\beta}, \tag{4.20}$$

where $0 < \delta < \widehat{M} - 8\pi\nu$. The initial value $M_-(s,0)$ is a stationary subsolution of the problem (3.1):

$$2\nu s M_{-ss} + \frac{1}{2\pi} M_- M_{-s} \geq 0, \;\; s > 0, \tag{4.21}$$

and, hence,

$$\frac{\partial}{\partial t} M_-(s,t) \geq 0, \;\; s > 0, \;\; t > 0. \tag{4.22}$$

Integrating the equation (3.1) in s we find

$$\frac{d}{dt}\int_0^1 M_-(s,t)ds = 2\nu s M_{-s}(s,t)|_{s=0+}^{s=1} + \frac{1}{4\pi}\left[M_-^2 - 8\pi\nu M_-\right]|_{s=0+}^{s=1}. \tag{4.23}$$

Using the estimate (4.22) we find that

$$\frac{\partial}{\partial s}\left(2\nu s M_{-s} + \frac{1}{4\pi}\left[M^2 - 8\pi\nu M\right]\right) \geq 0, \;\; s > 0, \;\; t > 0. \tag{4.24}$$

The last inequality together with the monotonicity of M in s yields the existence of the limit

$$A = \lim_{s\to 0+} s M_{-s}(s,t). \tag{4.25}$$

Since

$$\frac{1}{\varepsilon}\int_0^\varepsilon s M_{-s}(s,t)ds = M_-(\varepsilon,t) - \frac{1}{\varepsilon}\int_0^\varepsilon M_-(s,t)ds \underset{\varepsilon\to 0}{\to} 0,$$

then $A = 0$ and using the equation (4.23) we obtain

$$\frac{d}{dt}\int_0^1 M_-(s,t)ds > \frac{1}{4\pi}\left[M_-^2 - 8\pi\nu M_-\right]|_{s=0+}^{s=1}. \tag{4.26}$$

Applying the monotonicity of the function $M_-(s,t)$ in t we find that

$$\underline{\lim}_{t\to\infty}\frac{d}{dt}\int_0^1 M_-(s,t)ds \geq \frac{1}{4\pi}\lim_{t\to\infty}\left[M_-^2 - 8\pi\nu M_-\right]|_{s=0+}^{s=1} \geq 0,$$

and using Lemma 4. 3,we obtain that

$$\underline{\lim}_{t\to\infty}\frac{d}{dt}\int_0^1 M_-(s,t)ds \geq \frac{\widehat{M} - \delta - 8\pi\nu}{4\pi}\left(\widehat{M} - \delta - \lim_{t\to\infty} M_-(0+,t)\right) \geq 0. \tag{4.27}$$

Combining the last inequality with the estimate $\int_0^s M_-(s,t)ds \le \widehat{M} - \delta$ we deduce that

$$\lim_{t\to\infty} M_-(0+,t) = \widehat{M} - \delta. \tag{4.28}$$

For any $\delta > 0$, we find sufficiently large $\beta > 0$, such that

$$M(s,1) \ge \left(\widehat{M} - \delta\right) \frac{s}{s+\beta} = M_-(s,0),$$

and, hence,

$$M(s,t+1) \ge M_-(s,t), \; s > 0, \; t > 0. \tag{4.29}$$

Taking the limit $t \to \infty$ in the inequality (4.29), we obtain that

$$\underline{\lim} M(0+,t) \ge \widehat{M} - \delta, \tag{4.30}$$

for any $\delta > 0$. Finally, sending δ to 0 in the inequality (4.30) we complete the proof. □

REFERENCES

[1] W. Jager, S. Luckhaus, *On explosions of solutions to a system of partial differential equations modelling chemotaxis*, Trans. Amer. Math. Soc. 329, 2 (1992) 819-824
[2] T. Nagai, *Blow-up of radially symmetric solutions to a chemotaxis system,* Adv. Math. Sci. Appl. (1995) 1-21
[3] M.A. Herrero, M.A., J.J.L. Velazquez, *Chemotactic collapse for the Keller-Segel model,* J. Math. Biol. 35 (1996) 177-196
[4] M.A. Herrero, M.A., J.J.L. Velazquez, *Singularity patterns in a chemotaxis model,* Math. Ann. 306 (1996) 583-623
[5] M.A. Herrero, M.A., J.J.L. Velazquez, *A blow-up mechanism for a chemotaxis model,* Ann. Scuola Normale Sup. Pisa, IV, XXIV (1997) 633-683
[6] T. Senba, *Blow-up of radially symmetric solutions to some systems of partial differential equations modelling chemotaxis*, Adv. Math. Sci. Appl. 7 (1997) 79-92
[7] M.A. Herrero, M.A., E. Medina, J.J.L. Velazquez, *Self-similar blow-up for a reaction-diffusion system*, J. Comp. Appl. Math. 97, (1998) 99-119
[8] T. Nagai, T. Senba, *Global existence and blow-up of radial solutions to a parabolic-elliptic system of chemotaxis,* Adv. Math. Sci. Appl. 8 (1998) 145-156
[9] M.A. Herrero, *Asymptotic properties of reaction-diffusion systems modeling chemotaxis,* In "Applied and Industrial Mathematics, Venice 2" R. Spigler ed. Kluwer (2000) 89-107
[10] D. Horstmann, G. Wang, *Blow-up in a chemotaxis model without symmetry assumptions,* Euro. Jnl. of Applied Mathematics 12, (2001), pp. 159-177.
[11] G. Wolanski, *Comparison between two models of self-gravitating cluters: conditions for gravitational collapse,* Nonlinear Anal. Theory Methods Appl. 24 (1995) 1119-1129
[12] E.F. Keller, L.A. Segel, *Models for chemotaxis.* J, Theor. Biol. 30 (1971) 225-234
[13] T. Senba, T. Suzuki, *Parabolic system of chemotaxis: blow-up in a finite and the infinite time*, Methods Appl. Anal. 8 (2001), pp. 349–368.
[14] T. Senba, T. Suzuki, *Weak solutions to a parabolic system of chemotaxis,* J. Func. Anal. 191 (2002), pp. 17–51.
[15] O. A. Ladyzhenskaya, V. A. Solonnikov, N. N. Uraltseva, *Linear and quasilinear equations of parabolic type.* Nauka, Moscow (1967). (English ed.: Transl. Math. Monographs, Vol. 23, AMS, Providence (1968).)
[16] M. Primicerio, B. Zaltzman, *Free boundary in a radial symmetric chemotaxis*, in "Proceedings of WASCOM 2001" (R.Monaco et al. eds.), World Scientific (2002), pp.416-427
[17] M. Primicerio, B. Zaltzman, *A free boundary problem arising in chemotaxis*, Adv. Math. Sci. Appl, 12 (2002), pp. 685-708.